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Products of Distributions: Nonstandard Methods

M. Oberguggenberger

Es werden Hilfsmittel der Nonstandard-Analysis entwickelt, die zur Untersuchung der Multiplikation von Distributionen geeignet sind. Wir befassen uns mit Produkten, die durch Regularisierung und Grenzwertbildung definiert sind, und erhalten Nonstandard-Kriterien für die Existenz der Produkte (diese Kriterien erweisen sich als vorteilhaft bei der Berechnung konkreter Beispiele). Ferner leiten wir neue Vergleichsresultate für verschiedene Produktdefinitionen her. Als weitere Anwendung konstruieren wir Algebren von Distributionen — als Quotienten von externen Räumen glatter Funktionen —, die ähnliche Eigenschaften wie die Colombeauschen Algebren besitzen.

Развиваются вспомогательные средства нестандартного анализа пригодные к исследованию произведения обобщённых функций. Мы занимаемся произведениями определенными регуляризацией и образованием предела и получаем нестандартные критерии существования произведений (эти критерии оказываются выгодными при вычислении конкретных примеров). Выводятся также новые результаты сравнения для разных определений произведения. Как дальнейшее применение мы построим алгебры обобщенных функций — как фактор-алгебра внешних пространств гладких функций — имеющие сходные свойства с алгебрами Colombeau.

Nonstandard tools are developed which are suitable for studying products of distributions defined by regularization and passage to the limit. We obtain nonstandard criteria for the existence of the products (which are demonstrated to be useful for calculating standard examples) as well as new standard results clarifying the relationship between different types of such products. As an offspring we are able to construct algebras of distributions — as quotients of external spaces of smooth functions — which have properties similar to the Colombeau algebras.

1. Introduction

Let S and T be distributions on $\mathbb{R}^n$. One way to multiply S and T is to define their product as

$$\text{(P1)} \qquad \lim_{\varepsilon \to 0} (S * \varphi^\varepsilon)(T * \psi^\varepsilon)$$

provided the limit exists in $\mathcal{D}'(\mathbb{R}^n)$ for all nets $\{\varphi^\varepsilon\}_{\varepsilon>0}$, $\{\psi^\varepsilon\}_{\varepsilon>0}$ which vary in certain classes of nets of smooth functions and converge to the Dirac measure (called delta-nets). A strictly more general way is to take

$$\text{(P2)} \qquad \lim_{\varepsilon \to 0} (S * \varphi^\varepsilon)(T * \varphi^\varepsilon)$$

as the definition, and this is the product we shall be concerned here. Definition (P2) has become important recently because of its relation to Colombeau algebras: If the product of S and T in the sense of (P2) exists, then the element ST in the Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$ admits an associated distribution [2, Thm. 3.5.7].

We consider four successively smaller classes of delta-nets, leading to four successively more general definitions of the product (details in Section 3):

(1) delta-nets in the sense of HIRATA-OGATA [6] and MIKUSIŃSKI [11];

(2) restricted delta-nets in the sense of SHIRAISHI [18, p. 91] and ITANO [8];

(3) delta-nets in the sense of ANTOSIK-MIKUSIŃSKI-SIKORSKI [1, p. 116] and KAMIŃSKI [9, p. 85];

(4) model delta-nets in the sense of KAMIŃSKI [9, p. 89].

It is well known that for definition (P1), equivalent products are obtained by using any of the classes (2)—(4) (see ITANO [7], SHIRAISHI [18], KAMIŃSKI [9]), while class (1) produces a more stringent definition [13, Appx]. It is also known (see ITANO [7, p. 177]) that in dimension $n = 1$, the existence of the product (P1) of S and T with delta-nets of any of the types (1)—(4) implies the existence of the Tillmann product [20, p. 108] of S and T, which is defined by analytic regularization. What concerns product (P2), no comparable results have been available so far. We show here that the existence of product (P2) with delta-nets of type (2) implies the existence of the Tillmann product. We give a nonstandard criterion for the existence of the product (P2) with delta-nets of type (3), which enables one to conclude in many concrete examples that if product (P2) exists with delta-nets (4), then it also exists with delta-nets (3). The question of equivalence of the products (P2) obtained by employing classes (2)—(4) remains open; however, type (1) is seen to yield a less general product.

Colombeau has constructed (standard) commutative, associative differential algebras of generalized functions on open subsets Ω of $\mathbb{R}^n$ with the following properties:

(a) $\mathcal{D}'(\Omega)$ is a subspace;

(b) the derivation in the algebra extends differentiation in the sense of distributions;

(c) $\mathcal{E}^\infty(\Omega)$ is a differential subalgebra (with respect to the pointwise product on $\mathcal{E}^\infty(\Omega)$);

(d) the algebras are invariant under superposition by smooth maps of polynomial growth.

There are several possibilities to achieve such a construction [2—4]. For algebras with properties (a), (b), property (c) is optimal (for instance, the continuous functions cannot constitute a subalgebra [16]). Turning nonstandard, we observe that $\mathcal{E}^\infty(\Omega)$, viewed as an internal set, is an algebra into which the standard distributions may be imbedded. But this imbedding does not render the standard smooth functions a subalgebra. We show that a quotient of a certain external subalgebra of $\mathcal{E}^\infty(\mathbb{R}^n)$ does better: It contains the standard tempered distributions and has the standard smooth functions of polynomial growth as a subalgebra with respect to their pointwise product; it satisfies (b) and the standardized version of (d). A different nonstandard construction of an algebra with properties (a)—(d) has recently been given by TODOROV [22] using ultrapower methods.

We employ Nelson's version of nonstandard analysis: internal set theory [12]. In addition, we freely work with external sets, when appropriate. The plan of exposition is as follows:

Section 2 provides the necessary background on the nonstandard theory of distributions. We follow the ideas of STROYAN-LUXEMBURG [19, Chap. 104], translated into internal set theory, but develop some additional material (including structure theorems) which is frequently needed in the sequel. We found it useful to collect these results together with short proofs since they are not available in the literature in this form. Section 3 starts with a nonstandard definition of the product of any two standard distributions as an internal smooth function. This construction serves as a tool, and also relates our approach to LI BANG-HE's [10], RAJU's [15] and TODOROV's [21]. Then the nonstandard characterizations of product (P2) are given, and the comparison results are derived. Section 4 is devoted to the construction of the differential algebras containing the standard tempered distributions.

There are two appendices: In the first one we put the results of Section 3 to use, completing the investigation of the example in [13, Appx]. The notions we need from internal set theory — some of which go beyond Nelson's excellent introduction [12] — are collected in the second appendix.

2. Background on the nonstandard theory of distributions

For the following basic vocabulary the reader is referred to Nelson's article [12]: standard; internal, external formula; internal, external set; infinitesimal (real, complex) number; limited (real, complex) number; infinitely large (natural, real, complex) number; standard part (of a limited number). Let $a, b \in \mathbb{C}$. We write $a \sim b$ if $a - b$ is infinitesimal; $a \sim \infty$ if a is infinitely large; for limited $a \in \mathbb{C}$, ${}^0a$ denotes the standard part of a. We use the quantifiers

$$\exists^{\mathrm{st}} x \quad \text{for} \quad \exists x\,(x \text{ standard}) \text{ and } \forall^{\mathrm{st}} x \quad \text{for} \quad \forall x\,(x \text{ standard}).$$

By abuse of notation, we shall employ the set brackets { } and the elementhood "$\in$" for internal and external sets alike, stating only verbally when a set is to be considered as external. Finally, if X is an internal set, we define the external set

$${}^{\mathrm{st}}X = \{x \in X : x \text{ is standard}\}.$$

In what follows, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}^\infty$, $\mathcal{D}'$, $\mathcal{S}'$ are the usual spaces of functions and distributions on $\mathbb{R}^n$ (n a fixed standard natural number); for these and all other internal spaces of distributions we use the notation of SCHWARTZ [17]; $\mathcal{D}_k = \{\varphi \in \mathcal{D} : \varphi(x) = 0$ for $|x| \geq k\}$ for $k \in \mathbb{N}$. Following STROYAN-LUXEMBURG [19, Chap. 10.4] we introduce several external subsets (actually vector spaces over ${}^{\mathrm{st}}\mathbb{C}$) of the internal set $\mathcal{E}^\infty$ of smooth functions on $\mathbb{R}^n$.

Definition 2.1: (a) A function $\psi \in \mathcal{D}$ is called **D**-*limited* if $\psi \in \mathcal{D}_k$ for some standard $k \in \mathbb{N}$ and $\sup \{|\partial^\alpha \psi(x)| : x \in \mathbb{R}^n\}$ is limited for all standard $\alpha \in \mathbb{N}_0^n$. We define the external set $\mathbf{D} = \{\psi \in \mathcal{D} : \psi \text{ is } \mathbf{D}\text{-limited}\}$.

(b) $\psi \in \mathcal{D}$ is called **D**-*infinitesimal*, denoted as $\psi \approx_{\mathrm{D}} 0$, if $\psi \in \mathcal{D}_k$ for some standard $k \in \mathbb{N}$ and $\sup \{|\partial^\alpha \psi(x)| : x \in \mathbb{R}^n\} \sim 0$ for all standard $\alpha \in \mathbb{N}_0^n$.

(c) A function $\psi \in \mathcal{S}$ is called S-*limited* if $\sup \{(1 + |x|)^l\, |\partial^\alpha \psi(x)| : x \in \mathbb{R}^n\}$ is limited for all standard $l \in \mathbb{N}$ and all standard $\alpha \in \mathbb{N}_0^n$. We define the external set $\mathbf{S} = \{\psi \in \mathcal{S} : \psi \text{ is S-limited}\}$.

(d) $\psi \in \mathcal{S}$ is called S-*infinitesimal*, denoted as $\psi \approx_{\mathrm{S}} 0$, if $\sup \{(1 + |x|)^l\, |\partial^\alpha \psi(x)| : x \in \mathbb{R}^n\} \sim 0$ for all standard $l \in \mathbb{N}$ and all standard $\alpha \in \mathbb{N}_0^n$.

(e) An element $T \in \mathcal{E}^\infty$ is called a *limited distribution*, if $\int T(x)\,\psi(x)\,dx$ is limited for all $\psi \in \mathbf{D}$; T is called an *infinitesimal distribution*, if $\int T(x)\,\psi(x)\,dx \sim 0$ for all $\psi \in \mathbf{D}$. We define the external sets $\mathbf{D}' = \{T \in \mathcal{E}^\infty : T \text{ is a limited distribution}\}$ and $\mathbf{d}' = \{T \in \mathcal{E}^\infty : T \text{ is an infinitesimal distribution}\}$.

(f) An element $T \in \mathcal{E}^\infty$ is called a *limited tempered distribution* if $T\psi \in L^1(\mathbb{R}^n)$ and $\int T(x)\,\psi(x)\,dx$ is limited for all $\psi \in \mathbf{S}$; $\mathbf{S}' = \{T \in \mathcal{E}^\infty : T \text{ is a limited tempered distribution}\}$.

Notation: Given $\psi \in \mathbf{D}$, $T \in \mathbf{D}'$, we shall write $\langle T, \psi\rangle$ for $\int T(x)\,\psi(x)\,dx$.

Remark 2.2: It is clear that ${}^{\mathrm{st}}\mathscr{D} \subset \mathbf{D}$ and ${}^{\mathrm{st}}\mathscr{S} \subset \mathbf{S}$. Also, if $\psi \approx_{\mathrm{D}} 0$ (respectively $\psi \approx_{\mathrm{S}} 0$), then ψ is contained in every standard neighborhood of zero in $\mathscr{D}$ (respectively $\mathscr{S}$). In case of $\mathscr{D}$, the converse is not true. Indeed, using the characterization of the neighborhoods of zero of SCHWARTZ [17, p. 65] and the idealization axiom [12, p. 1166], one sees easily that for every infinitely large $\omega \in \mathbb{N}$ there is an element $\varphi \in \mathscr{D}$ with support $(\varphi) \subset \{x \in \mathbb{R}^n : \omega \leqq |x| \leqq \omega + 1\}$ which is contained in every standard neighborhood of zero in $\mathscr{D}$. On the other hand, if $\psi \in \mathscr{D}_k$ for some standard $k \in \mathbb{N}$ and is contained in every standard neighborhood of zero in $\mathscr{D}_k$, then $\psi \approx_{\mathrm{D}} 0$.

The next proposition characterizes the limited distributions as the "continuous" elements with respect to the infinitesimality relation introduced above.

Proposition 2.3: *Let $T \in \mathscr{E}^\infty$. Then the following are equivalent:*

(a) $T \in \mathbf{D}'$;

(b) *If $\psi \in \mathscr{D}$ and $\psi \approx_{\mathrm{D}} 0$, then $\langle T, \psi\rangle \sim 0$.*

Proof: (a) $\Rightarrow$ (b): If $\psi \in \mathscr{D}$ and $\psi \approx_{\mathrm{D}} 0$ then $w\psi \approx_{\mathrm{D}} 0$ for all standard $w \in \mathbb{N}$ and so there is an infinitely large $\omega \in \mathbb{N}$ such that $\omega\psi \approx_{\mathrm{D}} 0$ by Robinson's lemma (cf. Appx 2). In particular, $\omega\psi \in \mathbf{D}$, and so $|\langle T, \omega\psi\rangle| < \varepsilon\omega$ for all standard $\varepsilon > 0$, since $\varepsilon\omega$ is infinitely large. Thus $|\langle T, \psi\rangle| \sim 0$.

(b) $\Rightarrow$ (a): Let $\psi \in \mathbf{D}$. Then $\dfrac{1}{\varkappa}\psi \approx_{\mathrm{D}} 0$ for all infinitely large $\varkappa \in \mathbb{N}$, thus $\left|\left\langle T, \dfrac{1}{\varkappa}\psi\right\rangle\right| \leqq 1$ for all such $\varkappa$. By the permanence principle (cf. Appx 2) there is a standard $k \in \mathbb{N}$ such that $\left|\left\langle T, \dfrac{1}{k}\psi\right\rangle\right| \leqq 1$, that is, $T \in \mathbf{D}'$ ∎

Our next goal is to identify the standard distributions as elements of $\mathbf{D}'$. To this end we fix a standard "mollifier" $\theta \in {}^{\mathrm{st}}\mathscr{D}$ with $\int \theta(x)\,dx = 1$ and an infinitesimal real number $\varrho \sim 0$. We set

$$\theta_\varrho(x) = \varrho^{-n}\theta\left(\frac{x}{\varrho}\right) \tag{2.1}$$

and want to show that the map $T \to T * \theta_\varrho$ is an imbedding of ${}^{\mathrm{st}}\mathscr{D}'$ into $\mathbf{D}'$.

Lemma 2.4: *Let $\psi \in \mathbf{D}$ and $\theta \in {}^{\mathrm{st}}\mathscr{D}$, $\varrho \sim 0$ as above. Then $\psi * \theta_\varrho \in \mathbf{D}$ and $\psi * \theta_\varrho - \psi \approx_{\mathrm{D}} 0$.*

Proof: It is clear that $\psi * \theta_\varrho$ belongs to some $\mathscr{D}_k$ with k standard. Let $\alpha \in {}^{\mathrm{st}}\mathbb{N}_0{}^n$. Then

$$\partial^\alpha(\psi * \theta_\varrho)(x) = \int \theta(y)\,\partial^\alpha\psi(x - \varrho y)\,dy;$$

and this integral is limited independently of x, since $\sup\{|\partial^\alpha\psi(z)| : z \in \mathbb{R}^n\}$ is limited. Thus $\psi * \theta_\varrho \in \mathbf{D}$. Next,

$$\begin{aligned}|\partial^\alpha(\psi * \theta_\varrho - \psi)(x)| &= \left|\int \theta(y)\,(\partial^\alpha\psi(x - \varrho y) - \partial^\alpha\psi(x)\,dy\right| \\ &\leqq |\varrho| \int |y\theta(y)|\,dy \cdot \sup\{|\text{gradient } \partial^\alpha\psi(z)| : z \in \mathbb{R}^n\}\end{aligned}$$

which is infinitesimal independently of x ∎

Lemma 2.5: *Let $T \in {}^{\mathrm{st}}\mathscr{D}'$, $\psi \in \mathbf{D}$. Then $\langle T, \psi\rangle$ is limited. If $\psi \approx_{\mathrm{D}} 0$, then $\langle T, \psi\rangle \sim 0$.*

Proof: There is a standard $k \in \mathbb{N}$ such that $\psi \in \mathscr{D}_k$. On the other hand, since $T \in {}^{\mathrm{st}}\mathscr{D}'$, there is a standard $m \in \mathbb{N}$ such that if $\varphi \in \mathscr{D}_k$ and

$$s_m(\varphi) = \sup\{|\partial^\alpha\varphi(x)| : x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0{}^n, |\alpha| \leqq m\} \leqq \frac{1}{m},$$

then $|\langle T, \varphi\rangle| \leqq 1$. But $s_m(\psi) \leqq M$ for some standard $M \in \mathbb{N}$, thus we have that $|\langle T, \psi\rangle| \leqq mM$, which is a limited number.

If $\psi \approx_{\mathrm{D}} 0$, we take an infinitely large $\omega \in \mathbb{N}$ such that $\omega\psi \approx_{\mathrm{D}} 0$. With k and m as above we have that $s_m(\omega\psi) \leq \frac{1}{m}$, and thus $|\langle T, \psi\rangle| \leq \frac{1}{\omega} \sim 0$ ∎

Proposition 2.6: *Let $T \in {}^{\mathrm{st}}\mathcal{D}'$, $\theta \in {}^{\mathrm{st}}\mathcal{D}$, $\varrho \sim 0$ as above. Then we have:*

(a) $T * \theta_\varrho \in \mathbf{D}'$.

(b) $\langle T * \theta_\varrho, \psi\rangle \sim \langle T, \psi\rangle$ *for all* $\psi \in \mathbf{D}$.

(c) *If $\langle T * \theta_\varrho, \psi\rangle \sim 0$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$, then $T = 0$. In particular, if $T * \theta_\varrho \in \mathbf{d}'$, then $T = 0$.*

Proof: (a) Let $\psi \in \mathbf{D}$. Then $\langle T * \theta_\varrho, \psi\rangle = \langle T, \psi * \check{\theta}_\varrho\rangle$ where $\check{\theta}_\varrho(x) = \theta_\varrho(-x)$. By Lemma 2.4, $\psi * \check{\theta}_\varrho \in \mathbf{D}$, by Lemma 2.5, $\langle T, \psi * \check{\theta}_\varrho\rangle$ is limited. (b): $\langle T * \theta_\varrho, \psi\rangle - \langle T, \psi\rangle = \langle T, \psi * \check{\theta}_\varrho - \psi\rangle$. The conclusion follows again from Lemmas 2.4 and 2.6. (c): If $\langle T * \theta_\varrho, \psi\rangle \sim 0$, then $\langle T, \psi\rangle \sim 0$ by (b). But $\langle T, \psi\rangle$ is a standard complex number, thus $\langle T, \psi\rangle = 0$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$. By transfer (cf. Appx 2) this implies that $\langle T, \psi\rangle = 0$ for all $\psi \in \mathcal{D}$ ∎

Remark 2.7: The assertions of Prop. 2.3 through Prop. 2.6 remain valid in the setting of tempered distributions, as is seen by a straightforward modification of the proofs. Specifically, the following version of Prop. 2.6 will be needed in Section 4: *Let $T \in {}^{\mathrm{st}}\mathcal{S}'$ and $\theta \in {}^{\mathrm{st}}\mathcal{S}$ with $\int \theta(x)\,dx = 1$, $\varrho \sim 0$. Then* (a) $T * \theta_\varrho \in \mathbf{S}'$; (b) $\langle T * \theta_\varrho, \psi\rangle \sim \langle T, \psi\rangle$ *for all* $\psi \in \mathbf{S}$; (c) *if* $\langle T * \theta_\varrho, \psi\rangle \sim 0$ *for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$, then $T = 0$.*

Prop. 2.6 (c) shows that convolution by θ_ϱ produces an injective map ${}^{\mathrm{st}}\mathcal{D}' \to \mathbf{D}'/\mathbf{d}'$. We are going to show that this map is actually surjective.

Proposition 2.8: *Let $\psi \in \mathbf{D}$. Then there is a unique standard $\varphi \in \mathcal{D}$ with $\psi \approx_{\mathrm{D}} \varphi$.*

Proof: It follows from the mean value theorem and the fact that ψ belongs to $\mathbf{D}$ that ψ is s-continuous at every standard $a \in \mathbb{R}^n$. By the s-continuity theorem (cf. Appx 2), there is a standard, continuous function $\varphi: \mathbb{R}^n \to \mathbb{C}$ such that $\varphi(a) = {}^0\psi(a)$, for all $a \in {}^{\mathrm{st}}\mathbb{R}^n$. Moreover, we even have $\varphi(x) \sim \psi(x)$ for all $x \in \mathbb{R}^n$, because $\varphi(x) - \psi(x)$ attains its maximum on $\mathbb{R}^n$, and this maximum is infinitesimal. An analogous conclusion holds for all standard derivatives of ψ. Thus we have

$$\begin{aligned}&\forall^{\mathrm{st}}\alpha \in \mathbb{N}^n\ \exists^{\mathrm{st}}\varphi_\alpha: \mathbb{R}^n \to \mathbb{C},\ \varphi_\alpha \text{ continuous, with}\\ &\sup\{|\partial^\alpha\psi(x) - \varphi_\alpha(x)| : x \in \mathbb{R}^n\} \sim 0.\end{aligned} \tag{2.2}$$

It remains to show that $\varphi_\alpha = \partial^\alpha\varphi$. Let first $\alpha = (1, 0, \ldots, 0)$, $a \in {}^{\mathrm{st}}\mathbb{R}^n$, $x \in \mathbb{R}^n$, $x \sim a$. Then

$$\begin{aligned}\varphi(x) - \varphi(a) &\sim \psi(x) - \psi(a) \sim (x_1 - a_1)\,\partial^\alpha\psi(\xi)\\ &\sim (x_1 - a_1)\,\varphi_\alpha(a) + (x_1 - a_1)\left(\varphi_\alpha(\xi) - \varphi_\alpha(a)\right)\end{aligned}$$

for some $\xi \sim a$. Since φ_α is continuous and standard, we have $(x_1 - a_1)^{-1}\left(\varphi(x) - \varphi(a)\right) \sim \varphi_\alpha(a)$. Thus φ is differentiable in the direction $\alpha = (1, 0, \ldots, 0)$ at all standard $a \in \mathbb{R}^n$, and $\partial^\alpha\varphi(a) = \varphi_\alpha(a)$. By transfer, $\partial^\alpha\varphi = \varphi_\alpha$. The same argument works for all other standard derivatives, thus $\varphi \in {}^{\mathrm{st}}\mathcal{E}^\infty$. Since ψ belongs to some $\mathcal{D}_k$ with standard $k \in \mathbb{N}$, so does φ. That is, $\varphi \in {}^{\mathrm{st}}\mathcal{D}$, and by (2.2) $\psi \approx_{\mathrm{D}} \varphi$. Uniqueness is evident ∎

Proposition 2.9: *Let $T \in \mathbf{D}'$. Then there is a unique standard $U \in \mathcal{D}'$, denoted by ${}^0T$, such that $\langle T, \psi\rangle \sim \langle U, \psi\rangle$ for all $\psi \in \mathbf{D}$.*

Proof: Since $T \in \mathbf{D}'$, ${}^0\langle T, \psi\rangle$ exists for every $\psi \in \mathbf{D}$. By the construction principle for maps (cf. Appx 2), there is a unique standard map $U: \mathcal{D} \to \mathbb{C}$ such that $\langle U, \varphi\rangle = {}^0\langle T, \varphi\rangle$ for all $\varphi \in {}^{st}\mathcal{D}$. It is clear that U is linear (transfer). Fix $k \in {}^{st}\mathbb{N}$. Prop. 2.3 together with Remark 2.2 says that T is s-continuous at every standard element of $\mathcal{D}_k$. The s-continuity theorem implies that U is continuous on $\mathcal{D}_k$. Thus $U \in {}^{st}\mathcal{D}'$, and $\langle U, \varphi\rangle \sim \langle T, \varphi\rangle$ for all $\varphi \in {}^{st}\mathcal{D}$. If $\psi \in \mathbf{D}$, then there is a $\varphi \in {}^{st}\mathcal{D}$ with $\varphi \approx_{\mathrm{D}} \psi$ by Prop. 2.9. But then $\langle U, \psi\rangle \sim \langle U, \varphi\rangle \sim \langle T, \varphi\rangle \sim \langle T, \psi\rangle$ by Lemma 2.5 and Prop. 2.3 ∎

Fixing a $\theta \in {}^{st}\mathcal{D}$ and $\varrho \sim 0$ as in Prop. 2.6 we have the (noncanonical) inclusions

$$ {}^{st}\mathcal{D}' \hookrightarrow \mathbf{D}' \hookrightarrow \mathcal{E}^\infty \hookrightarrow \mathcal{D}' \tag{2.3}$$

where the first one is given by convolution with θ_ϱ, whereas the others are subspace relations. Moreover, convolution with θ_ϱ induces a bijection of

$$ {}^{st}\mathcal{D}' \simeq \mathbf{D}'/\mathbf{d}' $$

as follows from Prop. 2.6(c) and 2.9.

We now turn to structure theorems, which will be needed in Section 4. The first theorem is a counterpart to the classical structure theorem for $\mathcal{D}'$, asserting that limited distributions locally are finite derivatives of pointwise limited smooth functions.

Proposition 2.10: *Let $T \in \mathcal{E}^\infty$. The following are equivalent:*

(a) $T \in \mathbf{D}'$.

(b) *For all standard $k \in \mathbb{N}$ there exist an element $S \in \mathcal{E}^\infty$ with* $\sup\{|S(x)| : |x| \leqq k\}$ *limited and a standard $\alpha \in \mathbb{N}_0^n$ such that $T(x) = \partial^\alpha S(x)$ for all $x \in \mathbb{R}^n$, $|x| \leqq k$.*

Proof: (b) $\Rightarrow$ (a): Let $\psi \in \mathbf{D}$ and let $k \in {}^{st}\mathbb{N}$ such that $\psi \in \mathcal{D}_k$. Then

$$\langle T, \psi\rangle = \int \partial^\alpha S(x)\, \psi(x)\, dx = (-1)^{|\alpha|} \int S(x)\, \partial^\alpha \psi(x)\, dx$$

is limited. (a) $\Rightarrow$ (b): Let $k \in {}^{st}\mathbb{N}$, and ${}^0T \in {}^{st}\mathcal{D}'$ as given by Prop. 2.9. By the classical structure theorem [17, p. 82] and transfer, there is $\alpha \in {}^{st}\mathbb{N}_0^n$ and a standard, continuous function f with compact support, such that $\langle {}^0T, \psi\rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \psi\rangle$ for all $\psi \in \mathcal{D}_{k+1}$. Letting θ_ϱ be as in Prop. 2.6 we have that

$$\langle \partial^\alpha (f * \theta_\varrho), \psi\rangle \sim \langle T, \psi\rangle \tag{2.4}$$

for all $\psi \in \mathbf{D} \cap \mathcal{D}_{k+1}$ by Prop. 2.6 and 2.9. Let $g(x) = T(x) - \partial^\alpha (f * \theta_\varrho)(x)$. It follows from (2.4) that $\sup\{|g(x)| : |x| \leqq k\}$ is infinitesimal. Set

$$I^{(1,0,\ldots,0)} g(x) = \int_0^{x_1} g(\xi, x_2, \ldots, x_n)\, d\xi$$

and define $I^\beta g(x)$ inductively for all $\beta \in {}^{st}\mathbb{N}^n$. Clearly, $\sup\{|I^\alpha g(x)| : |x| \leqq k\}$ is infinitesimal as well; on the other hand, $\sup\{|f * \theta_\varrho(x)| : x \in \mathbb{R}^n\}$ is limited. If we set $S(x) = I^\alpha g(x) + f * \theta_\varrho(x)$, we have that $\sup\{|S(x)| : |x| \leqq k\}$ is limited and that $T(x) = \partial^\alpha S(x)$ for $|x| \leqq k$ ∎

Corollary 2.11: *Let $T \in \mathbf{D}'$ and $\varrho \in \mathbb{R}$ be a positive infinitesimal. Then*

$$\forall^{st} k \in \mathbb{N}\ \exists^{st} j \in \mathbb{N} \textit{ such that } \sup\{|T(x)| : |x| \leqq k\} \leqq \varrho^{-j}.$$

Proof: Fix $k \in {}^{st}\mathbb{N}$. Let f, g and α be as in the proof of Prop. 2.10. Then $T(x) = g(x) + \partial^\alpha(f * \theta_\varrho)(x)$ for $|x| \leq k$; $\sup\{|g(x)| : |x| \leq k\}$ is infinitesimal, and

$$|\partial^\alpha(f * \theta_\varrho)(x)| \leq \int |f(x - \varrho y)\, \varrho^{-|\alpha|}\, \partial^\alpha\theta(y)|\, dy \leq C\varrho^{-|\alpha|}$$

for some standard $C > 0$ and all $|x| \leq k$. Thus $\sup\{|T(x)| : |x| \leq k\} \leq \varrho^{-|\alpha|-1}$ ∎

We remark that the assertion of Cor. 2.11 is not void: the constant function $T(x) = \omega$ with ω infinitely large does not satisfy the assertion with $\varrho = (\log \omega)^{-1}$.

3. Products of distributions

We start this section by introducing a nonstandard product of any two standard distributions. Let $\theta \in {}^{st}\mathcal{D}$ with $\int \theta(x)\, dx = 1$ and fix a real infinitesimal number ϱ. Motivated by the inclusions (2.3) we make the following definition.

Definition 3.1: Let $S, T \in {}^{st}\mathcal{D}'$. Then

$$M_\varrho^\theta(S, T) = (S * \theta_\varrho)(T * \theta_\varrho)$$

is called the M_ϱ^θ-*product* of S and T (with θ_ϱ defined by (2.1)).

Remark 3.2: (a) $M_\varrho^\theta(S, T)$ belongs to $\mathcal{E}^\infty$, but not to $\mathbf{D}'$, in general. The M_ϱ^θ-product is commutative and satisfies the Leibniz rule. (b) If we allow θ to belong to ${}^{st}\mathcal{D}_{L^\infty} \cap L^1$, then the assertions of Prop. 2.6 are still true for $T \in {}^{st}\mathcal{D}'_{L^1}$. Thus in case $S, T \in {}^{st}\mathcal{D}'_{L^1}(\mathbb{R})$ it makes sense to define the product $M_\varrho^\Delta(S, T)$ where Δ is the Tillmann mollifier $\Delta(x) = [\pi(1 + x^2)]^{-1}$. This is Li Bang-He's product [10, p. 564] when applied to integrable distributions. (c) Raju's definition [15, p. 384] is in a similar spirit, but not related to ours. In our notation, Raju defines the product of two standard distributions S and T as $(S * \theta_\varrho)\, T$ with θ symmetric. The result is a noncommutative product valued in $\mathcal{D}'$. (d) In the framework of his "asymptotic functions", Todorov [21] has considered a product which leads to analogous formulas. Rather than choosing a fixed mollifier, Todorov works with certain classes of "kernels" representing a given distribution.

Example 3.3: For the square of the Dirac measure δ in one dimension we have

$$\langle M_\varrho^\theta(\delta, \delta), \psi\rangle \sim \left\langle \frac{1}{\varrho}\, c_0\delta + c_1\delta', \psi \right\rangle \quad \text{for all } \psi \in \mathbf{D},$$

where

$$c_0 = \int \theta^2(x)\, dx \quad \text{and} \quad c_1 = -\int x\theta^2(x)\, dx.$$

Indeed, $\langle M_\varrho^\theta(\delta, \delta), \psi\rangle = \dfrac{1}{\varrho} \displaystyle\int \theta^2(x)\, \psi(\varrho x)\, dx$. The result follows by Taylor-expanding ψ around zero up to order two and observing that the third term only contributes an infinitesimal to the product. Taking in particular the Tillmann mollifier Δ, one has

$$\langle M_\varrho^\Delta(\delta, \delta), \psi\rangle \sim \left\langle \frac{1}{2\pi\varrho}\, \delta, \psi \right\rangle \quad \text{for all} \quad \psi \in \mathbf{D},$$

because in this case $c_0 = 1/2\pi$ and $c_1 = 0$. This is precisely Li Bang-He's result [10, p. 579]. A related formula involving the value of δ at zero holds in certain distribution algebras introduced by Berg [1a, p. 267].

We now turn to investigate internal products of distributions defined by regularization and passage to the limit. To fix notation, we introduce several classes of delta-nets.

Definition 3.4: (a) A net $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1} \subset \mathcal{D}(\mathbb{R}^n)$ with

$$\varphi^\varepsilon \geq 0 \quad \text{and} \quad \int \varphi^\varepsilon(x)\,dx = 1 \quad \text{for all } \varepsilon$$

will be called a C_i-*delta-net* ($i = 1, 2, 3, 4$), provided it satisfies condition (C_i) as follows:

(C_1) support $(\varphi^\varepsilon) \to \{0\}$ as $\varepsilon \to 0$;

(C_2) support $(\varphi^\varepsilon) \to \{0\}$ as $\varepsilon \to 0$ and

$$\forall \alpha \in \mathbb{N}_0^n \exists A_\alpha > 0 \text{ such that } \int |x^{|\alpha|}|\,|\partial^\alpha \varphi^\varepsilon(x)|\,dx \leq A_\alpha \text{ for all } \varepsilon;$$

(C_3) support $(\varphi^\varepsilon) \subset \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$ and

$$\forall \alpha \in \mathbb{N}_0^n \exists A_\alpha > 0 \text{ such that } \varepsilon^{|\alpha|} \int |\partial^\alpha \varphi^\alpha(x)|\,dx \leq A_\alpha \text{ for all } \varepsilon;$$

(C_4) $\varphi^\varepsilon(x) = \varepsilon^{-n}\varphi\left(\dfrac{x}{\varepsilon}\right)$ for some $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) Let $S, T \in \mathcal{D}'(\mathbb{R}^n)$. We say that the M_i-*product* ($i = 1, 2, 3, 4$) of S and T exists if

$$\lim_{\varepsilon\to 0} (S * \varphi^\varepsilon)(T * \varphi^\varepsilon) = M_i(S, T)$$

exists in $\mathcal{D}'(\mathbb{R}^n)$ for all C_i-delta-nets $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1}$ and is independent of the particular C_i-delta-net chosen (the last sentence is redundant für $i \neq 4$).

Notation: The lower index notation φ_ε will be reserved for C_4-nets in accordance with (2.1), the upper index notation φ^ε for general delta-nets.

Remark 3.5: (a) C_1-nets were introduced by HIRATA-OGATA [6] and MIKUSIŃSKI [11], C_2-nets by SHIRAISHI [18], called "restricted delta-nets" there, the condition $\int \varphi^\varepsilon(x)\,dx = 1$ actually being replaced by $\lim\limits_{\varepsilon\to 0} \int \varphi^\varepsilon(x)dx = 1$, which obviously yields an equivalent product. The M_2-product was studied by ITANO [8]. C_3-nets were introduced by ANTOSIK-MIKUSIŃSKI-SIKORSKI [1] and were studied by KAMIŃSKI [9] together with C_4-nets (called "model nets" there), which appear at many places [2, 3, 5, 7]. We remark that all authors quoted above use sequences instead of nets, but our definitions are equivalent. (b) In [1, 9] C_3- and C_4-nets are defined without requiring $\varphi^\varepsilon \geq 0$. So it is important to note that the M_3- and M_4-product remains unchanged when this condition is deleted. For the M_4-product this follows immediately from [14, Prop.]; in the case of the M_3-product we observe that (C_3) implies that $\sup\{|\varphi^\varepsilon(x)| : x \in \mathbb{R}^n\} \leq \varepsilon^{-n}A_0$. Thus if we take $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \geq 0$, $\chi(x) = 0$ for $|x| \geq 2$, $\chi(x) \geq A_0$ for $|x| \leq 1$, then $\varphi^\varepsilon + \chi_\varepsilon \geq 0$ for all ε, and the same arguments as in the proof of the Prop. in [14] apply. (c) Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi \geq 0$, $\int \varphi(x)\,dx = 1$, $\varphi(0) \neq 0$, support $(\varphi) \subset [-1/2, 1/2]$. Then $\varphi^\varepsilon(x) = \dfrac{1}{2}\left(\dfrac{1}{\varepsilon^2}\varphi\left(\dfrac{x}{\varepsilon^2}\right) + \dfrac{1}{\varepsilon}\varphi\left(\dfrac{x-\varepsilon}{\varepsilon}\right)\right)$ is an example of a net which satisfies (C_2) but not (C_3), while $\psi^\varepsilon(x) = \dfrac{1}{\varepsilon^2}\varphi\left(\dfrac{x-\varepsilon}{\varepsilon^2}\right)$ is a net which satisfies (C_1) but not (C_2). On the other hand, $(C_{i+1}) \subset (C_i)$ for all i, and therefore the existence of the M_i-product implies the existence of the M_{i+1}-product.

The following nonstandard characterization relates the M_4-product and the M_ϱ^θ-product.

Proposition 3.6: *Let $S, T \in {}^{\mathrm{st}}\mathcal{D}'$. The following are equivalent:*

(a) *$M_4(S, T)$ exists.*

(b) *There is $W \in \mathbf{D}'$ such that $\langle M_\varrho^\theta(S, T), \psi\rangle \sim \langle W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$, all $\varrho \sim 0$, and all $\theta \in {}^{\mathrm{st}}\mathcal{D}$ with $\int \theta(x)\,dx = 1$ and $\theta \geq 0$.*

(c) *There is $W \in \mathbf{D}'$ such that $M_\varrho^\theta(S, T) - W \in \mathbf{d}'$ for all $\varrho \sim 0$ and all $\theta \in {}^{\mathrm{st}}\mathcal{D}$ with $\int \theta(x)\,dx = 1$ and $\theta \geq 0$.*

In this case we have $M_4(S, T) = {}^0W$.

Proof: (a) $\Rightarrow$ (b): Let $V = \mathrm{M}_4(S, T) = \lim_{\varepsilon\to 0} (S * \theta_\varepsilon)(T * \theta_\varepsilon)$. Then V is standard, and the characterization of the convergence of a standard net (cf. Appx 2) gives $\langle \mathrm{M}_\varrho^\theta(S, T), \psi\rangle \sim \langle V, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$ and all $\varrho \sim 0$. Thus (b) holds with $W = V * \theta_\varrho$. (b) $\Rightarrow$ (a): We show that $\lim_{\varepsilon\to 0} (S * \varphi_\varepsilon)(T * \varphi_\varepsilon) = {}^0W$ for all C_4-nets $\{\varphi_\varepsilon\}_{0<\varepsilon\leq 1}$. By transfer, we may assume that $\{\varphi_\varepsilon\}_{0<\varepsilon\leq 1}$ is standard. Then (b) holds with $\theta = \varphi_1$, and we have $\langle (S * \varphi_\varrho)(T * \varphi_\varrho), \psi\rangle \sim \langle W, \psi\rangle \sim \langle {}^0W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$ and all $\varrho \sim 0$ (see Prop. 2.9). By the characterization of convergence of a standard net this means $\lim_{\varepsilon\to 0} \langle (S * \varphi_\varepsilon) \times (T * \varphi_\varepsilon), \psi\rangle = \langle {}^0W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$. Applying transfer again, we have (a).

(c) means that (b) holds not only for all standard $\psi \in \mathcal{D}$, but for all $\psi \in \mathbf{D}$. Thus it remains to prove (b) $\Rightarrow$ (c). Let $\psi \in \mathbf{D}$. Since ψ is the sum of a standard test function and an infinitesimal one (Prop. 2.8) it suffices to prove the assertion (c) with $\psi \approx_{\mathbf{D}} 0$. First, the net $\{(S * \theta_\varepsilon)(T * \theta_\varepsilon)\}_{0<\varepsilon\leq 1} \subset \mathcal{D}'$ is pointwise bounded (this follows from its convergence and the support properties of θ_ε), and hence equicontinuous. By transfer this means

$$\forall^{\mathrm{st}}\delta > 0\ \exists^{\mathrm{st}} \text{ neighborhood } \mathcal{N} \text{ of zero in } \mathcal{D} \text{ such that}$$

$$\forall\, \varepsilon, 0 < \varepsilon \leqq 1, \forall\, \varphi \in \mathcal{N}: |\langle (S * \theta_\varepsilon)(T * \theta_\varepsilon), \varphi\rangle| < \delta.$$

But ψ belongs to every standard neighborhood of zero in $\mathcal{D}$. Thus we have

$$\forall^{\mathrm{st}}\, \delta > 0\ \forall\, \varepsilon, 0 < \varepsilon \leqq 1: |\langle (S * \theta_\varepsilon)(T * \theta_\varepsilon), \psi\rangle| < \delta,$$

implying that $\langle \mathrm{M}_\varrho^\theta(S, T), \psi\rangle \sim 0$. On the other hand we also have $\langle W, \psi\rangle \sim 0$ by Prop. 2.3 ∎

According to Remark 3.5 (b) the equivalence of (a) and (b), (c) in Prop. 3.6 remains valid when the condition $\theta \geqq 0$ is dropped. Enhancing criterion (b) by requiring it to be fulfilled for all $\mathbf{D}$-limited mollifiers θ we obtain the existence of the M_3-product.

Proposition 3.7: *Let $S, T \in {}^{\mathrm{st}}\mathcal{D}'$. If there exists a $W \in \mathbf{D}'$ such that $\langle \mathrm{M}_\varrho^\theta(S, T), \psi\rangle \sim \langle W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$, all $\varrho \sim 0$ and all $\theta \in \mathbf{D}$ with $\int \theta(x)\,dx = 1$ and $\theta \geqq 0$, then $\mathrm{M}_3(S, T)$ exists, and it equals ${}^0W$.*

Proof: By Prop. 3.6, we have that $\mathrm{M}_4(S, T) = {}^0W$ exists. Let $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1}$ be a standard C_3-net. We have to show that $\lim_{\varepsilon\to 0} (S * \varphi^\varepsilon)(T * \varphi^\varepsilon) = {}^0W$. As in the proof of Prop. 3.6 it suffices to show that $\langle (S * \varphi^\varrho)(T * \varphi^\varrho), \psi\rangle \sim \langle {}^0W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$ and all $\varrho \sim 0$. From (C_3) we obtain by transfer:

$$\forall^{\mathrm{st}}\, \alpha \in \mathbb{N}_0^{\,n} \exists^{\mathrm{st}} A_\alpha > 0 \text{ such that } \varepsilon^{|\alpha|} \int |\partial^\alpha \varphi^\varepsilon(x)|\, dx < A_\alpha, \text{ for } \forall\, \varepsilon, 0 < \varepsilon \leqq 1,$$

in particular, $\varrho^{|\alpha|} \int |\partial^\alpha \varphi^\varrho(x)|\, dx$ is limited for all $\alpha \in {}^{\mathrm{st}}\mathbb{N}_0^{\,n}$. Setting $\theta(x) = \varrho^n \varphi^\varrho(\varrho x)$ we infer that $\int |\partial^\alpha \theta(x)|\, dx$ is limited for all $\alpha \in {}^{\mathrm{st}}\mathbb{N}_0^{\,n}$. Since $\theta(x) = 0$ for $|x| \geqq 1$, this implies that $\theta \in \mathbf{D}$. But $\varphi^\varrho = \theta_\varrho$, and so the hypotheses imply that $\langle (S * \varphi^\varrho)(T * \varphi^\varrho), \psi\rangle \sim \langle {}^0W, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}$ ∎

Corollary 3.8: *Let $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. If $\mathrm{M}_4(\delta, f)$ exists, then $\mathrm{M}_3(\delta, f)$ exists, too.*

Proof: We may assume that f is standard. Let $\theta \in \mathbf{D}$ be as in Prop. 3.7. We write $\theta = \varphi + \eta$ with φ standard and η $\mathbf{D}$-infinitesimal. Let $\psi \in {}^{\mathrm{st}}\mathcal{D}$, $\varrho \sim 0$. We will show that $\langle \theta_\varrho (f * \theta_\varrho), \psi\rangle \sim \langle \varphi_\varrho(f * \varphi_\varrho), \psi\rangle$, from where the assertion follows by Prop. 3.6 and 3.7. This amounts to showing that

$$\langle \varphi_\varrho(f * \eta_\varrho) + \eta_\varrho(f * \varphi_\varrho) + \eta_\varrho(f * \eta_\varrho), \psi\rangle \sim 0.$$

But the first term

$$\langle \varphi_\varrho(f * \eta_\varrho), \psi\rangle = \int\int \varphi(x)\, f(\varrho y)\, \eta(x-y)\, \psi(\varrho x)\, dy\, dx$$

is infinitesimal, because integration extends over a standard compact region in $\mathbb{R}^{2n}$, φ, f, and ψ are bounded by a standard number, and η is infinitesimal. A similar estimate applies to the other terms ∎

To demonstrate that the criteria of Prop. 3.6 and 3.7 are useful in concrete calculations, we continue the investigation of the example of [13, Appx] with regard to the products $M_1 - M_4$ in Appx 1. It is seen that the M_2-product may exist while the M_1-product does not. No example distinguishing the products M_2, M_3, M_4 is known.

Our next goal will be to prove that the existence of the Shiraishi-Itano product M_2 implies the existence of the Tillmann product. For $S \in \mathcal{D}'(\mathbb{R})$ there exists a function $\hat{S}(z)$, analytic in $\mathbb{C} \setminus \text{support}(S)$, such that $S = \lim_{\varepsilon\to 0} \hat{S}_\varepsilon$ in $\mathcal{D}'(\mathbb{R})$, where $\hat{S}_\varepsilon(x) = \hat{S}(x + i\varepsilon) - \hat{S}(x - i\varepsilon)$, see [20]. $\hat{S}$ is unique up to an entire function.

Definition 3.9: Let $S, T \in \mathcal{D}'(\mathbb{R})$. The *Tillmann product* or M_5-*product* of S and T is said to exist if $\lim_{\varepsilon\to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon = M_5(S, T)$ exists in $\mathcal{D}'(\mathbb{R})$.

For $S \in \mathcal{D}'_{L^1}(\mathbb{R})$, a particular choice for $\hat{S}$ is

$$\hat{S}(z) = \frac{1}{2\pi i}\left\langle S(x), \frac{1}{x - z}\right\rangle.$$

Letting $\Delta(x) = [\pi(1 + x^2)]^{-1}$ it is easily seen that, for $S, T \in \mathcal{D}'_{L^1}(\mathbb{R})$, Def. 3.9 is equivalent to $\lim_{\varepsilon\to 0} (S * \Delta_\varepsilon)\,(T * \Delta_\varepsilon) = M_5(S, T)$ in $\mathcal{D}'(\mathbb{R})$.

Proposition 3.10: *Let $S, T \in \mathcal{D}'_{L^1}(\mathbb{R})$. If $U = \lim_{\varepsilon\to 0} (S * \varphi^\varepsilon)\,(T * \varphi^\varepsilon)$ exists for all symmetric C_2-nets $\{\varphi^\varepsilon\}_{0<\varepsilon\le 1}$, then the Tillmann product $M_5(S, T)$ exists also and coincides with U. In particular, if $M_2(S, T)$ exists, then so does $M_5(S, T)$.*

Proof: By transfer we may assume that S, T und U are standard. By the structure theorem for $\mathcal{D}'_{L^1}$, we have

$$S = \sum_{j=0}^{l} \partial^j f_j, \qquad T = \sum_{j=0}^{m} \partial^j g_j$$

for some standard $l, m \in \mathbb{N}$ and standard $f_j, g_j \in L^1(\mathbb{R})$. Let $\varrho \sim 0$. We wish to show that $\langle (S * \Delta_\varrho)\,(T * \Delta_\varrho), \psi\rangle \sim \langle U, \psi\rangle$ for all $\psi \in {}^{\mathrm{st}}\mathcal{D}(\mathbb{R})$. The idea is to construct a standard C_2-net $\{\varphi^\varepsilon\}_{0<\varepsilon\le 1}$ such that

$$\sup\left\{\left|\partial^j(\Delta_\varrho(\xi) - \varphi^\varrho(\xi))\right| : \xi \in \mathbb{R}\right\} \sim 0 \tag{3.1}$$

for $0 \le j \le n = l + m + 1$. We then have

$$\begin{aligned}
&\langle (S * \Delta_\varrho)\,(T * \Delta_\varrho), \psi\rangle - \langle U, \psi\rangle \\
&\sim \langle (S * \Delta_\varrho)\,(T * \Delta_\varrho), \psi\rangle - \langle (S * \varphi^\varrho)\,(T * \varphi^\varrho), \psi\rangle \\
&= \Big\langle (S * \Delta_\varrho)\,\big(T * (\Delta_\varrho - \varphi^\varrho)\big), \psi\Big\rangle + \Big\langle \big(S * (\Delta_\varrho - \varphi^\varrho)\big)\,(T * \varphi^\varrho, \psi\Big\rangle.
\end{aligned}$$

The first term of the last line equals

$$\sum_{i=0}^{l}\sum_{j=0}^{m}\iiint f_i(z)\,\partial_x{}^i\Delta_\varrho(x-z)\,g_j(y)\left(\partial_x{}^j\Delta_\varrho(x-y)-\partial_x{}^j\varphi^\varrho(x-y)\right)\psi(x)\,dx\,dy\,dz$$
$$=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{i+1}\binom{i+1}{k}\int\int\int f_i(z)\,\frac{1}{\pi}\arctan\left(\frac{x-z}{\varrho}\right)g_j(y)$$
$$\times\ \partial_x{}^{j+k}\left(\Delta_\varrho(x-y)-\varphi^\varrho(x-y)\right)\partial_x{}^{i+1-k}\psi(x)\,dx\,dy\,dz.$$

Given (3.1), the part $\partial_x{}^{j+k}\left(\Delta_\varrho(x-y)-\varphi^\varrho(x-y)\right)$ is infinitesimal independently of x, y, while all three integrals are limited, since f_i, g_j and $\partial^{i+1-k}\psi$ are standard L^1-functions. Thus the first term is infinitesimal. The second term is estimated similarly, the part $\frac{1}{\pi}\arctan\left(\frac{x-z}{\varrho}\right)$ being replaced by $\int_{-\infty}^{x-z}\varphi^\varrho(\xi)\,d\xi$ which is also bounded by one.

Thus it remains to construct $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1}$. We take a standard $\chi\in\mathcal{D}(\mathbb{R})$, χ symmetric, $0\leqq\chi\leqq 1$, $\chi(x)=1$ for $|x|\leqq 1$, $\chi(x)=0$ for $|x|\geqq 2$, and set

$$\varphi^\varepsilon(x)=\Delta_\varepsilon(x)\,\chi\left(\frac{x}{\lambda}\right).$$

We shall show that φ^ε has the desired properties if we choose $\lambda=\varepsilon^{1/(n+3)}$. To this end we need some preliminary estimates. We first observe that

$$\partial^i\Delta(x)=(1+x^2)^{-i-1}P_i(x),\qquad i\geqq 0,$$

for some polynomials P_i of degree i, and

$$\partial^i\Delta_\varepsilon(x)=(\varepsilon^2+x^2)^{-i-1}\varepsilon^{i+1}P_i\left(\frac{x}{\varepsilon}\right).$$

Using the fact that degree $(P_i)=i$ and that $\varepsilon<\lambda$ one deduces immediately the estimates $(0\leqq i\leqq n)$

$$|\partial^i\Delta_\varepsilon(x)|\leqq C(\varepsilon^2+\lambda^2)^{-i-1}\varepsilon\lambda^i\leqq C\varepsilon\lambda^{-i-2}\quad\text{for}\quad\lambda\leqq|x|\leqq 2\lambda \tag{3.2}$$

and

$$|\partial^i\Delta_\varepsilon(x)|\leqq|x|^{-i}(x^2+\varepsilon^2)^{-i/2-1}\varepsilon^{i+1}\left|P_i\left(\frac{x}{\varepsilon}\right)\right|$$
$$\leqq C\varepsilon\lambda^{-i-2}\quad\text{for}\quad|x|\geqq 2\lambda; \tag{3.3}$$

here and henceforth C denotes a generic positive constant. Next, $\partial^j\left(\Delta_\varepsilon(x)\left(1-\chi\left(\frac{x}{\lambda}\right)\right)\right)$ equals zero for $|x|\leqq\lambda$ and equals $\partial^j\Delta_\varepsilon(x)$ for $|x|\geqq 2\lambda$. For $\lambda\leqq|x|\leqq 2\lambda$ we infer from (3.2) that

$$\left|\partial^j\left(\Delta_\varepsilon(x)\left(1-\chi\left(\frac{x}{\lambda}\right)\right)\right)\right|=\left|\sum_{i=0}^{j}\binom{j}{i}\partial^i\Delta_\varepsilon(x)\,\partial^{j-i}\left(1-\chi\left(\frac{x}{\lambda}\right)\right)\right|$$
$$\leqq C\sum_{i=0}^{j}\binom{j}{i}(\varepsilon\lambda^{-i-2})\,\lambda^{i-j}\leqq C\varepsilon\lambda^{-j-2} \tag{3.4}$$

for $0\leqq j\leqq n$. For $|x|\geqq 2\lambda$ the expression is estimated by (3.3) and we have

$$\sup\{|\partial^j\Delta_\varepsilon(x)-\partial^j\varphi^\varepsilon(x)|:x\in\mathbb{R}\}\leqq C\varepsilon\lambda^{-n-2} \tag{3.5}$$

for $0\leqq j\leqq n$. Therefore, if we take $\lambda=\varepsilon^{1/(n+3)}$ and evaluate (3.5) at $\varepsilon=\varrho\sim 0$, we obtain the desired infinitesimality assertion (3.1).

It remains to prove that $\{\varphi^\varepsilon\}_{0<\varepsilon\leqq 1}$ is a C_2-net. First, $\varphi^\varepsilon \geqq 0$ and support $(\varphi^\varepsilon) \subset [-2\lambda, 2\lambda] \to \{0\}$ as $\varepsilon \to 0$. Second,

$$\int_{-\infty}^{\infty} \varphi^\varepsilon(x)\,dx = \int_{-\lambda}^{\lambda} \Delta_\varepsilon(x)\,dx + \int_{|x|\geqq\lambda} \Delta_\varepsilon(x)\,\chi\left(\frac{x}{\lambda}\right) dx.$$

Due to the relation between ε and λ, the first integral on the right-hand side tends to 1, the second to 0 as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \varphi^\varepsilon(x)\,dx = 1$$

which is not quite the condition required of C_2-nets, but one which leads to a product equivalent to the M_2-product (cf. Remark 3.5(a)). Finally, we have to show that for every $j \geqq 0$,

$$\sup_{0<\varepsilon\leqq 1} \int_{-\infty}^{\infty} |x^j\partial^j\varphi^\varepsilon(x)|\,dx < \infty. \tag{3.6}$$

For $|x| \geqq 2\lambda$ the integrand vanishes. For $\lambda \leqq |x| \leqq 2\lambda$ an estimate similar to (3.4) yields, for fixed $j \geqq 0$,

$$\left|\partial^j \left(\Delta_\varepsilon(x)\,\chi\left(\frac{x}{\lambda}\right)\right)\right| \leqq C\varepsilon\lambda^{-j-2},$$

thus

$$\int_{\lambda\leqq|x|\leqq 2\lambda} |x^j\partial^j\varphi^\varepsilon(x)|\,dx \leqq C\lambda^{j+1}\varepsilon\lambda^{-j-2} = C\,\frac{\varepsilon}{\lambda} \to 0$$

as $\varepsilon \to 0$. For $|x| \leqq \lambda$, the integrand equals $|x^j\partial^j\Delta_\varepsilon(x)|$, and

$$\int_{-\lambda}^{\lambda} |x^j\partial^j\Delta_\varepsilon(x)|\,dx = \int_{-\lambda}^{\lambda} \left|x^j(x^2+\varepsilon^2)^{-j-1}\,\varepsilon^{j+1}P_j\left(\frac{x}{\varepsilon}\right)\right| dx$$
$$\leqq \int_{-\infty}^{\infty} |x^j(x^2+1)^{-j-1}\,P_j(x)|\,dx < \infty$$

since degree $(P_j) = j$, and (3.6) is proved ∎

Remark 3.11: $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1}$ is not a C_3-net. Indeed,

$$2\lambda \int_{-\infty}^{\infty} |\partial^1\varphi^\varepsilon(x)|\,dx \geqq 2\lambda \int_{-\lambda}^{\lambda} |\partial^1\Delta_\varepsilon(x)|\,dx = \frac{2\lambda}{\pi\varepsilon} \int_{-\lambda/\varepsilon}^{\lambda/\varepsilon} \frac{|2x|}{(1+x^2)^2}\,dx \to \infty$$

as $\varepsilon \to 0$. It remains open whether the existence of the M_3- or M_4-product implies the existence of the Tillmann product.

Corollary 3.12: *Let $S, T \in \mathcal{D}'(\mathbb{R})$. If $U = M_2(S, T)$ exists, then the Tillmann product $M_5(S, T)$ exists also and coincides with U.*

Proof: We may assume that S, T, U are standard. Letting $\varrho \sim 0$ and $\psi \in {}^{st}\mathcal{D}(\mathbb{R})$, we have to show that $\langle \hat{S}_\varrho \hat{T}_\varrho, \psi\rangle \sim \langle U, \psi\rangle$. Take $\chi \in {}^{st}\mathcal{D}(\mathbb{R})$, $\chi \equiv 1$ in a standard neighborhood of support (ψ). Then both $T(1-\chi)\hat{}$ and $S(1-\chi)\hat{}$ are analytic in

$\mathbb{C} \setminus$ support $(1-\chi)$, so $T(1-\chi)_\varrho\hat{}$ and $S(1-\chi)_\varrho\hat{}$ vanish on support (ψ). Since $(\cdot)_\varrho\hat{}$ is a linear operation and $(S\chi)_\varrho\hat{} = (S\chi) * \Delta_\varrho$ we are reduced to showing that $\langle (S\chi * \Delta_\varrho)(T\chi * \Delta_\varrho), \psi\rangle \sim \langle U, \psi\rangle$. If we take $\{\varphi^\varepsilon\}_{0<\varepsilon\leq 1}$ as constructed in the proof of Prop. 3.10 with $S\chi$, $T\chi$ in the place of S, T, it follows from the proof of Prop. 3.10 that

$$\langle (S\chi * \Delta_\varrho)(T\chi * \Delta_\varrho), \psi\rangle \sim \langle (S\chi * \varphi^\varrho)(T\chi * \varphi^\varrho), \psi\rangle.$$

But $\chi \equiv 1$ in a standard neighborhood of the support of ψ, so

$$\langle (S\chi * \varphi^\varrho)(T\chi * \varphi^\varrho), \psi\rangle = \langle (S * \varphi^\varrho)(T * \varphi^\varrho), \psi\rangle \sim \langle U, \psi\rangle,$$

and the proof is complete ∎

4. Algebras containing the standard distributions

We fix a positive infinitesimal number ϱ and start by introducing an external space $\mathbf{E}_\varrho \subset \mathscr{E}^\infty(\mathbb{R}^n)$ of smooth functions as follows.

Definition 4.1: $\mathbf{E}_\varrho$ is the external set of all $T \in \mathscr{E}^\infty(\mathbb{R}^n)$ with the following property:

$$\forall^{st}\alpha \in \mathbb{N}_0{}^n\ \forall^{st}k \in \mathbb{N}\ \exists^{st} j \in \mathbb{N} \text{ such that } \sup\{|\partial^\alpha T(x)| : |x| \leqq k\} \leqq \varrho^{-j}.$$

It is clear from Cor. 2.11 that $\mathbf{D}' \subset \mathbf{E}_\varrho$; $\mathbf{E}_\varrho$ is a commutative and associative differential algebra over ${}^{st}\mathbb{C}$. If $\theta \in {}^{st}\mathscr{D}$ with $\int \theta(x)\,dx = 1$, then $S \to S * \theta_\varrho$ is an imbedding of ${}^{st}\mathscr{D}'$ into $\mathbf{E}_\varrho$. However, this imbedding does not preserve the pointwise product on ${}^{st}\mathscr{E}^\infty$, because $(f * \theta_\varrho)(g * \theta_\varrho) \neq (fg) * \theta_\varrho$ for $f, g \in {}^{st}\mathscr{E}^\infty$ in general. We shall now construct a quotient of $\mathbf{E}_\varrho$ and an imbedding of ${}^{st}\mathscr{S}'$ which turns ${}^{st}\mathcal{O}_M$, the standard smooth functions of polynomial growth, into a subalgebra. This is a nonstandard counterpart to Colombeau's construction of his algebra $\mathscr{G}_s(\mathbb{R}^n)$, see [4].

Definition 4.2: $\mathbf{N}_\varrho$ is the external set of all $T \in \mathscr{E}^\infty(\mathbb{R}^n)$ with the following property:

$$\forall^{st}\alpha \in \mathbb{N}_0{}^n\ \forall^{st}k \in \mathbb{N}\ \forall^{st}i \in \mathbb{N} : \sup\{|\partial^\alpha T(x)| : |x| \leqq k\} \leqq \varrho^i. \tag{4.1}$$

It is clear that $\mathbf{N}_\varrho$ is an ideal in $\mathbf{E}_\varrho$ closed under differentiation. Therefore,

$$\mathbf{G}_\varrho = \mathbf{E}_\varrho / \mathbf{N}_\varrho$$

is a differential algebra (commutative, associative) over ${}^{st}\mathbb{C}$. If $T \in \mathbf{E}_\varrho$, we shall write $[T]$ for its equivalence class in $\mathbf{G}_\varrho$. We now fix a standard $\theta \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} &\int \theta(x)\,dx = 1, \\ &\int x^\alpha \theta(x)\,dx = 0, \text{ for all } \alpha \in {}^{st}\mathbb{N}_0{}^n,\ |\alpha| \geqq 1. \end{aligned} \tag{4.2}$$

The existence of such a θ follows by Fourier transform and Borel's theorem (cf. Trèves [23, p. 390]).

Lemma 4.3: *If $f \in {}^{st}\mathcal{O}_M(\mathbb{R}^n)$, then $f * \theta_\varrho - f \in \mathbf{N}_\varrho$.*

Proof: We deduce condition (4.1) for $\alpha = (0, \ldots, 0)$, the proof for general α being similar. Let $i \in {}^{st}\mathbb{N}$. By Taylor's theorem and (4.2),

$$\begin{aligned} f * \theta_\varrho(x) - f(x) &= \int \big(f(x - \varrho y) - f(x)\big)\,\theta(y)\,dy \\ &= \sum_{|\beta| = i+1} \int \frac{(-\varrho y)^\beta}{\beta!}\,\partial^\beta f(\xi)\,\theta(y)\,dy \end{aligned}$$

with ξ between x and $x - \varrho y$; in particular, $|\xi| \leqq |x| + |\varrho y| \leqq |x| + |y|$. Since $f \in {}^{\mathrm{st}}\mathcal{O}_{\mathrm{M}}$, $|\partial^\beta f(\xi)|$ is bounded by $C(1 + |x|)^l (1 + |y|)^l$ for some standard $C > 0$ and some standard $l \in \mathbb{N}$. Since $\theta \in {}^{\mathrm{st}}\mathcal{S}$, the integrals are bounded by $C\varrho^{i+1}$ for some other standard $C > 0$, uniformly for $|x| \leqq k$ for every standard $k \in \mathbb{N}$. Since ϱ is infinitesimal we have $C\varrho^{i+1} \leqq \varrho^i$, proving the assertion ∎

Proposition 4.4: *Let ϱ be a positive infinitesimal and let $\theta \in {}^{\mathrm{st}}\mathcal{S}(\mathbb{R}^n)$ satisfy* (4.2). *Then:*

(a) *The map $S \to [S * \theta_\varrho]$ defines an imbedding of ${}^{\mathrm{st}}\mathcal{S}'(\mathbb{R}^n)$ into $\mathbf{G}_\varrho$ which preserves differentiation.*

(b) *${}^{\mathrm{st}}\mathcal{O}_{\mathrm{M}}(\mathbb{R}^n)$ is a subalgebra of $\mathbf{G}_\varrho$; more precisely*

$$[f * \theta_\varrho]\,[g * \theta_\varrho] = [(fg) * \theta_\varrho] \quad \text{for} \quad f, g \in {}^{\mathrm{st}}\mathcal{O}_{\mathrm{M}}(\mathbb{R}^n).$$

(c) *If $P \in {}^{\mathrm{st}}\mathcal{O}_{\mathrm{M}}(\mathbb{R}^m)$ and $[T_1], \ldots, [T_m] \in \mathbf{G}_\varrho$, then $[P(T_1, \ldots, T_m)]$ is a welldefined element of $\mathbf{G}_\varrho$.*

Proof: (a): We know from Remark 2.7 that $S \to S * \theta_\varrho$ is an injection of ${}^{\mathrm{st}}\mathcal{S}'$ into $\mathbf{E}_\varrho$ with $\partial^\alpha(S * \theta_\varrho) = (\partial^\alpha S) * \theta_\varrho$. Thus it remains to show that if $S \in {}^{\mathrm{st}}\mathcal{S}'$ and $S * \theta_\varrho \in \mathbf{N}_\varrho$, then $S = 0$. Let $\psi \in {}^{\mathrm{st}}\mathcal{D}$. If $S * \theta_\varrho \in \mathbf{N}_\varrho$, then $\langle S * \theta_\varrho, \psi\rangle \sim 0$ since $\psi \in \mathcal{D}_k$ for some standard k. By Remark 2.7, $S = 0$. (b) follows immediately from Lemma 4.3. (c) is a simple application of the definitions and the fact that $\mathbf{N}_\varrho$ is an ideal ∎

Finally, since $\mathbf{N}_\varrho \subset \mathbf{d}'$ we can introduce an infinitesimality relation on $\mathbf{G}_\varrho$ by calling $[T]$ $\mathbf{G}_\varrho$-*infinitesimal* if $T \in \mathbf{d}'$ for some representative T of $[T]$. This infinitesimality relation may serve the same purpose as the notion of an "associated distribution" in the Colombeau algebras [2, Def. 3.5.2]: Indeed, multiplication in $\mathbf{G}_\varrho$ generally does not preserve distributional products other than the multiplication of ${}^{\mathrm{st}}\mathcal{O}_{\mathrm{M}}$. For instance, $x\delta(x) = 0$ in the sense of distribution theory, but the product $[x]\,[\theta_\varrho(x)]$ of the images of its factors is not equal to zero in $\mathbf{G}_\varrho$. We have however, that $[x]\,[\theta_\varrho(x)]$ is $\mathbf{G}_\varrho$-infinitesimal. A general result showing that many distributional products concide with the corresponding product in $\mathbf{G}_\varrho$ on a macroscopic level will now be stated. Let $S, T \in \mathcal{S}'(\mathbb{R}^n)$. Call $U \in \mathcal{S}'(\mathbb{R}^n)$ the $\mathbf{M}_6$-*product* of S and T, if

$$\lim_{\varepsilon \to 0} (S * \varphi_\varepsilon)\,(T * \varphi_\varepsilon) = U$$

in $\mathcal{D}'(\mathbb{R}^n)$, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi(x)\,dx = 1$.

Proposition 4.5: *Let $S, T \in {}^{\mathrm{st}}\mathcal{S}'$. If the product $U = \mathbf{M}_6(S, T)$ exists, then $[S * \theta_\varrho] \times [T * \theta_\varrho] - [U * \theta_\varrho]$ is $\mathbf{G}_\varrho$-infinitesimal.*

Proof: Along the lines of the proof of Prop. 3.6, one first deduces that

$$\langle (S * \theta_\varrho)\,(T * \theta_\varrho), \psi\rangle \sim \langle U, \psi\rangle \sim \langle U * \theta_\varrho, \psi\rangle \text{ for all } \psi \in {}^{\mathrm{st}}\mathcal{D};$$

next one employs an equicontinuity argument to obtain $\langle (S * \theta_\varrho)\,(T * \theta_\varrho), \psi\rangle \sim 0$ for all $\psi \approx_{\mathrm{D}} 0$. Then Prop. 2.8 gives that $(S * \theta_\varrho)\,(T * \theta_\varrho) - U * \theta_\varrho \in \mathbf{d}'$ ∎

Appendix 1. We consider here — in one dimension — the products of the Dirac measure δ and the distributions T_r defined by

$$T_r(x) = \sum_{m=1}^{\infty} \frac{1}{m^r}\,\delta\left(x - \frac{1}{m}\right).$$

with $r > 1$. The following assertions hold:

(a) $M_4(\delta, T_r)$ *does not exist for* $r < 2$;

(b) $M_3(\delta, T_r)$ *exists for* $r \geqq 2$, *and we have*: $M_3(\delta, T_2) = \frac{1}{2}\,\delta$, $M_3(\delta, T_r) = 0$ *for* $r > 2$.

Proof: We begin by showing that $M_4(\delta, T_2) = \frac{1}{2}\,\delta$. Let $\theta \in {}^{st}\mathcal{D}$ be as in Prop. 3.6, $\varrho \sim 0$, $\psi \in {}^{st}\mathcal{D}$. We may assume that support $(\theta) \subset [-1/2, 1/2]$. First,

$$\langle M_\varrho'^{\theta}(\delta, T_2), \psi\rangle = \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\varrho^2 m^2}\,\theta\left(\frac{x}{\varrho}\right)\theta\left(\frac{x}{\varrho} - \frac{1}{\varrho m}\right)\psi(x)\,dx$$

$$= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^2}\,\theta(x)\,\theta\left(x - \frac{1}{\varrho m}\right)\psi(\varrho x)\,dx.$$

Writing $\psi(\varrho x) = \psi(0) + \varrho x \psi'(\xi)$ it suffices (Prop. 3.6) to show that

$$\sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \int_{-\infty}^{\infty} \theta(x)\,\theta\left(x - \frac{1}{\varrho m}\right)dx \sim \frac{1}{2} \tag{A 1}$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \int_{-\infty}^{\infty} |x|\,\theta(x)\,\theta\left(x - \frac{1}{\varrho^m}\right)dx \sim 0. \tag{A 2}$$

Since support $(\theta) \subset [-1/2, 1/2]$, the integrals vanish if $m \leqq [1/\varrho]$ where $[1/\varrho]$ denotes the largest integer in $1/\varrho$. But $[1/\varrho]$ is infinite, thus the expression (A.2) is infinitesimal (because $\sum m^{-2}$ converges). To estimate (A.1) we rewrite it as

$$\sum_{m=[1/\varrho]+1}^{\infty} \frac{1}{\varrho m^2}\,\theta * \check{\theta}\left(\frac{1}{\varrho m}\right) = \sum_{m=[1/\varrho]+1}^{\infty} \frac{1}{\varrho}\left(\frac{1}{m} - \frac{1}{m+1}\right)\theta * \check{\theta}\left(\frac{1}{\varrho m}\right) + \sum_{m=[1/\varrho]+1}^{\infty} \frac{1}{\varrho}\,\frac{1}{m^2(m+1)}\,\theta * \check{\theta}\left(\frac{1}{\varrho m}\right).$$

Again, the second term is infinitesimal, because

$$\sum_{m=[1/\varrho]+1}^{\infty} \frac{1}{\varrho m^3} \sim \int_{1/\varrho}^{\infty} \frac{dx}{\varrho x^3} \sim 0. \tag{A 3}$$

The first term is recognized as a step function with infinitesimal step size $\sup\left\{\frac{1}{\varrho}\left(\frac{1}{m} - \frac{1}{m+1}\right) : m \geqq \left[\frac{1}{\varrho}\right] + 1\right\} \leqq \varrho$, which may be interpreted as the ϱ-th member of a standard net of step functions converging to $\theta * \check{\theta}$ on $[0, 1]$. Thus the first term is infinitely close to

$$\int_0^1 \theta * \check{\theta}(y)\,dy.$$

A simple calculation using the fact that $\theta * \check{\theta}$ is an even function shows that this integral equals 1/2.

To show that $M_3(\delta, T)$ exists also, we use Prop. 3.7 and proceed as in the proof of Cor. 3.8. If $\theta \in \mathbf{D}$ is as in Prop. 3.7 we write $\theta = \varphi + \eta$ with φ standard and $\eta \approx_{\mathbf{D}} 0$. We have to show that all the three sums

$$\sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \varphi * \check{\eta}\left(\frac{1}{\varrho m}\right), \quad \sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \eta * \check{\varphi}\left(\frac{1}{\varrho m}\right), \quad \sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \eta * \check{\eta}\left(\frac{1}{\varrho m}\right)$$

are infinitesimal. But if the support of θ is contained in $[-1/2, 1/2]$ so are the supports of φ and η. So all sums actually start with $m = [1/\varrho] + 1$. But $\sum_{m=[1/\varrho]+1}^{\infty} 1/\varrho m^2$ is a limited number (similar to (A.3)) and all convolutions are uniformly bounded by an infinitesimal number. In the case $r > 2$, already $\sum_{m=[1/\varrho]+1}^{\infty} 1/\varrho m^r \sim 0$, so

$$\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^r} \theta(x)\, \theta\left(x - \frac{1}{\varrho m}\right) \psi(\varrho x)\, dx \sim 0 \quad \text{for every} \quad \theta \in \mathbf{D}.$$

By Prop. 3.7, $M_3(\delta, T_r) = 0$. In the case $r < 2$ we take $\psi \equiv 1$ near zero and $\theta \in {}^{\mathrm{st}}\mathcal{D}$ such that $\theta * \theta(y) \geqq c$ for some standard $c > 0$ and $|y| \leqq 1/2$. Then

$$\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^r} \theta(x)\, \theta\left(x - \frac{1}{\varrho m}\right) \psi(\varrho x)\, dx \geqq c \sum_{m=[2/\varrho]+1}^{\infty} \frac{1}{\varrho m^r} \sim \infty,$$

thus $M_4(\delta, T_r)$ does not exist. The proof of (a) and (b) is complete ∎

Additional remarks: (a) The product $M_2(\delta, T_r) = 0$ exists and equals zero for $r > 2$. This follows from the remarks in the Introduction concerning the product (P1) and the fact that the corresponding assertion for the product (P1) of δ and T_r has been verified in [13, Appx]. It is not clear whether $M_2(\delta, T_2)$ exists. However, the product (P1) of δ and T_2 does not exist, because $\delta * T_2$ does not have a value at zero in the sense of Lojasiewicz (cf. [18, Prop. 4 and 5]).

(b) The product $M_1(\delta, T_r)$ does not exist for any $r > 1$: We exhibit a sequence $\{\chi_j\}_{j \geqq 1}$ of type (C_1) such that $\langle(\delta * \chi_j)(T_r * \chi_j), \psi\rangle$ does not converge if $\psi \equiv 1$ near zero. Let $\chi \in \mathcal{D}$, support $(\chi) \subset [-1, 1]$, $\chi \geqq 0$, $\int \chi(x)\, dx = \frac{1}{2}$, and set

$$\chi_j(x) = j^{r+1}\left(\chi\left(j^{r+1}\left(x - \frac{1}{2j}\right)\right) + \chi\left(j^{r+1}\left(x + \frac{1}{2j}\right)\right)\right).$$

Then

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \int_{-\infty}^{\infty} \chi_j(x)\, \chi_j\left(x - \frac{1}{m}\right) dx \geqq \frac{1}{j^r} \int_{-\infty}^{\infty} \chi_j(x)\, \chi_j\left(x - \frac{1}{j}\right) dx$$

$$= j^{r+2} \int_{-\infty}^{\infty} \chi^2\left(j^{r+1}\left(x - \frac{1}{2j}\right)\right) dx = j \int_{-\infty}^{\infty} \chi^2(y)\, dy \to \infty \quad \text{as} \quad j \to \infty.$$

Appendix 2. We collect here some basic notions from Internal Set Theory which are frequently employed in this paper; otherwise we refer to Nelson's article [12].

Transfer axiom: *Let $A(x, t_1, \ldots, t_k)$ be an internal formula with the free variables $x, t_1, \ldots, t_k$ and no other free variables. Then*

$$\left(\forall^{\mathrm{st}} x\, A(x, t_1, \ldots, t_k)\right) \Rightarrow \left(\forall x\, A(x, t_1, \ldots, t_k)\right)$$

whenever the parameters $t_1, \ldots, t_k$ *take standard values. Equivalent to this is*

$$\left(\exists x\, A(x, t_1, \ldots, t_k)\right) \Rightarrow \left(\exists^{\mathrm{st}} x\, A(x, t_1, \ldots, t_k)\right)$$

provided $t_1, \ldots, t_k$ *take standard values.*

The other axioms — idealization and standardization — are rarely applied here except via the following principles. They can be found in [12, p. 1166].

Construction principle for maps: *Let* X, Y *be standard sets, let* $A(x, y)$ *be a formula, internal or external, with free variables* x, y *and possibly others. Suppose that for all standard* $x \in X$ *there is a unique standard* $y \in Y$ *such that* $A(x, y)$. *Then there is a unique standard function* $f\colon X \to Y$ *such that* $A(x, f(x))$ *holds for all standard* $x \in X$.

Proof: See [12, Thm. 1.3] ∎

Permanence principles: *Let* $A(n)$ *be an internal formula over* $n \in \mathbb{N}$, *possibly containing other free variables.*

(1) *If* $A(n)$ *holds for all standard* $n \in \mathbb{N}$, *then there is an infinitely large* $\omega \in \mathbb{N}$ *such that* $A(n)$ *holds for* $1 \leqq n \leqq \omega$.

(2) *If* $A(\nu)$ *holds for all infinitely large* $\nu \in \mathbb{N}$, *then there is a standard* $n_0 \in \mathbb{N}$ *such that* $A(n)$ *holds for all* $n \geqq n_0$.

Proof: (1): $S = \{n \in \mathbb{N} : A(k) \text{ holds for } 1 \leqq k \leqq n\}$ is an internal set with $S \supset {}^{\mathrm{st}}\mathbb{N}$. Since ${}^{\mathrm{st}}\mathbb{N}$ is not a set [12, Thm. 1.1], S must contain an infinitely large number ω. (2) is proved similarly, see also [12, Example 6, p. 1177] ∎

We also need more general versions of these permanence principles. Let $(\Lambda, \leqq)$ be a standard directed set. An element $\omega \in \Lambda$ is called *infinitely large* if $\lambda \leqq \omega$ for all standard $\lambda \in \Lambda$.

Robinson's lemma: (1) *Let* Λ *be as above and let* $A(\lambda)$ *be an internal formula or a formula of the form* $A(\lambda) \equiv \left(\forall^{\mathrm{st}} y\colon B(\lambda, y)\right)$ *with* B *internal* (A, B *may contain other free variables*). *If* $A(\lambda)$ *holds for all standard* $\lambda \in \Lambda$, *then there is an infinitely large* $\omega \in \Lambda$ *such that* $A(\omega)$.

(2) *Let* $C(\lambda)$ *be internal or of the form* $C(\lambda) \equiv \left(\exists^{\mathrm{st}} y\colon D(\lambda, y)\right)$ *with* D *internal. If* $C(\omega)$ *holds for all infinitely large* $\omega \in \Lambda$, *then there is a standard* $\lambda \in \Lambda$ *such that* $C(\lambda)$.

Proof: (1): In the more general case $A(\lambda) \equiv \left(\forall^{\mathrm{st}} y\colon B(\lambda, y)\right)$ we have to show that

$$\exists \omega \in \Lambda\left(\forall^{\mathrm{st}} \lambda \in \Lambda \forall^{\mathrm{st}} y\colon \lambda \leqq w \text{ and } B(\omega, y)\right).$$

The validity is obtained from the idealization axiom, the hypothesis, and [12, Thm. 1.1]. (2) follows from (1) by negation ∎

We now need some facts about topology. Let X be a standard topological space, let $a \in {}^{\mathrm{st}}X$, $x \in X$. We say that x is *infinitely close* to a, denoted as $x \approx_X a$, if x is contained in all standard neighborhoods of a. Let again $(\Lambda, \leqq)$ be a standard directed set.

Characterization of the convergence of a standard net: *Let* X, Λ *be as above, let* $\{a_\lambda\}_{\lambda \in \Lambda}$ *be a standard net, let* $a \in {}^{\mathrm{st}}X$. *Then,* a_λ *converges to* a *if and only if* $a_\omega \approx_X a$ *for all infinitely large* $\omega \in \Lambda$.

Proof: If $a_\lambda \to a$ and V is a standard neighborhood of a, then $a_\omega \in V$ for all infinitely large $\omega \in \Lambda$. Thus $a_\omega \approx_X a$. Conversely, let $a_\omega \sim a$ for all infinitely large ω, let

V be a standard neighborhood of a, and consider the assertion $\forall \mu \geqq \omega : a_\mu \in V$. This is an internal formula which holds for all infinitely large ω. By Robinson's lemma, there is a standard λ such that $\forall \mu \geqq \lambda : a_\mu \in V$. By transfer, $a_\lambda \to a$ ∎

Of course, by reversing $\leqq$, "infinitely large" may be replaced by "infinitely small". We remark that if a standard net converges, then its limit is standard.

Finally, we discuss the notion of s-continuity. Let X, Y be standard topological spaces, $g : X \to Y$ a (possibly nonstandard) map, $a \in {}^{\mathrm{st}}X$. Then g is called *s-continuous* at a iff

$$\exists^{\mathrm{st}} b \in Y \text{ such that } \forall x \in X : x \approx_X a \Rightarrow g(x) \approx_Y b .$$

In the case of $Y = \mathbf{C}$ this implies that $g(a)$ is limited and $b = {}^0g(a)$. We only need the following special version of

The s-continuity theorem: *Let X be a standard topological space, $g : X \to \mathbf{C}$ a map which is s-continuous at every standard $a \in X$. Then there is a unique standard map $f : X \to \mathbf{C}$ such that $f(a) = {}^0g(a)$ for all standard $a \in X$, and f is continuous.*

Proof: Consider the formula $A(x, y) \equiv (g(x) \sim y)$. By the construction principle for maps, there is a unique standard $f : X \to \mathbf{C}$ such that $g(a) \sim f(a)$ for all standard $a \in X$, i.e. $f(a) = {}^0g(a)$. By transfer, it suffices to prove that f is continuous at any standard $a \in X$. For this we let $(a_\lambda)_{\lambda \in \Lambda}$ be a standard net converging to a and show

$$\forall^{\mathrm{st}} \varepsilon > 0 \exists^{\mathrm{st}} \lambda \in \Lambda \forall^{\mathrm{st}} \mu \geqq \lambda : |f(a_\mu) - f(a)| \leqq \varepsilon . \tag{A 4}$$

By the s-continuity of g, $g(a_\mu) \sim f(a)$ for all infinitely large μ. In particular, if ε is standard and λ infinitely large, then $|g(a_\mu) - f(a)| < \varepsilon$ for all $\mu \geqq \lambda$. By Robinson's lemma the latter assertion holds for some standard λ. If $\mu \geqq \lambda$ is standard, then $g(x_\mu) \sim f(x_\mu)$ by the construction of f. Thus $|f(x_\mu) - f(a)| < \varepsilon$ which proves (A.4) ∎

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VERFASSER:

Dr. MICHAEL OBERGUGGENBERGER
Institut für Mathematik und Geometrie der Universität
Technikerstr. 13
A-6020 Innsbruck