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On the Nonlinear Boltzmann Equation of the Carrier Transport in Semiconductors: I: Existence and Uniqueness of Solutions[1])

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Es werden Existenz- und Unitätssätze für Lösungen einer stationären, räumlich homogenen Boltzmann-Gleichung hergeleitet, die den Ladungsträgertransport in Halbleitern beschreibt. Eine Besonderheit der hier behandelten Gleichung gegenüber bekannteren Formen, z. B. des Strahlungstransportes, besteht darin, daß das Stoßintegral mit δ-Funktionen behaftet ist, so daß glatte Funktionen dadurch im allgemeinen in unstetige Funktionen übergeführt werden. Die Untersuchung der die Boltzmann-Gleichung beschreibenden Operatoren erlaubt die Konstruktion geeigneter anisotroper Sobolevräume, in denen Existenz und Unität der Lösungen gesichert sind.

Доказываются теоремы существования и единственности решений стационарного пространственно однородного уравнения Больцмана, описывающего транспорт носителей заряда в полупроводниках. Особенность здесь рассмотренного уравнения, в отличие от более известных форм (например, теории излучения), состоит в том, что интеграл столкновения включает δ-функции, так что гладкие функции в общем преобразованы в разрывные функции. Подробное исследование свойств операторов, описывающих уравнение Больцмана, позволяет построение подходящих анизотропных пространств Соболева, в которых существование и единственность решений обеспечены.

There are proved propositions on the existence and uniqueness of solutions of a steady-state, spatially homogeneous nonlinear Boltzmann equation which describes the charge carrier transport in semiconductors. In contrast to more known kinds of the Boltzmann equation (e.g. in radiation transfer theory), the form in question contains δ-functions in the collision integral. Therefore, smooth functions are transformed by the collision operator into discontinuous ones in general. The precise investigation of the properties of the operators describing the Boltzmann equation leads to the construction of suitable anisotropic Sobolev spaces, in which existence and uniqueness of solutions can be shown.

Phenomena of the electron transport in semiconductors can be described by a nonlinear partial integro-differential equation, the so-called Boltzmann equation. The subject of this paper is the investigation of the existence and uniqueness of solutions to the steady-state, spatially homogeneous Boltzmann equation. A forthcoming paper will be concerned with the numerical approximation of the solutions. In contrast to other kinds of this equation which are used, e.g., in the theory of radiation transport or in the kinetic gas theory, the considered form contains Dirac's δ-functions in the kernel of the collision integral. Consequently, the integral operator transforms continuous functions into discontinuous ones in general. Therefore, the investigation of the properties of the integral operator plays an important role in the following considerations. In the case of small electron concentrations it is possible to

[1]) Der abschließende Teil II *Numerical Approximation of Solutions* wird im folgenden Heft dieser Zeitschrift erscheinen.

use a linearized form of the equation in order to describe the transport phenomena adequately. Some results concerning the analytic properties of this form can be found, e.g., in [10, 13—16]. In particular [15] gave rise to some ideas presented in this paper.

1. The Boltzmann equation of electron transport

We investigate the equation

$$F \frac{\partial}{\partial x} u + c_1 u = g + \int_G \{W(\cdot, k')(1 - u) u(k') - W(k', \cdot)(1 - u(k')) u\} z(k') \, dk' \tag{1.1}$$

subject to the boundary conditions

$$u(-l, t) = u(l, t) \qquad \text{for all} \qquad t \in \bar{G}_2, \tag{1.2}$$

where $G = I \times G_2$ is a domain of the n-dimensional crystal momentum space ($n > 1$). Here, $I = (-l, l) \subset \mathbf{R}$ with $l > 0$ and $G_2 \subset \mathbf{R}^{n-1}$ an open and bounded domain with a sufficiently smooth boundary. Furthermore, let $k = (x, t) \in G$.

The solution u of (1.1) describes the steady-state charge carrier distribution (of electrons or holes) in a spatially homogeneous problem with a constant homogeneous electrical field applied. The differential part $F \, \partial u/\partial x$ of (1.1) ($F \in \mathbf{R}$, $F > 0$) describes the influence of the electrical field. Here we assumed that this field is parallel to the basic vector (1, 0). Since G usually represents a small part of a Brillouin zone (e.g., a neighbourhood of the conduction or valence-bond band edge), this is no serious restriction. By imposing periodic boundary conditions (1.2) we assume that G is so large that the equilibrium distribution is not essentially disturbed by the electrical field near the boundary of G. The interactions of the charge carriers and the crystal lattice are described by the collision integral. The term $W(k, k')(1 - u(k))$ states the density of the quantum-mechanical transition probability of a particle to move from a state k into the state k'. The factor $1 - u(k)$ in this density takes the Pauli principle into account. z is the state density in G. The integral kernel W has the form

$$W(k, k') = \sum_{s=-r}^{r} K_s(k, k') \, \delta(E(k) - E(k') + w_0^s) \tag{1.3}$$

where E is a continuously differentiable function defined on $\bar{G}$ (band structure) reflecting the energy a particle would have if it were in the respective state $k \in \bar{G}$. Every term of (1.3) describes possible state transitions. The δ-function is a consequence of the energy conservation principle. w_0^s are constants giving the amount of energy which a charge carrier interchanges with the crystal lattice. The occurance of the δ-functions has several implications on the properties of the collision integral. In dependence on the shape of the level sets of E and of the boundary of G it may happen that the integral part transforms smooth functions into discontinuous ones in general. Hence, the solutions of (1.1) will not be continuously differentiable.

In the following we will distinguish between two cases:

(I) In G there are no carrier sources or sinks:

$c_1 \equiv 0, g \equiv 0.$

(II) In G there are sources or sinks:

$c_1(k) \geqq 0 \ (k \in \bar{G}), c_1 \not\equiv 0.$

Case (II) includes the possibility to take into account such processes as, e.g., band-to-band transition and impact ionization. If the particle concentration is small, $1 - u(k) \approx 1$, therefore it is sufficient to use the linearized form

$$F \frac{\partial}{\partial x} u + c_1 u$$
$$= g + \int_G \{W(\cdot, k')\, u(k') - W(k', \cdot)\, u\}\, z(k')\, dk' \tag{1.4}$$

instead of (1.1) in order to describe the charge carrier transport adequately. An extensive representation of the physical background concerning the Boltzmann equation in semiconductor theory can be found in [1, 9].

For the integral kernel W and the state density z we assume the following to be true:

(A1) (i) Let $D_s = \{(k, k') \in \bar{G} \times \bar{G} \mid E(k) - E(k') = w_0^s\}$. Then, for $s = -r, \ldots, r$, $w_0^s = -w_0^{-s}$, $K_s \in C(D_s)$, $K_s(k, k') > 0$ $(k, k' \in D_s)$.

(ii) $z \in C(\bar{G})$, $z(x, t) \equiv z(t)$, and $z(t) > 0$ almost everywhere.

Remarks: 1. By (A1)/(i) we assume the reversibility of the collision processes described by (1.3). 2. (A1)/(i) yields $w_0^0 = 0$, hence we assume the acoustic scattering to be taken into account. This is not necessary for the results to be valid but it simplifies the notation.

From (A1) we have $Q_s \in C(D_s)$ where

$$Q_s(k, k') = K_s(k, k')\, K_{-s}(k', k)^{-1} \tag{1.5}$$

and

$$q_s = \min \{Q_s(k, k') \mid (k, k') \in D_s\} > 0. \tag{1.6}$$

Notations: In $\mathbf{R}^N$ we denote the Euclidean norm by $|\cdot|$ and the Lebesgue measure by λ^N. If X, Y are Banach spaces, let $B(X, Y)$ denote the space of all continuous linear operators defined on X and mapping into Y. $B_0(X, Y) \subseteq B(X, Y)$ be the subspace of all compact operators. For $A \in B(X, Y)$, $N(A)$ and $R(A)$ denote the kernel and the range of A, respectively. For a compact set K and a Banach space Z, let $C(K, Z)$ denote the Banach space, endowed with the supremum norm, of all continuous mappings defined on K and mapping into Z.

2. Band structures and collision integrals

The kernel (1.3) of the collision operator leads to integrals of the form

$$\int_G u(k)\, \delta\big(E(k) - w\big)\, dk. \tag{2.1}$$

These integrals will be defined now. Furthermore, some properties of the integrals will be proved. For this, the properties of the underlying band structure E as well as the boundary of G play an important role. The following basic shapes of E are often used and adapted to a wide range of semiconductors:

$$E(k) = ak^2, \tag{2.2}$$

$$E(k) = ax^2 + bt^2, \tag{2.3}$$

$$E(k)\left(1+\varepsilon E(k)\right)=ak^2, \tag{2.4}$$

$$E(k)=ax^2+bt^2-(c^2+dx^2)^{1/2}+c. \tag{2.5}$$

Model (2.2) was used, e.g., for p-type germanium [8], n-type indium antimonide and n-type gallium arsenide [12]. (2.4) is another model for n-type gallium arsenide [2]. In [11] (2.2), (2.3) and (2.5) were used to describe p-type tellur. The surfaces of constant energy of the first three models are spheres and ellipsoids, respectively. (2.5) is the so-called camel-back structure. The qualitative behaviour of (2.5) is plotted in the following sketch.

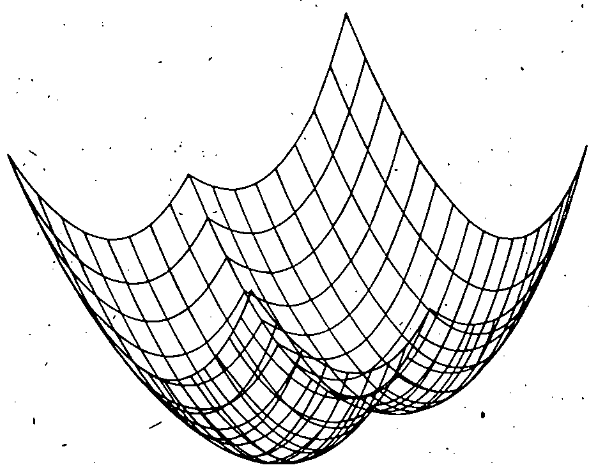

Fig. 1

Our investigations will be carried out for a sufficiently large class of structures containing all shapes (2.2)—(2.5).

Definition: Let there exist a domain $\Omega \subset \mathbf{R}^n$ and a diffeomorphism $\varphi\colon \Omega \to \tilde{G}:=\{k \in G \mid \operatorname{grad} E(k) \neq 0\}$ such that $E(\varphi(w,\zeta))=w$ ($w \in \mathbf{R}$, $\zeta \in \mathbf{R}^{n-1}$ with $(w,\zeta) \in \Omega$) and there exists an $M \in \mathbf{R}$ such that

$$\int_{\Omega(w)} |\det \varphi'(w,\zeta)|\, d\zeta \leqq M \quad (w \in \mathbf{R}). \tag{2.6}$$

For $w \in \mathbf{R}$ let $\Omega(w)=\{\zeta \in \mathbf{R}^{n-1} \mid (w,\zeta) \in \Omega\}$ and set, for $u \in C(\bar{G})$,

$$\int_G u(k)\,\delta\left(E(k)-w\right) dk = \int_{\Omega(w)} u\left(\varphi(w,\zeta)\right) |\det \varphi'(w,\zeta)|\, d\zeta. \tag{2.7}$$

Remarks: 1. It is easy to see that the definition (2.7) is independent of the choice of Ω and φ.

2. For $w \in \mathbf{R}$, let the functional δ_w on $C_0^\infty(G)$ be defined by

$$(\delta_w, u) = \int_G u(k)\,\delta\left(E(k)-w\right) dk.$$

Then $\delta_w \in \mathcal{D}'(G)$ is a well-known example of a generalized function (distribution) concentrated on a surface [3]. δ_w can be defined also under weaker assumptions. The condition grad $E(k) \neq 0$ for all $k \in G$ with $E(k) = w$ is sufficient (suppose that E is smooth). Our condition (2.6) guarantees that (δ_w, u) exists for all $u \in C(\overline{G})$ and $\delta_w \in C(\overline{G})^*$.

3. For the definition of $\delta_w \in C(\overline{G})^*$ for a fixed w it is obviously sufficient to use a neighbourhood of the surface of constant energy $E(k) = w$ instead of $\overline{G}$. The strong assumption is necessary in order to infer global propositions with respect to w.

Consider the function

$$u(k) = \int_G f(k, k')\, \delta\big(E(k) - E(k') + w_0\big)\, dk' \tag{2.8}$$

for some $f \in C(\overline{G} \times \overline{G})$. In order to show continuity properties of such functions we make the following assumption.

(A2) With the notations above let the following be true:

(i) Ω is bounded.

(ii) The transformation φ satisfies det $\varphi' \in C(\overline{\Omega})$.

(iii) There are exactly m ($m \geqq 0$) mutually different values $E_1, \ldots, E_m$ satisfying

$$E_{\min} := \inf \{E(k) \mid k \in G\} \leqq E_1 < E_2 < \cdots < E_m \leqq E_{\max} := \sup \{E(k) \mid k \in G\}$$

such that, for $w \in \mathbf{R}$ ($w \notin \{E_1, \ldots, E_m\}$), $\lambda^{n-1}\big(\Omega(w') \vartriangle \Omega(w)\big) \to 0$ for $w \to w'$. For $w = E_j$, measurable sets $\Omega_j^+, \Omega_j^- \subset \mathbf{R}^{n-1}$ exist such that $\lambda^{n-1}\big(\Omega(w') \vartriangle \Omega_j^{\pm}\big) \to 0$ for $w' \to E_j \pm 0$. Here, $\vartriangle$ denotes the symmetric difference.

Remark: This assumptions requires regularity properties of the energy structure as well as of the boundary of G. In the case of $n = 2$, (A2) is fulfilled for (2.2)–(2.5) with $m = 1$ and $E_1 = E_{\min}$.

Lemma 2.1: *Let* (A1)–(A2) *hold. Let u be defined by* (2.8) *with $f \in C(\overline{G} \times \overline{G})$. If $\mathbf{k} \in \overline{G}$ and $E(\mathbf{k}) + w_0 \notin \{E_1, \ldots, E_m\}$, then u is continuous at $\mathbf{k}$.*

Proof: Let $h(k, w, \zeta) = f\big(k, \varphi(w, \zeta)\big)\, |\det \varphi'(w, \zeta)|$. Because of (A2), $h \in C(\overline{G} \times \overline{\Omega})$. Let $(k_i)_{i \in \mathbf{N}} \subset \overline{G}$ be a sequence with $k_i \to \mathbf{k}$ for $i \to \infty$, and set

$$v_i(\zeta) = \begin{cases} h(k_i, E(k_i) + w_0, \zeta), & \zeta \in \Omega\big(E(k_i) + w_0\big), \\ 0, & \text{otherwise}, \end{cases}$$

$$v(\zeta) = \begin{cases} h\big(\mathbf{k}, E(\mathbf{k}) + w_0, \zeta\big), & \zeta \in \Omega\big(E(\mathbf{k}) + w_0\big), \\ 0, & \text{otherwise}. \end{cases}$$

We show that (v_i) converges in measure to u. Let $\varepsilon > 0$ be fixed and $\delta > 0$ such that $|h(k, w, \zeta) - h(k', w', \zeta')| < \varepsilon$ for all (k, w, ζ), $(k', w', \zeta') \in \overline{G} \times \overline{\Omega}$ satisfying $|(k, w, \zeta) - (k', w', \zeta')| < \delta$. Since E is continuous, there is an $i_0 \in \mathbf{N}$ such that $\big|\big(k_i, E(k_i) + w_0\big) - \big(\mathbf{k}, E(\mathbf{k}) + w_0\big)\big| < \delta$ $(i \geqq i_0)$. Regarding $v(\zeta) = v_i(\zeta) = 0$ if $\zeta \notin \Omega\big(E(k_i) + w_0\big) \cup \Omega\big(E(\mathbf{k}) + w_0\big)$, we obtain, for $i \geqq i_0$,

$$\begin{aligned} &\lambda^{n-1}(|v_i - v| \geqq \varepsilon) \\ &\leqq \lambda^{n-1}(\{\zeta \in \mathbf{R}^{n-1} \mid |v_i(\zeta) - v(\zeta)| \geqq \varepsilon\} \cap \Omega\big(E(k_i) + w_0\big) \cap \Omega\big(E(\mathbf{k}) + w_0\big)) \\ &\quad + \lambda^{n-1}\big(\Omega(E(k_i) + w_0) \vartriangle \Omega(E(\mathbf{k}) + w_0)\big) \\ &= \lambda^{n-1}\big(\Omega(E(k_i) + w_0) \vartriangle \Omega(E(\mathbf{k}) + w_0)\big). \end{aligned}$$

Hence, $\lim \lambda^{n-1}(|v_i - v| \geqq \varepsilon) = 0$. Since Ω and h are bounded,

$$\int_{\mathbf{R}^{n-1}} v_i(\zeta)\, d\zeta \to \int_{\mathbf{R}^{n-1}} v(\zeta)\, d\zeta$$

follows from Lebesgue's dominated convergence theorem. But this is equivalent to $u(k_i) \to u(\mathbf{k})$ ∎

Corollary: *Let the hypotheses of Lemma 2.1 be fulfilled, $E^j = E_j + w_0$, the indices j be selected in such a way that $E_{\min} \leqq E^j \leqq E_{\max}$ and $G^j = \{k \in \overline{G} \mid E^{j-1} < E(k) < E^j\}$ ($j = m', \ldots, m'' + 1$; $E^{m'-1} := E_{\min}$, $E^{m''+1} := E_{\max}$). Then, for u defined by (2.8), $u|_{G^j}$ is continuous and has a continuous extension onto $\overline{G^j}$.*

Further below we use extensively an analogue of Fubini's theorem. Indeed,

$$\iint_{GG} f(k, k')\, \delta\big(E(k) - E(k') + w_0\big)\, dk'\, dk$$
$$= \iint_{GG} f(k, k')\, \delta\big(E(k) - E(k') + w_0\big)\, dk\, dk'$$

for all $f \in C(\overline{G} \times \overline{G})$ and all $w_0 \in \mathbf{R}$.

3. On the solvability of the Boltzmann equation

In order to derive assertions on the existence and uniqueness of solutions we formulate the Boltzmann equation (1.1) as an operator equation in the Banach spaces $X = C(\overline{G}_2, H_p^1(I))$ and $Y = C(\overline{G}_2, L^2(I))$. We shall define A, B, C as the differential, the linear integral, and the nonlinear integral parts, respectively. Then $A, B \in B(X, Y)$ and A is bijective. We shall assume that B is even compact. Using the theory of positive operators in partially ordered Banach spaces [7] it is shown that in Case (II) the spectral radius $r(A^{-1}B)$ is less than 1 whereas in Case (I) one is a simple eigenvalue of $A^{-1}B$ which is in modulus strictly larger than the other eigenvalues and the associated eigenvector can be chosen to be strictly positive. Every physically relevant solution of the Boltzmann equation must satisfy the inequalities $0 \leqq u(k) \leqq 1$. It is possible to find constants $\sigma < 0$ and $\tau > 1$ which only depend on the functions K_s of (1.3) such that for all $u \in D := \{u \in X \mid \sigma < u(k) < \tau\}$ the derivative $(A - B - C)'(u) = A_u - B_u$ can be split into operators A_u and B_u so that the mentioned properties also hold for A_u and B_u. This fact is essentially used. In Case (I) the Boltzmann equation will be supplemented by a condition on the number of particles

$$\int_G u(k)\, z(k)\, dk = p.$$

Then we show the existence of a regular analytic solution path $u(p)$ of (1.1)—(1.2). In Case (II) every solution of the Boltzmann equation is isolated.

We introduce the following notations:

$$H_p^1(I) = \{v \in W^{1,2}(I) \mid v(-l) = v(l)\},$$
$$X = C\big(\overline{G}_2, H_p^1(I)\big), \qquad Y = C\big(\overline{G}_2, L^2(I)\big).$$

The following continuous imbeddings are valid: $X \to C(\overline{G}) \to Y \to L^2(G)$. Throughout this chapter we assume (A1)—(A2) to be fulfilled. Define the following operators (with $u \in X$, $k \in \bar{G}$):

$$Au(k) = F \frac{\partial}{\partial x} u(k) + c(k) u(k), \tag{3.1}$$

$$c = c_0 + c_1, \qquad c_0(k) = \int_G W(k', k) z(k') dk',$$

$$B_s u(k) = \int_G K_s(k, k') \delta(E(k) - E(k') + w_0^s) u(k') z(k') dk',$$

$$Bu = \sum_{s=-r}^{r} B_s u, \tag{3.2}$$

$$\bar{B}u(k) = \int_G (W(k', k) - W(k, k')) u(k') z(k') dk',$$

$$Cu(k) = u(k) \bar{B}u(k),$$

$$Tu = Au - Bu - Cu.$$

Since X is continuously imbedded into $C(\bar{G})$, the functions Bu, $\bar{B}u$, $Cu: \bar{G} \to \mathbf{R}$ are well-defined. Obviously, (1.1)–(1.2) is equivalent to $Tu = g$.

Proposition 3.1: *For all $u \in X$ and $v \in Y$, $\partial u/\partial x \in Y$, $vu \in Y$,* and $\|vu\|_Y \leqq \|v\|_Y \|u\|_{C(\bar{G})} \leqq \gamma \|v\|_Y \|u\|_X$ *for some $\gamma \in \mathbf{R}$ independent of u and v.*

In order to investigate the properties of the integral operators we introduce the Banach space $PC(G)$. According to Assumption (A2) let $E_{\min} \leqq E_1 < \cdots < E_m \leqq E_{\max}$ be defined. Let $\{E^1, \cdots, E^{m'}\} = \{w \in \mathbf{R} \mid w = E_j + w_0^s, -r \leqq s \leqq r, 1 \leqq j \leqq m\} \cap (E_{\min}, E_{\max})$. Assume $E^1 < \ldots < E^{m'}$ to hold. Set $G^j = \{k \in \bar{G} \mid E^j < E(k) < E^{j+1}\}$ ($j = 0, \ldots, m'$; $E^0 := E_{\min}$, $E^{m'+1} := E_{\max}$). Then let $PC(G)$ be the Banach space, equipped with the supremum norm, of all real valued bounded functions u defined on $G' = G^0 \cup \cdots \cup G^{m'}$ whose restrictions $u|_{G^j}$ are continuous and have continuous extensions onto $\overline{G^j}$. Obviously, $G^i \cap G^j = 0$ for $i \neq j$ and $\bar{G} = \overline{G^0} \cup \cdots \cup \overline{G^{m'}}$. Moreover, $PC(G)$ is isomorphic to the Banach space $C(G^0) \times \cdots \times C(G^{m'})$. Hence, the precompact subsets of $PC(G)$ can be characterized by the theorem of Arzela-Ascoli. From the corollary to Lemma 2.1 it becomes clear that the set $\partial G^0 \cup \cdots \cup \partial G^{m'}$ contains all discontinuity points of functions of the kinds Bu and $\bar{B}u$.

In the following we assume Assumption (A3) to be fulfilled:

(A3) (i) $c_1 \in Y$.

(ii) There exists a $d \in \mathbf{R}$ such that $\int_I c(x, t) dx \geqq d > 0$ for all $t \in \bar{G}_2$.

(iii) $\lambda^1(\{x \in I \mid E(x, t) = E^j\}) = 0$ for all $t \in \bar{G}_2$, $j = 0, \ldots, m' + 1$.

Proposition 3.2: *Let* (A1)–(A2) *be true. Then $B \in B(X, PC(G))$.*

Proof: By the corollary of Lemma 2.1, $Bu \in PC(G)$ for all $u \in X$. Because of (A1) there exists an $N > 0$ such that $K_s(k, k') z(k') \leqq N$ $((k, k') \in D_s)$. Then

$$\|B_s u\|_{PC(G)} = \sup_k \left| \int_G K_s(k, k') u(k') \delta(E(k) - E(k') + w_0^s) z(k') dk' \right|$$
$$\leqq \sup_{(k,k') \in D_s} K_s(k, k') z(k') \|u\|_{C(\bar{G})} \int_G \delta(E(k) - E(k') + w_0^s) dk'$$
$$\leqq N\gamma \|u\|_X M \blacksquare$$

This gives rise to the question which conditions have to be fulfilled such that $R(B) \subseteq Y$ holds. It turns out that Assumption (A3)/(iii) is sufficient for that.

Lemma 3.3: *Let (A3) hold. Then $PC(G)$ is continuously imbedded into Y.*

Proof: Let $u \in PC(G)$ and $v: \bar{G}_2 \to L^2(I)$ be defined by $v(t)\,(x) = u(x, t)$ $\big((x, t) \in G'\big)$. Because of (A3)/(iii) the measure of

$$V(t) = \bigcup_{j=0}^{m'+1} \{x \in \bar{I} \mid E(x, t) = E^j\}$$

vanishes for every $t \in \bar{G}_2$. Hence $v(t)$ is defined for almost every $x \in I$. Since $v(t)$ is continuous on every component of the open set $I \setminus V(t)$, $v(t)$ is measurable, and from the boundedness of u we get $v(t) \in L^2(I)$. Let $(t_i)_{i \in \mathbb{N}} \subset \bar{G}_2$ be a sequence with $t_i \to t$. Since u is continuous at every $k = (x, t) \in \big(I \setminus V(t)\big) \times \{t\}$, $v(t_i)\,(x) = u(x, t_i) \to u(x, t) = v(t)\,(x)$ for all $x \in I \setminus V(t)$. Using the boundedness of u, $v(t_i) \to v(t)$ in $L^2(I)$ follows from Lebesgue's dominated convergence theorem. Consequently, $v \in Y$. The continuity of the imbedding is now obvious ∎

Remark: (A2)/(iii) is also essentially necessary for the continuous imbedding $PC(G) \to Y$ to hold. For instance, (A2)/(iii) is fulfilled if the level sets $\{k \in \bar{G} \mid E(k) \leqq E^j\}$ are finite unions of strictly convex sets. This is the case for (2.2)—(2.5).

Corollary: *Let (A1)—(A3) hold. Then $B \in B(X, Y)$ and $\dot{c}_0 \in Y$.*

The inclusion $B \in B(X, Y)$ is too weak for our purposes. We suppose B to be even a compact operator: $B \in B_0(X, Y)$. Our previous assumptions are not sufficient to ensure this property. In order that $B \in B_0(X, Y)$ holds we need further assumptions on the band structure E. The sample structures (2.2)—(2.5) have this property.

(A4) The band structure E and the state density z are such that the integral operator $B \in B(X, Y)$ defined by (3.2) is compact for each integral kernel W satisfying (A1).

A proof of (A4) for a given band structure is loaded with technicalities. Wendt [15] has suggested a general scheme for proving the compactness. In Chapter 4 we illustrate this scheme by proving (A4) for a very simple band structure.

We summarize the properties of the operators A and C.

Lemma 3.4: *Let (A1)—(A3) hold. Then $A \in B(X, Y)$. Moreover, A is bijective and, consequently, continuously invertible on Y.*

Proof: Because of Proposition 3.1 and the corollary of Lemma 3.3, $A \in B(X, Y)$. A simple calculation shows that $Au = w$ if and only if

$$u(x, t) = \int_I G(x, t, \xi)\, w(\xi, t)\, d\xi,$$

$$G(x, t, \xi) = \frac{1}{F} \frac{e^{h(\xi,t) - h(x,t)}}{1 - e^{-h(l,t)}} \begin{cases} 1, & -l \leq \xi \leq x \leq l, \\ e^{-h(l,t)}, & -l \leq x < \xi \leq l, \end{cases} \tag{3.3}$$

$$h(x, t) = \frac{1}{F} \int_{-l}^{x} c(\xi, t)\, d\xi.$$

From this representation we get the estimate

$$0 < G(x, t, \xi) \leqq \alpha \quad \text{and} \quad \|A^{-1}\| \leqq \alpha \left((2l)^2 + \frac{2l}{F^2} \|c\|_Y{}^2\right)^{1/2}.$$

where

$$\alpha = \frac{\exp\left(F^{-1}(2l)^{1/2} \|c\|_Y\right)}{F\left(1 - \exp(-d/F)\right)}.$$

This yields the assertions ∎

Lemma 3.5: *Let* (A1)—(A3) *hold. Then:*

(i) *$Cu \in Y$ for all $u \in X$ and $C: X \to Y$ is analytical.*

(ii) *$C'(u)\, v = u\bar{B}v + v\bar{B}u$ $(u, v \in X)$ and $C' \in B(X, B(X, Y))$.*

Now we are in the position to prove our main results. At first we consider the linearized equation (1.4), (1.2). In operator notation it reads $(A - B)\, u = g$. In the following an *eigenvalue of* (A, B) be a $\lambda \in \mathbf{C}$ such that the complexified operator $A - \lambda B$ has a nontrivial nullspace.

Theorem 3.6: *Let* (A1)—(A4) *be true. Then we have:*

(i) *For all $z \in \mathbf{C}$, the complexified operator $A - zB$ is Fredholm with index zero. The eigenvalues have no finite point of accumulation.*

(ii) *There exists an eigenvalue $\lambda_0 \in \mathbf{R}$ having the properties*

a) *$\lambda_0 > 0$ and $|\lambda| > \lambda_0$ for all eigenvalues $\lambda \neq \lambda_0$ of (A, B).*

b) *The eigenvalue λ_0 is algebraically simple. The eigenvector $e \in X$ belonging to λ_0 can be chosen to be strictly positive, i.e., $e(k) > 0$ for all $k \in \bar{G}$.*

(iii) *In Case* (I) *it holds that $\lambda_0 = 1$, whereas $\lambda_0 > 1$ in Case* (II).

Proof: (i) Since A is bijective and B is compact, the assertion follows from Nikolskij's theorem [6: Theorem XIII.5.1].

(ii) Let $K_X = \{u \in X \mid u(k) \geqq 0 \text{ for all } k \in \bar{G}\}$ denote the cone of all nonnegative functions of X and K_Y the corresponding cone of all nonnegative functions of Y. The interior $\operatorname{int} K_X = \{u \in X \mid u(k) > 0 \text{ for all } k \in \bar{G}\}$ is nonempty. The operator $A^{-1}B \in B_0(X)$ is strictly positive, i.e., for every $u \in X$, $u \neq 0$, there exists an $n \in \mathbf{N}$ such that $(A^{-1}B)^n u \in \operatorname{int} K_X$ (cp. (A1), (3.3)). Theorems 2.5, 2.10, 2.13 of [7] imply the existence of an algebraically simple eigenvalue $\mu_0 \in \mathbf{R}$, $\mu_0 > 0$, and of an associated eigenvector $e \in \operatorname{int} K_X$ of $A^{-1}B$. Moreover, for all $\mu \in \sigma(A^{-1}B)$, $\mu \neq \mu_0$, we have $|\mu| < \mu_0$. Since, for $\lambda \neq 0$, λ is an eigenvalue of (A, B) if and only if $\lambda^{-1} \in \sigma(A^{-1}B)$, the assertion follows with $\lambda_0 = \mu_0^{-1}$.

(iii) For the eigenvalue λ_0 and the eigenvector e we have

$$F \frac{\partial}{\partial x} e + (c_0 + c_1)\, e = \lambda_0 \int_G W(\cdot, k')\, e(k')\, z(k')\, dk'. \tag{3.4}$$

Denote

$$\alpha = \iint_{GG} W(k, k')\, e(k')\, z(k')\, dk'\, z(k)\, dk = \int_G c_0(k')\, e(k')\, z(k')\, dk' > 0,$$

$$\alpha' = \int_G c_1(k)\, e(k)\, z(k)\, dk.$$

In Case (I) $\alpha' = 0$, whereas $\alpha' > 0$ in Case (II). Integrating (3.4) yields $\alpha + \alpha' = \lambda_0 \alpha$, and the assertion follows immediately ∎

From Theorem 3.6 we conclude the following corollary on the solvability of the linearized Boltzmann equation (1.4), (1.2).

Corollary: *Under the hypotheses of Theorem* 3.6 *we have:*

(i) *In Case* (I) *the equation* $Au - Bu = 0$ *has, except for a constant real factor, exactly one solution* $u \in X$. *This solution can be chosen to be strictly positive.*

(ii) *In Case* (II) $Au - Bu = g$ *has exactly one solution* $u \in X$ *for every right-hand side* $g \in Y$. *If* g *is a nonnegative function,* u *is so, too.*

Returning to the nonlinear equation (1.1), (1.2), from Lemma 3.5 we see that $T: X \to Y$ is analytical and $T'(u) = A_u - B_u$ where

$$A_u v = Av - v\bar{B}u, \qquad B_u v = Bv + u\bar{B}v \quad (u, v \in X). \tag{3.5}$$

A simple calculation shows that A_u and B_u have the following representations:

$$\begin{aligned} B_u v &= \int_G W_u(\cdot, k')\, v(k')\, z(k')\, dk', \\ W_u(k, k') &= W(k, k') + \big(W(k', k) - W(k, k')\big)\, u(k) \\ &= \sum_{s=-r}^{r} \big(K_s(k, k') + (K_{-s}(k', k) - K_s(k, k'))\, u(k)\big) \\ &\quad \times \delta\big(E(k) - E(k') + w_0{}^s\big), \end{aligned} \tag{3.6}$$

$$A_u v = F \cdot \frac{\partial}{\partial x} v + c_u v,$$

$$c_u = c_1 + c_{0,u}, \qquad c_{0,u} = \int_G W_u(k', \cdot)\, z(k')\, dk'.$$

Every physically reasonable solution of the Boltzmann equation must have the property $0 \leqq u(k) \leqq 1$. In the following we consider only solutions belonging to the open set $D \subseteq X$ defined below which contains all relevant solutions. With (1.6) let

$$\underline{q} = \min_{s=-r,\dots,r} \underline{q}_s, \qquad \bar{q} = \max_{s=-r,\dots,r} \max_{(k,k') \in D_s} Q_s(k, k').$$

Moreover, with

$$\sigma = \begin{cases} (1 - \bar{q})^{-1}, & \bar{q} \neq 1, \\ -\infty, & \bar{q} = 1, \end{cases} \qquad \tau = \begin{cases} (1 - \underline{q})^{-1}, & \underline{q} \neq 1, \\ +\infty, & \underline{q} = 1, \end{cases}$$

let $D = \{u \in X \mid \sigma < u(k) < \tau \text{ for all } k \in \bar{G}\}$. From the definition of Q_s it follows that $\underline{q} \leqq 1 \leqq \bar{q}$. Hence, $\sigma < 0$ and $\tau > 1$.

Let now $u \in D$ and $\bar{u} = \max\{u(k) \mid k \in \bar{G}\}$ and $\underline{u} = \min\{u(k) \mid k \in \bar{G}\}$. Then

$$\varepsilon = \min\{1 + (\underline{q} - 1)\,\bar{u}, \quad 1 + (\bar{q} - 1)\,\underline{u}\} > 0.$$

By (1.5), $K_s(k, k') + \big(K_{-s}(k', k) - K_s(k, k')\big)\, u(k) \geqq \varepsilon K_s(k, k')$. Now (A3), (3.6) yield

$$c_{0,u}(k) \geqq \varepsilon c_0(k), \quad k \in \bar{G}, \qquad \int_I c_u(x, t)\, dx \geqq \varepsilon d > 0, \quad t \in \bar{G}_2.$$

Hence we have shown the following essential result.

Lemma 3.7: *Let* (A1)–(A4) *be true. Then, for all* $u \in D$, *the statements of Theorem* 3.6 *hold for* $T'(u) = A_u - B_u$ *if* A *and* B *are replaced by* A_u *and* B_u, *respectively.*

As an immediate consequence we obtain

Theorem 3.8: *Consider Case* (II). *Let* (A1)–(A4) *be true. Then we have:*

(i) *Let* $u \in D$ *and* $Tu = g$. *Then there exist open neighbourhoods* $U \subseteq X$, $V \subseteq Y$ *of* u, g, *respectively, such that* $\tilde{T} = T|_U : U \to V$ *is bijective and* $\tilde{T}^{-1}$ *is continuously differentiable.*

(ii) *There exists a $\delta > 0$ such that, for all $g \in Y$ with $\|g\|_Y < \delta$ the equation $Tu = g$ has a solution $u \in D$.*

(iii) *Let $u \in D$ such that $Tu(k) \geqq 0$ for all $k \in \bar{G}$. Then u is a nonnegative function.*

Proof: (i) is a consequence of the implicit function theorem, and (ii) follows from (i) since $T0 = 0$.

(iii): We define a mapping $S: D \times X \to Y$ by $S(u, v) = A_{u/2}v - B_{u/2}v$. $A_{u/2}$ and $B_{u/2}$ are defined according to (3.5). Obviously, $S(u, u) = Tu$ $(u \in D)$. Since $u/2 \in D$, $(A_{u/2} - B_{u/2})^{-1} \in B(Y, X)$ exists and is positive. Therefore, $v \geqq 0$ follows from $S(u, v) = g \geqq 0$ for all $u \in D$. Setting $Tu = g$ we obtain the assertion ∎

Remark: Using the same methods it is possible to show that $u(k) \leqq 1$ follows from $Tu \leqq c_1$ for $u \in D$. But this proposition is useless since the condition $g \leqq c_1$ is often not fulfilled.

Lemma 3.9: *Let* (A1)—(A4) *be true and*

$$Y' = \left\{v \in Y \,\Big|\, \int_G v(k)\, z(k)\, dk = 0\right\}. \tag{3.7}$$

Then $T'(u)\, X = Y'$ for all $u \in D$ and $TX \subseteqq Y'$ in Case (I).

Proof: The inclusions $TX \subseteqq Y'$ and $T'(u)\, X \subseteqq Y'$ follow immediately from (1.1), (3.5) and $T'(u) = A_u - B_u$. Because of Lemma 3.7, $\dim N(T'(u)) = \operatorname{codim} R(T'(u)) = 1$ $(u \in D)$. Since $Y' \subset Y$ is closed and $\operatorname{codim} Y' = 1$, the identity $R(T'(u)) = Y'$ must hold ∎

In Case (I) the Boltzmann equation has no isolated solutions. But Lemma 3.9, Lemma 3.7 and Theorem 3.6 suggest the following consideration: For all $u \in D$, $N(T'(u)) = \operatorname{span}\{e_u\}$ where $e_u \in \operatorname{int} K_X$. If $g \in Y'$, the set L of solutions of the equation $T'(u)\, v = g$ has the representation $L = \{v_0 + \beta e_u \mid \beta \in \mathbf{R}\}$. If $e^* \in X^*$, $e^* \neq 0$, is a positive functional (i.e. $\langle e^*, u\rangle \geqq 0$ for all $u \in K_X$), for every $\alpha \in \mathbf{R}$ there exists exactly one solution v of the system $T'(u) = g$, $\langle e^*, v\rangle = \alpha$. For physical reasons it is advisable to choose e^* as the operator which assigns the number of particles to each distribution u. More precisely, let Y' be given by (3.7) and $H: X \times \mathbf{R} \to Y' \times \mathbf{R}$ be defined by

$$H(u, p) = \begin{pmatrix} Tu \\ \langle e^*, u\rangle - p \end{pmatrix} \quad \text{where} \quad \langle e^*, u\rangle = \int_G u(k)\, z(k)\, dk. \tag{3.8}$$

Instead of (1.1), (1.2) we consider the equation $H(u, p) = 0$.

Theorem 3.10: *Consider Case* (I). *Let* (A1)—(A4) *be true. With $p_{\max} = \langle e^*, 1\rangle$ and $D' = \{u \in D \mid 0 \leqq u(k) \leqq 1\}$ we have:*

(i) *$\dfrac{\partial}{\partial u} H(u, p)$ is bijective for all $(u, p) \in D \times \mathbf{R}$.*

(ii) *$H(0, 0) = H(1, p_{\max}) = 0$.*

(iii) *If $u: (p_1, p_2) \to D$ is a continuous solution path to the equation $H(u, p) = 0$, then it holds for all α, β satisfying $p_1 < \alpha < \beta < p_2$ that $u(\alpha)(k) < u(\beta)(k)$ for all $k \in \bar{G}$.*

(iv) *If $u \in D$, $p \in [0, p_{\max}]$, and $H(u, p) = 0$, then $u \geqq 0$ implies $u \leqq 1$.*

(v) *There exists a unique analytic solution path $u: [0, p_{\max}] \to D'$ of $H(u, p) = 0$ with $u(0) = 0$ and $u(p_{\max}) = 1$. Moreover, the equation $H(u, p) = 0$ has no further solutions in $D' \times [0, p_{\max}]$.*

Proof: We have, for $(u, p) \in X \times \mathbf{R}$,

$$\frac{\partial}{\partial u} H(u, p)\, v = \begin{pmatrix} T'(u)\, v \\ \langle e^*, v \rangle \end{pmatrix} \qquad (v \in X).$$

Assertion (i) follows from Lemma 3.9, Lemma 3.7 and Theorem 3.6; (ii) is obvious.

(iii): We remark first that H is an analytic mapping, and hence every continuous solution path is also analytic. Consequently, $u'(p)$ exists and

$$u'(p) = -\left[\frac{\partial}{\partial u} H\big(u(p), p\big)\right]^{-1} \frac{\partial}{\partial p} H\big(u(p), p\big), \qquad p \in (p_1, p_2).$$

Equivalently, $u'(p)$ is the solution of the system $T'\big(u(p)\big)\, u'(p) = 0$, $\langle e^*, u'(p)\rangle = 1$. Because of Lemma 3.7 and Theorem 3.6, $u'(p) \in \operatorname{int} K_X$, and the assertion follows.

(iv): Let $u \in D$ and $p \in [0, p_{\max}]$ such that $H(u, p) = 0$ and $u(k) > 0$ for all $k \in \overline{G}$. Then, $v = 1 - u$ is a solution of the system $\underline{A}v - \underline{B}v = 0$, $\langle e^*, v\rangle = p_{\max} - p$ where $\underline{A}$, $\underline{B}$ are defined according to (3.1), (3.2) using the integral kernel $\underline{W}(k, k') = W(k, k') \times u(k)$. $\underline{W}$ satisfies (A1). Hence, $v \geqq 0$ (i.e. $u \leqq 1$) by Theorem 3.6. Let now $u \geqq 0$. u lies on a continuous and strictly monotone solution path because of (i), (ii). Let $p < p_{\max}$. For $\alpha > p$ we have $u(\alpha) > u \geqq 0$. Hence, $u(\alpha) \leqq 1$. Since D' is closed in X, $u \leqq 1$ follows. If $p = p_{\max}$, we have $u(\alpha) > u$ for $\alpha > p_{\max}$. Furthermore, $v = v(\alpha) := 1 - u(\alpha)$ is a solution of the system $\underline{A}v - \underline{B}v = 0$, $\langle e^*, v\rangle = p_{\max} - \alpha < 0$. Using Theorem 3.6 we obtain $v \leqq 0$. Hence, $u(\alpha) \geqq 1$, and by the continuity of $u(\alpha)$, $u \geqq 1$. Since $\langle e^*, u\rangle = p_{\max}$, $u = 1$.

(v) This assertion is a consequence of (i)—(iv). A detailed proof is given in [5] ∎

According to Theorem 3.10/(v) the following representation of the Boltzmann equation is appropriate in Case (I):

$$\mathcal{X} = C([0, p_{\max}], X), \qquad \mathcal{Y} = C([0, p_{\max}], Y' \times \mathbf{R}),$$
$$\mathcal{T}: \mathcal{X} \to \mathcal{Y}, \quad \mathcal{T}u(p) = H\big(u(p), p\big), \qquad p \in [0, p_{\max}].$$

(1.1), (1.2) is described by $\mathcal{T}u = 0$. This equation has exactly one solution (which is analytical). Moreover, the derivative $\mathcal{T}'(u)$ is bijective for all $u \in \mathcal{D} := \{u \in \mathcal{X} \mid u(p) \in D\}$. This representation will be advantageous when investigating the convergence of numerical methods for the approximate solution of (1.1), (1.2).

Sometimes it happens that the domain G is not given in the cylindrical form $I \times G_2$. Due to physical considerations it is known that the probability that the charge carriers reach large energy values nearly vanishes. Hence, it is sufficient to consider the Boltzmann equation on such a domain G_0 where the energy does not exceed a given maximal value. To be more precise, let $G' \subseteq \mathbf{R}^n$ be a domain and $E_{\max}$ a given real constant (of maximal energy). Let E be a band structure defined on G'. Set

$$G_0 = \{k \in G' \mid E(k) < E_{\max}\},$$
$$G_2 = \{t \in \mathbf{R}^{n-1} \mid (x, t) \in G_0 \text{ for some } x \in \mathbf{R}\},$$
$$I = \{x \in \mathbf{R} \mid (x, t) \in G_0 \text{ for some } t \in \mathbf{R}^{n-1}\}.$$

We assume G_0 to be open, bounded, and convex. Set $G = I \times G_2$, and $G_0(t) = \{x \in \mathbf{R} \mid (x, t) \in G_0\}$, $x^1(t) = \min G_0(t)$, $x^2(t) = \max G_0(t)$. Let equation (1.1) be given on the domain G_0 with the modified boundary conditions

$$u(x^1(t), t) = u(x^2(t), t) \qquad (t \in \overline{G}_2). \tag{3.9}$$

We relate this equation to an equivalent equation (1.1) with the boundary conditions (1.2) defined on the domain G. The equivalence is to be understood in the following sense: For a given function $u \in C(\overline{G}_0)$ a function $\bar{u} \in C(\overline{G})$ is defined by

$$\bar{u}(x, t) = \begin{cases} u(x, t), & (x, t) \in \overline{G}_0, \\ u(x^1(t), t) & \text{otherwise.} \end{cases}$$

If the functions K_s are extended onto $\overline{G} \times \overline{G}$ by zero, then it holds: If u is a solution of (1.1), (3.9) on G_0, then $\bar{u}$ is a solution of (1.1), (1.2) on G and vice versa. Taking suitable subspaces $X'' \subseteq X$ and $Y'' \subseteq Y$ equipped with new norms it is possible to show results of the kind given above. A detailed representation can be found in [4]. The modified form becomes important when solutions are computed numerically. The approximation of the integral operators does not lead to full matrices because of the δ-functions. Using the modified formulation the number of nonvanishing elements decreases again.

4. A compactness proof

In our previous consideration we used the compactness assumption (A4) extensively. Here, we supply a proof of (A4) for a sample band structure. In an attempt to avoid as many technicalities as possible we choose the simplest case. Nevertheless, the essential ingredients are clearly seen.

In proving (A4) we follow the lines of WENDT [15]. He has suggested the following general scheme: For a fixed s, $B_s u(k)$ is considered to be a superposition of a linear operator and a linear functional. More precisely, let Z be a Banach space and $\{P_k \mid k \in \overline{G}\} \subseteq B(X, Z)$, $\{l_k \mid k \in \overline{G}\} \subseteq Z^*$ families of bounded linear operators and functionals, respectively, such that $B_s u(k) = l_k P_k u$, $k \in \overline{G}$. Assume the following to be true:

(i) The families $\{l_k\}$ and $\{P_k\}$ are bounded.

(ii) For every j $(0 \leqq j \leqq m')$ and every $\varepsilon > 0$ there exist a $\delta > 0$ such that, for every $k, k' \in G^j$ and $u \in X$, $|k - k'| < \delta$ implies $|(l_k - l_{k'}) P_{k'} u| \leqq \varepsilon \|u\|_X$.

(iii) The mapping $k \mapsto P_k$ is uniformly continuous on G^j $(0 \leqq j \leqq m')$.

Let j be fixed. For given $\varepsilon > 0$ we obtain

$$\begin{aligned} |B_s u(k) - B_s u(k')| &= |l_k P_k u - l_{k'} P_{k'} u| \\ &\leqq |l_k P_k u - l_k P_{k'} u| + |l_k P_{k'} u - l_{k'} P_{k'} u| \\ &\leqq \|l_k\| \, \|P_k - P_{k'}\| \, \|u\|_X + |(l_k - l_{k'}) P_{k'} u| \leqq C\varepsilon \|u\|_X \end{aligned}$$

if $k, k' \in G^j$ and $|k - k'|$ sufficiently small. Hence, for every bounded set $U \subset X$, $B_s U$ is equicontinuous on G^j. Therefore, $B_s U$ is precompact in $PC(G)$ and, consequently, in Y. We show using the simplest example how this algorithm works. Let $G_2 = (0, 1)$, $l = 1$, the band structure be given by (2.2) with $a = 1$, and the state density be $z(k) = 2\pi t$. This state density arises if a three-dimensional problem with cylindrical symmetry is transformed into a two-dimensional model. Let $G^+ = \{k \in G \mid x > 0\}$ and $G^- = \{k \in G \mid x < 0\}$. Then set

$$B_s^{\pm} u(k) = \int_{G^\pm} K_s(k, k') \, u(k') \, \delta\big(E(k) - E(k') + w_0^s\big) \, z(k') \, dk'.$$

Obviously, $R(B_s^{\pm}) \subseteq PC(G)$ and $B_s = B_s^+ + B_s^-$. We consider only B_s^+ since the proof can be done analogously for B_s^-. We choose φ and Ω as follows:

$$\Omega = \left\{(w, \zeta) \in \mathbf{R}^2 \mid 0 < w < 2, \begin{array}{ll} 0 < \zeta < w^{1/2} & (w \leqq 1) \\ (w - 1)^{1/2} < \zeta < 1 & (w > 1) \end{array}\right\},$$

$$\varphi(w, \zeta) = \begin{pmatrix} \varphi_1(w, \zeta) \\ \varphi_2(w, \zeta) \end{pmatrix} = \begin{pmatrix} (w - \zeta^2)^{1/2} \\ \zeta \end{pmatrix}.$$

The point in this choice is the fact that $t = \zeta$, i.e., ζ is orthogonal to the electrical field. Now, let $Z = L^1(G_2)$. Define $P_k \in B(X, Z)$ and $l_k \in Z^*$ by

$$P_k u(t') = \begin{cases} u(\varphi_1(w(k), t'), t') \dfrac{\pi t'}{\varphi_1(w(k), t')}, & t' \in \Omega(w(k)), \\ 0 & \text{otherwise}, \end{cases}$$

$$l_k v = \int_{\Omega(w(k))} K_s(k, \varphi_1(w(k), t')) \, v(t') \, dt',$$

$$w(k) = E(k) + w_0^s, \qquad k \in \bar{G}.$$

Then $B_s^+ u(k) = l_k P_k u$ $(k \in \bar{G})$. Let C be a generic constant independent of u, v, k, k' in the following. Denote for short $k_1 = \varphi(w(k), t'')$ and $k_1' = \varphi(w(k'), t'')$ $(k, k' \in \bar{G})$. A simple calculation shows that the families $\{l_k\}$ and $\{P_k\}$ are bounded. Moreover, for $(k, k' \in G)$

$$I(k, k') = \int_{\Omega(w(k')) \setminus \Omega(w(k))} \frac{\pi t''}{\varphi_1(w(k'), t'')} \, dt'',$$

$$J(k, k') = \int_{\Omega(w(k')) \cap \Omega(w(k))} \left| \varphi_1(w(k), t'')^{-1} - \varphi_1(w(k'), t'')^{-1} \right| \pi t'' \, dt''$$

we have $I, J \in C(\bar{G} \times \bar{G})$ and $I(k, k) = J(k, k) = 0$. Let now $\varepsilon > 0$ be given. Since $K_s \in C(D_s)$, $\varphi_1 \in C(\bar{\Omega})$, and $w \in C(\bar{G})$, a $\delta_1 > 0$ exists such that, for all $k, k' \in \bar{G}$ $(|k - k'| < \delta_1)$ and $t'' \in \Omega(w(k')) \cap \Omega(w(k))$, $|K_s(k, k_1) - K_s(k, k_1')| < \varepsilon$. Furthermore, there exists a $\delta_2 > 0$ such that $I(k, k') < \varepsilon$ for $|k - k'| < \delta_2$. Hence, for $|k - k'| < \delta := \min\{\delta_1, \delta_2\}$,

$$\begin{aligned} &|(l_k - l_{k'}) P_{k'} u| \\ &\leqq \int_{\Omega(w(k')) \cap \Omega(w(k))} |K_s(k, k_1) - K_s(k', k_1')| \, |u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} \, dt'' \\ &\quad + \int_{\Omega(w(k')) \setminus \Omega(w(k))} |K_s(k', k_1') \, u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} \, dt'' \\ &\leqq \varepsilon \, \|u\|_{C(\bar{G})} \, \pi w(k')^{1/2} + C \, \|u\|_{C(\bar{G})} \, I(k, k') \leqq \varepsilon C \, \|u\|_X . \end{aligned}$$

Hence, (ii) holds. In order to show (iii) we may estimate

$$\begin{aligned} \|P_k u - P_{k'} u\|_Z &= \int_{G_2} |P_k u(t'') - P_{k'} u(t'')| \, dt'' \\ &\leqq \int_{\Omega(w(k')) \cap \Omega(w(k))} \left| u(k_1) \frac{\pi t''}{\varphi_1(w(k), t'')} - u(k_1') \frac{\pi t''}{\varphi_1(w(k'), t'')} \right| dt'' \\ &\quad + \int_{\Omega(w(k)) \setminus \Omega(w(k'))} |u(k_1)| \frac{\pi t''}{\varphi_1(w(k), t'')} \, dt'' \\ &\quad + \int_{\Omega(w(k')) \setminus \Omega(w(k))} |u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} \, dt'' \end{aligned}$$

$$\leqq \int_{\Omega(w(k'))\cap\Omega(w(k))} |u(k_1) - u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''$$

$$+ \|u\|_{C(\bar{G})} J(k, k') + \|u\|_{C(\bar{G})}[I(k, k') + I(k', k)].$$

The first term (call it T) may be further estimated:

$$T = \int_{\Omega(w(k'))\cap\Omega(w(k))} \left| \int_{\varphi_1(w(k'),t'')}^{\varphi_1(w(k),t'')} \frac{\partial}{\partial x} u(\xi, t'')\, d\xi \right| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''$$

$$\leqq \int_{\Omega(w(k'))\cap\Omega(w(k))} \left|\varphi_1(w(k), t'') - \varphi_1(w(k'), t'')\right|^{1/2} \|u\|_X \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''. \quad (4.1)$$

Taking into account the continuity of I, J, u, φ_1 we obtain for sufficiently small $|k - k'|$ that $|P_k u - P_{k'} u|_Z \leqq \varepsilon \|u\|_X$, i.e. (iii). Now we conclude that B_s^+ is compact independent of the special choice of the kernel K_s. Treating B_s^- analogously we obtain (A4). Note that we used essentially the differentiability of u with respect to x in (4.1).

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