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2. Example: Find and three Anwendungen

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Products of Distributions: Nonstandard Methods

M. OBERGUGGEN BERG ER

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Es werden Hilfsmittel der Nonstandard-Analysis entwickelt, die zur Untersuchung der Multiplikation von Distributionen geeignet sind. Wir befassen uns mit Produkten, die durch Regularisierung und Grenzwertbildung definiert sind, und erhalten Nonstandard-Kriterien für die Existenz der Produkte (diese Kriterien erweisen sich als vorteilhaft bei der Berechnung konkreter Beispiele). Ferner leiten wir neue Vergleichsresultate für verschiedene Produktdefjni: tionen her. Als weitere Anwendung konstruieren wir Algebren von Distributionen - als Quotienten von externen Räumen glatter Funktionen -, die ähnliche Eigenschaften wie die Colombeauschen Algebren besitzen.

Развиваются вспомогательные средства нестандартного анализа пригодные к исследованию произведения обобщённых функций. Мы занимаемся произведениями определенными регуляризацией и образованием предела и получаем нестандартные критерии существования произведений (эти критерии оказываются выгодными при вычислении конкретных примеров). Выводятся также новые результаты сравнения для разных ¹определений произведения. Как дальнейшее применение мы построим алгебры обоб-Iценных функций - как фактор-алгебра внешних пространств гладких функций имеющие сходные свойства с алгебрами Colombeau.

Nonstandard tools are developed which are suitable for studying products of distributions defined by regularization and passage to the limit. We obtain nonstandard criteria for the existence of the products (which are demonstrated to be useful for calculating standard examples) as well as new stindard results clarifying the relationship between different types of such products. As an offspring we are able to construct algebras of distributions - as quotients of external spaces of smooth functions - which have properties similar to the Colombeau algebras. Nonstandard tools are developed which are suitable
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I. Introduction

Let, S and T be distributions on \mathbb{R}^n . One way to multiply S and T is to define their.

$$
(P1) \qquad \lim_{\epsilon \to 0} (S * \varphi^{\epsilon}) (T * \psi^{\epsilon})
$$

provided the limit exists in $\mathcal{D}'(\mathbb{R}^n)$ for all nets $\{\varphi^{\epsilon}\}_{{\epsilon}>0}$, $\{\psi^{\epsilon}\}_{{\epsilon}>0}$ which vary in certain classes of nets of smooth functions and converge to the Dirac measure (called deltanets). A strictly more general way is to take

$$
(P2) \qquad \lim_{\epsilon \to 0} (S * \varphi^{\epsilon}) (T * \varphi^{\epsilon})
$$

as the definition, and this is the product we shall be concerned here. Definition (P2) has become important recently because of its relation to Colombeau algebras: If the product of \tilde{S} and T in the sense of (P2) exists, then the element ST in the Colombeau algebra $\mathcal{S}(\mathbb{R}^n)$ admits an associated distribution [2, Thm. 3.5.7].

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We consider four successively smaller classes of delta-nets, leading to four succe sively more general definitions of the product (details in Section 3):

- (1) delta-nets in the sense of HIRATA-OGATA $[6]$ and MIKUSINSKI $[11]$;
- (2) restricted delta-nets in the sense of SHIRAISHI [18, p. 91] and ITANO [8];
- (3) delta-nets in the sense of ANTOSIK-MIKUSINSKI-SIKORSKI [1, p. 116] and KamiNSKI o
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(4) model delta-nets in the sense of KAMINSKI [9, p. 89].

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y of the classes (2)-(4 It is well known that for definition $(P1)$, equivalent products are obtained by using any of the classes $(2) - (4)$ (see ITANO [7], SHIRAISHI [18], KAMIŃSKI [9]), while class (1) produces. a more stringent definition [13, Appx]. it is also known (see **TTAN0** [7, p. 177]) that in dimension $n = 1$, the existence of the product (P1) of S and T with delta-nets of any of the types $(1) - (4)$ implies the existence of the Tillmann product $[20, p. 108]$ of S and T, which is defined by analytic regularization. What concerns product $(P2)$, no comparable results have been available so far. We show here that the existence of product $(P2)$ with delta-nets of type (2) implies the existence of the Tillmann product. We give a nonstandard criterion for the existence of the product (P2) with delta-nets of type (3), which enables one to conclude in many concrete examples that if product (P2) exists with delta-nets (4), then it also exists with delta-nets (3). The question of equivalence of the products (P2) obtained by employing classes (2) -(4) remains open; however, type (1) is seen to yield a less, general (4) model delta-nets in the sense
It is well known that for definit
any of the classes $(2) - (4)$ (see I
(1) produces a more stringent
 $(7, p. 177]$) that in dimension *n*
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delta-nets (3). The question of equivalence of the products (P2) obtained by employ
ing classes (2)-(4) remains open; however, type

product.
Colombeau has constructed (standard) commutative, associative differential algebras of generalized functions on open subsets Ω of \mathbb{R}^n with the following properties:

- (a) $\mathcal{D}'(\Omega)$ is a subspace;
- (b) the derivation in the algebra extends differentiation in the sense of distributions;
- (c) $\mathcal{E}^{\infty}(\Omega)$ is a differential subalgebra (with respect to the pointwise product on
- (d) the algebras are invariant under superposition by smooth maps-of polynomial

There are several possibilities to achieve such a construction $[2-4]$. For algebras with properties (a), (b), property (c) is optimal (for instance, the continuous functions cannot constitute a subalgebra [16]). Turning nonstandard, we observe that $\mathcal{C}^{\infty}(\Omega)$, viewed as an internal set, is an algebra into which the standard distributions may be. imbedded. But this imbedding does not render the standard smooth functions a subalgebra. We show that a quotient of a certain external subalgebra of $\mathcal{E}^{\infty}(\mathbb{R}^{n})$ does better: It contains the standard tempered distributions and has the standard smooth functions of polynomial growth as a subalgebra with respect to their pointwise product; it satisfies (b) and the standardized version of (d). A different nonstandard construction of an algebra with properties (a) - (d) has recently been given by TODOROV [22] using ultrapower methods.

We employ Nelson's version of nonstandard analysis: internal set theory [12]. In addition, we freely work with external sets, when appropriate. The plan of exposition isasfollows:

Section 2 provides the necessary background on the nonstandard theory of distributions. We follow the ideas of STROYAN-LUXEMBURG [19, Chap. 104], translated into internal set theory, but develop some additional material (including structure theo-. rems) which is frequently needed in the sequel. We found it useful to collect these rems) which is frequently needed in the sequel. We found it useful to conect these
results together with short proofs since they are not available in the literature in this form. Section-3 starts with a nonstandard definition of the product of any two stand-

ard distributions as an internal smooth function. This construction serves as a tool, and also relates our approach to Li BANG-HE'S [10], RAJU'S [15] and T0D0ROv's [21]. Then the nonstandard characterizations of product (P2) are given, and the comparison results are derived. Section 4 is devoted to the construction of the differential algebras containing the standard tempered distributions.

There are two appendices: In the first one we put the results of Section 3 to use, completing the investigation of the 'example in [13, Appx]. The 'notions we need from internal set theory - some of which go beyond Nelson's excellent introduction $[12]$ - are collected in the second appendix.

2. Background on the nonstandard theory of distributions

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For the following basic vocabulary the reader is referred to Nelson's article $[12]:$ standard; internal, external formula; internal, external set; infinitesimal (real, complex) nunber; limited (real, complex) number; infinitely large (natural, real, complex) number; standard part (of a limited number). Let $a, b \in \mathbb{C}$. We write $a \sim b$ if $a - b$ is infinitesimal; $a \sim \infty$ if a is infinitely large; for limited $a \in \mathbb{C}$, $^{\circ}a$ denotes the standard part of a . We use the quantifiers ential algebras containing the standard tempered distributions.

There are two appendices: In the first ofe we put the results of Section 3 to completing the investigation of the example in [13, Appx]. The notions we not

\exists stx for $\exists x$ (x standard) and \forall stx for $\forall x$ (x standard).

By abuse of notation, we shall employ the set brackets $\{\}$ and the elementhood " ζ " for internal and external sets alike, stating only verbally when a set is to be considered as external. Finally, if X is an internal set, we define the external set

$$
^{st}X = \{x \in X : x \text{ is standard}\}.
$$

In what follows, D, T, E , D', \cancel{S} are the usual spaces of functions and distributions on RP *(n* a fixed standard natural number); for these and all other internal spaces of distributions we use the notation of SCHWARTZ [17]; $\mathcal{D}_k = \{ \varphi \in \mathcal{D} : \varphi(x) = 0 \}$ *•for* $z \in \mathbb{R}^n$ is standard).

In what follows, $\mathcal{D}, \mathcal{J}, \mathcal{E}^{\infty}, \mathcal{D}', \mathcal{J}'$ are the usual spaces of functions and distribu-

tions on \mathbb{R}^n (*n* a fixed standard natural number); for these and all othe for $|x| \geq k$ for $k \in \mathbb{N}$. Following STROYAN-LUXEMBURG [19, Chap. 10.4] we introduce smooth functions on \mathbb{R}^n . spaces of distributions we use the notation of SCHWARTZ [17]; $\mathcal{D}_k = \{\varphi \in \mathcal{D} : \varphi(x) = 0$
for $|x| \ge k\}$ for $k \in \mathbb{N}$. Following STROYAN-LUXEMBURG [19, Chap. 10.4] we introduce
several external subsets (actually vec In what follows, $\mathcal{D}, \mathcal{J}, \mathcal{E}^{\infty}, \mathcal{D}', \mathcal{S}'$ are the usual spaces of fur
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spaces of distributions we use the notation of SCHWARTZ [17]; \mathcal

for $|z| \le k$; for $k \in \mathbb{N}$, Following STR
several external subsets (actually ve
smooth functions on \mathbb{R}^n .
Definition 2.1: (a) A function ψ
and $k \in \mathbb{N}$ and sup $\{|\partial^s \psi(x)| : x \in \mathbb{R}^n\}$
the external set $\$ the external set $\mathbf{D} = \{w \in \mathcal{D} : w \text{ is } \mathbf{D}\text{-limited}\}.$

(b) $\psi \in \mathcal{D}$ is called **D**-infinitesimal, denoted as $\psi \approx D_0$, if $\psi \in \mathcal{D}_k$ for some standard $k \in \mathbb{N}$ and sup $\{|\partial^{\alpha} \psi(x)| : x \in \mathbb{R}^n\} \sim 0$ for all standard $\alpha \in \mathbb{N}_0^n$.

(c) A function $\psi \in \mathcal{S}$ is called *S-limited* if $\sup \{(1 + |x|)^i | \partial^{\alpha} \psi(x) | : x \in \mathbb{R}^n \}$ is limited for all standard $l \in \mathbb{N}$ and all standard $\alpha \in \mathbb{N}_0^n$. We define the external set $S = \{ \psi \in \mathcal{S} : \psi \text{ is S-limited} \}.$ (d) $\psi \in \mathcal{D}$ is called *S-infinitesimal*, denoted as $\psi \approx_D 0$, if $\psi \in \mathcal{D}_k$ for some standard $\sum_{k=1}^{n} N$ and sup $\{|\partial^{\alpha}\psi(x)| : x \in \mathbb{R}^n\} \sim 0$ for all standard $\alpha \in \mathbb{N}_0^n$.

(c) A function $\psi \in \mathcal{F}$ is

(d) $\psi \in \mathcal{S}$ is called S-*infinitesimal*, denoted as $\psi \approx_S 0$, if $\sup \{(1 + |x|)^l | \partial^2 \psi(x) \}$:
 $x \in \mathbb{R}^n$ ~ 0 for all standard $l \in \mathbb{N}$ and all standard $\alpha \in \mathbb{N}_0$ ⁿ.

(e) An element $T \in \mathcal{C}^{\infty}$ is called a *limited distribution*, if $\int T(x) \psi(x) dx$ is limited for *all if* $\varphi \in \mathcal{S}$ is called *S-infinitesimal*, denoted as $\psi \approx_S 0$, if sup $\{(1 + |x|)^l | \partial^2 \psi(x) \}$: $x \in \mathbb{R}^n$ ~ 0 for all standard $l \in \mathbb{N}$ and all standard $\alpha \in \mathbb{N}_0^n$.
(e) An element $T \in \mathcal{E}^{\infty}$ i $\psi \in \mathbf{D}$. We define the external sets $\mathbf{D}' = \{T \in \mathcal{E}^{\infty} : T \text{ is a limited distribution}\}\$ and $\mathbf{d}' = \{T \in \mathcal{E}^{\infty} : T \text{ is an infinitesimal distribution}\}\$. (e) An element $T \in \mathcal{E}^{\infty}$ is called a *limited distribution*, if $\int T(x)\psi(x) dx$ is limited for $\psi \in \mathbf{D}$; T is called an *infinitesimal distribution*, if $\int T(x)\psi(x) dx \sim 0$ for all $\in \mathbf{D}$. We define the external all $\psi \in \mathbf{D}$; *T* is called an *infinitesimal distribution*, if $\int T(x) \psi(x) dx \sim 0$ for all $\psi \in \mathbf{D}$. We define the external sets $\mathbf{D}' = \{T \in \mathcal{E}^{\infty} : T \text{ is a limited distribution}\}$ and $\mathbf{d}' = \{T \in \mathcal{E}^{\infty} : T \text{ is an infinitesimal distribution}\}.$

(f) A (b) $\psi \in \mathcal{D}$ is called **D**-infinitesimal, denoted as $\psi \approx_D 0$, if $\psi \in \mathcal{D}_k$ for some sta
 $k \in \mathbb{N}$ and sup $\{|\partial^2 \psi(x)| : x \in \mathbb{R}^n\} \sim 0$ for all standard $\alpha \in \mathbb{N}_0^n$.

(c) A function $\psi \in \mathcal{P}$ is called

Notation: Given $\psi \in \mathbf{D}$, $T \in \mathbf{D}'$, we shall write $\langle T, \psi \rangle$ for $\int T(x) \psi(x) dx$.

Remark 2.2: It is clear that ${}^{st}\mathcal{D} \subset \mathbf{D}$ and ${}^{st}\mathcal{J} \subset \mathbf{S}$. Also, if $\psi \approx_{\mathbf{D}} 0$ (respectively $\psi \approx_{\mathbf{S}} 0$), then ψ is contained in every standard neighborhood of zero in $\mathcal D$ (respectively $\mathcal S$). In case of D , the converse is not true. Indeed, using the characterization of the neighborhoods of zero of **SCHWARTZ** [17, p. 651 and the idealization axiom [12, p. 11661, one sees easily that for every infinitely large $\omega \in \mathbb{N}$ there is an element $\varphi \in \mathcal{D}$ with support $(\varphi) \subset \{x \in \mathbb{R}^n : \omega \leq |x| \leq \omega + 1\}$ which is contained in every standard neighborhood of zero in \mathcal{D} . On the other hand, if $\psi \in \mathcal{D}_k$ for some standard $k \in \mathbb{N}$ and is contained in every standard neighborhood of zeroin \mathcal{D}_k , then 350 M. OBERGUGGENBERGER

Motation: Given $\psi \in \mathbf{D}$, $T \in \mathbf{D}'$, we shall write $\langle T, \psi \rangle$ for

Remark 2.2: It is clear that $\exists \mathcal{D} \subset \mathbf{D}$ and $\exists \mathcal{F} \subset \mathbf{S}$. Also, if

then ψ is contained in every standard

The next proposition characterizes the limited distributions as the "continuous" • elements with respect to the infinitesimality relation introduced above. The next proposition characterizes the

ments with respect to the infinitesimali

Proposition 2.3: Let $T \in \mathcal{E}^{\infty}$. Then the

(a) $T \in \mathbf{D}'$;

(b) If $\psi \in \mathcal{D}$ and $\psi \approx_{\mathbf{D}} 0$, then $\langle T, \psi \rangle \sim$

Proof: (a)

Proposition 2.3: Let $T \in \mathcal{E}^{\infty}$. Then the following are equivalent:

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0.

The next proposition characterizes the limited distributions as the "continuous"

ments with respect to the infinitesimality relation introduced above.

Proposition 2.3: Let $T \in \mathcal{E}^{\infty}$. Then the following are equiva **Proof:** (a) \Rightarrow (b): If $\psi \in \mathcal{D}$ and $\psi \approx_0 0$ then $w\psi \approx_0 0$ for all standard $w \in \mathbb{N}$ and so there is an infinitely large $\omega \in \mathbb{N}$ such that $\omega\psi \approx_0 0$ by Robinson's lemma (cf. Appx 2). In particular, $\omega \psi \in \mathbf{D}$, and so $|\langle T, \omega \psi \rangle| < \epsilon \omega$ for all standard $\epsilon > 0$, since *•* $\epsilon \omega$ is infinitely large. Thus $|\langle T, \psi \rangle| \sim 0$. (b) If $\psi \in \mathcal{D}$ and $\psi \approx_D 0$, then $\langle T, \psi \rangle \sim 0$
Proof: (a) \Rightarrow (b): If $\psi \in \mathcal{D}$ and $\psi \approx_D 0$ t
there is an infinitely large $\omega \in \mathbb{N}$ such
ppx 2). In particular, $\omega \psi \in \mathbf{D}$, and so $|\langle T \rangle$
is infinitel

0 for all infinitely large $x \in \mathbb{N}$, thus *Troon:* (a) \Rightarrow (b): *H* $\psi \in D$ and $\psi \approx_{D} 0$ then $\omega \psi \approx_{D} 0$ for an standard $\omega \in \mathbb{N}$ and
 ω in the permanence is an infinitely large $\omega \in \mathbb{N}$ such that $\omega \psi \approx_{D} 0$ by Robinson's lemma (cf
 ω is i *I* (b) \Rightarrow (a): Let $\psi \in \mathbf{D}$. Then $\frac{1}{\gamma}$
 $\left|\left\langle T, \frac{1}{\kappa} \psi \right\rangle\right| \leq 1$ for all such \varkappa . B a standard $k \in \mathbb{N}$ such that $\left|\left\langle T, \frac{1}{\gamma}\right\rangle\right|$ $\frac{1}{k} \psi$ α *i be following are equivalent:*
 α *c*_{*D*} 0 then $w\psi \approx_{D} 0$ for all standard $w \in \mathbb{N}$ a

such that $\omega\psi \approx_{D} 0$ by Robinson's lemma
 α $|\langle T, \omega\psi \rangle| < \varepsilon\omega$ for all standard $\varepsilon > 0$, sin
 α _D 0 f $\left|\left\langle T, \frac{1}{x} \psi\right\rangle\right| \leq 1$ for all such
a standard $k \in \mathbb{N}$ such that
Our next goal is to identify
end we fix a standard "moll
real number $\varrho \sim 0$. We set *m* $\psi \in \mathcal{D}$ and ψ is the dynamical state of \mathbb{D} , and ψ for \mathbb{D} , and \mathbb{D} . Thus $|\langle T, \psi \rangle| \sim$
 $\psi \in \mathbb{D}$. Then $\frac{1}{\kappa}$ is all such z. By the star of "mollifier" θ
We set θ ($\frac{x}{\varrho}$) $\left|\left\langle T, \frac{1}{x} \psi\right\rangle\right| \leq 1$ for all such z. By the permanence principle (cf. Appx 2)
a standard $k \in \mathbb{N}$ such that $\left|\left\langle T, \frac{1}{k} \psi\right\rangle\right| \leq 1$, that is, $T \in \mathbb{D}'$
Our next goal is to identify the standard dis $\begin{vmatrix} \Rightarrow (a): \text{ Let } \psi \in \mathbb{R} \end{vmatrix} \leq 1 \text{ for a}$

lard $k \in \mathbb{N}$ such

lard $k \in \mathbb{N}$ such

mber $\varrho \sim 0$. W
 $\theta_{\varrho}(x) = \varrho^{-n}\theta$

int to show that

ma 2.4: Let ψ

p 0.

f: It is clear t
 $\partial^2(\psi * \theta_{\varrho})$ (x)

' Our next goal is to identify the standard distributions as elements of *B'.* To this Our next goal is to identify the standard distributions as elements of **D'**. To this end we fix a standard "mollifier", $\theta \in {}^{st}\mathcal{D}$ with $\int \theta(x) dx = 1$ and an infinitesimal real number $\theta \sim 0$. We set $\theta_{\theta}(x) = e^{-n\theta} \$ Uur next goal is to identify the standard d
d we fix a standard "mollifier". $\theta \in {}^{s_1}\mathcal{D}$ w
al number $\theta \sim 0$. We set
 $\theta_e(x) = e^{-n\theta} \left(\frac{x}{\theta}\right)$
d want to show that the map $T \rightarrow T * \theta_e$ if
Lemma 2.4: Let $\psi \in \mathbf{D}$ a $\left|\left\langle \frac{I}{x}, \frac{V}{x}\right\rangle\right| \leq 1$

a standard $k \in \mathbb{N}$

our next goal if

end we fix a star

real number $\varrho \sim 0$
 $\theta_{\varrho}(x) = \varrho$

and want to show

Lemma 2.4: L
 $-\psi \approx 0$

Proof: It is clear

Then
 $\partial^2(\psi * \theta_{\varrho})$
 $\begin{cases} \n\vdots \ni 1, \text{ that is, } T \in \mathbf{D}' \n\end{cases}$
 $\text{and distributions as } \text{elementwise} \text{ and } \text{distributions}$
 $\text{and } \text{distributions} \text{ as } \text{elementwise} \text{ and } \text{and} \text{ } \text{and} \$

$$
\theta_e(x) = e^{-n\theta} \left(\frac{x}{\varrho}\right)
$$
\n(2.1)

\nand want to show that the map $T \to T * \theta_e$ is an imbedding of ${}^{\text{st}}\mathcal{D}'$ into **D'**.

\nLemma 2.4: Let $\psi \in \mathbf{D}$ and $\theta \in {}^{\text{st}}\mathcal{D}$, $\varrho \sim 0$ as above. Then $\psi * \theta_e \in \mathbf{D}$ and $\psi * \theta_e$

\n $-\psi \approx_{\text{D}} 0$.

\nProof: It is clear that $\psi * \theta_e$ belongs to some \mathcal{D}_k with k standard. Let $\alpha \in {}^{\text{st}}\mathbb{N}_0^n$.

\nThen,

\n
$$
\mathcal{E}(\psi * \theta_1)(x) = \int \theta(\psi) \, \partial^2 \psi(x - \varrho u) \, du
$$

0 *as above. Then* $\psi * \theta_e \in \mathbf{D}$ *and* $\psi *$ 2.4: Let $\psi \in \mathbf{D}$ and $\theta \in {}^{st}\mathcal{D}$, $\varrho \sim 0$
is clear that $\psi * \theta_{\varrho}$ belongs to sor
 $\varphi * \theta_{\varrho}$ (x) = $\int \theta(y) \partial^s \psi(x - \varrho y) dy$;

$$
\partial^{\alpha}(\psi * \theta_{\varrho}) (x) = \int \theta(y) \partial^{\alpha} \psi(x - \varrho y) dy,
$$

and this integral is limited independently of *x*, since sup $\{\frac{\partial^2 \psi(z)}{z \in \mathbb{R}^n\}}$ is limited. Thus $\psi * \theta_{\rho} \in D$. Next,

- ' *0 —* **p)** (x)I = If *0(y) (e(x —* y) --- *x) dy* kl *I y0(y) dy .* sup (Igradient ,p(z)l: *z €* 1R'}

which is infinitesimal independently of x

Let TE FOR *LEX***c**,
 $|\partial^{\alpha}(\psi * \theta_{e} - \psi)(x)| = |\int \theta(y) (\partial^{\alpha} \psi(x - \varrho y) - \partial^{\alpha} \psi(x) dy|$
 $\leq |\varrho| \int |y\theta(y)| dy \cdot \sup \{|\text{gradient } \partial^{\alpha} \psi(z)| : z \in \mathbb{R}^{n}\}$

which is infinitesimal independently of x ■

Lemma 2.5: *Let T* \in st D', $\psi \in \$ $T \in$ st \mathcal{D}' , there is a standard $m \in \mathbb{N}$ such that if $\varphi \in \mathcal{D}_k$ and *m*(initesimal independently of x **a**
 a 2.5: Let $T \in {}^{st}\mathcal{D}'$, $\psi \in \mathbf{D}$. Then $\langle T, \psi \rangle$ is limited. If $\psi \in \mathbf{D}$. There is a standard $k \in \mathbb{N}$ such that $\psi \in \mathcal{D}_k$. On there is a standard $m \in \mathbb{N}$ $|\psi(z)|: z \in \mathbb{R}^n$
 $|\psi(z)|: z \in \mathbb{R}$
 $\psi \approx_0 0$, then $\langle z \rangle$
 $\psi \approx_0 0$, then $\langle z \rangle$
 $\frac{1}{m}$,

$$
s_m(\varphi) = \sup \left\{ |\partial^{\alpha} \varphi(x)| : x \in \mathbb{R}^n \right\} \alpha \in \mathbb{N}_0^n, |\alpha| \leq m \right\} \leq \frac{1}{m},
$$

then $|T, \varphi\rangle| \leq 1$. But $s_m(\psi) \leq M$ for some standard $M \in \mathbb{N}$, thus we have that $|\langle T, \psi \rangle| \leq mM$, which is a limited number.
If $\psi \approx_0 0$, we take an infinitely large $\omega \in \mathbb{N}$ such that $\omega \psi \approx_0 0$. With k and m as

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If *y* ≈ *D* (*N*, which is a limited number.

If *y* ≈ *D* (*N*, we take an infinitely large $\omega \in \mathbb{N}$ such that $\omega \psi \approx_D 0$ Froducts of
then $|\langle T, \varphi \rangle| \leq 1$. But $s_m(\psi) \leq M$ for some standard $M \in \mathbb{N}$,
 $|\langle T, \psi \rangle| \leq mM$, which is a limited number.
If $\psi \approx_0 0$, we take an infinitely large $\omega \in \mathbb{N}$ such that $\omega \psi \approx$
above we have that **Products of Distributi**
 Products of Distributi
 Proposition 2.6: Let $T \in \mathbb{R}^2$ **, and thus** $|\langle T, \varphi \rangle| \leq \frac{1}{\omega} \sim 0$ **. With

Proposition 2.6: Let** $T \in \mathbb{R}^2$ **, and thus** $|\langle T, \psi \rangle| \leq \frac{1}{\omega} \sim 0$ **II**
 Proposit *en* $|\langle T, \varphi \rangle|$ ≤ **1**. But $s_m(\psi)$, $T, \psi \rangle$ ≤ mM , which is a line If $\psi \approx_D 0$, we take an infi
 If $\psi \approx_D 0$, we take an infi
 ove we have that $s_m(\omega \psi)$ ≤
 Proposition 2.6: Let T ∈

(a) *T* * θ_e ∈ **D'**.

(b) $\langle T * \theta_e, \psi \rangle \sim \langle T, \psi \rangle$ for all $\psi \in \mathbf{D}$.

Proposition 2.6:

(a) $T * \theta_e \in \mathbf{D}'$.

(b) $\langle T * \theta_e, \psi \rangle \sim \langle T \rangle$

(c) If $\langle T * \theta_e, \psi \rangle \sim$
 $= 0$. (c) If $\langle T * \theta_e, \psi \rangle \sim 0$ for all $\psi \in {}^{st}\mathcal{D}$, then $T = 0$. In particular, if $T * \theta_e \in d'$, then $T = 0$.

Proof: (a) Let $\psi \in \mathbf{D}$. Then $\langle T * \theta_e, \psi \rangle = \langle T, \psi * \check{\theta}_e \rangle$ where $\check{\theta}_e(x) = \theta_e(-x)$. By Lemma 2.4, $\psi * \check{\theta}_e \in \mathbf{D}$, by Lemma 2.5, $\langle T, \psi * \check{\theta}_e \rangle$ is limited. (b): $\langle T * \theta_e, \psi \rangle - \langle T, \psi \rangle$ $=\langle T, \psi * \check{\theta}_e - \psi \rangle$. The conclusion follows again from Lemmas 2.4 and 2.6. (c): If **Proof:** (a) Let $\psi \in \mathbf{D}$. Then $\langle T * \theta_e, \psi \rangle = \langle T, \psi * \check{\theta}_e \rangle$ where $\check{\theta}_e(x) = \theta_e(-x)$. By Lemma 2.4, $\psi * \check{\theta}_e \in \mathbf{D}$, by Lemma 2.5, $\langle T, \psi * \check{\theta}_e \rangle$ is limited. (b): $\langle T * \theta_e, \psi \rangle - \langle T, \psi \rangle = \langle T, \psi * \check{\theta}_e - \psi \rangle$. T Thus'(A) Let $\psi \in D$. Hen $\langle T * \theta_e, \psi \rangle = \langle T, \psi * \theta_e \rangle$ where $\theta_e(x) = \theta_e(-x)$. By
Lemma 2.4, $\psi * \theta_e \in D$, by Lemma 2.5, $\langle T, \psi * \theta_e \rangle$ is limited. (b): $\langle T * \theta_e, \psi \rangle - \langle T, \psi \rangle$
= $\langle T, \psi * \theta_e - \psi \rangle$. The conclusion follows agai Proof: (a) Let ψ
Lemma 2.4, $\psi * \check{\theta}_e \in \mathbb{R}$
= $\langle T, \psi * \check{\theta}_e - \psi \rangle$. Then $\langle T * \theta_e, \psi \rangle \sim 0$, then thus $\langle T, \psi \rangle = 0$ for a for all $\psi \in \mathcal{D}$ if Remark 2.7: The as

Remark 2.7: The assertions of Prop. 2.3 through Prop. 2.6 remain valid in the setting of tempered distributions, as is seen by a straightforward modification of the proofs. Specifically, the following version of Prop. 2.6 will be needed in Section 4: Let $T \in \mathbb{S}^4$ and $\theta \in \mathbb{S}^4$ *with* $\int \theta(x) dx = 1$, $\theta \sim 0$. *Then* (a) $T * \theta_e \in S'$; (b) $\langle T * \theta_e, \psi \rangle \sim \langle T, \psi \rangle$ for all $\psi \in S$; (c) if $\langle T * \theta_e, \psi \rangle$ ~ 0 *for all* $\psi \in {}^{st}D$, *then* $T = 0$. tempered distributions, as is seen by a straightforward modification of the proofs. Specific the following version of Prop. 2.6 will be needed in Section 4: Let $T \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^d$ $\int \theta(x) dx = 1$, $\varrho \sim 0$. Th = $\langle T, \psi * \theta_e - \psi \rangle$. The conclusion follows again from Lemmas 2.4
 $\langle T * \theta_e, \psi \rangle \sim 0$, then $\langle T, \psi \rangle \sim 0$ by (b). But $\langle T, \psi \rangle$ is a standard c

thus $\langle T, \psi \rangle = 0$ for all $\psi \in {}^{st}\mathcal{D}$. By transfer (cf. Appx 2) this

Prop. 2.6 (c) shows that convolution by θ_e produces an injective map ${}^{st}\mathcal{D}' \to D'/d'$.

Proposition 2.8: Let $\psi \in D$. Then there is a unique standard $\varphi \in \mathcal{D}$ with $\psi \approx_{D} \varphi$.

Proof: It follows from the mean value theorem and the fact that ψ belongs to **D** that ψ is s-continuous at every standard $a \in \mathbb{R}^n$. By the s-continuity theorem (cf. Appx 2), there is a standard, continuous function $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi(a) = {}^0\psi(a)$, for all $a \in {}^{st} \mathbb{R}^n$. Moreover, we even have $\varphi(x) \sim \psi(x)$ for all $x \in \mathbb{R}^n$, because $\varphi(x) - \psi(x)$ for all $\psi \in \mathcal{D}$. By earlisted (cf. Appx 2) this implies that (T, ψ)
for all $\psi \in \mathcal{D}$. Beneark 2.7: The assertions of Prop. 2.3 through Prop. 2.6 remain valid in the sett
tempered distributions, as is seen by a s attains its maximum on \mathbb{R}^n , and this maximum is infinitesimal. An analogous conclusion holds for all standard derivatives of ψ . Thus we have between the pair of Properties of $\mathcal{V}^* \circ \mathcal{V}^*$ and $\mathcal{V}^* \circ \mathcal{V}^*$ and $\mathcal{V}^* \circ \mathcal{V}^* \circ \mathcal{V}^*$ and $\mathcal{V}^* \circ \mathcal{V}^* \circ \mathcal{V}^*$ and $\mathcal{V}^* \circ \mathcal{V}^* \circ \mathcal{V}^*$ for all $\psi \in \mathcal{V}^*$ for all $\psi \in$ **o** sition 2.8: Let $\psi \in \mathbf{D}$. Then there is a unique standard $\varphi \in \mathcal{D}$ with $\psi \approx_{\mathbf{D}} \varphi$.
 continuous at every standard $a \in \mathbb{R}^n$ **.** By the s-continuity theorem (of single in a set a standard, continuous

$$
\forall \, {}^{st}\alpha \in \mathbb{N}^n \exists \, {}^{st}\varphi_a \colon \mathbb{R}^n \to \mathbb{C}, \, \varphi_a \text{ continuous, with}
$$
\n
$$
\sup \, \{|\partial^a \psi(x) - \varphi_a(x)| : x \in \mathbb{R}^n\} \sim 0. \tag{2.2}
$$

It remains to show that $\varphi_a = \partial^a \varphi$. Let first $\alpha = (1, 0, ..., 0), a \in \text{stR}^n, x \in \mathbb{R}^n, x \sim a$. Then

$$
\varphi(x) - \varphi(a) \sim \psi(x) - \psi(a) \sim (x_1 - a_1) \partial^{\alpha} \psi(\xi)
$$

$$
\sim (x_1 - a_1) \varphi_{\alpha}(a) + (x_1 - a_1) (\varphi_{\alpha}(\xi) - \varphi_{\alpha}(a))
$$

for some $\xi \sim a$. Since φ_a is continuous and standard, we have $(x_1 - a_1)^{-1} (\varphi(x))$ Figure 1. Moreover

attains its maximum on

clusion holds for all stan
 $\forall s^t \alpha \in \mathbb{N}^n$ and stan
 $\forall s^t \alpha \in \mathbb{N}^n$ and stan
 $\forall s^t \alpha \in \mathbb{N}^n$ and $\exists s^t \varphi_s$
 $\exists s^t \varphi_s$
 $\exists s^t \varphi_s$
 $\exists s^t \varphi_s$
 $\exists t$ remains **If remains to show that** $\varphi_a = \partial^a \varphi$ **.** Let first $\alpha = (1, 0, ..., 0)$, $a \in {}^{st} \mathbb{R}^n$, $x \in \mathbb{R}^n$, $x \sim a$.

Then
 $\varphi(x) - \varphi(a) \sim \psi(x) - \psi(a) \sim (x_1 - a_1) \partial^a \psi(\xi)$
 $\sim (x_1 - a_1) \varphi_a(a) + (x_1 - a_1) (\varphi_a(\xi) - \varphi_a(a))$

for some ξ standard $a \in \mathbb{R}^n$, and $\partial^{\alpha} \varphi(a) = \varphi_{\alpha}(a)$. By transfer, $\partial^{\alpha} \varphi = \varphi_{\alpha}$. The same argument Then
 $\varphi(x) - \varphi(a) \sim \psi(x) - \psi(a) \sim (x_1 - a_1) \partial^2 \psi(\xi)$
 $\sim (x_1 - a_1) \varphi_a(a) + (x_1 - a_1) (\varphi_a(\xi) - \varphi_a(a))$

for some $\xi \sim a$. Since φ_a is continuous and standard, we have $(x_1 - a_1)^{-1} \left(\frac{\pi}{2} - \varphi(a) \right) \sim \varphi_a(a)$. Thus φ is d with standard $k \in \mathbb{N}$, so does φ . That is, $\varphi \in {}^{st}\mathcal{D}$, and by (2.2) $\psi \approx_{\mathcal{D}} \varphi$. Uniqueness is evident \blacksquare for some $\xi \sim a$. Since φ_a is continuous
 $-\varphi(a)\rangle \sim \varphi_a(a)$. Thus φ is differentiable

standard $a \in \mathbb{R}^n$, and $\partial^a \varphi(a) = \varphi_a(a)$. By

works for all other standard derivatives,

with standard $k \in \mathbb{N}$, so do

Proposition 2.9: Let $T \in D'$. Then there is a unique standard $U \in \mathcal{D}'$, denoted by ${}^{\circ}T$, such that $\langle T, \psi \rangle \sim \langle U, \psi \rangle$ for all $\psi \in D$.

'Proof: Since $T \in D'$, $\sqrt[q]{T}$, ψ exists for every $\psi \in D$. By the construction principle for maps (cf. Appx 2), there is a unique standard map $U: \mathcal{D} \to \mathbb{C}$ such that $\langle U, \varphi \rangle$ $=$ $\mathfrak{O}(T, \varphi)$ for all $\varphi \in$ st \mathcal{D} . It is clear that *U* is linear (transfer). Fix $k \in$ stN. Prop. 2.3 together with Remark 2.2 says that *T* is s-continuous at every standard element of gether with Remark 2.2 says that *T* is s-continuous at every standard element of μ_k . Thus $U \in \mathbb{S}^d$, \mathcal{D}_k . 352 M. OBERGOGENBERGER

Proof: Since $T \in \mathbf{D}'$, $\mathbb{P}(T, \psi)$ exists for every $\psi \in \mathbf{D}$. By the construction principle

for maps (cf. Appx 2), there is a unique standard map $U : \mathcal{D} \to \mathbb{C}$ such that $\langle U, \varphi \rangle$
 and $\langle U, \varphi \rangle \sim \langle T, \varphi \rangle$ for all $\varphi \in {}^{s_1}\mathcal{D}$. If $\psi \in \mathbf{D}$, then there is a $\varphi \in {}^{s_1}\mathcal{D}$ with $\varphi \approx {}_{D}\psi$ by Prop. 2.9. But then $\langle U, \psi \rangle \sim \langle U, \varphi \rangle \sim \langle T, \varphi \rangle \sim \langle T, \psi \rangle$ by Lemma 2.5 and Prop. **2.31.** (cf. Appx 2), there is a unique standard map $U: \mathcal{D} \to \mathbb{C}$ such that $\langle U, \varphi \rangle$

(cf. Appx 2), there is a unique standard map $U: \mathcal{D} \to \mathbb{C}$ such that $\langle U, \varphi \rangle$

for all $\varphi \in {}^{4}D$. It is clear that U is lin

$$
st\mathfrak{D}' \hookrightarrow \mathbf{D}' \hookrightarrow \mathcal{E}^{\infty} \hookrightarrow \mathcal{D}' \tag{2.3}
$$

where the first one is given by convolution with θ_e , whereas the others are subspace relations. Moreover, convolution with θ_{ϵ} induces a bijection of st $\mathcal{D}' \hookrightarrow \mathbf{D}' \hookrightarrow \mathcal{E}^{\infty}$

e first one is give

Moreover, convo

st $\mathcal{D}' \simeq \mathbf{D}'/d'$

$$
{}^{\mathrm{st}}\mathcal{D}'\simeq \mathbf{D}'/\mathbf{d}'~.
$$

as follows from Prop. 2.6(c) and 2.9.

We now turn to structure theorems, which will be needed in Section 4. The first theorem is a counterpart to the classical structure theorem for \mathcal{D}' , asserting that limited distributions locally are finite derivatives of pointwise limited smooth functions. **Prop.** 2.6(c) and 2.9.

to structure theorems, which will be needed in Section 4. The first

unterpart to the classical structure theorem for \mathcal{D}' , asserting that

tions locally are finite derivatives of pointwise li

Proposition 2.10: Let $T \in \mathcal{E}^{\infty}$. The following are equivalent:

(b) For all standard $k \in \mathbb{N}$ -there exist an element $S \in \mathcal{E}^{\infty}$ with $\sup \{|S(x)| : |x| \leq k\}$ (b) For all standard $k \in \mathbb{N}$ there exist an element $S \in \mathbb{C}^{\infty}$ with $\sup_{\{x\}} \{S(x)\}$: μ
limited and a standard $\alpha \in \mathbb{N}_0$ ⁿ such that $T(x) = \partial^s S(x)$ for all $x \in \mathbb{R}^n$, $|x| \leq k$. (a) $T \in \mathbf{D}'$.

(b) For all standard $k \in \mathbb{N}$ -there exist an element $S \in \mathcal{C}^{\infty}$ with sup $\{ |k \text{ and } a \text{ standard } \alpha \in \mathbb{N}_0 \}$ such that $T(x) = \partial^{\alpha} S(x)$ for all $x \in \mathbb{R}^n$, $\{ \text{Proof } \colon (b) \Rightarrow (a) \colon \text{Let } \psi \in \mathbf{D} \text{ and let }$

$$
\langle T,\psi\rangle=\int \partial^{\alpha}S(x)\,\psi(x)\,dx=(-1)^{|\alpha|}\int S(x)\,\partial^{\alpha}\psi(x)\,dx
$$

is limited. (a) \Rightarrow (b): Let,k \in s^tN, and ^oT \in stD' as given by Prop. 2.9. By the classical structure theorem [17, p. 82] and transfer, there is, $\alpha \in {}^{st} \mathbb{N}_0$ ⁿ and a standard, continuous function *f* with compact support, such that $\langle {}^0T, \psi \rangle = (-1)^{|a|} \langle f, \partial^a \psi \rangle$ for all $\psi \in \mathcal{D}_{k+1}$. Letting θ_e be as in Prop. 2.6 we have that *limited and a standard* $\alpha \in \mathbb{N}_0$ ⁿ such that $T(x) = \partial^{\alpha}S(x)$ for
 $\hat{r}(\alpha) = \hat{r}(\alpha) \Rightarrow \hat{r}(\alpha) = \hat{r}$ *following are equivalent:*
 it an element $S \in \mathcal{E}^{\infty}$ *with* sup $\{|S(x)| : |x| \leq k\}$
 t $T(x) = \partial^{\alpha}S(x)$ *for all* $x \in \mathbb{R}^{n}$, $|x| \leq k$.
 $k \in \text{stN}$ such that $\psi \in \mathcal{D}_k$. Then
 $-1)^{|\alpha|} \int S(x) \partial^{\alpha} \psi(x) dx$
 and let $k \in {}^{st}N$ such
 $lx = (-1)^{|a|} \int S(x) \partial$

, and ${}^{0}T \in {}^{st}\mathcal{D}'$ as giv

d transfer, there is α

support, such that

p. 2.6 we have that

is and 2.9. Let $g(x) =$

k) is infinitesimal. S

..., \hat{x}_n) $d\xi$

...,

$$
\langle \partial^a (f * \theta_\varrho), \psi \rangle \sim \langle T, \psi \rangle
$$

for all $\psi \in \mathbf{D} \cap \mathcal{D}_{k+1}$ by Prop. 2.6 and 2.9. Let $g(x) = T(x) - \partial^2 (f * \varrho_{\epsilon})$ (x). It follows from (2.4) that $\sup \{|g(x)| : |x| \leq k\}$ is infinitesimal. Set
I^{(1,0,...,0) $g(x) = \int_a^x g(\xi, x_2, ..., x_n) d\xi$} from (2.4) that sup $\{ |g(x)| : |x| \leq k \}$ is infinitesimal. Set

$$
I^{(1,0,\ldots,0)}g(x)=\int\limits_{0}^{x_1}g(\xi,x_2,\ldots,\hat{x}_n)\,d\xi
$$

 $I^{(1,0,\ldots,0)}g(x) = \int_{0}^{x_1} g(\xi, x_2, \ldots, x_n) d\xi$
and define $I^{\beta}g(x)$ inductively for all $\beta \in {}^{st}\mathbb{N}^n$. Clearly, sup $\{|I^{\alpha}g(x)| : |x| \leq \text{n} \}$
intesimal as well; on the other hand, sup $\{|f * \theta_e(x)| : x \in \mathbb{R}^n\}$ is lim *k}* is infi is the other hand, sup $\{f * \theta_{\rho}(x) | : x \in \mathbb{R}^n\}$ is limited. If we set $I^{(1,0,\ldots,0)}g(x) = \int_{0}^{1} g(\xi, x_2, \ldots, x_n) d\xi$

and define $I^{\beta}g(x)$ inductively for all $\beta \in {}^{st}\mathbb{N}^n$. Clearly, sup {

intesimal as well; on the other hand, sup { $|f * \theta_e(x)| : x \in \mathbb{R}$
 $S(x) = I^{\alpha}g(x) + f * \theta_e(x)$, we have t $S(x) = I^{\alpha}g(x) + I * \theta_{\alpha}(x)$, we have that sup $\{ |S(x)| : |x| \leq k \}$ is limited and that and define $I^{\beta}g(x)$ inductively for all $\beta \in {}^{st}\mathbb{N}^n$. Clearly,

mitesimal as well; on the other hand, sup $\{|f * \theta_{\varrho}(x)| : S(x) = I^{\alpha}g(x) + f * \theta_{\varrho}(x)\}$, we have that $\sup \{|S(x)| : |x| \leq T(x) = \partial^{\alpha}S(x)$ for $|x| \leq k$ **I**

Coro $T(x) = \partial^x S(x)$ for $|x| \leq k$

Corollary 2.11: Let $T \in D'$ and $\rho \in \mathbb{R}$ be a positive infinitesimal. Then.

 \forall st $k \in \mathbb{N}$ \exists st $j \in \mathbb{N}$ such that $\sup \{|T(x)| : |x| \leq k\} \leq \varrho^{-j}$.

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Proof: Fix $k \in \text{N}$. Let f, g and α be as in the proof of Prop. 2.10. Then $T(x)$ $= g(x) + \partial^2(i * \theta_e)(x)$ for $|x| \leq k$; sup $\{|g(x)| : |x| \leq k\}$ is infinitesimal, and $k \in$ st**N**. Le
* θ_{ϱ} (x) for
* θ_{ϱ} (x) \leq $f(x) = g(x) + \partial^2(f * \theta_e)$ (x) for $|x| \leq k$; sup
 $|\partial^2(f * \theta_e) (x)| \leq \int |f(x - ey)|^2$
for some standard $C > 0$ and all $|x| \leq$
Me noneals that the execution of Co. **Froducts of Distributi**
 he as in the proof of Prop. 2.10.
 $\{|g(x)| : |x| \le k\}$ is infinitesimal, a
 $|g^{-|a|} \partial^a \theta(y)| dy \le C \rho^{-|a|}$
 k. Thus sup $\{|T(x)| : |x| \le k\} \le \rho^{-1}$
 \therefore 2.11 is not void: the constant fu

$$
|\partial^{\alpha}(f * \theta_{\varrho}) (x)| \leq \int |f(x - \varrho y) \varrho^{-|\alpha|} \partial^{\alpha} \theta(y)| dy \leq C \varrho^{-|\alpha|}
$$

|
|-1 ||

We remark that the assertion of Cor. 2.11 is not void: the constant function $T(x)$ $=\omega$ with ω infinitely large does not satisfy the assertion with $\rho = (\log \omega)^{-1}$.

3. Products of distributions

We start this section by introducing a nonstandard product of any two standard $=\omega$ with ω infinitely large does not satisfy the assertion with $\rho = (\log \omega)^{-1}$.
3. Products of distributions.
We start this section by introducing a nonstandard product of any two standard
distributions. Let $\theta \in {}^{st}\math$ Motivated by the inclusions (2.3) we make the following definition. **Model 3** $C > 0$ and all $|x| \leq k$. Thus a

nark that the assertion of Cor. 2.11 is
 α *i* of initialy large does not satisfy the

tts of distributions

this section by introducing a nonsta

ions. Let $\theta \in {}^{st}\mathcal{D}$ wi

Definition 3.1: Let $S, T \in \mathbb{S}\mathcal{D}'$. Then

$$
M_{\rho}^{\theta}(S, T) = (S * \theta_{\varrho}) (T * \theta_{\varrho})
$$

is called the M_o^{θ} -product of S and T (with θ_o defined by (2.1)).

Remark 3.2: (a) $M_{\rho}^{\theta}(S, T)$ belongs to \mathcal{E}^{∞} , but not to D', in general. The M_{ρ}^{θ} -product is commutative and satisfies the Leibniz rule. (b) If we allow θ to belong to $\mathfrak{sl}_{L^{\infty}} \cap L^1$, then the assertions of Prop. 2.6 are still true for $T \in {}^{st}\mathcal{D}'_L$. Thus in case S, $T \in {}^{st}\mathcal{D}'_L(\mathbb{R})$ it makes sense to define the product $M_a{}^d(S, T)$ where Δ is the Tillmann mollifier $\Delta(x) = [\pi(1 + x^2)]^{-1}$. This is Li Bang-He's product [10, p. 5641 when applied to integrable distributions. (c) Raju's definition [15, p. 384] is in a similar spirit, but not related to ours. In our notation, Raju defines the product of two standard distributions S and T as $(S * \theta_o)$ T with θ symmetric. The result is a noncommutative product valued in \mathcal{D}' . (d) In the framework of his "asymptotic functions", ToDoRov [21] has considered a product which leads to analogous formulas. Rather than choosing a fixed mollifier, Todorov works with certain classes of "kernels" representing a given Definition 3.1: Le
 $M_e^{\theta}(S, T) = ($

is called the M_e^{θ} -produ

Remark 3.2: (a) M_e^{θ}

commutative and satisfie

assertions of Prop. 2.6 are

to define the product M_e

is Li Bang-He's product

inition [15, p. 384] of *S* and *T* (with θ_e defined by (2.1)).
 T) belongs to \mathcal{E}^{∞} , but not to **D**', in general. The M_e^0 .

le Leibniz rule. (b) If we allow θ to belong to $\pi \mathcal{D}_{L^{\infty}} \cap L$

il true for $T \in \pi \mathcal{D}'_{L^1}$

Example 3.3: For the square of the Dirac measure δ in one dimension we have

on.
\n
$$
\langle M_e^{\theta}(\delta, \delta), \psi \rangle \sim \left\langle \frac{1}{\rho} c_0 \delta + c_1 \delta', \psi \right\rangle \text{ for all } \psi \in \mathbf{D},
$$

where

$$
c_0 = \int \theta^2(x) dx \quad \text{and} \quad c_1 = -\int x \theta^2(x) dx.
$$

Indeed, $\langle M_e^{\theta}(\delta, \delta), \psi \rangle = \frac{1}{\rho} \int \theta^2(x) \psi(\varrho x) dx$. The result follows by Taylor-expanding around zero up to order two and observing that the third term only contributes an infinitesimal to the product. Taking in particular the Tillmann mollifier \varDelta , one has α_e (c, c, e, $\gamma - \beta$
zero up to order
mal to the produc
 $\langle M_e^d(\delta, \delta), \psi \rangle \sim$

$$
\langle M_e^d(\delta, \delta), \psi \rangle \sim \left\langle \frac{1}{2\pi\varrho} \delta, \psi \right\rangle
$$
 for all $\psi \in \mathbf{D}$,

because in this case $c_0 = 1/2\pi$ and $c_1 = 0$. This is precisely Li Bang-He's result [10, p. 579]. A related formula involving the value of δ at zero holds in certain distribution algebras introduced by BERG [I a, p. 267].

We now turn to investigate internal products of distributions defined by regularization and passage to the limit. To fix notation, we introduce several classes of deltanets.

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Definition 3.4: (a) A net ${\varphi}^{\epsilon}{}_{0\leq \epsilon \leq 1} \subset \mathcal{D}(\mathbb{R}^{n})$ with For determinance $\varphi^{\epsilon} \geq 0$ and $\int \varphi^{\epsilon}(x) dx = 1$ for all ϵ .

$$
\varphi^{\epsilon} \geq 0
$$
 and $\int \varphi^{\epsilon}(x) dx = 1$ for all ϵ .

will be called a C_i -delta-net $(i = 1, 2, 3, 4)$, provided it satisfies condition (C_i) as 'follows: will be called a C_i-delta-net $(i = 1,$
follows:
(C₁) support $(\varphi^{\epsilon}) \rightarrow \{0\}$ as $\varepsilon \rightarrow 0$;
(C₂) support $(\varphi^{\epsilon}) \rightarrow \{0\}$ as $\varepsilon \rightarrow 0$ and $\varphi^{\epsilon} \geq 0$ and $\int \varphi^{\epsilon}(x) dx = 1$ for all ϵ

will be called a C_i-delta-net $(i = 1, 2, 3, 4)$, provides

follows:

(C₁) support $(\varphi^{\epsilon}) \rightarrow \{0\}$ as $\epsilon \rightarrow 0$;
 (ζ_2) support $(\varphi^{\epsilon}) \rightarrow \{0\}$ as $\epsilon \rightarrow 0$ and
 \forall

 (C_1) support $(\varphi^{\epsilon}) \rightarrow \{0\}$ as $\varepsilon \rightarrow 0$;

 $\forall \alpha \in \mathbb{N}_0^n \exists A_\alpha > 0$ such that $\int |x^{|\alpha|}|\partial^{\alpha} \varphi(x)| dx \leq A_\alpha$ for all ε ; (C₁) support (φ^{ϵ}) \rightarrow {0} as $\varepsilon \rightarrow 0$;

(C₂) support (φ^{ϵ}) \rightarrow {0} as $\varepsilon \rightarrow 0$ an
 $\forall \alpha \in \mathbb{N}_0^n \exists A_{\alpha} > 0$ such there (C₃) support (φ^{ϵ}) \subset { $x \in \mathbb{R}^n : |x| \le$
 $\forall \alpha \in \mathbb{N}_0^n \exists A_{\alpha} > 0$ (C_3) support $(\varphi^{\epsilon}) \subset \{x \in \mathbb{R}^n : |x| \leq \epsilon\}$ and

$$
\forall \alpha \in \mathbb{N}_0^n \exists A_\alpha > 0 \text{ such that } \varepsilon^{|\alpha|} \int |\partial^\alpha \varphi^\alpha(x)| dx \leq A_\alpha \text{ for all } \varepsilon;
$$

(C₄)
$$
\varphi^{t}(x) = \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)
$$
 for some $\varphi \in \mathcal{D}(\mathbb{R}^{n})$.

follows:
 (C_1) support $(\varphi^{\epsilon}) \to \{0\}$ as $\varepsilon \to 0$;
 (C_2) support $(\varphi^{\epsilon}) \to \{0\}$ as $\varepsilon \to 0$ and
 $\forall \alpha \in \mathbb{N}_0^n \exists A_{\alpha} > 0$ such that $\int |x^{|\alpha|}|\left|\frac{\partial \varphi(x)}{\partial x} \leq A_{\alpha} \text{ for all } \varepsilon$;
 (C_3) support $(\varphi^{\epsilon}) \subset \{x \in$ $\nabla \alpha \in \mathbb{N}_0^n \exists A_a > 0$ such that $\int |\mathcal{D}^{\alpha} \varphi^{\epsilon}(x)| dx \leq A_a$

ort $(\varphi^{\epsilon}) \subset (x \in \mathbb{R}^n : |x| \leq \epsilon)$ and
 $\nabla \alpha \in \mathbb{N}_0^n \exists A_a > 0$ such that $\epsilon^{|\alpha|} \int |\partial^{\alpha} \varphi^{\alpha}(x)| dx \leq A_a$ for
 $= \epsilon^{-n} \varphi \left(\frac{x}{\epsilon} \right)$ for some $\varphi \$

$$
\lim_{\epsilon \to 0} (S * \varphi^{\epsilon}) (T * \varphi^{\epsilon}) = M_i(S, T)
$$

exists in $\mathcal{D}'(\mathbb{R}^n)$ for all C_i-delta-nets\{ φ^{ϵ} ₀ \lt_{ϵ} \leq ₁ and is independent of the particular C, delta-net chosen (the last sentence is redundant für $i \neq 4$).

(b) Let $S, T \in \mathcal{D}(\mathbb{R}^n)$. We say that the M_1 -*product* $(i = 1, 2, 3, 4)$ or S and T
exists if
 $\lim_{\epsilon \to 0} (S * \varphi^{\epsilon}) (T * \varphi^{\epsilon}) = M_i(S, T)$
exists in $\mathcal{D}'(\mathbb{R}^n)$ for all C_i -delta-nets $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ *Notation:* The lower index notation φ_{ϵ} will be reserved for C₄-nets in accordance with (2.1), the upper index notation φ^{ϵ} for general delta-nets.

Remark 3.5: (a) C₁ nets were introduced by HIRATA-OGATA [6] and MIRUSINSKI [11], C₂-*Notation*: The lower index notation φ_{ϵ} will be reserved for C₄-nets in accordance with (2.1), the upper index notation φ^{ϵ} for general delta-nets.
Remark 3.5: (a) C₁-nets were introduced by HIRATA-OGATA nets by SHIRAISHI [18], called "restricted delta-nets" there, the condition $\int \varphi^{\epsilon}(x) dx = 1$
actually being replaced by lim $\int \varphi^{\epsilon}(x) dx = 1$, which obviously yields an equivalent product. The M₂ product was studied by ITANO [8]. C₃-nets were introduced by ANTOSIK-MIKU-SINSKI-SIKORSKI ^[1] and were studied by **KAMINSKI** ^[9] together with C₄-nets (called "model nets" there), which appear at many places [2, 3, 5, 7]. We remark that all authors quoted above use sequences instead of nets, but our definitions are equivalent. (b) In [1, 9] C_3 - C_i -delta-net chosen
 Notation: The low

with (2.1), the uppe

Remark 3.5: (a) C

nets by SHRAISHI [1]

actually being replaced

The M_2 -product was

sixski-SKKORSKI [1] a

nets'' there), which a

above use sequences The M₂-product was studied by ITANO [8]. C_3 -nets were introduced by ANTOSIR-MIKU-
sinSKI-SIKORSKI [1] and were studied by KAMINSKI [9] together with C_4 -nets (called "model
nets" there), which appear at many places uct this follows immediately from [14, Prop.]; in the case of the M₃-product'we observe
that (C_3) implies that sup $\{\varphi^{\epsilon}(x): x \in \mathbb{R}^n\} \leq \epsilon^{-n}A_0$. Thus if we take $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \geq 0$, $\chi(x)$
 $\chi(x)$ that (C_3) implies that $\sup \ |\varphi(x)| : x \in \mathbb{R}^n \le \varepsilon^{-n} A_0$. Thus if we take $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \ge 0$, $\chi(x)$
= 0 for $|x| \ge 2$, $\chi(x) \ge A_0$ for $|x| \le 1$, then $\varphi^t + \chi_t \ge 0$ for all ϵ , and the same arguments as in the proof of the Prop. in [14] apply. (c) Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi \geq 0$, $\varphi(x) = 0$ for $|x| \geq 2$, $\chi(x) \geq A_0$ for $|x| \leq 1$, then $\varphi' + \chi_t \geq 0$ for all ϵ , and the same arguments as in the proof of the P **support (q)** \subset [-1/2, 1/2]. Then $\varphi^{\epsilon}(x) = \frac{1}{2} \left(\frac{1}{\epsilon^2} \varphi \left(\frac{x}{\epsilon^2} \right) + \frac{1}{\epsilon} \varphi \left(\frac{x-\epsilon}{\epsilon} \right) \right)$ is an example of a support (φ) \subset [-1/2, 1/2]. Then $\varphi^{\epsilon}(x) = \frac{1}{2} \left(\frac{1}{\epsilon^2} \varphi \left(\frac{x}{\epsilon^2$ that (C_3) implies that $\sup \{|\varphi^{\epsilon}(x)| : x \in \mathbb{R}^n\} \leq \epsilon^{-n} A_0$. Thus if we take $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \geq 0$, $\chi(x) = 0$ for $|x| \geq 2$, $\chi(x) \geq A_0$ for $|x| \leq 1$, then $\varphi^{\epsilon} + \chi_{\epsilon} \geq 0$ for all ϵ , and the sa are equ

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 φ $\left(\frac{x}{\varepsilon} - \frac{\varepsilon}{\varepsilon}\right)$ is $\lceil \cdot \rceil$ is an example of a $\left(\frac{x}{\varepsilon^2}\right) + \frac{1}{\varepsilon} \varphi$
 $\frac{1}{2} \varphi \left(\frac{x-\varepsilon}{\varepsilon^2}\right)$
i, and the net which satisfies (C₂) but not (C₃), while $\psi^{t}(x) = \frac{1}{\epsilon^{2}} \varphi \left(\frac{x - \epsilon}{\epsilon^{2}} \right)$ is a net which satisfies (C₁) support (φ) \subset [-1/2, 1/2]. Then $\varphi^{\epsilon}(x) = \frac{1}{2} \left(\frac{1}{\epsilon^2} \varphi \left(\frac{x}{\epsilon^2} \right) + \frac{1}{\epsilon} \varphi \left(\frac{x-\epsilon}{\epsilon} \right) \right)$ is an example of a
net which satisfies (C₂) but not (C₃), while $\psi^{\epsilon}(x) = \frac{1}{\epsilon^2} \varphi \left(\frac{x-\epsilon}{\$ $2 \left(\frac{\varepsilon^2}{\varepsilon^2} \right) \left(\frac{\varepsilon}{\varepsilon^2} \right) \left(\frac{\varepsilon}{\varepsilon^2} \right) \left(\frac{\varepsilon}{\varepsilon} \right)$

t which satisfies (C_2) but not (C_3) , while $\psi^{\varepsilon}(x) = \frac{1}{\varepsilon^2} \varphi \left(\frac{x - \varepsilon}{\varepsilon^2} \right)$ is a net

t not (C_2) . On the other h that (C_3) implies that $\sup_{\{|x| \leq x\}} \sum_{\emptyset} e^{-\alpha} A_0$ for $|x| \leq x \leq R$, $\lim_{\emptyset} \sum_{\emptyset} e^{-\alpha} A_0$. Intis if we take χ $\epsilon \geq 0$ for $|x| \leq 2$, $\chi(x) \geq A_0$ for $|x| \leq 1$, then $\psi^{\epsilon} + \chi_{\epsilon} \geq 0$ for all ϵ , and as exists in $\mathcal{D}'(\mathbb{R}^n)$ for all G_1 delu-inetially ψ_{Pd+2} and ψ independent of the particular \mathcal{C}_r delta-interior (he last sentence is redundant fir $i = 4$).
 Motation: The lower index notation φ ,

but not (C_2) . On the other hand, $(C_{i+1}) \subset (C_i)$ for all *i*, and therefore the existence of the M_i -
product implies the existence of the M_{i+1} -product.

The following nonstandard characterization relates the M_{4} -product and the M_{ρ}^{0} .

(a) $M_4(S, T)$ *exists.*

o

 $\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right) \left$

. Proposition 3.6: Let $S, T \in \mathbb{R} \mathcal{D}'$. The following are equivalent:

(a) $M_4(S, T)$ exists.

(b) There is $W \in D'$ such that $\langle M_e^{\theta}(S, T), \psi \rangle \sim \langle W, \psi \rangle$ for all $\psi \in \mathbb{R} \mathcal{D}$, all $\varrho \sim 0$, and $\theta \in \mathbb{R} \mathcal$

(c) \exists *There is* $W \in \mathbf{D}'$ such that $\mathbf{M}_{\mathbf{e}}^{\theta}(S, T) - W \in \mathbf{d}'$ for all $\mathbf{e} \sim 0$ and all $\theta \in {}^{\text{st}}\mathcal{D}$ with $\int \theta(x) dx = 1$ and $\theta \geq 0$.

In this case we have $M_4(S, T) = \varnothing W$.

Proof: (a) \Rightarrow (b): Let $V = M_4(S, T) = \lim (S * \theta) (T * \theta)$. Then *V* is standard, .
- ^

and the characterization of the convergence of 'a standard net (cf. Appx 2) gives Froof: (a) \Rightarrow (b): Let $V = M_4(S, T) = \lim_{\epsilon \to 0} (S * \theta_{\epsilon}) (T * \theta_{\epsilon})$. Then *V* is standard,
and the characterization of the convergence of 'a standard net (cf. Appx 2) gives
 $\langle M_e^{\theta}(S, T), \psi \rangle \sim \langle V, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D$ Froducts of Distributions 355

Proof: (a) \Rightarrow (b): Let $V = M_4(S, T) = \lim_{r \to 0} (S * \theta_r) (T * \theta_r)$. Then V is standard

and the characterization of the convergence of 'a standard net (cf. Appx 2) gives
 $(M_e^*(S, T), \psi) \sim \langle V, \psi \rangle$ f fer, we may assume that $\{\varphi_i\}_{0 < \epsilon \leq 1}$ is standard. Then (b) holds with $\theta = \varphi_1$, and we have **Proof:** (a) \Rightarrow (b): Let $V = M_4(S, T) = \lim_{\epsilon \to 0} (S * \theta_{\epsilon}) (T * \theta_{\epsilon})$. Then V is standard,
and the characterization of the convergence of a standard net (cf. Appx 2) gives
 $\langle M_e^s(S, T), \psi \rangle \sim \langle V, \psi \rangle$ for all $\psi \in {}^{s\dagger}\mathcal{D}$ $\langle (S * \varphi_e) (\dot{T} * \varphi_e), \psi \rangle \sim \langle \dot{W}, \psi \rangle \sim \langle {}^0W, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$ and all $\varrho \sim 0$ (see Prop.
2.9). By the characterization of convergence of a standard net this means lim $\langle (S * \varphi_e)$ characterization of the convergence of a standard net (cf. Appx 2), ψ) $\sim \langle V, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$ and all $\varrho \sim 0$. Thus (b) holds with $W =$
 \therefore We show that $\lim_{\epsilon \to 0} (S * \varphi_{\epsilon}) (T * \varphi_{\epsilon}) = {}^{0}W$ for all

 \times $(T * \varphi_t), \psi$ = $\langle 0W, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$. Applying transfer again, we have (a).

(c) means that (b) holds not only for all standard $\psi \in \mathcal{D}$, but for all $\psi \in \mathbf{D}$. Thus it remains to prove (b) \Rightarrow (c). Let $\psi \in \mathbf{D}$. Since ψ is the sum of a standard test function and an infinitesimal one (Prop. 2.8) it suffices to prove the assertion (c) with $\psi \approx D$. First, the net $\{(S * \theta_t) (T * \theta_t)\}_{0 \leq t \leq 1} \subset \mathcal{D}'$ is pointwise bounded (this follows from its convergence and the support properties of θ_i , and hence equicontinuous. By transfer this means First $\{(\mathbf{S} * \mathbf{U}_{\epsilon}) \mid \mathbf{I} * \mathbf{U}_{\epsilon}\}\$ is pointwise bound
there and the support properties of θ_{ϵ}), and hence equals
 $\forall s \in \mathbf{S} > 0$ \exists^{st} neighborhood $\mathcal N$ of zero in $\mathcal D$ such that
 $\forall s, 0 < \epsilon \leq 1$, Formalling to prove (b) \Rightarrow (c). Let ψ
and an infinitesimal one (Prop. 2.
First, the net $\{\left(S * \theta_{\epsilon}\right) (T * \theta_{\epsilon})\}_{0 \leq \epsilon}$
convergence and the support prop
this means
 $\forall {}^{st}\delta > 0$ $\exists {}^{st}$ neighborhood
 $\forall {}_{\epsilon}, 0 < \epsilon \le$

$$
\forall \epsilon, 0 < \epsilon \leq 1, \forall \varphi \in \mathcal{N}: |\langle (S * \theta_{\epsilon}) (T * \theta_{\epsilon}), \varphi \rangle| < \delta.
$$

But ν belongs to every standard neighborhood of zero. in \mathcal{D} . Thus we have

$$
\forall^{\text{st}}\,\delta > 0\,\forall\,\epsilon,\,0 < \epsilon \leq 1: |\langle (S * \theta_{\epsilon})\,(T * \theta_{\epsilon}),\,\psi \rangle| < \delta,
$$

But ψ belongs to every standard
 $\forall^{st} \delta > 0 \ \forall \ \epsilon, 0 < \epsilon \le$

implying that $\langle M_e^{\theta}(S, T), \psi \rangle \sim$

Prop. 2.3 0. On the other hand we also have $\langle W, \psi \rangle \sim 0$ by

According to Remark 3.5 (b) the equivalence of (a) and (b), (c) in Prop. 3.6 remains valid when the condition $\theta \geq 0$ is dropped. Enhancing criterion (b) by requiring it to be fulfilled for all **D**-limited mollifiers θ we obtain the existence of the M_3 -product. *for B.1*, *for B.15 (b) the equivalence of (a) and (b), (c) in Prop. 3.6 remains* **then the condition** $\theta \ge 0$ **is dropped. Enhancing criterion (b) by requiring it Ifilled for all D**-limited mollifiers θ we obtain t $\forall \varepsilon, 0 < \varepsilon \leq 1$, $\forall \varphi \in \mathcal{N}: |\langle (S * \theta_{\varepsilon}) (T * \theta_{\varepsilon}), \varphi \rangle| < \delta$.
But ψ belongs to every standard neighborhood of zero in \mathcal{D} . Thus w
 $\forall s : \delta > 0 \forall \varepsilon, 0 < \varepsilon \leq 1: |\langle (S * \theta_{\varepsilon}) (T * \theta_{\varepsilon}), \psi \rangle| < \delta$,
implyin

Proposition 3.7: Let S, $T \in \mathbb{S}^1$. If there exists a $W \in \mathbb{D}'$ such that $\langle M_o^o(S, T), \psi \rangle$. $\sim \langle W, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$, all $\varrho \sim 0$ and all $\theta \in \mathbf{D}$ with $\int \theta(x) dx = 1$ and $\theta \geq 0$, then $M_3(S,T)$ exists, and it equals 0W .

Proof: By Prop. 3.6, we have that $M_4(S, T) = {}^0W$ exists. Let $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ be a standard C₃-net. We have to show that $\lim_{x \to \infty} (S * \varphi^c)$ $(T * \varphi^c) = {}^0W$. As in the proof of '—+0 Proof: By Prop. 3.6, we have that $M_4(S, T) = {}^0W$ existed C₃-net. We have to show that $\lim_{i\to 0} (S * \varphi^i) (T * \varphi^i)$
Prop. 3.6 it suffices to show that $\langle (S * \varphi^e) (T * \varphi^e), \psi \rangle \sim$ all $\rho \sim 0$. From (C_2) we obtain by t **Prop.** 3.6 it suffices to show that $\langle (S * \varphi^e) (T * \varphi^e), \psi \rangle \sim \langle^0 W, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$ and valid when the condition $\theta \ge 0$ is dropped. Enhance

to be fulfilled for all **D**-limited mollifiers θ we obtain t

Proposition 3.7: Let S, $T \in {^{st}\mathcal{D}}'$. If there exists $\sim \langle W, \psi \rangle$ for all $\psi \in {^{st}\mathcal{D}}$, all $\$ According to Remark 3.5

valid when the condition θ

to be fulfilled for all D-limit

Proposition 3.7: Let S
 $\sim \langle W, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$, at
 $M_3(S, T)$ exists, and it equal

Proof: By Prop. 3.6, we

dard C₃ According to Remark 3.5 (b) the equivalence of (a) and (b), (c) in Prop. 3.6 remains
valid when the condition $\theta \ge 0$ is dropped. Enhancing criterion (b) by requiring it
to be fulfilled for all D-initied mollifiers θ **Proposition 3.7:** *Let S*, $T \in \mathbb{R}^5$. *If* there exists a $W \in \mathbb{D}'$ such that $\langle M_v^0(\langle W, \psi \rangle)$ for all $\psi \in \mathbb{R}^5$, all $\varrho \sim 0$ and all $\theta \in \mathbb{D}$ with $\int \theta(x) dx = 1$ and $\theta \in$
 (S, T) exists, and it equal

all $\varrho \sim 0$. From (C₃) we obtain by transfer:
 $\forall^{st} \alpha \in \mathbb{N}_0^{n}$ ^{3st} $A_s > 0$ such that $\varepsilon^{|\alpha|} \int |\partial^s \varphi^{\varepsilon}(x)| dx < A_s$, for $\forall \varepsilon, 0 < \varepsilon \leq 1$,

all $\varrho \sim 0$. From (C_3) we obtain by transier:
 $\forall^{st} \alpha \in \mathbb{N}_0^n \exists^{st} A_s > 0$ such that $\varepsilon^{|\alpha|} \int |\partial^s \varphi^{\varepsilon}(x)| dx < A_s$, for $\forall \varepsilon, 0 < \varepsilon \le 1$,

in particular, $\varrho^{|\alpha|} \int |\partial^{\alpha} \varphi^{\varepsilon}(x)| dx$ is limited for all $\alpha \in$ infer that $\int |\partial^{\alpha}\theta(x)| dx$ is limited for all $\alpha \in {\rm^{st}}\mathbb{N}_0^n$. Since $\theta(x) = 0$ for $|x| \ge 1$, this implies that $\theta \in \mathbb{D}$. But $\varphi^{\varrho} = \theta_{\varrho}$, and so the hypotheses imply that $\langle (S * \varphi^{\varrho}) (T * \varphi^{\varrho}), \psi \rangle$ $\sim \langle$

Proof: We may assume that *f* is standard. Let $\theta \in D$ be as in Prop. 3.7. We write *Proof:* We may assume that *f* is standard. Let $v \in D$ be as in Prop. 3.7. We write $\theta = \varphi + \eta$ with φ standard and η **D**-infinitesimal. Let $\psi \in {}^{st}\mathcal{D}$, $\varrho \sim 0$. We will show that $\langle \theta_e(f * \theta_e), \psi \rangle \sim \langle \varphi_e(f *$ that $\langle \theta_e(f * \theta_e), \psi \rangle \sim \langle \varphi_e(f * \varphi_e), \psi \rangle$, from where the assertion follows by Prop. 3.6 and 3.7. This amounts to showing that - $\begin{array}{l} \rho \sim \rho \sim \rho \ \mathrm{or} \ \mathrm{of:} \ \rho \sim \rho \ \mathrm{of:} \ \rho \sim \rho \ \mathrm{of:} \$ is initial for all $\alpha \in \text{Hilb}_0^n$,
ited for all $\alpha \in \text{Hilb}_0^n$,
 $\{(\mathbb{R}^n)$. If $M_4(\delta, f)$ exists
the f is standard. Let θ
 α η **D**-infinitesimal.
 α , ψ , from where the
ving that
 $\theta + \eta_e(f * \eta_e), \psi \rangle \sim 0$.

$$
\langle \varphi_{\rm e}(f * \eta_{\rm e}) + \eta_{\rm e}(f * \varphi_{\rm e}) + \eta_{\rm e}(f * \eta_{\rm e}), \psi \rangle \sim 0.
$$

 $23*$

But the first term.

first term
\n
$$
\langle \varphi_e(f * \eta_e), \psi \rangle = \int \int \varphi(x) f(\varrho y) \eta(x - y) \psi(\varrho x) dy dx
$$

is infinitesimal, because integration extends over a standard compact region in \mathbb{R}^{2n} , φ , *f*, and ψ are bounded by a standard number, and η is infinitesimal. A similar esti-
mate applies to the other terms \blacksquare 356 M. OBERGUGGENBERGER

But the first term
 $\langle \varphi_{\theta}(f * \eta_{e}), \psi \rangle = \iint \varphi(x) f(\varrho y) \eta(x - y) \psi(\varrho x) dy dx$

is infinitesimal, because integration extends over a standard compact regio
 φ , *f*, and ψ are bounded by a standar

To demonstrate that the criteria of Prop. 3.6 and 3.7 are useful in concrete calculations, we continue the investigation of the example of [13, Appx] with regard to the products $M_1 - M_4$ in Appx 1.1t is seen that the M_2 -product may exist while the M_1 product does not. No example distinguishing the products M_2 , M_3 , M_4 is known. ate applies to the other terms **I**

To demonstrate that the criteria of Prop. 3.6 and 3.7 are useful in concrete calcula-

ms, we continue the investigation of the example of [13, Appx] with regard to the

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12]. $\hat{M}_5(S$

Our next goal will be to prove that the existence of the Shiraishi-Itano product M_2 implies the existence of the Tillmann product. For $S \in \mathcal{D}'(\mathbb{R})$ there exists a function from the support of the example of [15, Appx] with regard to the
roducts $M_1 - M_4$ in Appx 1. It is seen that the M_2 -product may exist while the M_1 -
roduct does not. No example distinguishing the products M_2 , $M_$ Our next goal will be to prove that the existence of the Shiraishi-Itan
implies the existence of the Tillmann product. For $S \in \mathcal{D}'(\mathbb{R})$ there exis
 $\hat{S}(z)$, analytic in $\mathbb{C} \setminus \text{support}(S)$, such that $S = \lim_{\epsilon \to 0} S_{\$ coducts M_1-M_4 in Appx 1..1

coduct does not. No example

Our next goal will be to pro

oplies the existence of the Ti

z), analytic in $\mathbb{C} \setminus$ support
 $\hat{S}(x + i\epsilon) - \hat{S}(x - i\epsilon)$, see [

Definition 3.9: Let $S, T \in$

Definition 3.9: Let $S, T \in \mathcal{D}'(\mathbb{R})$. The *Tillmann product* or M_5 -product of S and *T* is said to exist if $\lim \hat{S}_i \hat{T}_i = M_s(S, T)$ exists in $\mathcal{D}'(\mathbb{R})$.

$$
\hat{S}(z) = \frac{1}{2\pi i} \left\langle S(x), \frac{1}{x-z} \right\rangle.
$$

Letting $\Delta(x) = [\pi(1 + x^2)]^{-1}$ it is easily seen that, for S, $T \in \mathcal{D}'_{L^1}(\mathbb{R})$, Def. 3.9 is $\text{equivalent to } \lim_{\epsilon \to 0} \left(S * A_{\epsilon} \right) (T * A_{\epsilon}) = M_{5}(S, T) \text{ in } \mathcal{D}'(\mathbb{R}).$

Fig. 7.10: $\begin{cases} \n\mathcal{L}(x) = \frac{1}{2\pi i} \left(\frac{\mathcal{L}(x)}{x} - \frac{1}{z} \right) \n\end{cases}$

2.11. $\mathcal{L}(x) = [\pi(1 + x^2)]^{-1}$ it is easily seen that, for $S, T \in \mathcal{D}'_L(\mathbb{R})$, Def. 3.9 is

1.11. If $\mathcal{L}(X, T) = \lim_{\epsilon \to 0} (S * \varphi)$ $(T * \varphi)$ exi all symmetric C₂-nets ${\varphi^e}_{0\leq \epsilon \leq 1}$, then the Tillmann product $M_5(S, T)$ exists also and *coincides with U. In particular, if* $M_2(S, T)$ exists, then so does $M_5(S, T)$. Froposition 3.10: Let'S

all symmetric C₂-nets { φ^* }_{0<t'} coincides with U. In particular

Proof: By transfer we may

ture theorem for \mathcal{D}'_L , we have lim $(S * A_t)(T * A_t) = M_5(S, T)$ in $\mathcal{D}'(\mathbb{R})$.

on 3.10: Let $S, T \in \mathcal{D}'_L(\mathbb{R})$. If $U = \lim_{\epsilon \to 0} (S * \varphi^{\epsilon})$ (2
 C_2 -nets $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$, then the Tillmann product $M_5(S, T$

U. In particular, if $M_2(S, T)$ exist such that
 $\{S, T \in \mathcal{D}'_{L^1}(\mathbb{R})\}$. If $U = \lim_{\epsilon \to 0} (S * \varphi^{\epsilon})$
 $\{e_{\epsilon \leq 1}, \text{ then the Tillmann product } M_5(S, \text{lar, if } M_2(S, T) \text{ exists, then so does } M_5(\text{car, if } M_2(S, T) \text{ exists, then so does } M_5(\text{car, if } M_2(S, T) \text{ exists, then so does } M_5(\text{car, if } M_2(S, T) \text{ exists, then so does } M_5(\text{car, if } M_2(S, T) \text{ exists,$ $\mathcal{D}'_{L^1}(\mathbb{R})$. If $U = \lim_{\epsilon \to 0} (S * \varphi^{\epsilon}) (T * \varphi^{\epsilon})$ exists for
en the Tillmann product $M_5(S, T)$ exists also and
 $M_2(S, T)$ exists, then so does $M_5(S, T)$.
ume that S, T und U are standard. By the struc-
 g_j
 $\mathcal{G$

Proof: By transfer we may assume that *S*, *T* und *U* are standard. By the struc-
ture theorem for \mathcal{D}'_L , we have

$$
netric C_2-nets \{ \varphi^s \}_{0 \leq t \leq 1}, then t.
$$

with U. In particular, if M₂(S
f: By transfer we may assume
orem for \mathcal{D}'_L , we have

$$
S = \sum_{i=0}^l \partial^i f_i, \qquad T = \sum_{i=0}^m \partial^i g_i
$$

for some standard $l, m \in \mathbb{N}$ and standard $f_i, g_j \in L^1(\mathbb{R})$. Let $\varrho \sim 0$. We wish to show that $\langle (S * \Lambda_{\ell}) (T * \Lambda_{\ell}), \psi \rangle \sim \langle U, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}(\mathbb{R})$. The idea is to construct ture theorem for \mathcal{D}'_L , we hat
 $S = \sum_{j=0}^l \partial^j f_j$, T

for some standard $l, m \in \mathbb{R}$

show that $\langle (S * \Lambda_e) (T * \Lambda_e) \rangle$

a standard C_2 -net $\{\varphi^i\}_{0 \leq \epsilon \leq 1}$

sup $\frac{\int |\partial^i (A \setminus F) - \varphi|^2}{\epsilon}$ a standard C_2 -net $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ such that for some standard $l, m \in \mathbb{N}$ and standard
show that $\langle (S * A_{\varrho}) (T * A_{\varrho}), \psi \rangle \sim \langle U, \psi \rangle$ for
a standard C₂-net $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ such that
 $\sup \{|\partial^{i}(A_{\varrho}(\xi) - \varphi^{\varrho}(\xi))| : \xi \in \mathbb{R}\} \sim 0$
for $0 \leq j \leq n = l + m + 1$

$$
\sup \left\{ \left| \partial^j (\varDelta_\varrho(\xi) - \varphi^\varrho(\xi)) \right| : \xi \in \mathbb{R} \right\} \sim 0 \tag{3.1}
$$

• .

with U. In particular, if
$$
M_2(S, T)
$$
 exists, then so does $M_5(S, T)$.
\n: By transfer we may assume that S, T and U are standard. By the str-
\norem for \mathcal{D}'_{L^1} , we have
\n
$$
S = \sum_{j=0}^{l} \partial^j f_j, \qquad T = \sum_{j=0}^{m} \partial^j g_j
$$
\n: standard $l, m \in \mathbb{N}$ and standard $f_j, g_j \in L^1(\mathbb{R})$. Let $\varrho \sim 0$. We wish
\nt $\langle (S * \varDelta_{\varrho}) (T * \varDelta_{\varrho}), \psi \rangle \sim \langle U, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}(\mathbb{R})$. The idea is to construct
\nrd C_2 -net $\{\varphi^{\epsilon}\}_{\infty \leq 1}$ such that
\n
$$
\sup \{ |\partial^j (\varDelta_{\varrho}(\xi) - \varphi^{\varrho}(\xi))| : \xi \in \mathbb{R} \} \sim 0
$$
\n
$$
i \leq n = l + m + 1.
$$
 We then have
\n
$$
\langle (S * \varDelta_{\varrho}) (T * \varDelta_{\varrho}), \psi \rangle - \langle U, \psi \rangle
$$
\n
$$
\sim \langle (S * \varDelta_{\varrho}) (T * \varDelta_{\varrho}), \psi \rangle - \langle (S * \varrho^{\varrho}) (T * \varphi^{\varrho}), \psi \rangle
$$
\n
$$
= \langle (S * \varDelta_{\varrho}) (T * (\varDelta_{\varrho} - \varphi^{\varrho})), \psi \rangle + \langle (S * (\varDelta_{\varrho} - \varphi^{\varrho})) (T * \varphi^{\varrho}, \psi \rangle.
$$

The first term of the last line equals.

Proof 357

\nThe first term of the last line equals

\n
$$
\sum_{i=0}^{L} \sum_{j=0}^{m} \int \int \int f_i(z) \, \partial_z^i \Delta_{\theta}(x-z) \, g_j(y) \left(\partial_z^j \Delta_{\theta}(x-y) - \partial_z^j \varphi^{\varrho}(x-y) \right) \psi(x) \, dx \, dy \, dz
$$
\n
$$
= \sum_{i=0}^{L} \sum_{j=0}^{m} \sum_{k=0}^{i+1} {i + 1 \choose k} \int \int \int f_i(z) \, \frac{1}{\pi} \arctan\left(\frac{x-z}{\varrho}\right) g_j(y) \cdot
$$
\n
$$
\times \partial_z^{j+k} \left\{ \Delta_{\theta}(x-y) - \varphi^{\varrho}(x-y) \right\} \partial_z^{j+1-k} \psi(x) \, dx \, dy \, dz.
$$
\nGiven (3.1), the part $\partial_z^{j+k} \left\{ \Delta_{\theta}(x-y) - \varphi^{\varrho}(x-y) \right\}$ is infinitesimal independently of *x*, *y*, while all three integrals' are limited, since *f*_i, *g*_j and *∂^{i+1-k}y* are standard *L*-functions. Thus the first term is infinitesimal. The second term is estimated similarly, the part $\frac{1}{\pi}$ arctan $\left(\frac{x-z}{\varrho}\right)$ being replaced by $\int_{-\infty}^{\infty} \varphi(\xi) \, d\xi$ which is also bounded by Thus it remains to construct $\langle \varphi^{\epsilon} \rangle_{0 < \epsilon \leq 1}$. We take a standard *χ* ∈ *D*(R), *χ* symmetric, $0 \leq \chi \leq 1$, *χ*(*x*) = 1 for $|x| \leq 1$, *χ*(*x*) = 0 for $|x| \geq 2$, and set

functions. Thus the first term is infinitesimal. The second term is estimated similarly,

functions. Thus the first term is infinitesimal. The second term is estimated similarly,
the part $\frac{1}{\pi}$ arctan $\left(\frac{x-z}{a}\right)$ being replaced by $\int_{a}^{x-z} \varphi(\xi) d\xi$ which is also bounded by ven (3.1), the part $\partial_z t^{+\kappa} (A_{\varrho}$, while all three integrals
notions. Thus the first term
ne part $\frac{1}{\pi} \arctan \left(\frac{x - z}{\varrho} \right)$ be
e.
Thus it remains to construc
 $\leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq$
 $\varphi^{\epsilon}(x) = \Delta_{\epsilon}($

Thus it remains to construct ${\varphi^{\epsilon}}|_{0 \leq \epsilon \leq 1}$. We take a standard $\chi \in \mathcal{D}(\mathbb{R})$, χ symmetric, $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$, and set Thus it remains to construct $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$. We take a standard χ
 $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$, and set
 $\varphi^{\epsilon}(x) = \varDelta_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right)$.

We shall show that φ^{ϵ} ha

$$
p^{\epsilon}(x) = \Delta_{\epsilon}(x) \chi \left(\frac{x}{\lambda} \right).
$$

We shall show that φ^{ϵ} has the desired properties if we choose $\lambda = \epsilon^{1/(n+3)}$. To this end $\varphi^{\epsilon}(x) = \Delta_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right).$
We shall show that φ^{ϵ} has the desired properties if we choose $\lambda = \epsilon^{1/\epsilon}$
we need some preliminary estimates. We first observe that
 $\partial^{\epsilon} \Delta(x) = (1 + x^{2})^{-\epsilon - 1} P_{\epsilon}(x), \qquad i \geq 0,$ we need some preliminary estimates. We first observe that
 $\partial^i A(x) = (1 + x^2)^{-i-1} P_i(x), \quad i \ge 0,$

for some polynomials P_i of degree *i*, and

$$
\partial^{i} \Delta(x) = (1 + \dot{x}^{2})^{-i-1} P_{i}(x), \quad i \geq 0,
$$

it remains to construct
$$
\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}
$$
. We take
\n $[1, \chi(x) = 1 \text{ for } |x| \leq 1, \chi(x) = 0 \text{ for } |x|$
\n $\varphi^{\epsilon}(x) = \Delta_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right)$.
\n $[1, \chi(x)] = \Delta_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right)$.
\n $[2, \chi(x)] = (1 + x^{2})^{-i-1} P_{i}(x), \quad i \geq 0$
\n $[2, \chi(x)] = (1 + x^{2})^{-i-1} P_{i}(x), \quad i \geq 0$
\n $[2, \chi(x)] = (e^{2} + x^{2})^{-i-1} e^{i+1} P_{i} \left(\frac{x}{\epsilon}\right)$.
\n $[2, \chi(x)] = (e^{2} + x^{2})^{-i-1} e^{i+1} P_{i} \left(\frac{x}{\epsilon}\right)$
\n $[2, \chi(x)] \leq C(e^{2} + x^{2})^{-i-1} e^{i} \leq C e^{2} e^{2} e^{2}$

Using the fact that degree $(P_i) = i$ and that $\varepsilon < \lambda$ one deduces immediately the $\partial^i \Delta_i(x) = (\varepsilon^2)$
Using the fact that destimates $(0 \leq i \leq n)$ we choose $\lambda = \varepsilon^{1/(n+3)}$. To this end

rive that

for $\lambda \leq |x| \leq 2\lambda$ (3.2)
 $\begin{cases} \n\lambda & \text{if } 2\lambda \leq |x| \leq 2\lambda \n\end{cases}$

$$
\partial^i \Lambda_{\epsilon}(x) | \leq C(\epsilon^2 + \lambda^2)^{-i-1} \epsilon \lambda^i \leq C \epsilon \lambda^{-i-2} \quad \text{for} \quad \lambda \leq |x| \leq 2\lambda \tag{3.2}
$$

$$
\varphi^{\epsilon}(x) = \Delta_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right).
$$
\nWe shall show that φ^{ϵ} has the desired properties if we choose $\lambda = \epsilon^{1/(n+3)}$. To this end
\nwe need some preliminary estimates. We first observe that\n
$$
\partial^i \Delta(x) = (1 + \dot{x}^2)^{-i-1} P_i(x), \quad i \geq 0,
$$
\nfor some polynomials P_i of degree *i*, and\n
$$
\partial^i \Delta_{\epsilon}(x) = (\epsilon^2 + x^2)^{-i-1} \epsilon^{i+1} P_i \left(\frac{x}{\epsilon}\right).
$$
\nUsing the fact that degree $(P_i) = i$ and that $\epsilon < \lambda$ one deduces immediately the
\nestimates $(0 \leq i \leq n)$ \n
$$
|\partial^i \Delta_{\epsilon}(x)| \leq C(\epsilon^2 + \lambda^2)^{-i-1} \epsilon \lambda^i \leq \tilde{C} \epsilon \lambda^{-i-2}
$$
 for $\lambda \leq |x| \leq 2\lambda$ \n(3.2)\nand\n
$$
|\partial^i \Delta_{\epsilon}(x)| \leq |x|^{-i} (x^2 + \epsilon^2)^{-i/2-1} \epsilon^{i+1} |P_i \left(\frac{x}{\epsilon}\right)|
$$
\n
$$
\leq C\epsilon \lambda^{-i-2} \text{ for } |x| \geq 2\lambda;
$$
\nthere are an element of the elements of $|x| \geq 2\lambda$ and $\tilde{C}^i \Delta_{\epsilon}(x)$ for $|x| \geq 2\lambda$. For $\lambda \leq |x| \leq 2\lambda$ we infer

here and henceforth C denotes a generic positive constant. Next, $\partial^j\Big(A_\epsilon(x)\Big(1-\chi\Big(\frac{x}{\lambda}\Big)\Big)\Big)$ $\partial^i \Delta_i(x) = (\varepsilon^2 + x^2)^{-i-1} \varepsilon^{i+1} P_i\left(\frac{x}{\varepsilon}\right).$
Using the fact that degree $(P_i) = i$ and that
estimates $(0 \le i \le n)$
 $|\partial^i \Delta_i(x)| \le C(\varepsilon^2 + \lambda^2)^{-i-1} \varepsilon \lambda^i \le C\varepsilon \lambda^{-1}$
and
 $|\partial^i \Delta_i(x)| \le |x|^{-i}(x^2 + \varepsilon^2)^{-i/2-1} \varepsilon^{i+1}| P$ equals zero for $|x| \leq \lambda$ and equals $\partial^i A_i(x)$ for $|x| \geq 2\lambda$. For $\lambda \leq |x| \leq 2\lambda$ we infer

$$
|\partial^{i} A_{i}(x)| \leq C(\varepsilon^{2} + 2^{2})^{-i-1} \varepsilon \lambda^{i} \leq C\varepsilon \lambda^{-i-2} \quad \text{for} \quad \lambda \leq |x| \leq 2\lambda \tag{3.2}
$$

and

$$
|\partial^{i} A_{i}(x)| \leq |x|^{-i}(x^{2} + \varepsilon^{2})^{-i|2-1} \varepsilon^{i+1} |P_{i}(\frac{x}{\varepsilon})|
$$

$$
\leq C\varepsilon \lambda^{-i-2} \quad \text{for} \quad |x| \geq 2\lambda; \tag{3.3}
$$

here and henceforth *C* denotes a generic positive constant. Next, $\partial^{j} (A_{i}(x) (1 - \chi(\frac{x}{\lambda})))$
equals zero for $|x| \leq \lambda$ and equals $\partial^{j} A_{i}(x)$ for $|x| \geq 2\lambda$. For $\lambda \leq |x| \leq 2\lambda$ we infer
from (3.2) that

$$
\left|\partial^{j} (A_{i}(x) (1 - \chi(\frac{x}{\lambda})))\right| = \left|\sum_{i=0}^{j} {j \choose i} \partial^{i} A_{i}(x) \partial^{j-i} (1 - \chi(\frac{x}{\lambda}))\right|
$$

$$
\leq C \sum_{i=0}^{j} {j \choose i} (\varepsilon \lambda^{-i-2}) \lambda^{j-i} \leq C\varepsilon \lambda^{-j-2} \tag{3.4}
$$

for $0 \leq j \leq n$. For $|x| \geq 2\lambda$ the expression is estimated by (3.3) and we have

$$
\sup \{|\partial^{j} A_{i}(x) - \partial^{j} \varphi(x)| : x \in \mathbb{R}\} \leq C\varepsilon \lambda^{-n-2} \tag{3.5}
$$

for $0 \leq j \leq n$. Therefore, if we take $\lambda = \varepsilon^{1/(n+3)}$ and evaluate (3.5) at $\varepsilon = \varrho \sim 0$, we
obtain the desired infinitesimality assertion (3.1).

22 the expression is estimated by (3.3) and we have $\sup \{|\partial^i A_i(x) - \partial^i \varphi^i(x)| : x \in \mathbb{R}\}\leq C \varepsilon \lambda^{-n-2}$ (3.5)
for $0 \leq j \leq n$. Therefore, if we take $\lambda = \varepsilon^{1/(n+3)}$ and evaluate (3.5) at $\varepsilon = \varrho \sim 0$, we

obtain the desired infinitesimality assertion (3.1).

 $\ddot{}$

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8 M. OBERGUGGENBERGER
It remains to prove that ${\varphi^{\epsilon}}_{0 < \epsilon \leq 1}$
 \rightarrow {0} as $\varepsilon \rightarrow 0$. Second, is a C₂-net. First, $\varphi^{\epsilon} \ge 0$ and support $(\varphi^{\epsilon}) \subset [-2\lambda, 1]$ 358 M. OBERGUGGENBER

It remains to prove that {
 2λ] \rightarrow {0} as $\varepsilon \rightarrow 0$. Second,

358 M. OBERQUGGENBEROER
\nIt remains to prove that
$$
\{\varphi^{\epsilon}\}_{0<\epsilon\leq 1}
$$
 is a C₂-net. First, $\varphi^{\epsilon} \geq$
\n22] $\rightarrow \{0\}$ as $\epsilon \rightarrow 0$. Second,
\n
$$
\int_{-\infty}^{\infty} \varphi^{\epsilon}(x) dx = \int_{-2}^{2} \Delta_{\epsilon}(x) dx + \int_{|x| \geq 1} \Delta_{\epsilon}(x) \chi \left(\frac{x}{\lambda}\right) dx.
$$
\nDue to the relation between ϵ and λ , the first integral on t
\n1, the second to 0 as $\epsilon \rightarrow 0$. Therefore,
\n
$$
\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \varphi^{\epsilon}(x) dx = 1
$$
\nwhich is not quite the condition required of C₂-nets, but or
\nequivalent to the M₂-product (cf. Remark 3.5(a)). Finally

Due to the relation between ε and λ , the first-integral on the right-hand side tends to **1**

$$
\lim_{t\to 0}\int\limits_{-\infty}^{\infty}\varphi^t(x)\,dx=1
$$

which is not quite the condition required of C_2 -nets, but one which leads to a product equivalent to the M_2 -product (cf. Remark 3.5(a)). Finally, we have to show that for $every \, j \geq 0$, lim $\int_{-\infty}^{\infty} \varphi^{\epsilon}(x) dx = 1$
 $\mapsto 0 - \infty$

hich is not quite the condition required of C₂-nets, but one which lead

quivalent to the M₂-product (cf. Remark 3.5(a)). Finally, we have to

cery $j \ge 0$,

sup $\int_{0 < \epsilon \$ lim $\int_{-\infty}^{\infty} \varphi^t(x) dx = 1$
 $\longleftrightarrow -\infty$

which is not quite the condition required of C₂-nets, but one which lea

equivalent to the M₂-product (cf. Remark 3.5(a)). Finally, we have t

svery $j \ge 0$,
 $\sup_{0 < \epsilon \le 1} \int_{-\$ Due to the relation between ε and λ

1, the second to 0 as $\varepsilon \to 0$. Thereform
 $\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \varphi^{\epsilon}(x) dx = 1$

which is not quite the condition requivalent to the M₂-product (cf. Revery $j \ge 0$,

sup \int

$$
\sup_{\epsilon \le 1} \int_{-\infty}^{\infty} |x^j \partial^j \varphi^{\epsilon}(x)| dx < \infty. \tag{3.6}
$$
\n
$$
2\lambda \text{ the integrand vanishes. For } \lambda \le |x| \le 2\lambda \text{ an estimate similar to (3.4)}
$$
\n
$$
\text{fixed } j \ge 0,
$$
\n
$$
i \left(\Delta_{\epsilon}(x) \chi \left(\frac{x}{\lambda} \right) \right) \le C\epsilon \lambda^{-j-2},
$$
\n
$$
\int_{\{|x| \le 2\lambda}^{\infty}} |x^j \partial^j \varphi^{\epsilon}(x)| dx \le C\lambda^{j+1} \epsilon \lambda^{-j-2} = C \frac{\epsilon}{\lambda} \to 0
$$

S **/**

every $j \ge 0$,

sup $\int_{0 < t \le 1}^{\infty} |x^j \partial^j \varphi^t(x)| dx < \infty$.

For $|x| \ge 2\lambda$ the integrand vanishes. For $\lambda \le |x| \le$

yields, for fixed $j \ge 0$, *22* an estimate similar to (3.4) \mathbb{R}

thus
$$
\left|\frac{\partial^j \left(\varDelta_\epsilon(x) \chi\left(\frac{x}{\lambda}\right)\right)}{z}\right|
$$

$$
|\int_{\lambda \leq |x| \leq 2\lambda} |x^{j}\partial^{j}\varphi^{\epsilon}(x)| dx \leq C\lambda^{j+1} \epsilon \lambda^{-j-2} = C \frac{\epsilon}{\lambda} \to 0
$$

 $C_{\cal{E}}$ i-i- 2 .

as $\varepsilon \to 0$. For $|x| \leq \lambda$, the integrand equals $|x^i \partial^i \Delta_i(x)|$, and

$$
\left|\frac{\partial^j \left(A_{\epsilon}(x) \chi\left(\frac{x}{\lambda}\right)\right)}{x \leq |x|^2} \right| \leq C \varepsilon \lambda^{-j-2},
$$
\n
$$
\int_{x \leq |x| \leq 2\lambda} |x^j \partial^j \varphi^{\epsilon}(x)| dx \leq C \lambda^{j+1} \varepsilon \lambda^{-j-2} = C \frac{\varepsilon}{\lambda} \to 0
$$
\n
$$
\int_{x \leq |x| \leq 2\lambda} \left| x^j \partial^j A_{\epsilon}(x) \right| dx = \int_{-\lambda}^{\lambda} \left| x^j (x^2 + \varepsilon^2)^{-j-1} \varepsilon^{j+1} P_j\left(\frac{x}{\varepsilon}\right) \right| dx
$$
\n
$$
\leq \int_{-\infty}^{\infty} |x^j (x^2 + 1)^{-j-1} P_j(x)| dx < \infty
$$
\ngree $(P_j) = j$, and (3.6) is proved \blacksquare \n
$$
\text{tr } k \cdot 3.11 : \left\{ \varphi^{\epsilon} \right\}_{0 \leq \epsilon \leq 1} \text{ is not a } C_3 \text{-net. Indeed,}
$$
\n
$$
2\lambda \int_{-\infty}^{\infty} |\partial^j \varphi^{\epsilon}(x)| dx \geq 2\lambda \int_{-\lambda}^{\lambda} |\partial^j A_{\epsilon}(x)| dx = \frac{2\lambda}{\pi \varepsilon} \int_{-\lambda/\epsilon}^{\lambda/\epsilon} \frac{|2x|}{(1+x^2)^2} dx
$$

since degree $(P_i) = j$, and (3.6) is proved **I**

Remark 3.11: $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ is not a C₃-net. Indeed,

(X)I dx = *f* 12x1 A *dx* -^ oc raj (1±x2) - **00** —1 *—Alt product* M5 (S, *T) exists also and coincides with U.*

as $\varepsilon \to 0$. It remains open whether the existence of the M₃ or M₄ product implies the existence of the Tillmann product.

Corollary 3.12: Let $S, T \in \mathcal{D}'(\mathbb{R})$. *If* $U = M_2(S, T)$ exists, then the Tillmann product $M_5(S, T)$ exists also and coincides with U.

 $e \rightarrow 0$. It remains open whether the existence of the M_3 or M_4 -product implies the existence
the Tillmann product.
Corollary 3.12: Let *S*, $T \in \mathcal{D}'(\mathbb{R})$. If $U = M_2(S, T)$ exists, then the Tillmann
oduct $M_5(S, T)$ as $\varepsilon \to 0$. It remains open whether the existence of the M_3 or M_4 -product implies the existence
of the Tillmann product.
Corollary 3.12: Let S, $T \in \mathcal{D}'(\mathbb{R})$. If $U = M_2(S, T)$ exists, then the Tillmann
product of the Tilmann product.

Corollary 3.12: Let S, $T \in \mathcal{D}'(\mathbb{R})$. If $U = M_2(S, T)$ exists, then

product $M_5(S, T)$ exists also and coincides with U.

Proof: We may assume that S, T, U are standard. Letting $\rho \sim 0$ a

we h neighborhood of support (ψ) . Then both $T(1-\chi)^2$ and $S(1-\chi)^2$ are analytic in product $M_5(S, T)$ exists also and coincides with U.

Proof: We may assume that S, T, U are standard. Letting ρ

we have to show that $\langle S_e \hat{T}_e, \psi \rangle \sim \langle U, \psi \rangle$. Take $\chi \in {}^{st}\mathcal{D}(\mathbb{R})$, preighborhood of support (ψ)

Products of Distributions 359

 $\frac{1}{2}$

 $\mathbb{C}\setminus \text{support } (1-\chi),$
 $\left(\cdot\right)_e^*$ is a linear operation so $T(1 - \chi)e^{\lambda}$ and $S(1 - \chi)e^{\lambda}$ vanish on support (*v*). Since $(\cdot)_e$ is a linear operation and $(S_X)_e$ = $(S_X) * J_e$ we are reduced to showing that is a linear operation and $(S_{\chi})_e^{\gamma} = (S_{\chi}) * \Lambda_e$ we are reduced to snowing that $* \Lambda_e$ $(T_{\chi} * \Lambda_e)$, ψ $\sim \langle U, \psi \rangle$. If we take $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ as constructed in the proof of 3.10 with S_{χ} , T_{χ} in the Prop. 3.10 with S_χ , T_χ in the place of S, T, it follows from the proof of Prop. 3.10 that Froates of 1
 e^* and $S(1 - \chi)e^*$ vanish on
 $\chi e^* = (S\chi) * \chi_e$ we are reduce

if we take $\{\varphi^{\epsilon}\}_{0 \leq \epsilon \leq 1}$ as construte of S, T, it follows from the
 $\langle (S\chi * \varphi^{\varrho}) (T\chi * \varphi^{\varrho}), \psi \rangle$.

nood of the support of ψ

$$
\langle (S\chi * \varDelta_e) (T\chi * \varDelta_e), \psi \rangle \sim \langle (S\chi * \varphi^{\varrho}) (T\chi * \varphi^{\varrho}), \psi \rangle.
$$

1 in a standard neighborhood of the support of $\langle (S\chi * \varphi^{\varrho}) (T\chi * \varphi^{\varrho}), \psi \rangle = \langle (S * \varphi^{\varrho}) (T * \varphi^{\varrho}), \psi \rangle \sim 0$
proof is complete.

But $\chi = 1$ in a standard neighborhood of the support of ψ , so

$$
\langle (S\chi \ast \varphi^\varrho) \ (T\chi \ast \varphi^\varrho), \psi \rangle = \langle (S \ast \varphi^\varrho) \ (T \ast \varphi^\varrho), \psi \rangle \sim \langle U, \psi \rangle,
$$

and the proof is complete **I**

 $\frac{1}{2}$

│
│ │
│ │ │ │ │ │
│ │ │ │ │ │

4. Algebras containing the standard distributions

We fix a positive infinitesimal number ϱ and start by introducing an external space

 $\langle (S\chi * \varphi^e) (T\chi * \varphi^e), \psi \rangle = \langle (S * \varphi^e) (T * \varphi^e), \psi \rangle$
and the proof is complete **I**
4. Algebras containing the standard distributions
We fix a positive infinitesimal number ρ and start by i
 $\mathbf{E}_{\varrho} \subset \mathcal{E}^{\infty}(\math$ Definition 4.1: E_e is the external set of all $T \in \mathcal{E}^{\infty}(\mathbb{R}^n)$ with the following property:

 \forall st $\alpha \in \mathbb{N}_0$ ⁿ \forall st $k \in \mathbb{N}$ \exists st $j \in \mathbb{N}$ such that sup $\{|\partial^{\alpha}T(x)| : |x| \leq k\} \leq e^{-j}$.

It is clear from Cor. 2.11 that $D' \subset E_{\rho}$; E_{ρ} is a commutative and associative differential algebra over ${}^{\text{st}}\mathbb{C}$. If $\theta \in {}^{\text{st}}\mathcal{D}$ with $\int \theta(x) dx = 1$, then $S \to S * \theta_{\text{e}}$ is an imbedding of $\mathbf{F}_{\mathbf{Q}}$ into \mathbf{E}_{e} . However, this imbedding does not preserve the pointwise product on $s \circ \mathcal{C}$, because $(f * 0_e)$ $(g * \theta_e) + (fg) * \theta_e$ for $f, g \in s \circ \mathcal{C}$ in general. We shall product on ${}^{\circ}{}_{6}{}^{\circ}{}_{\circ}$, because $(f * v_e)$ $(g * v_e) = (g) * v_e$ for $f, g \in {}^{\circ}{}_{6}{}^{\circ}{}_{\circ}{}^{_{6}}$ in general. We shand-
now construct a quotient of E_e and an imbedding of ${}^{\circ}{}_{5}{}^{\circ}{}_{\circ}{}^{\prime}$ which turns ${}^{\circ}$ ard smooth functions of polynomial growth, into a subalgebra. This is a nonstand-
ard counterpart to Colombeau's construction of his algebra $\mathcal{S}_\gamma(\mathbb{R}^n)$, see [4]. It is clear from Cor. 2.11 that $\mathbf{D}' \subset \mathbf{E}_e$; \mathbf{E}_e is a commutative and associat
differential algebra over st. If $\theta \in {}^{st}\mathcal{D}$ with $\int \theta(x) dx = 1$, then $S \to S * \theta_e$ is
imbedding of ${}^{st}\mathcal{D}'$ into \mathbf{E}_e . Ho $\mathbf{E}_{e} \subset \mathcal{E}^{\infty}(\mathbb{R}^{n})$ of

Definition 4

erty:
 $\forall^{st} \alpha \in \mathbb{I}$

It is clear from

differential alge

imbedding of $^{st}\mathcal{E}^{\infty}$

product on $^{st}\mathcal{E}^{\infty}$,

now construct a

ard smooth func

ard counterpar $\forall^{\text{st}}\alpha \in \mathbb{N}_0^n \ \forall^{\text{st}}k \in \mathbb{N} \exists^{\text{st}} j \in \mathbb{N}$ such that $\text{sup } \{|\partial^{\alpha}T(x)| : |x| \leq k\} \leq \varrho^{-j}$.

lear from Cor. 2.11 that $\mathbf{D}' \subset \mathbf{E}_e$; \mathbf{E}_e is a commutative and associative

al algebra over stC. If *g* of ${}^{\tilde{u}}\mathcal{D}'$ into \mathbf{E}_e . However, this imbedding does not preserve the point
 on ${}^{\tilde{u}}\mathcal{D}''$ into \mathbf{E}_e . However, this imbedding does not preserve the point
 truct a quotient of \mathbf{E}_e and

(R") with the following prop- $\frac{1}{2}$ th -

$$
\forall^{\text{st}}\alpha\in\mathbb{N}_{0}^{\mathfrak{n}}\ \forall^{\text{st}}k\in\mathbb{N}\ \forall^{\text{st}}i\in\mathbb{N}\colon\text{sup}\ |\partial^{\text{st}}T(x)|\cdot|x|\leq k\}\leq\varrho^{i}.\tag{4.1}
$$

It is clear that N_e is an ideal in E_e closed under differentiation. Therefore,

$$
G_{\circ} = E_{\circ}/N_{\circ}
$$

is a differential algebra (commutative, associative) over ${}^{\text{st}}\mathbb{C}$. If $T \in \mathbb{E}_{e}$, we shall

Now consider a quotient of
$$
L_{\theta}
$$
 and an discontinuity of a subalgebra. This is a nonstandard component to Colombia growth, into a subalgebra $\mathcal{S}_{\eta}(\mathbb{R}^n)$, see [4]. Definition 4.2: \mathbb{N}_{e} is the external set of all $T \in \mathcal{E}^{\infty}(\mathbb{R}^n)$ with the following property: $\forall^{st}\alpha \in \mathbb{N}_{0}^{n}$ $\forall^{st}k \in \mathbb{N}$ $\forall^{st}i \in \mathbb{N}$: $\sup \{|\partial^s T(x)| : |x| \leq k\} \leq \varrho^i$. (4.1) It is clear that \mathbb{N}_{e} is an ideal in \mathbb{E}_{e} closed under differentiation. Therefore, $G_{e} = \mathbb{E}_{e}/\mathbb{N}_{e}$ is a differential algebra (commutative, associative) over $^{st}\mathbb{C}$. If $T \in \mathbb{E}_{e}$, we shall write [T] for its equivalence class in G_{e} . We now fix a standard $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $\int \theta(x) dx = 1$, $\int x^s \theta(x) dx = 0$, for all $\alpha \in {}^{st}\mathbb{N}_{0}^{n}$, $|\alpha| \geq 1$. (4.2) $\int x^s \theta(x) dx = 0$, for all $\alpha \in {}^{st}\mathbb{N}_{0}^{n}$, $|\alpha| \geq 1$. The existence of such a θ follows by Fourier transform and Borel's theorem (cf. $\text{Tr}(\mathbb{R}^n) \times \mathbb{R}^n$

TREVES [23, p. 390]).

Lemma 4.3: If
$$
f \in {}^{st}O_M^{\prime}(\mathbb{R}^n)
$$
, then $f * \theta_e - f \in N_e$.

Proof: We deduce condition (4.1) for $\alpha = (0, ..., 0)$, the proof for general α being nilar. Let $i \in \text{thN}$. By Taylor's theorem and (4.2),
 $\qquad \qquad f * \theta_e(x) - f(x) = \int (f(x - ey) - f(x)) \theta(y) dy$
 $= \sum \int \frac{(-ey)^{\beta}}{\beta} \partial^{\beta} f(\xi) \theta(y) dy$ similar. Let $i \in \text{N}$. By Taylor's theorem and (4.2), emm
roof
lar. I
/

/ ***** 0(X) - */(x)* **=.** *f (/(x-----* y) — /(x)) *^O (y) dy r (—y) afi/()O(y).dy* I=i+tJ - 'S

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with ξ between x and $x - \varrho y$; in particular, $|\xi| \le |x| + |\varrho y| \le |x| + |y|$. Since
 $f \in {}^{st}O_M$, $|\partial^{\beta}f(\xi)|$ is bounded by $C(1 + |x|)^{l} (1 + |y|)^{l}$ for some standard $C > 0$ and some standard $l \in \mathbb{N}$. Since $\theta \in \mathbb{S}^{t}$, the integrals are bounded by $C e^{t+1}$ for some other standard $C > 0$, uniformly for $|x| \leq k$ for every standard $k \in \mathbb{N}$. Since ϱ is infinites-
imal we have $C\varrho^{i+1} \leq \varrho^i$, proving the assertion with ξ between x and $x - \varrho y$; in particular,
 $f \in {}^{st}O_M$, $|\partial^{\beta}f(\xi)|$ is bounded by $C(1 + |x|)^{l} (1 +$

some standard $l \in \mathbb{N}$. Since $\theta \in {}^{st}\mathcal{S}$, the integrals a

standard $C > 0$, uniformly for $|x| \leq k$ for

Proposition 4.4: Let ϱ *be a positive infinitesimal and let* $\theta \in {}^{st}\mathscr{S}(\mathbb{R}^n)$ *satisfy (4.2). Then:*

(a) The map $S \to [S * \theta_o]$ defines an imbedding of ${}^{st}\mathcal{S}'(\mathbb{R}^n)$ into G_e which preserves *differentiation.*

(b) ${}^{st}O_M(\mathbb{R}^n)$ *is a subalgebra of* G_e *; more precisely*

 $[f * \theta_{\rho}]$ $[g * \theta_{\rho}] = [(fg) * \theta_{\rho}]$ for $f, g \in {}^{st} \mathcal{O}_M(\mathbb{R}^n)$.

(c) If $P \in {}^{st}O_M(\mathbb{R}^m)$ *and* $[T_1], \ldots, [T_m] \in G_e$, *then* $[P(T_1, \ldots, T_m)]$ *is a welldefined element of (.*

Proof: (a): We know from Remark 2.7 that $S \to S * \theta_{\ell}$ is an injection of $^{st}\mathcal{F}'$ into \mathbf{E}_{ρ} with $\partial^{\mathfrak{a}}(S * \theta_{\rho}) = (\partial^{\mathfrak{a}}S) * \theta_{\rho}$. Thus it remains to show that if $S \in {}^{\text{st}}\mathcal{S}'$ and $S * \theta_{\rho} \in N_{\rho}$, then $S = 0$. Let $\psi \in {}^{st}\mathcal{D}$. If $S * \theta_{\varrho} \in N_{\varrho}$, then $\langle S * \theta_{\varrho}, \psi \rangle \sim 0$ since $\psi \in \mathcal{D}_k$ for some standard *k*. By Remark 2.7, $S = 0$. (b) follows immediately from Lemma 4.3. (c) is a simple application of the definitions and the fact that N_e is an ideal

Finally, since $N_e \subset d'$ we can introduce an infinitesimality relation on G_e by calling $[T]$ G_e -infinitesimal if $T \in d'$ for some representative T of $[T]$. This infinitesimality relation may serve the same purpose as the notion of an "associated distri-, bution" in the Colombeau algebras [2, Def. 3.5.2]: Indeed, multiplication in G_e generally does not preserve distributional products other than the multiplication of ${}^{st}O_M$. For instance, $x\delta(x) = 0$ in the sense of distribution theory, but the product $[x]$ $[\theta_{\varrho}(x)]$ of the images of its factors is not equal to zero in G_{ϱ} . We have however, that $[x][\theta_{\varrho}(x)]$ is G_{ϱ} -infinitesimal. A general result showing that many distributional products concide with the corresponding product in G_e on a macroscopic level will now be stated. Let S, $T \in \mathcal{S}'(\mathbb{R}^n)$. Call $U \in \mathcal{S}'(\mathbb{R}^n)$ the \tilde{M}_6 -product of S and *T*, if
 $\lim_{\epsilon \to 0} (S * \varphi_\epsilon) (T * \varphi_\epsilon) = U$

in $\mathcal{D}'(\mathbb{R}^n)$, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi(x) dx = 1$.

$$
\lim_{\epsilon \to 0} (S * \varphi_{\epsilon}) (T * \varphi_{\epsilon}) = U
$$

I

Proposition 4.5: Let S, T \in stS.' If the product $U = M_6(S, T)$ exists, then $[S * 0]$ \times $[T * \theta_o] - [U * \theta_o]$ is G_o -infinitesimal. ((S) $\left\{U * \theta_e\right\}$ is G_e -infinitesimal.
 \therefore Along the lines of the proof of Prop. 3.6, one first deduc
 $\left\langle (S * \theta_e) (T * \theta_e), \psi \right\rangle \sim \left\langle U, \psi \right\rangle \sim \left\langle U * \theta_e, \psi \right\rangle$ for all $\psi \in {}^{st}\mathcal{D}$;

Proof: Along the lines of the proof of Prop. 3.6, one first deduces that

$$
\langle (S * \theta_{\varrho}) (T * \theta_{\varrho}), \psi \rangle \sim \langle U, \psi \rangle \sim \langle U * \theta_{\varrho}, \psi \rangle \text{ for all } \psi \in {}^{\text{st}}\mathcal{D};
$$

Froot: Along the lines of the proof of Prop. 3.6, one first deduces that
 $\langle (S * \theta_{\varrho}) (T * \theta_{\varrho}), \psi \rangle \sim \langle U, \psi \rangle \sim \langle U * \theta_{\varrho}, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$;

next one employs an equicontinuity argument to obtain $\langle (S * \theta_{\var$ $0\ {\rm for}$ Proof: Along the lines of the proof of Prop. 3.6, one first deduces that
 $\langle (S * \theta_{\varrho}) (T * \theta_{\varrho}), \psi \rangle \sim \langle U, \psi \rangle \sim \langle U * \theta_{\varrho}, \psi \rangle$ for all $\psi \in {}^{st}\mathcal{D}$;

next one employs an equicontinuity argument to obtain $\langle (S * \theta_{\var$

Appendix 1. We consider here $-$ in one dimension $-$ the products of the Dirac measure δ and the distributions T_r defined by

$$
T_r(x) = \sum_{m=1}^{\infty} \frac{1}{m^r} \delta\left(x - \frac{1}{m}\right).
$$

- with $r>1$. The following assertions hold:
	-

(b) $r > 1$. The following assertions hold:

(a) $M_4(\delta, T_r)$ does not exist for $r < 2$;

(b) $M_3(\delta, T_r)$ exists for $r \geq 2$, and we have: $M_3(\delta, T_s)$
 > 2 . (a) $M_4(\delta, T_r)$ does not exist for $r < 2$;

(b) $M_3(\delta, T_r)$ exists for $r \ge 2$, and we have: $M_3(\delta, T_2) = \frac{1}{2} \delta$, $M_3(\delta, T_r) = 0$ for. $r>2$.

Proof. We begin by showing that $M_4(\delta, T_2) = \frac{1}{\delta} \delta$. Let $\theta \in {}^{st}\mathcal{D}$ be as in Prop. 3.6, $\rho \sim 0$, $\psi \in$ st \mathcal{D} . We may assume that support $(\theta) \subset [-1/2, 1/2]$. First, (a) $M_4(\delta, T_r)$ does not exists for
 $r > 2$.
 Proof: We begin by show
 $\rho \sim 0$, $\psi \in {}^{st}\mathcal{D}$. We may a
 $\langle M^{'}_{\varrho}(\delta, T_2), \psi \rangle =$

$$
\langle M_{\varrho}^{\prime}(\theta, T_{2}), \psi \rangle = \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\varrho^{2}m^{2}} \theta\left(\frac{x}{\varrho}\right) \theta\left(\frac{x}{\varrho} - \frac{1}{\varrho m}\right) \psi(x) dx
$$

\n
$$
= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^{2}} \theta(x) \theta\left(x - \frac{1}{\varrho m}\right) \psi(\varrho x) dx.
$$

\nWriting $\psi(\varrho x) = \psi(0) + \varrho x \psi'(\xi)$ it suffices (Prop. 3.6) to show that
\n
$$
\sum_{m=1}^{\infty} \frac{1}{\varrho m^{2}} \int_{-\infty}^{\infty} \theta(x) \theta\left(x - \frac{1}{\varrho m}\right) dx \sim \frac{1}{2}
$$

\nand
\n
$$
\sum_{m=1}^{\infty} \frac{1}{m^{2}} \int_{-\infty}^{\infty} |x| \theta(x) \theta\left(x - \frac{1}{\varrho m}\right) dx \sim 0.
$$

\nSince support $(\theta) \subset [-1/2, 1/2]$, the integrals vanish if $m \leq [1/\varrho]$ where $[1/\varrho]$ denotes the largest integer in 1/ ϱ . But [1/ ϱ] is infinitesimal

$$
m=1 \t{J \atop -\infty} \t{om^2} \t{om^2} \t{om}
$$

\n
$$
\varphi(\varrho x) = \psi(0) + \varrho x \psi'(\xi) \text{ it suffices (Prop. 3.6) to show that}
$$

\n
$$
\sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \int_{-\infty}^{\infty} \theta(x) \theta(x) - \frac{1}{\varrho m} dx \sim \frac{1}{2}
$$
\n(A1)

and .

$$
\sum_{m=1}^{\infty} \frac{1}{m^2} \int_{-\infty}^{\infty} |x| \theta(x) \theta\left(x - \frac{1}{2^m}\right) dx \sim 0.
$$
\n
$$
\sum_{m=1}^{\infty} \frac{1}{m^2} \int_{-\infty}^{\infty} |x| \theta(x) \theta\left(x - \frac{1}{2^m}\right) dx \sim 0.
$$
\n(A2)\n\noport $(\theta) \subset [-1/2, 1/2]$, the integrals vanish if $m \leq [1/\varrho]$ where $[1/\varrho]$ denotes
st integer in $1/\varrho$. But $[1/\varrho]$ is infinite, thus the expression (A.2) is infinitesimal

$$
\sum m^{-2} \text{ converges). To estimate (A.1) we rewrite it as
$$
\n
$$
\sum_{m=1}^{\infty} \frac{1}{1/\varrho!+1} \frac{\varrho}{\varrho m^2} \theta * \theta\left(\frac{1}{2^m}\right) = \sum_{m=1}^{\infty} \frac{1}{1/\varrho!+1} \frac{\left(\frac{1}{m} - \frac{1}{m+1}\right)}{m+1} \theta * \theta\left(\frac{1}{2^m}\right)
$$
\n
$$
\approx 1 \qquad 1 \qquad \times (1)
$$

Since support $(\theta) \subset [-1/2, 1/2]$, the integrals vanish if $m \leq [1/e]$ where $[1/e]$ denotes the largest integer in $1/\rho$. But $[1/\rho]$ is infinite, thus the expression (A.2) is infinitesimal (because $\sum m^{-2}$ converges). To estimate (A.1) we rewrite it as

$$
\sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \int_{-\infty}^{\infty} \theta(x) \theta(x) \left(x - \frac{1}{\varrho m}\right) dx \sim \frac{1}{2}
$$
\n
$$
\sum_{m=1}^{\infty} \frac{1}{m^2} \int_{-\infty}^{\infty} |x| \theta(x) \theta(x) - \frac{1}{\varrho m} dx \sim 0.
$$
\npoint (0) $\subset [-1/2, 1/2]$, the integrals vanish if $m \leq [1/\varrho]$ where
st integer in 1/ ϱ . But [1/ ϱ] is infinite, thus the expression (A.2) is
 $\sum m^{-2}$ converges). To estimate (A.1) we rewrite it as

$$
\sum_{m=11/\varrho+1}^{\infty} \frac{1}{\varrho m^2} \theta * \tilde{\theta} \left(\frac{1}{\varrho m}\right) = \sum_{m=(1/\varrho)+1}^{\infty} \frac{1}{\varrho} \left(\frac{1}{m} - \frac{1}{m+1}\right) \theta * \tilde{\theta} \left(\frac{1}{\varrho m}\right).
$$
\nThe second term is infinitesimal, because.

\n
$$
\sum_{m=(1/\varrho)+1}^{\infty} \frac{1}{\varrho m^3} \sim \int_{1/\varrho}^{\infty} \frac{dx}{\varrho x^3} \sim 0.
$$
\nit term is recognized as a step function with infinitesimal

Again, the second term is infinitesimal, because.

$$
+\sum_{m=\lfloor 1/e \rfloor+1} \frac{1}{\varrho} \frac{1}{m^2(m+1)} \theta * \theta \left(\frac{1}{\varrho m} \right).
$$

the second term is infinitesimal, because.

$$
\sum_{m=\lfloor 1/e \rfloor+1}^{\infty} \frac{1}{\varrho m^3} \sim \int_{1/e}^{\infty} \frac{dx}{\varrho x^3} \sim 0.
$$
 (A 3)

The first term is recognized as a step function with infinitesimal step size \sup The first term is recognized as a step function with infinitesimal step size
 $\sup \left\{ \frac{1}{\varrho} \left(\frac{1}{m} - \frac{1}{m+1} \right) : m \ge \left[\frac{1}{\varrho} \right] + 1 \right\} \le \varrho$, which may be interpreted as the
 ϱ -th member of a standard net of $\left(\frac{1}{3} \sim \int_{1/e}^{1/e} \frac{1}{e} \right)$
cognized
 $\left(\frac{1}{2} m \right) \ge m$ $\left|\frac{1}{-}\right| + 1$ \leq *e*, which may be interpreted as the the first term, is infinitely close to gain, the second term is infinities
 $\sum_{m=11/e}^{\infty} \frac{1}{2m^3} \sim \int_{1/e}^{\infty} \frac{dx}{e^{x^3}}$

The first term is recognized as
 $\lim_{n \to \infty} \left[\frac{1}{e} \left(\frac{1}{m} - \frac{1}{m+1} \right) : m \right] \ge \left[\frac{1}{e} \right]$
 $\lim_{n \to \infty} \frac{1}{n} \lim_{n \to \infty} \frac{1$ form is infinitely close to
 $\int_0^1 \theta * \theta(y) dy$. sup $\left[\frac{1}{\varrho}\right]$ $\left(\frac{m}{m} - \frac{m}{m+1}\right)$.
 ϱ -th member of a stand

the first term is infinite
 $\int_{0}^{1} \theta * \theta(y) dy$.

A simple calculation usignal equals 1/2.

$$
\int\limits_{0}^{1}\theta\ast\check\theta(y)\,dy\ldots
$$

A simple calculation using the fact that $\theta * \dot{\theta}$ is an even function shows that this inte-

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. To show that $M_3(\delta, T)$ exists also, we use Prop. 3.7 and proceed as in the proof of Cor. 3.8. If $\theta \in \mathbf{D}$ is as in Prop. 3.7 we write $\theta = \varphi + \eta$ with φ standard and $\eta \approx_{\mathbf{D}} 0$. We have to show that all the three sums w that $M_3(\delta, T)$ exists

if $\theta \in D$ is as in Prop.

ito show that all the the dependence of $\sum_{n=1}^{\infty} \frac{1}{\varrho m^2} \varphi * \check{\eta} \left(\frac{1}{\varrho m}\right)$,

$$
\sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \varphi * \check{\eta} \left(\frac{1}{\varrho m} \right), \quad \sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \eta * \check{\varphi} \left(\frac{1}{\varrho m} \right), \quad \sum_{m=1}^{\infty} \frac{1}{\varrho m^2} \eta * \check{\eta} \left(\frac{1}{\varrho m} \right)
$$

are infinitesimal. But if the support of θ is contained in $[-1/2, 1/2]$ so are the supports of φ and η . So all sums actually start with $m = [1/\varrho] + 1$. But $\sum_{n=1}^{\infty} 1/m^2$ is a limited is equally start with $m = [1/q] + 1$. But $\sum_{m=11/4}^{\infty} 1$
and all convolutions are uniformly bounded number (similar to $(A.3)$) and all convolutions are uniformly bounded, by an infinite-, simal number. In the case $r > 2$, already $\sum_{m=\lfloor 1/\varrho\rfloor+1}^{\infty} 1/\varrho m^r \sim 0$, so similar to (A.3)) and all convolutions are uniformly bounded by an infinite-

similar to (A.3)) and all convolutions are uniformly bounded by an infinite-

mether. In the case $r > 2$, already $\sum_{m=1}^{\infty} \frac{1}{\rho m^r} \sim 0$ 362 M. OBEROUGOEXBEROER

To show that $M_3(\delta, T)$ exists also, we use Prop. 3.7 and I

Cor. 3.8. If $\theta \in \mathbf{D}$ is as in Prop. 3.7 are writte $\theta = \varphi + \eta$ with

We have to show that all the three sums
 $\sum_{p=1}^{\infty} \frac{1}{\rho$ *om on o b b* **s contrained** in $[-1, 0, 0]$
i convolutions are uniform
i already $\sum_{m=1}^{\infty} 1/m^r \sim 0$, s
m= $(1/e)+1$
pm $\left(\frac{1}{2}\right)\psi(\varrho x) dx \sim 0$ for every tandard $c > 0$ and $|y| \leq 1$
i and ϱx *d*

$$
\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^r} \theta(x) \theta\left(x - \frac{1}{\varrho m}\right) \psi(\varrho x) dx \sim 0 \quad \text{for every} \quad \theta \in \mathbf{D},
$$

By Prop. 3.7, $M_3(\delta, T_r) = 0$. In the case $r < 2$ we take $\psi = 1$ near zero and $\theta \in \mathbb{R}$
such that $\theta * \theta(\psi) \ge c$ for some standard $c > 0$ and $|y| \le 1/2$. Then

$$
\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\varrho m^r} \theta(x) \theta\left(x - \frac{1}{\varrho m}\right) \psi(\varrho x) dx \ge c \sum_{m=[2/\varrho]+1}^{\infty} \frac{1}{\varrho m^r} \sim \infty,
$$

$$
\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varrho m^r}\,\theta(x)\,\theta\left(x-\frac{1}{\varrho m}\right)\psi(\varrho x)\,dx\geq c\sum_{m=(2/\varrho)+1}^{\infty}\frac{1}{\varrho m^r}\sim\infty,
$$

thus $M_4(\delta, T_r)$ does not exist. The proof of (a) and (b) is complete \blacksquare

Additional remarks: (a) The product $M_2(\delta, T_r) = 0$ exists and equals zero for $r > 2$. This • follows from the remarks in the Introduction 'concerning the product (P1) and the fact that the corresponding assertion for the product (P1) of δ and T_r has been verified in [13, Appx]. It is not clear whether $M_2(\delta, T_2)$ exists. However, the product (P1) of δ and T_2 does not exist, because $\delta * T_2$ does not have a value at zero in the sense of Lojasiewicz (cf. [18, Prop. 4 and 5]). corresponding assertion for the product (P1) of δ and \tilde{T}_r has been verified in [13, Appx]. It is
not clear whether $M_2(\delta, T_2)$ exists. However, the product (P1) of δ and T_2 does not exist, be-
cause $\delta * T_2$ thus $M_4(\delta, T_r)$ does not exist. The product

Additional remarks: (a) The product

follows from the remarks in the Introductio

corresponding assertion for the product (P)

not clear whether $M_2(\delta, T_2)$ exists. Howeve

c **thus** $M_4(\delta, T_r)$ **does not exist. The proof of (a) and (b) is

Additional remarks: (a) The product** $M_2(\delta, T_r) = 0$ **exists

follows from the remarks in the Introduction concerning the product

corresponding assertion for th**

(b) The product $M_1(\delta, T_r)$ does not exist for any $r > 1$: We exhibit a sequence $\{x_i\}_{i \geq 1}$ of type bes not have a
 iluct $M_1(\delta, T_r)$ d
 $\langle (\delta * \chi_j) (T_r * \chi_j) \rangle$
 $\geq 0, \int \chi(x) dx$
 $= j^{r+1} \left(\chi \int_j r \right)$

$$
\chi_j(x) = j^{r+1} \left(\chi \left(j^{r+1} \left(x - \frac{1}{2j} \right) \right) + \chi \left(j^{r+1} \left(x + \frac{1}{2j} \right) \right) \right).
$$

$$
\begin{aligned}\n&= [-1, 1], \, \chi \geq 0, \int \chi(x) \, dx = \frac{1}{2}, \text{ and set} \\
& \chi_i(x) = j^{r+1} \left(\chi \left(j^{r+1} \left(x - \frac{1}{2j} \right) \right) + \chi \left(j^{r+1} \left(x + \frac{1}{2j} \right) \right) \right).\n\end{aligned}
$$
\nThen\n
$$
\sum_{m=1}^{\infty} \frac{1}{m^r} \int_{-\infty}^{\infty} \chi_i(x) \chi_i \left(x - \frac{1}{m} \right) dx \geq \frac{1}{j^r} \int_{-\infty}^{\infty} \chi_i(x) \chi_i \left(x - \frac{1}{j} \right) dx
$$
\n
$$
= j^{r+2} \int_{-\infty}^{\infty} \chi^2 \left(j^{r+1} \left(x - \frac{1}{2j} \right) \right) dx = j \int \chi^2(y) \, dy \to \infty \text{ as } j \to \infty.
$$
\nAppendix 2. We collect here, some basic notions from Internal Set Theo which are frequently employed in this paper; otherwise we refer to Nelson's arti [12].\n\nTransfer axiom: Let $A(x, t_1, ..., t_k)$ be an internal formula with the free variabian $(\chi^{4}x, A(x, t_1, ..., t_k)) \Rightarrow (\forall x \, A(x, t_1, ..., t_k))$

Appendix 2. We collect here some basic notions from Internal Set Theory **'** which are frequently employed in this paper; otherwise we refer to Nelson's article **Appendix 2.** We collect here some basic notions for $-\infty$
 Appendix 2. We collect here some basic notions for the requently employed in this paper; otherwise 2].

Transfer axiom: Let $A(x, t_1, ..., t_k)$ be an internal $t_1, ...,$

Transfer axiom: Let $A(x, t_1, ..., t_k)$ be an internal formula with the free variables $x, t_1, ..., t_k$ and no other free variables. Then

$$
(\forall^{\text{st}} x \ A(x, t_1, \ldots, t_k)) \Rightarrow (\forall x \ A(x, t_1, \ldots, t_k))
$$

whenever the parameters ^t ¹ , ...,tk take standard values. Equivalent to this is

the parameters
$$
t_1, ..., t_k
$$
 take standard valt
\n
$$
(\exists x \ A(x, t_1, ..., t_k)) \Rightarrow (\exists^{\text{st}} \uparrow A(x, t_1, ..., t_k))
$$

provided t₁ \ldots , t_k *take s*
 $(\exists x \ A(x, t_1, \ldots, t_k)) \Rightarrow (\exists^{st} x \ A(t_1, \ldots, t_k) \Rightarrow (\exists^{st} x \ A(t_k, t_1, \ldots, t_k))$
 provided t₁ \ldots , t_k *take standard values.*

The other axioms — idealization as
 $\exists x \ B(t_k)$ with the followi $\left(\exists x \ A(x, t_1, ..., t_k)\right) \Rightarrow \left(\exists^{\text{st}} x \ A(x, t_1, ..., t_k)\right)$
 provided $t_1, ..., t_k$ take standard values.

• The other axioms — idealization and standardization — are rarely applied here • except via the following principles. They can be found in [12, p. 1166].

Construction principle for maps: Let X, Y be standard sets, let $A(x, y)$ be a *formula, internal or external, with free variables x, y azd posibly others.' Supp9se that . for all standard* $x \in X$ *there is a unique standard* $y \in Y$ *such that* $A(x, y)$ *. Then there is a* $unique\ standard\ function\ f\colon X\to Y\ such\ that\ A(x,f(x))\ holds\ for\ all\ standard\ x\in X.$ **Products of Distributions**
 Products is
 $(\exists x \ A(x, t_1, ..., t_k) \Rightarrow (\exists^k x \ A(x, t_1, ...,$ $(\exists x \ A(x, t_1, ..., t_k)) \Rightarrow (\exists^5 x \ A(x, t_1, ..., t_k))$
 contained t₁, ..., t_k take standard values.

The other axioms $-$ idealization and standardization $-$ are rarely applie

except via the following principles. They can be found provided t_1, \ldots, t_k take standard values.

The other axioms \rightarrow idealization and standardize

except via the following principles. They can be four

Construction principle for maps: Let X, Y

for all standard $x \in X$ th $\begin{array}{ccc}\n & \text{for} & \text{for} \\
 & \text{for} & \text{un} \\
 & \text{on} & \text{the} \\
 & & \text{in} & \text{the} \\
 & & \text{in} & \text{the}\n\end{array}$ The other axioms — idealization and
except via the following principles. They
Construction principle for map
formula, internal or external, with free varia-
for all standard $x \in X$ there is a unique state
unique standard

Permanence principles: Let $A(n)$ be an internal formula over $n \in \mathbb{N}$, possibly

(1) If $A(n)$ holds for all standard $n \in \mathbb{N}$, then there is an infinitely large- $\omega \in \mathbb{N}$ such

(2) *If* $A(v)$ holds for all infinitely large $v \in \mathbb{N}$, then there is a standard $n_0 \in \mathbb{N}$ such

Proof: (1): $S = \{n \in \mathbb{N} : A(k) \text{ holds for } 1 \leq k \leq n\}$ is an internal set with $S = \mathbb{N} \mathbb{N}$. Since \mathbf{H} is not a set [12, Thm. 1.1], S must contain an infinitely large number ω . (2) is proved similarly, see also $[12, \text{Example } 6, \text{ p. } 1177]$ that $A(n)$ holds for all $n \ge n_0$.

Proof: (1): $S = \{n \in \mathbb{N} : A(k) \text{ holds for } 1 \le k \le n\}$ is an internal set with S :

Since stN is not a set [12, Thm. 1.1], S must contain an infinitely large num

(2) is proved similarly, s From an ence principle $\{12, 1111, 1601\}$. Been that $A(n)$ holds for that $A(n)$ holds for $\{12\}$ (2) If $A(v)$ holds for $\{12\}$ (2) If $A(v)$ holds for $\{11\}$. Proof: (1): $S = \{n \in \mathbb{S} \mid n \in \mathbb{S} \}$ (2) is proved

We also need more general versions of these permanence principles. Let (A, \leq) be a standard directed set. An element $\omega \in A$ is called *infinitely large* if $\lambda \leq \omega$ for all standard $\lambda \in \Lambda$.

Robinson's lemma: (1) Let Λ be as above and let $A(\lambda)$ be $\hat{a}n$ internal formula or *a formula of the form* $A(\lambda) \equiv (\forall^{\text{st}} y : B(\lambda, y))$ *with B internal (A, B may contain other free variables). If A (A) holds for all standard A* € *A, then there is an infinitely large* $\omega \in A$ such that $A(\omega)$. bo need more general versions of these permanence p

o need more general versions of these permanence p

d directed set. An element $\omega \in A$ is called *infinite*
 $\lambda \in A$.

son's lemma: (1) Let Λ be as above and let $A(\$ a standard directed set. An element $\omega \in A$ is called

standard $\lambda \in A$.

Robinson's lemma: (1) Let Λ be as above and

a formula of the form $A(\lambda) \equiv (\forall s y : B(\lambda, y))$ with B

free variables). If $A(\lambda)$ holds for all standa

(2) Let $C(\lambda)$ be internal or of the form $C(\lambda) \equiv (\exists^{st}y: D(\lambda, y))$ with D internal. If $C(\omega)$ holds for all infinitely large $\omega \in A$, then there is a standard $\lambda \in A$ such that $C(\lambda)$. E A such that $A(\omega)$.

(2) Let $C(\lambda)$ be internal or of the form $C(\lambda) \equiv \begin{cases} \exists \\ \text{lds} \text{ for all infinitely large } \omega \in A \text{, then there is a} \end{cases}$

Proof: (1): In the more general case $A(\lambda) \equiv$

 $(\forall^{\text{st}} y : B(\lambda, y))$ we have to show that θ .

The validity is obtained from the idealization axiom, the hypothesis, and [12, Thm.

We now need some facts about topology. Let X be a standard topological space, let $a \in \mathbb{S}^1 X$, $x \in X$. We say that x is *infinitely close* to a, denoted as $x \approx_X a$, if x is contained in all standard neighborhoods of a. Let again (A, \leq) be a standard directed structure between $\lambda(x) = \lambda(x, y)$. $\lambda(x, y) = \lambda(x, y)$

the variables). If $A(\lambda)$ holds for all standard $\lambda \in \Lambda$, then
 $\omega \in \Lambda$ such that $A(\omega)$.

(2) Let $C(\lambda)$ be internal or of the form $C(\lambda) \equiv (\exists^s y : D$

holds for all infin *a* \in st*X*, *x* \in *X*. We say that *x* is *infinitely close* to *a*, denoted as $x \approx_{X} a$, if *x* is contained in all standard neighborhoods of *a*. Let again (A, \leq) be a standard directed set.
Characterization *re* is a stand

(λ) = $(\forall$ sty:
 $\{(a, y)\}$.
 $\{(a, y)\}$.
 \therefore Let X be a close to a, c
 \therefore Let again
 \therefore Let $a_{\omega} \sim$

Characterization of the convergence of a standard net: *LetX, A be as* • Characterization of the convergence of a standard net: Let X, Λ be as
above, let $\{a_i\}_{i\in\Lambda}$ be a standard net, let $a \in {}^{st}X$. Then, a_i converges to a if and only if
 $a_{\omega} \approx_{\chi} a$ for all infinitely large $\omega \in \$

Proof: If $a_2 \rightarrow a$ and *V* is a standard neighborhood of *a*, then $a_\omega \in V$ for all infi-

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V be a standard neighborhood of a, and consider the assertion $\forall \mu \geq \omega : a_{\mu} \in V$. This is an internal formula which holds for all infinitely large ω . By Robinson's lemma, **there is a standard neighborhood of a, and consider the assertion** $\forall \mu \ge \omega$ **
is an internal formula which holds for all infinitely large** ω **. By Robir
there is a standard** λ **such that** $\forall \mu \ge \lambda : a_{\mu} \in V$ **. By transfer,**

Of course, by reversing \leq , "infinitely large" may be replaced by "infinitely small". We remark that if a standard net converges, then its limit is standard.

Finally, we discuss the notion of s-continuity. Let X, Y be standard topological spaces, $g: X \to Y$ a (possibly nonstandard) map, $a \in {}^{st}X$. Then g is called *s-continuous* at a iff

$$
\exists^{\text{st}} b \in Y \text{ such that } \forall x \in X : x \approx_X a \Rightarrow g(x) \approx_Y b.
$$

In the case of $Y = \mathbb{C}$ this implies that $g(a)$ is limited and $b = {}^{\circ}g(a)$. We only need the following special version of

The s-continuity theorem: Let X be a standard topological space, $g: X \to \mathbb{C}$ α map which is s-continuous at *every standard* $a \in X$. Then there is a unique standard *map f*: $X \rightarrow \mathbb{C}$ *such that* $f(a) = {}^{\circ}g(a)$ *for all standard a* $\in X$ *, and <i>f is continuous. f*st*b* \in *Y* such that $\forall x \in X : x \approx_X a \Rightarrow g(x) \approx_Y b$.

In the case of $Y = \mathbb{C}$ this implies that $g(a)$ is limited and $b' = {}^0g(a)$. We only need

the following special version of

The s-continuity theorem: Let *X* be a

Froof: Consider the formula $A(x, y) \equiv (g(x) \sim y)$. By the construction principle for maps, there is a unique standard $f: X \to \mathbb{C}$ such that $g(a) \sim f(a)$ for all standard $a \in X$, i.e. $f(a) = {}^0g(a)$. By transfer, it suffices to prove that f is continuous at any standard $a \in X$. For this we let $\{a_i\}_{i \in A}$ be a standard net converging to a and show ³⁴^s $\ell \in Y$ such that $\forall x \in X : x \approx_X a \Rightarrow g(x) \approx_Y b$.

se of $Y = \mathbb{C}$ this implies that $g(a)$ is limited and $b = {}^0g(a)$. We only need

ving special version of

continuity theorem: Let X be a standard topological space,

$$
\forall^{\text{st}}\varepsilon > 0 \exists^{\text{st}}\lambda \in \Lambda \forall^{\text{st}}\mu \geq \lambda : |f(a_{\mu}) - f(a)| \leq \varepsilon. \tag{A 4}
$$

By the s-continuity of g, $g(a_\mu) \sim f(a)$ for all infinitely large μ . In particular, if ε is Solution $a \in X$, i.e. $f(a) = {}^0g(a)$. By transfer, it suffices to prove that f is continuous at any standard $a \in X$. For this we let $\{a_i\}_{i \in A}$ be a standard net converging to a and show $\forall^{st} \varepsilon > 0.$ $\exists^{st} \lambda \in \Lambda \forall^{st}$ By the s-continuity of g , $g(a_{\mu}) \sim f(a)$ for all infinitely large μ . In particular, if a standard and λ infinitely large, then $|g(a_{\mu}) - f(a)| < \varepsilon$ for all $\mu \geq \lambda$. By Robinso lemma the latter assertion holds for s Proof: Consider the formula $A(x, y) \equiv (g(x) \sim y)$. By then maps, there is a unique standard $f: X \to \mathbb{C}$ such that g $a \in X$, i.e. $f(a) = {}^0g(a)$. By transfer, it suffices to prove the standard $a \in X$. For this we let $\{a_i\}_{$

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