On the Nonlinear Boltzmann Equation of the Carrier Transport in Semiconductors. I: Existence and Uniqueness of Solutions¹)

M. HANKE

Es werden Existenz- und Unitätssätze für Lösungen einer stationären, räumlich homogenen Boltzmann-Gleichung hergeleitet, die den Ladungsträgertränsport in Halbleitern beschreibt. Eine Besonderheit der hier behandelten Gleichung gegenüber bekannteren Formen, z. B. des Strahlungstransportes, besteht darin, daß das Stoßintegral mit δ -Funktionen behaftet ist, so daß glatte Funktionen dadurch im allgemeinen in unstetige Funktionen übergeführt werden. Die Untersuchung der die Boltz mann-Gleichung beschreibenden Operatoren erlaubt die Kon struktion geeigneter anisotroper Sobolevräume, in denen Existenz und Unität der Lösungen gesichert sind.

Доказываются теоремы существования и единственности решений стационарного пространственно однородного уравнения Больцмана, описывающего транспорт носителей заряда в полупроводниках. Особенность здесь рассмотренного уравнения, в отличие от более известных форм (например, теории излучения), состоит в том, что интеграл столкновения включает δ-функции, так что гладкие функции в общем преo6pa3013aini B paapiuuie yiiiunit. [Iop06110e 11ccJIe1013aHHe cBOtteTB onepaTopoB, описывающих уравнение Больцмана, позволяет построение подходящих анизотропных пространств Соболева, в которых существование и единственность решений обеспечены.

There are proved propositions on the existence and uniqueness of solutions of a steady-state, spatially homogeneous nonlinear Boltzmann equation which describes the charge carrier transport in semiconductors. In contrast to more known kinds of the Boltzmann equation (e.g. in radiation transfer theory), the form in question contains 6-functions in the collision integral. Therefore, smooth functions are transformed by the collision operator into discontinuous ones in general. The precise investigation of the properties of the operators describing the Boltzmann equation leads to the construction of suitable anisotropic Sobolev spaces, in which existence and uniqueness of solutions can be shown.

Phenomena of the electron transport in semiconductors can be described by a nonlinear partial integro-differential equation, the so-called Boltzmann equation. The subject of this paper is the investigation of the existence and uniqueness of solutions to the steady-state, spatially homogeneous Boltzmann equation. A forthcoming paper will be concerned with the numerical approximation of the solutions. In contrast to other kinds of this equation which are used, e.g., in the theory of radiation transport or in the kinetic gas theory,'the considered form contains Dirac's δ -functions in-the kernel of the collision integral. Consequently, the integral operator, transforms continuous functions into discontinuous ones in general. Therefore, the, investigation of the properties of the integral operator plays an important role in the following considerations. In the case of small electron concentrations it is possible to

¹) Der abschließende Teil II Numerical Approximation of Solutions wird im folgenden Heft dieser Zeitschrift erscheinen.

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use a linearized form of the equation in order to describe the transport phenomena adequately. Some results concerning the analytic properties of this form can be found, e.g., in $[10, 13-16]$. In particular $[15]$ gave rise to some ideas presented in this paper.

I. The Boltzmann equation of electron transport

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1. The Boltzmann equation of electron transport
We investigate the equation

$$
F \frac{\partial}{\partial x} u + c_1 u
$$

$$
= g + \int_{G} \{W(\cdot, k') (1 - u) u(k') - W(k', \cdot) (1 - u(k')) u\} z(k') dk'
$$
subject to the boundary conditions

$$
u(-l, t) = u(l, t) \quad \text{for all} \quad t \in \bar{G}_2,
$$

$$
u(c) = 1 \times G_2
$$
 is a domain of the *n*-dimensional crystal momentum space $(n > 1)$.
Here, $I = (-l, l) \subset \mathbb{R}$ with $l > 0$ and $G_2 \subset \mathbb{R}^{n-1}$ an open and bounded domain with
a sufficiently smooth boundary. Furthermore, let $k = (x, t) \in G$.

$$
u(-l, t) = u(l, t) \qquad \text{for all} \qquad t \in \bar{G}_2,
$$
\n
$$
(1.2)
$$

-I

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Here, $I = (-l, l) \subset \mathbb{R}$ with $l > 0$ and $G_2 \subset \mathbb{R}^{n-1$ a sufficiently smooth boundary. Furthermore, let $k = (x, t) \in G$.

The solution *u* of (1.1) describes the steady-state charge carrier distribution (of electrons or holes) in a spatially homogeneous problem with a constant homogeneous electrical field applied. The differential part F $\partial u/\partial x$ of (1.1) $(F \in {\bf R}, F>0)$ describes the influence of the electrical field. Here we assumed that this field is parallel to the basic vector (1, 0). Since *0* usually represents a small part of a Brillouin zone (e.g., a neighbourhood of the conduction or valence-bond band edge), this is no serious restriction. By imposing periodic boundary conditions (1.2) we assume that G is so large that the equilibrium distribution is not essentially disturbed by the electrical field near the boundary of G. The interactions of the charge carriers,and the crystal lattice are described by the collision integral. The term $W(k, k')$ ($i - u(k)$) states the density of the quantum-mechanical transition probability of a particle to move from a state *k* into the state *k'*. The factor $1 - u(k)$ in this density takes the Pauli principle into account. z is the state density in G. The integral kernel *W* has the form nce of the electrical field. Here we assumed that this field is parallel to the
tor (1, 0). Since G usually represents a small part of a Brillouin zone (e.g.,
ourhood of the conduction or valence-bond band edge), this is

$$
W(k, k') = \sum_{s=-r}^{r} K_s(k, k') \, \delta(E(k) - E(k') + w_0^s)
$$
 (1.3)

where E is a continuously differentiable function defined on \overline{G} (band structure) reflecting the energy a particle would have if it were in the respective state $k \in \overline{G}$. Every term of (1.3) describes possible state transitions. The δ -function is a consequence of the energy conservation principle. w_0 ⁸ are constants giving the amount of energy which a charge carrier interchanges with the crystal lattice. The occurance of the δ -functions has several implications on the properties of the collision integral. In dependence on the shape of the level sets of E and of the boundary of G it may happen that the integral part transforms smooth functions into discontinuous ones in general. Hence, the solutions of (1.1) will not be continuously differentiable Every term of (1.3) describes possible state transition
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In the following we will distinguish be
(I) In G there are no carrier source
 $c_1 \equiv 0, g \equiv 0$.
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In the following we will distinguish between two cases:

- In *G* there are no carrier sources or sinks:
 $c_1 = 0, g = 0.$
	-

In *G* there are sources or sinks:

$$
\langle c_1(k) \geq 0 \ (k \in \overline{G}), c_1 \not\equiv 0.
$$

Case '(11) includes the possibility to take into account such processes as, e.g., bandto-band transition and impact ionization. If the particle concentration is small, 0n the Boltzmann Equation of Carrier Transport
Case (II) includes the possibility to take into account such processes as, e.g
to-band transition and impact ionization. If the particle concentration
 $1 - u(k) \approx 1$, therefore

On the Boltzmann Equation of Carrier Transport 323
\n) includes the possibility to take into account such processes as, e.g., band-
\ntransition and impact ionization. If the particle concentration is small,
\n
$$
\approx 1, \text{ therefore it is sufficient to use the linearized form}
$$
\n
$$
F \frac{\partial}{\partial x} u + c_1 u
$$
\n
$$
= g + \int_{C} \{W(\cdot, k') u(k') - W(k', \cdot) u\} z(k') dk' \qquad (1.4)
$$
\nof (1.1) in order to describe the charge carrier transport adequately. An
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instead of (1.1) in order to describe the charge carrier transport adequately. An extensive representation of the physical background concerning the Boltzmann equation in semiconductor theory can be found in [1, 9]. instead of (1.1) in order to describe the charge carrier transport adequately. An extensive representation of the physical background concerning the Boltzmann equation in semiconductor theory can be found in [1, 9].

For

For the integral kernel *W* and the state density *z* we assume the following to be true:

(A1) (i) Let
$$
D_s = \{(k, k') \in \overline{G} \times \overline{G} \mid E(k) - E(k') = w_0^s\}
$$
. Then, for $s' = -r, \ldots, r, w_0^s = -w_0^{-s}, K_s \in C(D_s), K_s(k, k') > 0$ $(k, k' \in D_s)$.
\n(ii) $z \in C(\overline{G}), z(x, t) \equiv z(t), \text{ and } z(t) > 0$ almost everywhere.

Remarks: 1. By $(A1)/i$) we assume the reversibility of the collision processes described by (1.3). 2. (A1)/(i) yields $w_0^0 = 0$, hence we assume the acoustic scattering to be taken into account. This is not necessary for the results to be valid but it simplifies the notation. (a) (1) Let $D_s = \{(k, k') \in G \times G | 1 \}$
 $r, w_0^s = -w_0^{-s}, K_s \in C(D_s),$

(ii) $z \in C(\overline{G}), z(x, t) \equiv z(t),$ and

Remarks: 1. By (A1)/(i) we assume

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to be taken into account. This is not n *true:*

(A1) (i) Let $D_s = \{(k, k') \in \overline{G} \times \overline{G} \mid E(k) - E(k') = w_0^* \}$. Then, for $s' = -r, ...,$
 $r, w_0^* = -w_0^{-s}, K_s \in C(D_s), K_s(k, k') > 0$ ($k, k' \in D_s$).

(ii) $z \in C(\overline{G}), z(x, t) = z(t),$ and $z(t) > 0$ almost everywhere.

Remarks: 1. By (A1)/(i

From (A1) we have $Q_s \in C(D_s)$ where

$$
Q_s(k, k') = K_s(k, k') K_{-s}(k', k)^{-1}
$$
\n(1.5)

Notations: In ${\bf R}^{\boldsymbol{N}}$ we denote the Euclidean norm by $\left| \cdot \right|$ and the Lebesgue measure by λ^N . If X, Y'are Banach spaces, let $B(X, Y)$ denote the space of all continuous linear operators defined on X and mapping into Y: $B_0(X, Y) \subseteq B(X, Y)$ be the subspace of all compact operators. For $\overrightarrow{A} \in \overrightarrow{B}(X, Y)$, $N(\overrightarrow{A})$ and $\overrightarrow{R}(A)$ denote the kernel and the range of *A,* respectively. For a compact set *K* and a Banach space *Z,* let *C(K, Z)* denote the Banach space, endowed with the supremum norm, of all contin-uous mappings defined on *K* and mapping into *Z.* $Q_s(k, k') = K_s(k, k') K_{-s}(k', k)^{-1}$

and
 $q_s = \min \{Q_s(k, k') | (k, k') \in D_s\} > 0.$

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 f and the Lebesgue mother space of all continuate the space of all continuate $F \subseteq B(X, Y)$ be the land a Banach space emum norm, of all continuate of all continuation of the form continuation of the form properties of the int

The kernel (1.3) of the collision operator leads to integrals of the form

$$
\int_{G} u(k) \, \delta(E(k) - w) \, dk. \tag{2.1}
$$

-These integrals will he defined now. Furthermore, some properties of the integrals will be proved. For this, the properties of the underlying band structure *E* as well as the boundary of G play an important role. The following basic shapes of E are often used and adapted to a wide range of semiconductors: *E(k) = ak2 , - . -* (2.2) *Equals will be defined now. Furthermore, some properties of the integrals oved. For this, the properties of the underlying band structure <i>E* as well as dary of *G* play an important role. The following basic shapes of ttor leads to
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E(k) = ak^2,
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\n
$$
E(k) = ax^2 + bt^2.
$$
\n(2.2)

 $21*$.

and

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\n
$$
E(k) (1 + \varepsilon E(k)) = ak^2,
$$
\n(2.4)
\n
$$
E(k) = ax^2 + bt^2 - (c^2 + dx^2)^{1/2} + c.
$$
\n(2.5)
\n(2.6)
\n2.9 was used, e.g., for *p*-type germanium [8], *n*-type indium antimonide and

$$
E(k) = ax^2 + bt^2 - (c^2 + dx^2)^{1/2} + c. \tag{2.5}
$$

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Model (2.2) was used, e.g., for *p*-type germanium [8], *n*-type indium antimonide and *n*-type gallium arsenide [12]. (2.4) is another model for *n*-type gallium arsenide [2]. In $[11]$ (2.2) , (2.3) and (2.5) were used to describe p-type tellur. The surfaces of con- 324 M. HANI
 $E(k)$ (1
 $E(k) =$

Model (2.2) was

n-type gallium a

n [11] (2.2), (2.5

stant energy of is the so-called complete the following ske stant energy of the first three models are spheres and ellipsoids, respectively. (2.5) is the so-called camel-back structure. The qualitative behaviour of (2.5) is plotted in 324 M. HANKE
 $E(k) (1 + \varepsilon E(k)) = ak^2$, (3
 $E(k) = ax^2 + bt^2 - (c^2 + dx^2)^{1/2} + c$. (5)

Model (2.2) was used, e.g., for p-type germanium [8], n-type indium antimonide ε

n-type gallium arsenide [12]. (2.4) is another model for n-

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Our investigations will be carried out for a sufficiently large class of structures containing all shapes $(2.2) - (2.5)$.

Definition: Let there exist a domain $\Omega \subset \mathbb{R}^n$ and a diffeomorphism $\varphi: \Omega \to \tilde{G}$:
 $= \{k \in G \mid \text{grad } E(k) = 0\}$ such that $E(\varphi(w, \zeta)) = w \ (w \in \mathbb{R}, \ \zeta \in \mathbb{R}^{n-1} \text{ with } (w, \zeta) \in \Omega)$

and there exists an $M \in \mathbb{R$ **i** taining all shapes $(2.2) - (2.5)$.
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 $\int_{\Omega(w$ *f***u** is tigations will be carried out for a sufficiently large class of stru
 f shapes $(2.2) - (2.5)$.
 f tion: Let there exist a domain $\Omega \subset \mathbb{R}^n$ and a diffeomorphism
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 $= \{k \in G \mid \text{grad } E(k) \neq 0\}$ such that $E(p(w, \zeta)) = w \ (w$

$$
\int_{(w)} |\det \varphi'(w, \zeta)| d\zeta \leq M (w \in \mathbf{R}). \tag{2.6}
$$

$$
\int_{\Omega(w)} |\det \varphi'(w, \zeta)| d\zeta \leq M (w \in \mathbf{R}).
$$
\n(2.6)
\n
$$
\mathbf{R} \det \Omega(w) = \{\zeta \in \mathbf{R}^{n-1} \mid (w, \zeta) \in \Omega\} \text{ and set, for } u \in C(\overline{G}),
$$
\n
$$
\int_{G} u(k) \delta(E(k) - w) dk = \int_{\Omega(w)} u(\varphi(w, \zeta)) |\det \varphi'(w, \zeta)| d\zeta.
$$
\n(2.7)

Remarks: 1. It is easy to see that the definition (2.7) is independent of the choice of Ω and φ . For $w \in \mathbb{R}$
Rema
f Ω and
2. For
Then δ_w

2. For $w \in \mathbf{R}$, let the functional δ_w on $C^\infty_0(G)$ be defined by

$$
\varphi.
$$
\n
$$
w \in \mathbf{R}, \text{ let the functional } \delta_w \text{ on } \delta
$$
\n
$$
(\delta_w, u) = \int_a u(k) \, \delta(E(k) - w) \, dk.
$$
\n
$$
\delta_w(\delta(w)) = \delta_w(w) \text{ using a complex form}
$$

Then $\delta_{\omega} \in \mathcal{D}'(G)$ is a well-known example of a generalized function (distribution) concentrated on a surface [3]. δ_{ω} can be defined also under weaker assumptions. The

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condition grad $E(k)$ + 0 for all $k \in G$ with $E(k) = w$ is sufficient (suppose that *E* is smooth). Our condition (2.6) guarantees that (δ_u, u) exists for all $u \in C(G)$ and $\delta_m \in C(\overline{G})^*$.

3. For the definition of $\delta_w \in C(\overline{G})^*$ for a fixed w it is obviously sufficient to use a neighbourhood of the surface of constant energy $E(k) = w$ instead of \tilde{G} . The strong assumption is necessary in order to infer global propositions with respect to *W.* On the Boltzmann Equation of Carrier Transport 325

grad $E(k) \neq 0$ for all $k \in G$ with $E(k) = w$ is sufficient (suppose that E is

Our condition (2.6) guarantees that (δ_w, u) exists for all $u \in C(\overline{G})$ and

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Our condition (2.6) guarantees that (δ_w, u) e

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n is necessary *e* definition of $\delta_w \in C(\overline{G})^*$ for a fixed *w* it is obvood of the surface of constant energy $E(k) = w$
is necessary in order to infer global proposition
the function
 $k) = \int f(k, k') \delta(E(k) - E(k') + w_0) dk'$
 $\in C(\overline{G} \times \overline{G})$. In

Consider the function

^Emin *:=* inf {E(k)I *^k*€ G} *^E¹* <*E2* < ... ^S

for some $f\in C(\overline{G}\times \overline{G}).$ In order to show continuity properties of such functions we make the following assumption. $u(k) = \int_{G} f(k, k') \, \delta(E(k) - E(k') + w_0) \, dk'$
for some $f \in C(\overline{G} \times \overline{G})$. In order to show continuity propertion
make the following assumption.
(A2) With the notations above let the following be true:
(i) Ω is bounded.

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- (ii) The transformation φ satisfies det $\varphi' \in C(\overline{\Omega})$.
- With the notations above let the following be true:

(i) Ω is bounded.

(ii) The transformation φ satisfies det $\varphi' \in C(\overline{\Omega})$.

(iii) There are exactly m ($m \ge 0$) mutually different values $E_1, ..., E_m$

satisfyin

There are exactly
$$
m \ (m \geq 0)
$$
 mutual satisfying\n
$$
E_{\min} := \inf \{ E(k) \mid k \in G \} \leq E_1 < E_2 < \cdots
$$
\n
$$
E_m \leq E_{\max} := \sup \{ E(k) \mid k \in G \}
$$
\nwhich holds for $m \in \mathbf{R}$, for $G \in \mathbf{R}$.

such that, for $w \in \mathbb{R}$ $(w \notin \{E_1, ..., E_m\}), \lambda^{n-1}(\Omega(w') \wedge \Omega(w)) \to 0$ for *w* $m: m \geq 0$ *meating* different values E_1 , ..., E_m and $E_{\text{min}} := \inf \{ E(k) \mid k \in G \}$
 $\leq E_m \leq E_{\text{max}} := \sup \{ E(k) \mid k \in G \}$
 $\leq E_m \leq E_{\text{max}} := \sup \{ E(k) \mid k \in G \}$
 $\leq w \to w'$. For $w \in \mathbb{R}$ ($w \notin \{ E_1, ..., E_m \}$), $\lambda^{n-1}(\Omega(w') \$ $\mathbb{Z}(\overline{G} \times \overline{G})$. In order to show continuity properties of such functions we
wing assumption.

the notations above let the following be true:
 Q is bounded.

The transformation φ satisfies det $\varphi' \in C(\overline{\Omega})$

Remark: This assumptions requires regularity properties of the energy structure as well as of the boundary of G. In the case of $n = 2$, (A2) is fulfilled for $(2.2) - (2.5)$ with $m = 1$ and $E_1 = E_{\min}$.

Lemma 2.1: Let $(A1) - (A2)$, hold. Let u be defined by (2.8) with $f \in C(\overline{G} \times \overline{G})$. If $\mathbf{k} \in \overline{G}$ and $E(\mathbf{k}) + w_0 \in \{E_1, \ldots, E_m\}$, then u is continuous at \mathbf{k} .

Lemma 2.1: Let $(A1) - (A2)$, hold. Let u be defined by (2.8) with $f \in C(\overline{G} \times \overline{G})$. If
 $k \in \overline{G}$ and $E(k) + w_0 \notin \{E_1, ..., E_m\}$, then u is continuous at k.

Proof: Let $h(k, w, \zeta) = f(k, \varphi(w, \zeta)) |\text{det } \varphi'(w, \zeta)|$. Because of

 $v_i(\zeta)=\Big\{$ Frence.

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is assumptions requires regular

boundary of G. In the case of
 $E_1 = E_{\min}$.

Let $(A1) - (A2)$, hold. Let u be
 $+ w_0 \in \{E_1, ..., E_m\}$, then u is c
 k, w, ζ = $f(k, \varphi(w, \zeta)) |\text{det } \varphi'(v)$

ee a sequence with $\in \varOmega(E(k_{i})\,+\,w_{0}),$ *Let* $(w \in \{E_1, ..., E_m\})$
 w'. For $w = E_j$, measurable sets Ω_j
 $(\Omega(w') \triangle \Omega_j^{\pm}) \rightarrow 0$ for $w' \rightarrow E_j \pm 0$. If

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coundary of *G*. In the case of $n = 2$, ($E_1 = E_{\min}$.
 Let (A1)-(A *1* **0,** assumptions requires regula

oundary of G. In the case of
 $E_1 = E_{\text{min}}$.

Let $(A1) - (A2)$, *hold.* Let u be
 $w_0 \notin \{E_1, ..., E_m\}$, then u is c
 $w_0 \notin \{E_1, ..., E_m\}$, then u is c
 $w_0 \notin \{E_1, ..., E_m\}$, then u is c
 $w_0 \notin \$ $\in \varOmega(E(\mathbf{k})\,+\,w_{\mathbf{0}})$, \circ therwise \setminus

We show that (v_i) converges in measure to *u*. Let $\varepsilon > 0$ be fixed and $\delta > 0$ such that $|h(k, w, \zeta) - h(k', w', \zeta')] < \varepsilon$ for all (k, w, ζ) , $(k', w', \zeta') \in \overline{G} \times \overline{\Omega}$ satisfying (k, w, ζ) – (k', w', ζ') | $\lt \delta$. Since *E* is continuous, there is an $i_0 \in N$ such that $|(k_i, E(k_i) + w_0) - (k, E(k) + w_0)| < \delta$ $(i \geq i_0)$. Regarding $v(\zeta) = v_i(\zeta) = 0$ if *Q* and $E(\mathbf{k}) + w_0 \in (E_1, ..., E_m)$, then *u* is continuous at **k**.

Proof: Let $h(k, w, \zeta) = f(k, \varphi(w, \zeta)) |\det \varphi'(w, \zeta)|$. Because of (A2), $h(k_i)_{i\in\mathbb{N}} \subset \overline{G}$ be a sequence with $k_i \to \mathbf{k}$ for $i \to \infty$, and set
 $v_i(\zeta) = \begin{$ $\zeta \notin \Omega(E(k_i) + w_0) \cup \Omega(E(k) + w_0)$, we obtain, for $i \geq i_0$,

$$
\lambda^{n-1}(|v_i - v| \geq \varepsilon)
$$

\n
$$
\leq \lambda^{n-1}(\langle \zeta \in \mathbf{R}^{n-1} | v_i(\zeta) - v(\zeta) | \geq \varepsilon \rangle \cap \Omega(E(k_i) + w_0) \cap \Omega(E(\mathbf{k}) + w_0)
$$

\n
$$
+ \lambda^{n-1}(\Omega(E(k_i) + w_0) \wedge \Omega(E(\mathbf{k}) + w_0))
$$

\n
$$
= \lambda^{n-1}(\Omega(E(k_i) + w_0) \wedge \Omega(E(\mathbf{k}) + w_0)).
$$

 $\frac{1}{2}$

•
•
•
•
•

 $\label{eq:2} \frac{1}{\sqrt{2}}\sum_{i=1}^n \frac{1}{\sqrt{2}}\sum_{j=1}^n \frac{1}{j!} \sum_{j=1}^n \frac{1}{$

-

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Hence, $\lim_{\lambda} \lambda^{n-1}(|v_i - v| \ge \varepsilon) = 0$. Since Ω and *h* are bounded, $\frac{M}{100}$
 $\frac{1}{100}$

$$
\int_{\mathbf{R}^{n-1}} v_i(\zeta) d\zeta \to \int_{\mathbf{R}^{n-1}} v(\zeta) d\zeta
$$

follows from Lebesgue's dominated convergence theorem. But this is equivalent to $u(k_i) \rightarrow u(k)$ **I**

Corollary: Let the hypotheses of Lemma 2.1 be fulfilled, $E^j = E_j + w_0$, the indices *j* 326 M. HANKE

Hence, $\lim_{R \to 1} \lambda^{n-1}(|v_i - v| \ge \varepsilon) = 0$. Since Ω and h are bounded,
 $\int_{R^{n-1}} v_i(\zeta) d\zeta \to \int_{R^{n-1}} v(\zeta) d\zeta$
 $\int_{R^{n-1}} v_i(\zeta) d\zeta$

follows from Lebesgue's dominated convergence theorem. But this is $u|_G$ is continuous and has a continuous extension onto \overline{G} . *(i)* $a\zeta \rightarrow \int v(\zeta) d\zeta$
 Photosof R^{n-1}
 R_{RA} i *(k) k E* $\liminf_{m \to \infty} \frac{2.1}{m}$ *be fulfilled*, $E^i = E_j + ch$ *a way that* $E_{\min} \leq E^j \leq E_{\max}$ and $G^j = \{k \in G \mid E^j + 1\}$; $E^{m-1} := E_{\min}, E^{m'+1} := E_{\max}$. Then,

Further below we use extensively an analogue of Fubini's theorem. Indeed,

$$
\iint_{GC} f(k, k') \, \delta(E(k) - E(k') + w_0) \, dk' \, dk
$$
\n
$$
= \iint_{GC} f(k, k') \, \delta(E(k) - E(k') + w_0) \, dk \, dk'
$$

for all $f \in C(\overline{G} \times \overline{G})$ and all $w_0 \in \mathbf{R}$.

3. On the solvability of the Boltzmann equation

In order to derive assertions on the existence and uniqueness of solutions we formulate the Boltzmann equation (1.1) as an operator equation in the Banach spaces $X = C(\overline{G}_2, H_n^1(I))$ and $Y = C(\overline{G}_2, L^2(I))$. We shall define A, B, C as the differential, the linear integral, and the nonlinear integral parts, respectively. Then $A, B \in B(X, Y)$ and A is bijective. We shall assume that *B* is even compact. Using the theory of positive operators in partially ordered Banach spaces [7] it is shown that in Case (II) the spectral radius $r(A^{-1}B)$ is less than 1 whereas in Case (I) one is a simple eigenvalue of *A'B* which is in modulus strictly larger than the other • eigenvalues and the associated eigenvector can be chosen to be strictly positive. Every physithat B is even compact. Using the theory of positive operators in partially ordered Banach spaces [7] it is shown that in Case (II) the spectral radius $r(A^{-1}B)$ is less than 1 whereas in Case (I) one is a simple eigenval **ISB** $\mathbf{M} + \mathbf{M} \times \mathbf{K} \times \mathbf{K}$
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 IERONG CONSTANTS A CONSTANTS A CONSTANTS A CONSTANTS A CONSTANTS \mathbf{M}(\mathbf{k}) It is possible to find constants $\sigma < 0$ and $\tau > 1$ which only depend on the functions K_s of (1.3) such that for all $u \in D := \{u \in X \mid \sigma < u(k) < \tau\}$ the derivative $(A - B - C)'(u) = A_u - B_u$ can be split into operators A_u and B_u so that the mentioned properties also hold for A_u and B_u . This fact is essentially used. In Case (1) the Boltzmann equation will be supplemented by a condition on the number of particles Moreover that for all $u \in D$. $|u \in X|$ or $\in \text{Out}(I)$ and $\tau > 1$ which only is possible to find constants $\sigma < 0$ and $\tau > 1$ which only σ is the following the following notation of A_u and B_u so that the mentioned **Fully** become

such that:

can be spling

This fact

condition

 Fully and the severy solu

We int it is shown that in Case (II) the spectral radius $r(A^{-1}B)$ is

ne is a simple eigenvalue of $A^{-1}B$ which is in modulus strict

as and the associated eigenvector can be chosen to be estrictly

can trisolution of the Bolt

$$
\int\limits_G u(k) z(k) dk = p.
$$

Then we show the existence of a regular analytic solution path $u(p)$ of $(1.1) - (1.2)$. In Case (11) every solution of the Boltzmann equation is isolated.

•.

• **S -**

• **S**

Then we show the existence of a regular analytic solution path

\nery solution of the Boltzmann equation is isolated.

\nWe introduce the following notations:

\n
$$
H_p^1(I) = \{v \in W^{1,2}(I) \mid v(-l) = v(l)\},
$$
\n
$$
X = C(\bar{G}_2, H_p^1(I)), \qquad Y = C(\bar{G}_2, L^2(I)).
$$
\nso following continuous in the dimension σ and σ .

 $H_p^1(I) = \{v \in W^{1,2}(I) \mid v(-l) = v(l)\},$
 $X = C(\bar{G}_2, H_p^1(I)),$ $Y = C(\bar{G}_2, L^2(I)).$

The following continuous imbeddings are valid: $X \to C(\bar{G}) \to Y \to L^2(G)$. Throughout

this chapter we assume (A1)—(A2) to be fulfilled. Define the follo

 $\begin{aligned} \mathcal{L}_{\text{max}}(\mathcal{L}_{\text{max}}) = \mathcal{L}_{\text{max}}(\mathcal{L}_{\text{max}}) \end{aligned}$

0n the Boltzmann Equation of Carrier Trausport 327
\n(with
$$
u \in X
$$
, $k \in \overline{G}$):
\n
$$
Au(k) = F \frac{\partial}{\partial x} u(k) + c(k) u(k),
$$
\n(3.1)
\n
$$
c = c_0 + c_1, \qquad c_0(k) = \int W(k, k) z(k') dk',
$$
\n(3.2)
\n
$$
B_u(k) = \int_S K_s(k, k') \delta[E(k) - E(k') + w_0^*] u(k') z(k') dk',
$$
\n(3.2)
\n
$$
Bu = \sum_{s=-r}^{r} B_s u,
$$
\n(3.2)
\n
$$
\overline{B}u(k) = \int_S (W(k, k) - W(k, k')) u(k') z(k') dk',
$$
\n
$$
Cu(k) = u(k) \overline{B}u(k),
$$
\n
$$
Tu = du - Bu - Cu.
$$
\nSince X is continuously imbedded into $C(\overline{G})$, the functions Bu , $\overline{B}u$, $Cu : \overline{G} \rightarrow \mathbf{R}$ are well-defined. Obviously, (1.1)–(1.2) is equivalent to $Tu = g$.
\nProportion 3.1: For all $u \in X$ and $u \in X$ and $v \in Y$, $\partial u/\partial z \in Y$, $vu \in Y$, and $||vu||_Y$
\n
$$
= ||v||_Y ||u||_{C(\overline{S})} \le r ||v||_Y ||u||_X
$$
 for some $y \in \mathbf{R}$ in $d e$ and v .
\nIn order to investigate the properties of the integral operators we introduce the Banach space, $C(G)$, according to Assumption (42) let E_{int} and $\overline{E} = (x - x) < \overline{E} = \overline{E}_{\text{int}}$ for $\overline{E} = \overline{E}_{\text{int}}$ and $\overline{E} = \overline{E}_{\text{int}}$ and $\overline{E} = \overline{E}_{\text{int}}$ and $\overline{E} = \overline{E}_{\text{int}}$.
\n $(\overline{E}_{\text{min}} E_{\text{max}})$.

Since X is continuously imbedded into $C(\overline{G})$, the functions Bu, $\overline{B}u$, $Cu: \overline{G} \rightarrow \mathbf{R}$ are well-defined. Obviously, (1.1) – (1.2) is equivalent to $Tu = g$.

Proposition 3.1: For all $u \in X$ and $v \in Y$, $\partial u/\partial x \in Y$, $vu \in Y$, and $||vu||_Y$ \leq $||v||_Y||u||_{C(\bar{G})} \leq \gamma ||v||_Y||u||_X$ for some $\gamma \in \mathbf{R}$ independent of u and v.

In order to investigate the properties of the integral operators we introduce the properties of the integral operators we introduce the properties of the integral operators $E_n \le E_1 < \cdots < E_m \le E_{\text{max}}$ Since X is continuously imbedded into $C(\bar{G})$, the functions Bu , $\bar{B}u$, $Cu : \bar{G} \to \mathbf{R}$ are
well-defined. Obviously, $(1.1) - (1.2)$ is equivalent to $Tu = g$.
Proposition 3.1: For all $u \in X$ and $v \in Y$, $\partial u/\partial x \in Y$, vu $f(x) = \begin{cases} \nE_1 & \text{if } E \in \{E_1, \dots, E_n\} \\
E_2 & \text{if } E \in \{E_1, \dots, E_n\} \\
E_3 & \text{if } E \in \{E_4, \dots, E_n\} \\
E_4 & \text{if } E \in \{E_5, \dots, E_n\} \\
E_5 & \text{if } E \in \{E_6\} \\
E_6 & \text{if } E \in \{E_7\} \\
E_7 & \text{if } E \in \{E_7\} \\
E_8 & \text{if } E \in \{E_7\} \\
E_9 & \text{if } E \in \{E_8\} \\
E_1 & \$ $(j = 0, \ldots, m'$; $E^0 := E_{\min}, E^{m'+1} := E_{\max}$. Then let $PC(G)$ be the Banach space, equipped with the supremum norm, of all real valued bounded functions *u* defined on $G' = G^0 \cup \cdots \cup G^{m'}$ whose restrictions $u|_{G'}$ are continuous and have continuous be defined. Let $\{E^1, \dots, E^{n'}\} = \{w \in \mathbf{R} \mid w = E_j + w_0^s, -r \leq s \leq r, 1 \leq j \leq m\}$
 $\cap (E_{\min}, E_{\max})$. Assume $E^1 < \dots < E^{m'}$ to hold. Set $G^j = \{k \in \overline{G} \mid E^j < E(k) < E^{j+1}\}$
 $(j = 0, \dots, m'; E^0 := E_{\min}, E^{m'+1} := E_{\max})$. Then let $PC(G)$ b equipped with the supremum norm, of all real valued bounded functions u defined
on $G' = G^0 \cup \cdots \cup G^{m'}$ whose restrictions $u|_G$, are continuous and have continuous
extensions onto \overline{G} . Obviously, $G^i \cap G^j = 0$ for Moreover, $PC(G)$ is isomorphic to the Banach space $C(G^0) \times \cdots \times C(G^m)$. Hence, the precompact subsets of $PC(G)$ can be characterized by the theorem of Arzela-Ascoli. From the corollary to Lemma 2.1 it becomes clear that the on $G' = G^0 \cup \cdots \cup G^{m'}$ whose restrictions $u|_{G'}$ are continuous and have continuous extensions onto \overline{G} . Obviously, $G^i \cap G^j = 0$ for $i + j$ and $\overline{G} = \overline{G^0} \cup \cdots \cup$
Moreover, $PC(G)$ is isomorphic to the Banach sp contains all discontinuity points of functions of the kinds *Bu* and *Bu:* extensions onto \overline{G} . Obvious

Moreover, $PC(G)$ is isomorphic

precompact subsets of $PC(G)$

From the corollary to Lem

contains all discontinuity point

In the following we assume

(A3) (i) $c_1 \in Y$.

(ii) There exists $\begin{array}{l} \hbox{Bana} \\ \hbox{Bean} \\ \hbox{the} \\ \hbox{the} \\ \hbox{(} f = (\\ \hbox{equip}) \\ \hbox{on} \\ G' \\ \hbox{ex} \\ \hbox{Thereo} \\ \hbox{for} \\ \hbox{non} \\ \hbox{for} \\ \hbox{on} \\ \hbox{on} \\ \hbox{(} A3) \\ \hbox{for} \\ \hbox{for} \\ \hbox{on} \\ \hbox{on} \\ \hbox{for} \\ \hbox{on} \\ \hbox{on} \\ \hbox{for} \\ \hbox{on} \\ \hbox{on} \\ \hbox{on} \\ \hbox{on} \\ \hbox{on} \\ \hbox{on} \\$

In the following we assume Assumption $(A3)$ to be fulfilled:

$$
(A3)
$$

(i) $c_1 \in Y$.
(ii) There exists a $d \in \mathbb{R}$ such that $\int c(x, t) dx \geq d > 0$ for all $t \in \overline{G}_2$.

(iii)
$$
\lambda^1(\{x \in I \mid E(x, t) = E^j\}) = 0
$$
 for all $t \in \overline{G}_2$, $j = 0, ..., m' + 1$.

Proposition 3.2: Let $(A1) - (A2)$ *be true. Then* $B \in B(X, PC(G))$.

Proof: By the corollary, of Lemma 2.1, $Bu \in PC(G)$ for all $u \in X$. Because of $(A3)$
Prop
Prop
Proo

Proposition 3.2: Let (A1)–(A2) be true. Then
$$
B \in B(X, PC(G))
$$
.
\nProof: By the corollary of Lemma 2.1, $Bu \in PC(G)$ for all $u \in X$. Bec
\n(A1) there exists an $N > 0$ such that $K_s(k, k') z(k') \leq N((k, k') \in D_s)$. Then
\n
$$
||B_s u||_{PC(G)} = \sup_{k} \left| \int_{G} K_s(k, k') u(k') \delta(E(k) - E(k') + w_0^s) z(k') dk' \right|
$$
\n
$$
\leq \sup_{(k,k') \in D_s} K_s(k, k') z(k') ||u||_{C(\overline{G})} \int_{G} \delta(E(k) - E(k') + w_0^s) dk'
$$
\n
$$
\leq N \gamma ||u||_X M \blacksquare
$$

 \cdot

This gives rise to the question which conditions have to be fulfilled such that $R(B) \subseteq Y$ holds. It turns out that Assumption (A3)/(iii) is sufficient for that. 328 M. HANKE

This gives rise to the question which conditions have to be fulfilled suc
 $R(B) \subseteq Y$ holds. It turns out that Assumption (A3)/(iii) is sufficient for that.

 \mathbf{v}

Lemma 3.3: Let (A3) hold. Then PC(0) is continuously imbedded into Y.

Proof: Let $u \in PC(G)$ and $v: \overline{G}_2 \to L^2(I)$ be defined by $v(t)$ $(x) = u(x, t)$ $((x, t) \in G')$. Because of (A3)/(iii) the measure of M. HANKE

his gives rise to

his gives rise to
 P holds. It

emma 3.3: Let

emma 3.3: Let
 P

roof: Let $u \in P$

ause of $(A3)/$
 $(m'+1)$
 $V(t) = \bigcup_{j=0}^{m'+1}$

ishes for every the strain

$$
V(t) = \bigcup_{j=0}^{m+1} \{x \in \overline{I} \mid E(x,t) = E^j\}
$$

t
 t *t t t t t t <i>t <i>t t c <i>t t <i>t***** *<i>t <i>t* vanishes for every $t \in \overline{G}_2$. Hence $v(t)$ is defined for almost every $x \in I$. Since $v(t)$ is continuous on every component of the open set $I \setminus V(t)$, $v(t)$ is measurable, and from the boundedness of *u* we get $v(t) \in L^2(I)$. Let $(t_i)_{i\in N} \subset \overline{G}_2$ be a sequence with $t_i \rightarrow t$. Since *u* is continuous at every $k = (x, t) \in (I \setminus V(t)) \times (t_i, v(t_i) (x) = u(x, t_i)$ $\rightarrow u(x, t) = v(t)(x)$ for all $x \in I \setminus V(t)$. Using the boundedness of *u, v(t_i)* $\rightarrow v(t)$ in $L^2(I)$ follows from Lebesgue's dominated convergence theorem. Consequently, $v \in Y$. The continuity of the imbedding is now obvious $| \mathbf{I} |$

Remark: $(A2)/$ (iii) is also essentially necessary for the continuous imbedding $PC(G) \rightarrow Y$ to hold. For instance, $(A2)/$ (iii) is fulfilled if the level sets $\{k \in \overline{G} \mid E(\overline{k})\}$ $\leq E^{j}$ are finite unions of strictly convex sets. This is the case for (2.2)–(2.5).

Corollary: Let (A1)-(A3) hold. Then $B \in B(X, Y)$ and $c_0 \in Y$.

The inclusion $B \in B(X, Y)$ is too weak for our purposes. We suppose B to be even. a compact operator: $B \in B_0(X, Y)$. Our previous assumptions are not sufficient to ensure this property. In order that $B \in B_0(X, Y)$ holds we need further assumptions ensure this property. In order that $B \in B_0(X, Y)$ holds we need further assumptions on the band structure *E*. The sample structures (2.2) – (2.5) have this property. COFOILATY: Let $(A1)$ —(A3) hold. Then $B \in B(X, Y)$ and $c_0 \in Y$.
The inclusion $B \in B(X, Y)$ is too weak for our purposes. We suppose B to be even
a compact operator: $B \in B_0(X, Y)$. Our previous assumptions are not sufficient to

operator $B \in B(X, Y)$ defined by (3.2) is compact for each integral kernel W satisfying (A1).

A proof of (A4) for a given band structure is loaded with technicalities. **WENDT** [15] has suggested a general scheme for proving the compactness. In Chapter 4 we illustrate this scheme by proving (A4) for a very simple band structure.

We summarize the properties of the operators *A* and *C.*

• Lemma 3.4: *Let (A1)—(A3) hold. Then A* E *B(X,* Y). *Moreover, A'is bijective and, consequently, continuously invertible on* Y.

Proof: Because of Proposition 3.1 and the corollary of Lemma 3.3, $A \in B(X, Y)$. A simple calculation shows that $Au = w$ if and only if

and structure E. The sample structures (2.2)–(2.5) have this property.
\nThe band structure E and the state density z are such that the integral
\noperator B
$$
\in B(X, Y)
$$
 defined by (3.2) is compact for each integral kernel
\nW satisfying (A1).
\nof of (A4) for a given band structure is loaded with technicalities. WENDT
\nsuggested a general scheme for proving the compactness. In Chapter 4 we
\nthe this scheme by proving (A4) for a very simple band structure.
\nmmmarize the properties of the operators A and C.
\nna 3.4: Let (A1)–(A3) hold. Then $A \in B(X, Y)$. Moreover, A is bijective
\nsequently, continuously invertible on Y.
\nIf: Because of Proposition 3.1 and the corollary of Lemma 3.3, $A \in B(X, Y)$.
\ne calculation shows that $Au = w$ if and only if
\n $u(x, t) = \int G(x, t, \xi) w(\xi, t) d\xi$,
\n $G(x, t, \xi) = \frac{1}{F} \int_{-t}^{e^{h(\xi, t) - h(x, t)}} \begin{cases} 1, & -l \leq \xi \leq x \leq l, \\ e^{-h(l, t)}, & -l \leq x < \xi \leq l, \end{cases}$ (3.3)
\n $h(x, t) = \frac{1}{F} \int_{-t}^{x} c(\xi, t) d\xi$.
\nis representation we get the estimate
\n $0 < G(x, t, \xi) \leq \alpha$ and $||A^{-1}|| \leq \alpha \left((2l)^2 + \frac{2l}{F^2} ||c||_Y^2 \right)^{1/2}$

From this representation we get the estimate

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on the Boltzmann Equation of Carrier Transport 329

where where
-
This **v**

On the Boltzmann Equa
\nhere
\n
$$
\alpha = \frac{\exp (F^{-1}(2l)^{1/2} ||c||_Y)}{F(1 - \exp(-d/F))}.
$$
\nThis yields the assertions
\n
$$
\blacksquare
$$
\n
$$
\text{Lemma 3.5: } Let (A1) - (A3) hold. Then:
$$
\n(i) $Cu \in Y$ for all $u \in X$ and $C: X \to Y$ is analytic
\n(ii) $C'(u) v = u\overline{B}v + v\overline{B}u$ ($u, v \in X$) and $C' \in B(X)$

This yields the assertions *^I*

(i) $Cu \in Y$ for all $u \in X$ and $C: X \rightarrow Y$ is analytical.

On the Boltzmann Equation of Carrier Transport 329

are $\alpha = \frac{\exp (F^{-1}(2l)^{1/2} ||c||_Y)}{F(1 - \exp(-d/F))}$.

is yields the assertions

Lemma 3.5: Let (A1) – (A3) hold. Then:

(i) Cu e Y for all $u \in X$ and $C: X \to Y$ is analytical.
 Now we are in the position to prove our main results. At first we consider the lineal rized equation (1.4), (1.2). In operator notation it reads $(A - B) u = g$. In the following an *eigenvalue of* (A, B) be a $\lambda \in \mathbb{C}$ such that the complexified operator $A - \lambda B$ has a nontrivial nullspace.

Theorem 3.6: Let $(A1) - (A4)$ be true. Then we have:

(i) For all $z \in \mathbb{C}$, the complexified operat has a nontrivial nulispace.

Theorem 3.6: Let (A1)—(A4) be true. Then we have:

eigenvalues have no finite point of accumulation. (i) For all $z \in C$, the complexified operator $A - zB$ is Fredholm with index zero. The
genvalues have no finite point of accumulation.
(ii) There exists an eigenvalue $\lambda_0 \in \mathbb{R}$ having the properties
a) $\lambda_0 > 0$ and

(ii) There exists an eigenvalue $\lambda_0 \in \mathbb{R}$ having the properties

.b) The eigenvalue λ_0 is algebraically simple. The eigenvector $e \in X$ belonging to λ_0 *can be chosen to be strictly positive, i.e.,* $e(k) > 0$ *for all* $k \in \overline{G}$ *.*

(iii) In Case (I) it holds that $\lambda_0 = 1$, whereas' $\lambda_0 > 1$ in Case (II).

Proof: (i) Since *A* is bijective and *B* is compact, the assertion follows from Nikolskij's theorem [6: Theorem XIII.5.1].

Theorem 3.6: Let (A1)-(A4) be true. Then i

(i) For all $z \in C$, the complexified operator A -

eigenvalues have no finite point of accumulation.

(ii) There exists an eigenvalue $\lambda_0 \in \mathbb{R}$ having th

a) $\lambda_0 > 0$ a (ii) Let $K_X = \{u \in X \mid u(k) \geq 0 \text{ for all } k \in \overline{G}\}\$ denote the cone of all nonnegative functions of X and K_Y the corresponding cone of all nonnegative functions of Y. The interior int $K_X = \{u \in X \mid u(k) > 0 \text{ for all } k \in \overline{G}\}$ is nonempty. The operator $A^{-1}B$ *E* Bookship's theorem [6: Theorem XIII.5.1].

(ii) Let $K_X = \{u \in X \mid u(k) \ge 0 \text{ for all } k \in \overline{G}\}$ denote the cone of all nonnegative

functions of X and K_Y the corresponding cone of all nonnegative functions of Y. The

inter that $(A^{-1}B)^n u \in \text{int } K_X$ (cp. (A1), (3.3)). Theorems 2.5, 2.10, 2.13 of [7] imply the functions of X and K_Y the corresponding cone of all nonnegative functions of \tilde{Y} . The interior int $K_X = \{u \in X \mid u(k) > 0 \text{ for all } k \in \overline{G}\}$ is nonempty. The operator $A^{-1}B \in B_0(X)$ is strictly positive, i.e., for every Froof: (i) Since A is bijective and B is compact, the assertion follows from Nikolskij's theorem [6: Theorem XIII.5.1].

(ii) Let $K_X = \{u \in X \mid u(k) \ge 0 \text{ for all } k \in \overline{G}\}$ denote the cone of all nonnegative functions of X and $|\mu| < \mu_0$. Since, for $\lambda \neq 0$, λ is an eigenvalue of (A, B) if and only if $\lambda^{-1} \in \sigma(A^{-1}B)$, existence of an algebraically simple eigenvalue $\mu_0 \in \mathbb{R}$, $\mu_0 > 0$, and of an associated eigenvector $e \in \text{int } K_X$ of $A^{-1}B$. Moreover, for all $\mu \in \sigma(A^{-1}B)$, $\mu \neq \mu_0$, we have the assertion follows with $\lambda_0 =$ For $e \in \text{int } K_X$

Since, for $\lambda =$
 λ ion follows where the eigenval
 $F \frac{\partial}{\partial x} e + (c_0)$ **s** bijective and *B* is compact, the assertion follows from Nikolorem XHII.5.1].
 X | $u(k) \ge 0$ for all $k \in \overline{G}$ } denote the cone of all nonnegative
 the corresponding cone of all nonnegative functions of *Y*. The skips since in $(X_i = \{u \in X | u(k) \ge 0\})$ for all $k \in \overline{G}$ denote the cone of a functions of X and K_Y the corresponding cone of all nonnegative functions of X and K_Y the corresponding cone of all nonnegative functio

(iii) For the eigenvalue λ_0 and the eigenvector *e* we have

$$
F\frac{\partial}{\partial x}e + (c_0 + c_1)e = \lambda_0 \int\limits_{G} W(\cdot, k') e(k') z(k') dk'.
$$
 (3.4)

$$
|\mu| < \mu_0
$$
. Since, for $\lambda \neq 0$, λ is an eigenvalue of (A, B) if and only if $\lambda^{-1} \in \sigma(A^{-1}B)$, the assertion follows with $\lambda_0 = \mu_0^{-1}$. (iii) For the eigenvalue λ_0 and the eigenvector e we have\n
$$
F \frac{\partial}{\partial x} e + (c_0 + c_1) e = \lambda_0 \int_0^1 W(\cdot, k') e(k') z(k') dk'
$$
 (3.4)\nDenote\n
$$
\alpha = \int_0^1 W(k, k') e(k') z(k') dk' z(k) dk = \int_0^1 c_0(k') e(k') z(k') dk' > 0,
$$
\n
$$
\alpha' = \int_0^1 c_1(k) e(k) z(k) dk.
$$
\nIn Case (I) $\alpha' = 0$, whereas $\alpha' > 0$ in Case (II). Integrating (3.4) yields $\alpha + \alpha' = \lambda_0 \alpha$, and the assertion follows immediately \blacksquare \nFrom Theorem 3.6 we conclude the following corollary on the solvability of the linearized Boltzmann equation (1.4), (1.2).

In Case (I) $\alpha' = 0$, whereas $\alpha' > 0$ in Case (II). Integrating (3.4) yields $\alpha + \alpha' = \lambda_0 \alpha$, and the assertion follows immediately \blacksquare

From Theorem 3.6 we conclude the following corollary on the solvability of the linearized Boltzmann equation (1.4) , (1.2) .

Corollary: *Under the hypotheses 0/Theorem* 3.6 *we have:*

(i) In Case (I) the equation $Au - Bu = 0$ has, except for a constant real factor; *exactly one solution* $u \in X$. This solution can be chosen to be strictly positive.

(ii) In Case (II) $Au - Bu = q$ has exactly one solution $u \in X$ for every right-hand $side g \in Y$. If g is a nonnegative function, u is so, too.

Returning to the nonlinear equation (1.1), (1.2), from Lemma 3.5 we see that $T: X \to Y$ is analytical and $T'(u) = A_u - B_u$ where **II**ATY: Under the hypotheses of Theorem 3.6 we have:
 i Case (I) the equation $Au - Bu = 0$ has, except for a constant real factor,

one solution $u \in X$. This solution can be chosen to be strictly positive.
 Case (II) $Au -$

$$
A_{u}v = Av - v\overline{B}u, \qquad B_{u}v = Bv + u\overline{B}v \qquad (u, v \in X). \tag{3.5}
$$

A simple calculation shows that A_u and B_u have the following representations:

Returning to the nonlinear equation (1.1), (1.2), from Lemma 3.5 we see that
\n
$$
T: X \rightarrow Y
$$
 is analytical and $T'(u) = A_u - B_u$ where
\n $A_u v = Av - v\overline{B}u$, $B_u v = Bv + u\overline{B}v$ (u, $v \in X$).
\nA simple calculation shows that A_u and B_u have the following representations:
\n $B_u v = \int_{c} W_u(k, k') v(k') z(k') dk'$,
\n $W_u(k, k') = W(k, k') + (W(k', k) - W(k, k')) u(k)$
\n $= \sum_{s=-r}^{r} (K_s(k, k') + (K_{-s}(k', k) - K_s(k, k')) u(k))$
\n $\times \delta(E(k) - E(k') + w_0^s)$, (3.6)
\n $A_u v = F \cdot \frac{\partial}{\partial x} v + c_u v$,
\n $c_u = c_1 + c_{0,u}$, $c_{0,u} = \int_{c} W_u(k', \cdot) z(k') dk'$.
\nEvery physically reasonable solution of the Boltzmann equation must have the prop-
\nerty $0 \le u(k) \le 1$. In the following, we consider only solutions, belonging to the
\nopen set $D \subseteq X$ defined below which contains all relevant solutions. With (1.6) let
\n $q = \sum_{s=-r...r}^{r} q_s$, $\overline{q} = \max_{s=-r...r} \max_{(k,k') \in D_s} Q_s(k, k')$.
\nMoreover, with
\n $\sigma = \begin{cases} (1 - \overline{q})^{-1}, \overline{q} = 1, \\ -\infty, \overline{q} = 1, \\ -\infty, \overline{q} = 1, \end{cases} + \sum_{s=-r...r} (1 - \frac{q}{2})^{-1}, \underline{q} = 1, \\ +\infty, \underline{q} = 1, \\ \text{let } D = \{u \in X \mid \sigma < u(k) < \tau \text{ for all } k \in \overline{G}\}$. From the definition of Q , it follows that
\n $q \le 1 \le \overline{q}$. Hence, $\sigma < 0$ and $\tau > 1$.
\nLet now $u \in D$ and $\overline{u} = \max \{u(k) \$

Every physically reasonable solution of the Boltzmann equation must have the prop $e_u = c_1 + c_{0,u},$ $c_{0,u} = \int_G W_u(k', \cdot) z(k') dk'.$

Every physically reasonable solution of the Boltzmann equation must have the property $0 \le u(k) \le 1$. In the following we consider only solutions belonging to the open set $D \subseteq X$ defi $c_u = c_1 + c_{0,u},$ $c_{0,u} = \int_a^b W_u(k, \cdot) z(k) dk.$

Every physically reasonable solution of the Boltzmann equation must have the pierty $0 \le u(k) \le 1$. In the following we consider only solutions. belonging to open set $D \subseteq X$ defined $A_u v = F \cdot \frac{\partial}{\partial x} v + c_u v,$
 $c_u = c_1 + c_{0,u},$ $c_{0,u} = \int_{G} W_u(k', \cdot) z(k') dk'.$
 ysically reasonable solution of the Boltzmann e
 $u(k) \leq 1$. In the following we consider only
 $D \subseteq X$ defined below which contains all relevan
 $q = \min_{$

$$
\underline{q} = \min_{s=-r,\ldots,r} \underline{q}_s, \quad \bar{q} = \max_{s=-r,\ldots,r} \max_{(k,k')\in D_s} Q_s(k,k').
$$

Moreover, with

$$
\underline{q} = \min_{s=-r,\ldots,r} q_s, \quad \bar{q} = \max_{s=-r,\ldots,r} \max_{(k,k')\in D_s} Q_s(k,k').
$$

Moreover, with

$$
\sigma = \begin{cases} (1-\bar{q})^{-1}, \bar{q} = 1, & \tau = \begin{cases} (1-q)^{-1}, q \neq 1, \\ +\infty, q \neq 1, \end{cases} \\ \text{let } D = \{u \in X \mid \sigma < u(k) < \tau \text{ for all } k \in \bar{G}\}. \text{ From the definition } \underline{q} \leq 1 \leq \bar{q}. \text{ Hence, } \sigma < 0 \text{ and } \tau > 1. \\ \text{Let now } u \in D \text{ and } \bar{u} = \max \{u(k) \mid k \in \bar{G}\} \text{ and } \underline{u} = \min \{u(k) \mid k \in \bar{G}\} \end{cases}
$$

let $D = \{u \in X \mid \sigma' < u(k) < \tau \text{ for all } k \in \overline{G}\}$. From the definition of Q_s it follows that $q \leq 1 \leq \overline{q}$. Hence, $\sigma < 0$ and $\tau > 1$. $\sigma = \begin{cases} (1-q)^{-1}, q+1, & \tau = \begin{cases} (1-q)^{-1}, q+1, \\ +\infty, q=1, \end{cases} \end{cases}$
 $\sigma = \begin{cases} 1, & \tau = \begin{cases} (1-q)^{-1}, q+1, \\ +\infty, q=1, \end{cases} \end{cases}$
 $\sigma = \begin{cases} 1, & \tau = \begin{cases} (1-q)^{-1}, q+1, \\ +\infty, q=1, \end{cases} \end{cases}$
 $\sigma = \begin{cases} 1, & \tau > 0 \\ \tau = \tau$

Let now $u \in D$ and $\overline{u} = \max \{u(k) \mid k \in \overline{G}\}\$ and $\underline{u} = \min \{u(k) \mid k \in \overline{G}\}\$. Then

$$
\varepsilon = \min \{ 1 + (q-1) \, \overline{u}, \qquad 1 + (\overline{q} - 1) \, \underline{u} \} > 0.
$$

 $\begin{align*} \text{erty } 0 &\leq u(k) \leq \text{open set } D \subseteq X \ \text{open set } D &\leq X \ \frac{q}{s} &\leq \frac{n}{s} \end{align*} \begin{align*} \text{Moreover, with} \qquad \sigma &= \begin{cases} (& & \text{if } 0 < x \leq \frac{n}{2} \leq 1 \leq \bar{q}. \text{ Hence } M &\leq 1 \end{cases} \end{align*}$ Moreover, with
 $\sigma = \begin{cases} (1 - \bar{q})^{-1}, \bar{q} = 1, & \tau = \begin{cases} (1 - q)^{-1}, q + 1, \\ + \infty, & q = 1, \end{cases} \end{cases}$
 $\tau = \begin{cases} (1 - \bar{q})^{-1}, q + 1, & \tau = \begin{cases} (1 - q)^{-1}, q + 1, \\ + \infty, & q = 1, \end{cases} \end{cases}$
 $\tau \leq 1 \leq \bar{q}$. Hence, $\sigma < 0$ and $\tau > 1$.

Let now

$$
c_{0,u}(k) \geqq \varepsilon c_0(k), \quad k \in \overline{G}, \quad \int_l c_u(x,t) dx \geqq \varepsilon d > 0, \quad t \in \overline{G}_2.
$$

Hence we have shown the following essential result.

Lemma 3.7: Let $(A1) - (A4)$ be true. Then, for all $u \in D$, the statements of Theorem 3.6 *hold for* $T'(u) = A_u - B_u$ *if A and B are replaced by* A_u *and* B_u *, respectively.*

As an Immediate consequence we obtain

Theorem 3.8: *Consider Case* (II). Let $(A1) - (A4)$ be true. Then we have:

Lemma 3.7: Let (A1)-(A4) be true. Then, for all $u \in D$, the statements of Theorem

i hold for $T'(u) = A_u - B_u$ if A and B are replaced by A_u and B_u , respectively.

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Theorem 3.8: Consi 3.6 hold for $T'(u) = A_u - B_u$ if A and B are replaced by A_u and B_u , respectively.

As an immediate consequence we obtain

Theorem 3.8: *Consider Case* (II). Let $(A1) - (A4)$ be true. Then we have:

(i) Let $u \in D$ and $Tu = g$. *differentiable.* 1.5), $K_s(k,$
 $c_{0,u}(k) \ge$
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 $m \le 3.7: L$
 ld for $T'(u)$
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iii) There exists a $\delta > 0$ such that, for all $g \in Y$ with $||g||_Y < \delta$ the equation $Tu = g$ *has a solution* $u \in D$. (ii) There exists $a \cdot \delta > 0$ such that, for all $g \in Y$ with $||g||_Y < \delta$ the equation Tu is a solution $u \in D$.
(iii) Let $u \in D$ such that $Tu(k) \geq 0$ for all $k \in \overline{G}$. Then u is a nonnegative function.
Passet (i) is a se

Proof: (i) is a consequence of the implicit function theorem, and (ii) follows from (i) since $T0 = 0$.

(iii): We define a mapping $S: D \times X \to Y$ by $S(u, v) = A_{u/2}v - B_{u/2}v$. $A_{u/2}$ and (iii): We define a mapping $S: D \times X \to Y$ by $S(u, v) = A_{u/2}v - B_{u/2}v$. $A_{u/2}$ and $B_{u/2}$ are defined according to (3.5). Obviously, $S(u, u) = Tu$ $(u \in D)$. Since $u/2 \in D$, $(A_{u/2} - B_{u/2})^{-1} \in B(Y, X)$ exists and is positive. Ther $S(u, v) = g \ge 0$ for all $u \in D$. Setting $Tu = g$ we obtain the assertion **I** (ii) There exists
has a solution $u \in$
(iii) Let $u \in D$ so
Proof: (i) is a
from (i) since TO
(iii): We define
 $B_{u/2}$ are defined a
 $(A_{u/2} - B_{u/2})^{-1} \in$
 $S(u, v) = g \ge 0$ fo
Remark: Usir
from $Tu \le c_1$ for
often not fulfilled ED such that $Tu(k) \ge 0$ for all $k \in \overline{G}$. Then u is a nonnegative function.

is a consequence of the implicit function theorem, and (ii) follows
 $e \ T0 = 0$.

define a mapping $S: D \times X \to Y$ by $S(u, v) = A_{u/2}v - B_{u/2}v$. A_{u

 $(A_{u/2} - B_{u/2})^{-1} \in B(Y, X)$ exists and is positive. Therefore, $v \ge 0$ follows from $S(u, v) = g \ge 0$ for all $u \in D$. Setting $Tu = g$ we obtain the assertion **E**
Remark: Using the same methods it is possible to show that $u(k) \le 1$

Lemma 3.9: Let (A1)—(A4) *be true and*

-

0

$$
Y'=\left\{v\in Y\,\Big|\,\int\limits_G\,v(k)\,z(k)\,dk\,=\,0\right\}.
$$

Then T'(u) $X = Y'$ -for all $u \in D$ and $TX \subseteq Y'$ in Case (I).

Proof: The inclusions $TX \subseteq Y'$ and $T'(u)$ $X \subseteq Y'$ follow immediately from (1.1), $S(u, v) = g \ge 0$ for all $u \in D$. Setting $Tu = g$ we obtain the assertion **I**

Remark: Using the same methods it is possible to show that $u(k) \le 1$ follows

from $Tu \le c_1$ for $u \in D$. But this proposition is useless since the c (3.5) and $T'(u) = A_u - B_u$. Because of Lemma 3.7, dim $N(T'(u)) = \text{codim } R(T'(u))$ $Y' = 1$ *(u* $\in D$ *).* Since $Y' \subset Y$ is closed and codim $Y' = 1$, the identity $R(T'(u)) = Y'$ ${\color{black} \text{must hold}}$ ${\color{black} \blacksquare}$

. In Case (I) the Boltzmann equation 'has no isolated solutions. But Lemma 3.9, Lemma 3.7 and Theorem 3.6 suggest the following consideration: For all $u \in D$, $N(T'(u)) = \text{span } \{e_u\}$ where $e_u \in \text{int}'K_X$. If $g \in Y'$, the set *L* of solutions of the equation $T'(u)v = g$ has the representation $L = \{v_0 + \beta e_u \mid \beta \in \mathbb{R}\}$. If $e^* \in X^*$, $e^* \neq 0$, = 1 (*u* ∈ *D*). Since $Y' \subseteq Y$ is closed and codim $Y' = 1$, the identity $R(T'(u)) = Y'$
must hold **i**
L In Case (I) the Boltzmann equation has no isolated solutions. But Lemma 3.9,
Lemma 3.7 and Theorem 3.6 suggest the fol is a positive functional (i.e. $\langle e^*, u \rangle \ge 0$ for all $u \in K_X$), for every $\alpha \in \mathbb{R}$ there exists exactly one solution *v* of the system $T'(u) = g$, $\langle e^*, v \rangle = \alpha$. For physical reasons it is advisable to choose *e** as the operator which assigns the number of particles to each distribution *u*. More precisely, let *Y'* be given by (3.7) and $H: X \times \mathbf{R} \to Y' \times \mathbf{R}$ be. Then $T'(u) X = Y'$ for all $u \in D$ and $TX \subseteq Y'$

Proof: The inclusions $TX \subseteq Y'$ and $T'(u) X$
 (3.5) and $T'(u) = A_u - B_u$. Because of Lemma
 $= 1$ $(u \in D)$. Since $Y' \subseteq Y$ is closed and codim

must hold \blacksquare

In Case (I) the Boltzm In Case (I) the Boltzmann e
Lemma 3.7 and Theorem 3.6 s
 $N(T'(u)) = \text{span } \{e_u\}$ where $e_u \in$
ion $T'(u) = g$ has the represses a positive functional (i.e. $\langle e^* \rangle$,
exactly one solution v of the system and visable to choose $e^$ ation $L = \{v_0 + \beta e_u \mid \beta \in \mathbb{R}\}$. If $e^* \in X^*, e^* \neq 0$,
 ≥ 0 for all $u \in K_X$), for every $\alpha \in \mathbb{R}$ there exists
 $m T'(u) = g$, $\langle e^*, v \rangle = \alpha$. For physical reasons it is

tor which assigns the number of particles to $N(T'(u)) = \text{span } \{e_u\}$ where $e_u \in \text{int}'K_X$. If $g \in Y'$, the set *L* of solution $T'(u)v = g$ has the representation $L = \{v_0 + \beta e_u \mid \beta \in \mathbb{R}\}$. If $e^{\frac{1}{2}}$ is a positive functional (i.e. $\langle e^*, u \rangle \ge 0$ for all $u \in K_X$), for *a and and beorem 3.6 suggest the following*
and $N(T'(u)) =$ *span* (e_u) where $e_u \in \text{int } K_X$. If $g \in Y'$, the
tion $T'(u)v = g$ has the representation $L = \{v_0 + \beta \}$
is a positive functional (i.e. $\langle e^*, u \rangle \ge 0$ for all $u \$ a positive functional (i.e. $\langle e^*, u \rangle \ge 0$ for
actly one solution v of the system $T'(u) = v$
visable to choose e^* as the operator whic
stribution u. More precisely, let Y' be gi
fined by
 $\hat{H}(u, p) = \begin{pmatrix} Tu \\ \langle e^*, u \rangle - p \end{pm$

$$
H(u, p) = \begin{pmatrix} Tu \\ \langle e^*, u \rangle - p \end{pmatrix} \text{ where } \langle e^*, u \rangle = \int_G u(k) z(k) dk. \tag{3.8}
$$

Theorem 3.10: *Consider Case* (I). Let $(A1)$ – $(A4)$ be true. With $p_{max} = \langle e^*, 1 \rangle$
and $D' = \{u \in D \mid 0 \leq u(k) \leq 1\}$ we have: stead of (1.1), (1.2) we consider the equation *H*

Theorem 3.10: *Consider Case* (I). Let (A1)-
 $d D' = {u \in D | 0 \le u(k) \le 1}$ we have:

(i) $\frac{\partial}{\partial u} H(u, p)$ is bijective for all $(u, p) \in D \times \mathbb{R}$.

(ii) $H(0, 0) = H(1, n_{min}) = 0$

au

U

(iii) *If* $u: (p_1, p_2) \rightarrow D$ *is a continuous solution path to the equation* $H(u, p) = 0$ *, then it holds for all* α , β *satisfying* $p_1 < \alpha < \beta < p_2$ *that* $u(\alpha)$ (k) $\langle u(\beta) |k \rangle$ for all $k \in \overline{G}$. *(i)* $\frac{\partial u}{\partial u} H(u, p)$ is bijective for all $(u, p) \in D \times \mathbb{R}$.
 (ii) $H(0, 0) = H(1, p_{\text{max}}) = 0$.
 (iii) $If(u: (p_1, p_2) \to D$ is a continuous solution path to the equation $H(u, p) = 0$,
 n it holds for all α , β sat

with $u(0) = 0$ *and* $u(p_{max}) = 1$. Moreover, the equation $H(u,p) = 0$ has no further *solutions in* $D' \times [0, p_{\text{max}}]$.

•

Proof: We have, for $(u, p) \in X \times \mathbf{R}$,

: We have, for
$$
(u, p) \in X \times \mathbf{R}
$$
,
\n
$$
\frac{\partial}{\partial u} H(u, p) v = \begin{pmatrix} T'(u) v \\ \langle e^*, v \rangle \end{pmatrix} \qquad (v \in X).
$$

Assertion (i) follows from Lemma 3.9, Lemma 3.7 and Theorem 3.6; (ii) is obvious.

(iii): We remark first that H is an analytic mapping, and hence ϵ very continuous solution path is also analytic. Consequently, $u'(p)$ exists and - */*

We remark first that H is an analytic mapping, and hence éve
path is also analytic. Consequently,
$$
u'(p)
$$
 exists and

$$
u'(p) = -\left[\frac{\partial}{\partial u} H(u(p), p)\right]^{-1} \frac{\partial}{\partial p} H(u(p), p), \qquad p \in (p_1, p_2).
$$

Equivalently, $u'(p)$ is the solution of the system $T'(u(p)) u'(p) = 0$, $\langle e^*, u'(p) \rangle = 1$. Because of Lemma 3.7 and Theorem 3.6, $u'(p) \in \text{int } K_X$, and the assertion follows.

(iv): Let $u \in D$ and $p \in [0, p_{\text{max}}]$ such that $H(u, p) = 0$ and $u(k) > 0$ for all $k \in \overline{G}$. *v* $u'(p) = -\left[\frac{\partial}{\partial u}H(u(p), p)\right]$ $\frac{\partial}{\partial p}H(u(p), p)$, $p \in (p_1, p_2)$.

Equivalently, $u'(p)$ is the solution of the system $T'(u(p))u'(p) = 0$, $\langle e^*, u'(p)\rangle = 1$.

Because of Lemma 3.7 and Theorem 3.6, $u'(p) \in \text{int } K_X$, and the asser Δ , Δ are defined according to (3.1), (3.2) using the integral kernel $\Psi(k, k') = W(k, k')$ *w* of Lemma 3.7 and Theorem 3.6, $u'(p) \in \text{int } K_X$, and the assertion follows.

Let $u \in D$ and $p \in [0, p_{\text{max}}]$ such that $H(u, p) = 0$ and $u(k) > 0$ for all $k \in \overline{G}$.
 $= 1 - u$ is a solution of the system $Av - Bv = 0$, $\langle e^*, v$ *u* lies on a continuous and strictly monotone solution path. because of (i), (ii). Let Then, $v = 1 - u$ is a solution of the system $\Delta v - \underline{B}v = 0$, $\langle e^*, v \rangle = p_{\text{max}} - p$ where Δ , \underline{B} are defined according to (3.1), (3.2) using the integral kernel $\underline{W}(k, k') = W(k, k')$
 $\times u(k)$. \underline{W} satisfies (A1). He $u \leq 1$ follows. If $p = p_{max}$, we have $u(\alpha) > u$ for $\alpha > p_{max}$. Furthermore, $v = v(\alpha)$ Then, $v = 1 - u$ is a solution of the system $\Delta v - \underline{B}v = 0$, $\langle e^*, v \rangle = p_{\text{max}} - p$ where Δ , \underline{B} are defined according to (3.1), (3.2) using the integral kernel $\underline{W}(k, k') = W(k, k')$ $\times u(k)$. W satisfies (A1). Hence, v using, Theorem 3.6 we obtain $v \leq 0$. Hence, $u(\alpha) \geq 1$, and by the continuity of $u(\alpha)$, $u \geq 1$, $v(\alpha)$, $v(\alpha) \leq 1$ follows. If $p = p_{\max}$, we have $u(\alpha) > u$ for α p_{max} . For $\alpha > p$ we have $u(\alpha) > u$
 ≤ 1 follows. If $p = p_{\text{max}}$, we have $u(\alpha) = u(\alpha)$ is a solution of the syster.
 u (u/α) is a solution of the system is the system in the system is a consequence of (i)
 u $\$ $u(\alpha)$, $u \ge 1$. Since $\langle e^*, u \rangle = p_{\text{max}}$, $u = 1$.
(v) This assertion is a consequence of (i)—(iv). A detailed proof is given in [5] $\chi u(k)$. W satisfies (A1). Hence, $v \ge 0$ (i.e. $u \le 1$) by Theorem 3.6. Let now $u \ge 0$.
 u lies on a continuous and strictly monotone solution path because of (i), (ii). Let
 $p < p_{\text{max}}$. For $\alpha > p$ we have $u(\alpha) > u \ge$ is a solution of the system $\underline{A}v - \underline{B}v = 0$,

in 3.6 we obtain $v \leq 0$. Hence, $u(\alpha) \geq 1$, an

nee $\langle e^*, u \rangle = p_{\text{max}}, u = 1$.

rtion is a consequence of (i)-(iv). A detailed pr

o Theorem 3.10/(v) the following represe

equation is appropriate **iii** Case (I): According to Theorem $3.10/(v)$ the following representation of the Boltzmann

$$
\mathscr{L} = C([0, p_{\text{max}}], X), \qquad \mathscr{Y} = C([0, p_{\text{max}}], Y' \times \mathbf{R}),
$$

$$
\mathscr{F}: \mathscr{X} \to \mathscr{Y}, \qquad \mathscr{F}u(p) = H(u(p), p), \qquad p \in [0, p_{\text{max}}].
$$

 (1.1) , (1.2) is described by $\mathcal{T}u = 0$. This equation has exactly one solution (which is analytical). Moreover, the derivative $\mathcal{T}'(u)$ is bijective for all $u \in \mathcal{D} := \{u \in \mathcal{X} \mid \mathcal{I}\}$ $u(p) \in D$. This representation will be advantageous when investigating the convergence of numerical methods for the approximate solution of (1.1), (1.2).

Sometimes it happens that the domain G is not given in the cylindrical form $I \times G_2$. Due to physical considerations it is known that the probability that the charge carriers reach large energy values nearly vanishes. Hence, it is sufficient to consider the Boltzmann equation on such a domain G_0 where the energy does not exceed a given maximal value. To be more precise, let $G' \subseteq \mathbb{R}^n$ be a domain and E_{max} a given real constant (of maximal energy). Let E be a band structure defined on G' . Set physical considerations it is known that the probability that the charge contrary values nearly vanishes. Hence, it is sufficient to consider the Boltz such a domain G_0 where the energy does not exceed a given maximal unain G_0 where the energy does not exceed a given maximal value. To be more pre-
main G_0 where the energy does not exceed a given maximal value. To be more pre-
 $C \subseteq \mathbb{R}^n$ be a domain and E_{max} a given real c

$$
G_0 = \{k \in G' \mid E(k) < E_{\text{max}}\},
$$
\n
$$
G_2 = \{t \in \mathbb{R}^{n-1} \mid (x, t) \in G_0 \text{ for some } x \in \mathbb{R}\},
$$
\n
$$
I = \{x \in \mathbb{R} \mid (x, t) \in G_0 \text{ for some } t \in \mathbb{R}^{n-1}\}.
$$

We assume G_0 to be open, bounded, and convex. Set $G = I \times G_2$, and $G_0(t) = \{x \in \mathbb{R} \mid (x, t)\}$ $\in G_0$, $x^1(t) = \min G_0(t)$, $x^2(t) = \max G_0(t)$. Let equation (1.1) be given on the domain G_0 with the modified boundary conditions

$$
u(x^1(t), t) = u(x^2(t), t) \qquad (t \in \overline{G}_2).
$$

We relate this equation to an equivalent equation (1.1) with the boundary conditions (1.2) defined on the domain G. The equivalence is to be understood in the following sense: For a

$$
(3.9)
$$

On the Boltzmann Equa
\ngiven function
$$
u \in C(\overline{G}_0)
$$
 a function $\overline{u} \in C(\overline{G})$ is defined by
\n
$$
\overline{u}(x, t) = \begin{cases} u(x, t), & (x, t) \in \overline{G}_0, \\ u(x^1(t), t) & \text{otherwise.} \end{cases}
$$
\nIf the functions K, are extended onto $\overline{G} \times \overline{G}$ by zero, then

On the Boltzmann Equation of Carrier Transport
 J(\overline{G}_0) a function $\overline{u} \in C(\overline{G})$ is defined by
 $\begin{cases} u(x,t), & (x,t) \in \overline{G}_0, \\ u(x^1(t),t) & \text{otherwise.} \end{cases}$

are extended onto $\overline{G} \times \overline{G}$ by zero, then it holds: If *u* If the functions K_s are extended onto $\overline{G}\times\overline{G}$ by zero, then it holds: If u is a solution of.(1.1), (3.9) on G_0 , then \bar{u} is a solution of (1.1), (1.2) on G and vice versa. Taking suitable subspaces $X'' \subseteq X$ and $Y'' \subseteq Y$, equipped with new norms it is possible to show results of the kind given above. A detailed representation can be found in [4]. The modified form becomes important when solutions are computed numerically. The approximation of the integral operators does not lead to full matrices because of the δ -functions. Using the modified formulation the number of nonvanishing elements decreases again. given function $u \in C(\overline{G}_0)$ a function $\overline{u} \in C(\overline{G})$ is d
 $\overline{u}(x, t) = \begin{cases} u(x, t), & (x, t) \in \overline{G}_0, \\ u(x^1(t), t) & \text{otherwise.} \end{cases}$

If the functions K_s are extended onto $\overline{G} \times \overline{G}$ by

(3.9) on G_0 , then \overline{u} is

In our previous consideration we used the compactness assumption (A4) extensively. Here, we supply a proof of $(A4)$ for a sample band structure. In an attempt to avoid as many technicalities as possible we choose the simplest case. Nevertheless, the essential ingredients are clearly seen.

In proving $(A4)$ we follow the lines of WENDT [15]. He has suggested the following general scheme: For a fixed s , $B_xu(k)$ is considered to be a superposition of a linear operator and a linear functional. More precisely, let Z be a Banach space and $\{P_k\}$ *k* essential ingredients are clearly s

In proving (A4) we follow the l

general scheme: For a fixed s, B

operator and a linear functional.
 $k \in \overline{G}$ $\subseteq B(X, Z)$, $\{l_k \mid k \in \overline{G}\} \subseteq$

tionals, respectively, such that $k \in \overline{G}$ $\subseteq B(X, Z)$, $\{l_k \mid k \in \overline{G}\}\subseteq Z^*$ families of bounded. linear operators and functionals, respectively, such that $B_i u(k) = l_k P_k u$, $k \in \overline{G}$. Assume the following to be true: not lead to full matrices because of the δ -functions. Using the modified formulation the num
ber of nonvanishing elements decreases again.

4. A compactness proof

In our previous consideration we used the compactness (i) The families $\{l_k\}$ are bounded.

(ii) The mapping $k \mapsto P_k$ is uniformly the mapping $k \mapsto P_k$

(iii) The mapping $k \mapsto P_k$ is uniformly and $\{l_k\}$
 $\{l_k\}$ are bounded are clearly seen.
 $\{l_k\}$ are bounded are c $k \in G$ $\subseteq B(X, Z)$, $\{l_k | k \in G\} \subseteq Z^*$ families of bounded linear operators and functionals, respectively, such that $B_s u(k) = l_k P_k u$, $k \in \overline{G}$. Assume the following to be true:
(i) The families $\{l_k\}$ and $\{P_k\}$ are bo *e* \overline{G} \subseteq $B(X, Z)$, $\{l_k | k \in \overline{G}\} \subseteq Z^*$ families of bounded linear operators
tionals, respectively, such that $B_i u(k) = l_k P_k u$, $k \in \overline{G}$. Assume the follo
true:
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neral scheme: For a fixe pre, we supply a proof of (A4) for a sample band structure. I
many technicalities as possible we choose the simplest c
sential ingredients are clearly seen.
In proving (A4) we follow the lines of WENDT [15]. He has s
nera

Let *j* be fixed. For given
$$
\varepsilon > 0
$$
 we obtain
\n
$$
|B_s u(k) - B_s u(k')| = |l_k P_k u - l_k P_k u|
$$
\n
$$
\leq |l_k P_k u - l_k P_k u| + |l_k P_k u - l_k P_k u|
$$
\n
$$
\leq ||l_k|| \, ||P_k - P_k|| \, ||u||_X + |(l_k - l_k) P_k u| \leq C \varepsilon ||u||_X
$$
\nif *k*, *k'* \in *G^j* and $|k - k'|$ sufficiently small. Hence, for every bounded set $U \subset X$,

 B_sU is equicontinuous on G^j . Therefore, B_sU is precompact in $PC(G)$ and, consequently, in *Y.* We show using the simplest example how this algorithm works. Let $G_2 = (0, 1), l = 1$, the band structure be given by (2.2) with $a = 1$, and the state density be $z(k) = 2\pi t$. This state density arises if a three-dimensional problem with cylindrical symmetry is transformed into a two-dimensional model. Let G^+
= { $k \in G | x > 0$ } and $G^- = \{k \in G | x < 0\}$. Then set
 $B_s^{\perp}u(k) = \int K_s(k, k') u(k') \delta(E(k) - E(k') + w_0') z(k') dk'.$

$$
B_s^{\perp}u(k)=\int\limits_{G^{\perp}}K_s(k, k') u(k') \,\delta(E(k)-E(k')+w_0^{*}) z(k') \,dk'.
$$

Obviously, $R(B_s^{\pm}) \subseteq PC(G)$ and $B_s = B_s^+ + B_s^-$. We consider only B_s^+ since the proof can be done analogously for B_8^- . We choose φ and Ω as follows:

$$
\begin{aligned}\n\epsilon G \mid x > 0\n\end{aligned}\n\text{ and } G^- = \{k \in G \mid x < 0\}. \text{ Then } \epsilon \text{ set } \\
B_s \pm u(k) &= \int_{G^\pm} K_s(k, k') \, u(k') \, \delta(E(k) - E(k') + w_0^*) \, z(k') \, dk'.
$$
\n
$$
\text{ using } K(B_s \pm) \subseteq PC(G) \text{ and } B_s = B_s^+ + B_s^-. \text{ We consider only } B_s^+ \text{ can be done analogously for } B_s^-. \text{ We choose } \varphi \text{ and } \Omega \text{ as follows:}
$$
\n
$$
\Omega = \left\{ (w, \zeta) \in \mathbb{R}^2 \mid 0 < w < 2, \frac{0 < \zeta < w^{1/2}}{(w-1)^{1/2} < \zeta < 1 \quad (w > 1) \right\},
$$
\n
$$
\varphi(w, \zeta) = \left(\frac{\varphi_1(w, \zeta)}{\varphi_2(w, \zeta)} \right) = \left(\frac{(w - \zeta^2)^{1/2}}{\zeta} \right).
$$

• *^I*

•

 334° M. HANKE / The point in this choice is the fact that $t = \zeta$, i.e., ζ is orthogonal to the electrical *field.* Now, let $Z = L^1(G_2)$. Define $P_k \in B(X, Z)$ and $l_k \in Z^*$ by

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\nAt in this choice is the fact that
$$
t = \zeta
$$
, i.e., ζ is orthogonal to
\n
$$
P_k u(t') = \begin{cases} u(\varphi_1(w(k), t'), t') \frac{\pi t'}{\varphi_1(w(k), t')}, & t' \in \Omega(w(k)), \\ 0 & \text{otherwise}, \end{cases}
$$
\n
$$
l_k v = \int_{Q(w(k))} K_s(k, \varphi_1(w(k), t')) v(t') dt',
$$
\n
$$
w(k) = E(k) + w_0^*, \quad k \in \bar{G}.
$$
\n
$$
u(k) = l_k P_k u \ (k \in \bar{G}). \text{ Let } C \text{ be a-generic constant independence}
$$
\nfollowing. Denote for short $k_1 = \varphi(w(k), t')$ and $k_1' = \varphi(w(k')$, calculate the values $\{l_k\}$ and $\{P_k\}$ are bounded \bar{Y} .
\n
$$
I(k, k') = \int_{Q(w(k'))} \frac{\pi t''}{\varphi_1(w(k'), t'')} dt'',
$$
\n
$$
J(k, k') = \int_{Q(w(k'))} \frac{\pi t''}{\varphi_1(w(k'), t'')} - \varphi_1(w(k'), t'')^{-1} \pi t'' dt'
$$

Then $B_{\rho}^* u(k) = l_k P_k u$ ($k \in \overline{G}$). Let C be a generic constant independent of *u, v, k, k'* in the following. Denote for short $k_1 = \varphi(w(k), t'')$ and $k_1' = \varphi(w(k'), t'')$ (k, $k' \in \overline{G}$). $(k, k' \in G)$ $g(w(k))$
 $w(k) = E(k) + w_0^*$,
 $u(k) = l_k P_k u \ (k \in \overline{G})$. L

collowing. Denote for shot

calculation shows that t
 $f(k, k') = \int_{Q(w(k'))\setminus Q(w(k))} \overline{\varphi}$

$$
l_k v = \int K_s(k, \varphi_1(w(k), t')) v(t') dt,
$$
\n
$$
w(k) = E(k) + w_0^s, \qquad k \in \overline{G}.
$$
\nThen $B_s^+ u(k) = l_k P_k u$ $(k \in \overline{G})$. Let C be a generic constant independent of u, v, k, k' in the following. Denote for short $k_1 = \varphi(w(k), t'')$ and $k_1' = \varphi(w(k'), t'')$ $(k, k' \in \overline{G})$.
\nA simple calculation shows that the families $\{l_k\}$ and $\{P_k\}$ are bounded. Moreover, for $(k, k' \in G)$
\n
$$
I(k, k') = \int_{Q(w(k'))\setminus Q(w(k))} \frac{\pi t''}{\varphi_1(w(k'), t'')} dt'',
$$
\n
$$
J(k, k') = \int_{Q(w(k'))\setminus Q(w(k))} |\varphi_1(w(k), t')|^{-1} - \varphi_1(w(k'), t')^{-1}| \pi t'' dt''
$$
\nwe have $\dot{I}, J \in C(\overline{G} \times \overline{G})$ and $I(k, k) = J(k, k) = 0$. Let now $\varepsilon > 0$ be given. Since $K_s \in C(D_s)$, $\varphi_1 \in C(\overline{\Omega})$, and $w \in C(\overline{G})$, a $\delta_1 > 0$ exists such that, for all $k, k' \in \overline{G}$, $K_s \in C(D_s)$, $\varphi_1 \in C(\overline{\Omega})$, and $w \in C(\overline{G})$, a $\delta_1 > 0$ exists such that, for all $k, k' \in \overline{G}$.

Finally $\mu(k) = E(k) + w_0$ **,** $k \in G$.
 Finally $k + u(k) = l_k P_k u(k \in \overline{G})$ **. Let** C be a generic constant independent of u, v, k ,
 k' in the following. Denote for short $k_1 = \varphi(w(k), t')$ and $k'_1 = \varphi(w(k'), t')$ ($k, k' \in \overline{G}$).

A simp $(|k - k'| < \delta_1)$ and $\gamma t'' \in \Omega(w(k')) \cap \Omega(w(k)), |K_s(k, k_1) - K_s(k, k_1')| < \varepsilon$. Furthermore, there exists a $\delta_2 > 0$ such that $I(k, k') < \varepsilon$ for $|k - k'| < \delta_2$. Hence, for $|k - k'| < \delta := \min{\{\delta_1, \delta_2\}}$,
 $|l_k - l_k| P_{k'} u|$ First $D_s^{-1}u(k) = \frac{1}{k}F_ku(k \in G)$. Let C be a generic constant
 k' in the following. Denote for short $k_1 = \varphi(w(k), t')$ and k_1
 \therefore A simple calculation shows that the families $\{l_k\}$ and $\{P_k\}$ a
 $\langle k, k' \in G \rangle$
 we have $I, J \in C(\overline{G} \times \overline{G})$ and $I(k, k) = J(k, k) = 0$. Let now $\varepsilon > 0$ be given. Since $K_s \in C(D_s)$, $\varphi_1 \in C(\overline{\Omega})$, and $w \in C(\overline{G})$, a $\delta_1 > 0$ exists such that, for all $k, k' \in \overline{G}$ $(|k - k'| < \delta_1)$ and $i' \in \Omega(w(k')) \cap \Omega(w(k$ ven. Since
 $\lfloor k, k' \in \bar{G} \rfloor$

Further-

Hence, for $\begin{aligned} |k_1\rangle &|<\varepsilon. \text{ Further}, \\ |k_1\rangle &|<\varepsilon. \text{ Further}, \\ |k_1\rangle &|<\delta_2. \text{ Hence}, \end{aligned}$

$$
I(k, k) = \iint_{\Omega(w(k'))} \overline{\varphi_1(w(k'), t')} dt,
$$
\n
$$
J(k, k') = \iint_{\Omega(w(k'))} \varphi_1(w(k), t')^{-1} - \varphi_1(w(k', t')^{-1} | \pi t' dt' + \varphi_2(w(k), t')^{-1} | \pi t' dt''
$$
\nwe have $\hat{I}, J, \hat{\epsilon} C(\overline{G} \times \overline{G})$ and $I(k, k) = J(k, k) = 0$. Let now $\hat{\epsilon} > 0$ be given. Si
\n $K_s \in C(D_s), \varphi_1 \in C(\overline{\Omega}),$ and $w \in C(\overline{G}), a \delta_1 > 0$ exists such that, for all $k, k' \in (|k - k'| < \delta_1)$ and $\gamma t' \in \Omega(w(k')) \cap \Omega(w(k)), |K_s(k, k_1) - K_s(k, k_1')| < \varepsilon$. Further
\nmore, there exists a $\delta_2 > 0$ such that $I(k, k') < \varepsilon$ for $|k - k'| < \delta_2$. Hence,
\n $|k - k'| < \delta := \min{\delta_1, \delta_2}$,
\n $|l|_k - l_k| P_k u|$
\n $\leq \iint_{\Omega(w(k')) \cap \Omega(w(k))} |K_s(k, k_1) - K_s(k', k_1')| |u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''$
\n $\varphi(w(k')) \cap \Omega(w(k))$
\n $+ \iint_{\Omega(w(k')) \cap \Omega(w(k))} |K_s(k', k_1') u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''$
\n $\leq \varepsilon ||u||_{C(\overline{\delta})} \pi w(k')|^{1/2} + C ||u||_{C(\overline{\delta})} I(k, k') \leq \varepsilon C ||u||_X$.
\nHence, (ii) holds. In order to show (iii) we may estimate
\n $||P_k u - P_k u||_Z = \iint_{\overline{\delta_1}} |u(k') - P_k u(l'')| dt''$
\n $\leq \iint_{\Omega(w(k'), t'')} \frac{\pi t''}{\varphi_1(w(k), t'')} - u$

• *•*

$$
\leq \int_{\Omega(w(k'))\cap \Omega(w(k))} |K_s(k, k_1) - K_s(k', k'_1)| |u(k'_1)| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$
\n
$$
+ \int_{\Omega(w(k'))\Omega(w(k))} |K_s(k', k_1') u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$
\n
$$
\leq \varepsilon ||u||_{C(\overline{G})} \pi w(k')|^{1/2} + C ||u||_{C(\overline{G})} I(k, k') \leq \varepsilon C ||u||_X.
$$
\nHence, (ii) holds. In order to show (iii) we may estimate\n
$$
||P_k u - P_k u||_Z = \int_{G_s} |P_k u(t'') - P_k u(t'')| dt''
$$
\n
$$
\leq \int_{\Omega(w(k'))\cap \Omega(w(k))} |u(k_1) \frac{\pi t''}{\varphi_1(w(k), t'')} - u(k_1') \frac{\pi t''}{\varphi_1(w(k'), t'')}| dt''
$$
\n
$$
+ \int_{\Omega(w(k))\setminus \Omega(w(k))} |u(k_1)| \frac{\pi t''}{\varphi_1(w(k), t'')} dt''
$$
\n
$$
+ \int_{\Omega(w(k'))\setminus \Omega(w(k))} |u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$

- S -

On the Boltzmann Equation
\n
$$
\leq \int_{\Omega(w(k'))\cap \Omega(w(k))} |u(k_1) - u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$
\n
$$
+ \|u\|_{C(\bar{G})} J(k, k') + \|u\|_{C(\bar{G})}[I(k, k') + I(k', k')]
$$

$$
+ ||u||_{C(\bar{G})} J(k, k') + ||u||_{C(\bar{G})}[I(k, k') + I(k', k)].
$$

The first term (call it T) may be further estimated:

On the Boltzmann Equation of Carrier Transport 335
\n
$$
\leq \int_{\Omega(w(k'))\cap \Omega(w(k))} |u(k_1) - u(k_1')| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$
\n
$$
+ \|u\|_{C(\bar{G})} J(k, k') + \|u\|_{C(\bar{G})} [I(k, k') + I(k', k)].
$$
\nThe first term (call it *T*) may be further estimated:
\n
$$
T = \int_{\Omega(w(k'))\cap \Omega(w(k))} \left| \int_{\varphi_1(w(k), t'')}^{\varphi_1(w(k), t'')} \frac{\partial}{\partial x} u(\xi, t'') d\xi \right| \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''
$$
\n
$$
\leq \int_{\Omega(w(k'))\cap \Omega(w(k))} |\varphi_1(w(k), t'') - \varphi_1(w(k'), t'')|^{1/2} ||u||_X \frac{\pi t''}{\varphi_1(w(k'), t'')} dt''.
$$
\n(4.1)
\nTaking into account the continuity of *I*, *J*, *u*, φ_1 we obtain for sufficiently small $|k - k'|$ that $|P_k u - P_k u|_Z \leq \varepsilon ||u||_X$, i.e. (iii). Now we conclude that B_s^+ is compact

Taking into account the continuity of *I*, *J*, *u*, φ_1 we obtain for sufficiently small $|k - k'|$ that $|P_k u - P_k u|_Z \leq \varepsilon ||u||_X$, i.e. (iii). Now we conclude that B_s^+ is compact independent of the special choice of the kernel K_s . Treating B_s ⁻ analogously we obtain (A4). Note that we used essentially the differentiability of u with respect to x in (4.1). $\mathcal{Q}(w(k))$

ing into account the continuit
 k' that $|P_ku - P_{k'}u|_2 \leq \varepsilon ||u||$

pendent of the special choice of

. Note that we used essentially
 \vdots

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hematics and especially to W.

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