A Quantitative Bohman-Korovkin Theorem and its Sharpness for Riemann Integrable Functions

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Ausgehend von einer Arbeit von Pólya (1933) über die Konvergenz von Quadraturverfahren haben wir kürzlich einen Folgenkonvergenzbegriff im Raum der Riemann-integrierbaren Funktionen eingeführt, unter dem sich dieser klassische Raum als Vervollständigung der stetigen Funktionen darstellt. In der sich dann anschließenden Diskussion möglicher Auswirkungen auf die Approximationstheorie wurde unter anderem der Satz von Bohman-Korovkin (mit und ohne Ordnung) von stetigen auf Riemann-integrierbare Funktionen übertragen. Die dabei erzielten Schranken wurden über den ersten τ-Modul ausgedrückt. Ein Ziel dieser Arbeit ist es, entsprechende Abschätzungen gegen den zweiten τ-Modul herzuleiten. Die Schärfe dieser Resultate wird dann mittels eines quantitativen Beschränktheitsprinzips gezeigt. Schließlich werden die allgemeinen Ergebnisse im Zusammenhang mit Bernstein-Polynomen und linearen, interpolierenden Splines getestet.

Исходя от одной работы Полиа (1933) о сходимости методов квадратур мы ввели недавно понятие сходимости последовательностей в пространстве интегрируемых по Риману функций, при котором это классическое пространство оказалось дополнением непрерывных функций. В ходе последующей дискуссии о возможных последствиях на теорию аппроксимации переносилась, между прочим, теорема Бомана-Коровкина (с порядком и без него) с непрерывных на интегрируемые по Риману функций. Достигнутые при этом границы погрешности выражались через первый τ-модуль. Одна из целей этой работы — выводить соответствующие оценки относительно второго τ-модуля. Потом посредством количественного принципа ограниченности показывается что эти результаты являются точными. Наконец, общие результаты проверяются в свяаи с полиномами Бернштейна и линейными интерполирующими сплайнами.

Inspired by work of Pólya (1933) on the convergence of quadrature formulas, we previously introduced a concept of sequential convergence in the space of Riemann integrable functions under which this classical space is in fact the completion of continuous functions. Discussing its consequences to approximation theory, among others we already extended the (qualitative as well as quantitative) Bohman-Korovkin theorem from continuous to Riemann integrable functions. The error bounds obtained were given in terms of the first τ-modul. One purpose of this paper is to derive corresponding results, involving the second τ-modul. As a consequence of our previous quantitative uniform boundedness principle it is then shown that these estimates are indeed sharp. The general results obtained are illustrated in connection with Bernstein polynomials and linear interpolating splines.

1. Introduction

Let $B = B[a, b]$ and $R = R[a, b]$ be the spaces of functions, everywhere defined on the compact interval $[a, b]$ of the real axis $\mathbf{R}$ which are bounded and Riemann integrable, respectively. Though B and R are Banach spaces under the norm $\|f\|_B := \sup\{|f(u)|: u \in [a, b]\}$, uniform convergence is not appropriate to approximate in R since then suitable dense subsets are not available. This is avoided by the following concept of sequential convergence: A sequence $\{f_n\} \subset B$ is called (*Riemann*) *R-con-*

vergent to $f \in B$ if for $n \to \infty$

$$\|f_n\|_B = \mathcal{O}(1), \qquad \overline{\int} \sup_{k \geq n} |f_k - f| = o(1)$$

with upper Riemann integral $\overline{\int} f := \int_a^b f(u)\,du$. It turns out that B as well as R are still (sequentially) R-complete. From the point of view of approximation, the most important feature then is that the R-closure of standard classes of smooth functions (e.g., polynomials) yields R. In fact, Banach-Steinhaus theorems can be established in this framework, giving necessary and sufficient conditions for the R-convergence of operators. See [6] for the details.

Turning to positive operators, it was shown in [12] that the well-known Bohman-Korovkin theorem remains valid in the present context. Indeed, a sequence $\{T_n\}$ of positive linear operators of R into itself is R-convergent on R if and only if the sequence is R-convergent for the three test functions $p_i(u) := u^i$, $i = 0, 1, 2$.

Concerning quantitative extensions, an appropriate measure of smoothness is given by the k-th τ-*modul* $(f \in B, \delta > 0, k \in \mathbb{N})$

$$\tau_k(f, \delta) = \overline{\int_a^b} \omega_k(f, x, \delta)\,dx, \qquad \omega_k(f, x, \delta) = \sup\left\{|\Delta_h^k f(y)| : y \pm \frac{k}{2} h \in U_\delta(x)\right\},$$

$$\Delta_h^k f(y) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f\left(y + \left(j - \frac{k}{2}\right) h\right),$$

$$U_\delta(x) = [x - \delta, x + \delta)] \cap [a, b],$$

employed by the Bulgarian school of approximation during the last decade (see [15], there for functions f, measurable and bounded). Indeed, $\tau_1(f, \delta) = o(1)$ for $\delta \to 0+$ if and only if $f \in R$. The following error bound was established in [12].

Theorem 1: *Let $\{T_n\}$ be a sequence of positive linear operators, mapping R into itself such that $T_n p_0 = p_0$. Setting*

$$\mu_n := \sup_{k \geq n} \left\| T_k((u - x)^2; x) \right\|_B, \tag{1}$$

for each $f \in R$ there hold true the estimates

$$\|T_n f\|_B \leq \|f\|_B; \tag{2}$$

$$\overline{\int} \sup_{k \geq n} |T_k f - f| \leq 5\tau_1(f, \mu_n^{1/2}). \tag{3}$$

Obviously, this result is again completely parallel to the classical one, concerned with continuous functions and uniform convergence (see [3, p. 28] and the literature cited there). In Section 2 we improve (3) to an estimate involving the second τ-modul, again quite parallel to the procedure in the continuous case (see [8; 9] and the literature cited there). This, however, can only be done for a special class of positive operators, characterized by condition (9) on the fourth moment (see the comments, given at the end of Section 2). Applying our previous quantitative extensions of the uniform boundedness principle, it is then shown in Section 3 that the preceding estimates are indeed sharp. Finally, in Section 4 we test the general results in connection with Bernstein polynomials and linear interpolating splines (see also the hints to the literature, given there).

2. The Bohman-Korovkin theorem

To derive the estimate mentioned we proceed along the standard interpolation argument. To this end, define a norm for $f \in R$ and $\delta > 0$ via

$$\|f\|_\delta = \int_a^b M(f, x, \delta)\, dx, \qquad M(f, x, \delta) = \sup_{y \in U_\delta(x)} |f(y)|,$$

and let $R^2 = R^2[a, b]$ be the space of functions g, twice differentiable on $[a, b]$ with $g'' \in R$, endowed with the seminorm $|g|_2 := \int_a^b |g''(x)|\, dx$. The following lemma is well-known and may be found in [1; 2; 13]. Let us include a proof for the sake of completeness (and simplicity).

Lemma 2: *For $f \in R$, $0 < \delta < b - a$ there exists $g_\delta \in R^2$ such that*

$$\|f - g_\delta\|_\delta \leqq 10\tau_2(f, \delta), \qquad |g_\delta|_2 \leqq 20\delta^{-2}\tau_2(f, \delta). \tag{4}$$

Proof: First of all note that

$$\|f\|_\delta \leqq 2\, \|f\|_{\delta/2} \qquad (f \in R). \tag{5}$$

Indeed, one has

$$M(f, x, \delta) \leqq \begin{cases} M(f, x - \delta/2, \delta/2) + M(f, x + \delta/2, \delta/2), & a + \delta/2 \leqq x \leqq b - \delta/2, \\ M(f, x + \delta/2, \delta/2) + M(f, x, \delta/2), & a \leqq x \leqq a + \delta/2, \\ M(f, x - \delta/2, \delta/2) + M(f, x, \delta/2), & b - \delta/2 \leqq x \leqq b, \end{cases}$$

which yields

$$\|f\|_\delta \leqq \left(\int_a^{b-\delta} + \int_{a+\delta}^b + \int_{a+\delta/2}^{a+\delta} + \int_a^{a+\delta/2} + \int_{b-\delta}^{b-\delta/2} + \int_{b-\delta/2}^b \right) M(f, x, \delta/2)\, dx = 2\, \|f\|_{\delta/2}.$$

Now define Steklov means for $f \in R$ via

$$g_\delta(x) = 4\delta^{-2} \iint_{-\delta/4}^{\delta/4} \bar{f}(x + s + t)\, ds\, dt = 4\delta^{-2} \iint_{-\delta/4}^{\delta/4} \bar{f}(x - s - t)\, ds\, dt \tag{6}$$

where $\bar{f}$ is given as the extension (cf. [16, p. 121])

$$\bar{f}(x) = \begin{cases} f(x), & a \leqq x \leqq b, \\ -f(2a - x) + 2f(a), & 2a - b \leqq x \leqq a, \\ -f(2b - x) + 2f(b), & b \leqq x \leqq 2b - a. \end{cases}$$

Since for $y \pm h \in [x - \delta, x + \delta]$ the difference $\Delta_h^2 \bar{f}(y)$ only differs from $\Delta_h^2 f(y)$ if, e.g., $y - h < a \leqq y \leqq y + h$, and since in this special case

$$\Delta_h^2 \bar{f}(y) = \Delta_{y-a}^2 f(a + h) - 2\Delta_{y-a-h/2}^2 f(a + h/2) + 2\Delta_{h/2}^2 f(a + h/2),$$

it follows that for $x \in [a, b]$

$$\bar{\omega}_2(\bar{f}, x, \delta) \leqq 5\omega_2(f, x, \delta), \qquad \bar{\omega}_2(\bar{f}, x, \delta) := \sup_{y, y \pm h \in [x-\delta, x+\delta]} |\Delta_h^2 \bar{f}(y)|. \tag{7}$$

In view of (6) one obtains for $y \in U_{\delta/2}(x)$

$$|g_\delta(y) - f(y)| = 2\delta^{-2} \left| \iint_{-\delta/4}^{\delta/4} \Delta_{s+t}^2 \bar{f}(y)\, ds\, dt \right| \leqq \bar{\omega}_2(\bar{f}, x, \delta),$$

which delivers (cf. (5), (7)) $M(g_\delta - f, x, \delta/2) \leqq 5\omega_2(f, x, \delta)$, $\|f - g_\delta\|_\delta \leqq 10\tau_2(f, \delta)$. Moreover, $g_\delta \in R^2$ with $|g_\delta''(x)| = 4\delta^{-2}|\Delta_\delta^2 \bar{f}(x)| \leqq 4\delta^{-2}\overline{\omega}_2(\bar{f}, x, \delta)$, which implies (4) in view of (7) ∎

We are now in the position to prove the following quantitative Bohman-Korovkin theorem.

Theorem 3: *Let $\{T_n\}$ be a sequence of positive linear operators, mapping R into itself such that (cf. (1))*

$$T_n p_i = p_i \qquad (i = 0, 1), \tag{8}$$

$$\|T_n((u - x)^4; x)\|_B \leqq K\mu_n^2 \tag{9}$$

for some constant $K > 0$. Then there holds true the estimate

$$\overline{\int} \sup_{k \geq n} |T_k f - f| \leqq C\tau_2(f, \mu_n^{1/2}) \qquad (f \in R). \tag{10}$$

Proof: In view of Lemma 2 and

$$\overline{\int} \sup_{k \geq n} |T_k f - f| \leqq \overline{\int} \sup_{k \geq n} |T_k(f - g_\delta) - (f - g_\delta)| + \overline{\int} \sup_{k \geq n} |T_k g_\delta - g_\delta|$$

it is sufficient to show that ($\delta = \mu_n^{1/2}$)

$$\overline{\int} \sup_{k \geq n} |T_k(f - g_\delta) - (f - g_\delta)| \leqq 10 \|f - g_\delta\|_{\mu_n^{1/2}}, \tag{11}$$

$$\overline{\int} \sup_{k \geq n} |T_k g_\delta - g_\delta| \leqq C\mu_n |g_\delta|_2. \tag{12}$$

Obviously, (11) is a consequence of Theorem 1 since $\tau_1(f, \delta) \leqq 2 \|f\|_\delta$. To establish (12) consider the expansion ($g = g_\delta$, $u, x \in [a, b]$)

$$g(u) = g(x) + (u - x) g'(x) + \int_x^u (u - t) g''(t) \, dt. \tag{13}$$

Concerning the remainder one has

$$I := \left| \int_x^u (u - t) g''(t) \, dt \right| \leqq |u - x| \left| \int_x^u |g''(t)| \, dt \right|$$

$$\leqq \left(\delta + \frac{(u - x)^2}{\delta} \right) \int_{|t-x|<\delta} |g''(t)| \, dt + (u - x)^4 \int_{|t-x|\geq\delta} |g''(t)| \frac{dt}{|t - x|^3} =: I_1 + I_2.$$

It follows that for $k \geq n$ (cf. (8))

$$T_k(I_1; x) = \left(\delta + \frac{T_k((u - x)^2; x)}{\delta} \right) \int_{|t-x|<\delta} |g''(t)| \, dt \leqq \left(\delta + \frac{\mu_n}{\delta} \right) \int_{|t-x|<\delta} g''(t)| \, dt,$$

and therefore

$$\overline{\int} \sup_{k \geq n} T_k I_1 \leqq \left(\delta + \frac{\mu_n}{\delta} \right) \int \int_{|t-x|<\delta} |g''(t)| \, dt \, dx = \left(\delta + \frac{\mu_n}{\delta} \right) 2\delta \, |g|_2.$$

Furthermore, in view of (9) one obtains that for $k \geqq n$

$$T_k(I_2; x) \leqq K\mu_n{}^2 \int\limits_{|t-x|\geqq\delta} |g''(t)| \frac{dt}{|t-x|^3},$$

$$\overline{\int} \sup_{k\geqq n} T_k I_2 \leqq K\mu_n{}^2 \int\int\limits_{|t-x|\geqq\delta} |g''(t)| \frac{dt}{|t-x|^3}\, dx \leqq K\mu_n{}^2\delta^{-2}\, |g|_2.$$

Together with (8), (13) this delivers

$$\overline{\int} \sup_{k\geqq n} |T_k g - g| \leqq \left(\delta + \frac{\mu_n}{\delta}\right) 2\delta\, |g|_2 + K\mu_n{}^2\delta^{-2}\, |g|_2,$$

which establishes (12) by setting $\delta = \mu_n{}^{1/2}$ ∎

In view of (10) condition (8) is somehow natural (cf. [14, p. 18]) since then

$$\overline{\int} \sup_{k\geqq n} |T_k p_i - p_i| \leqq C\tau_2(p_i, \mu_n{}^{1/2}) = 0.$$

Thus the relevant assumption is condition (9), well-known in the literature in connection with a treatment of related problems such as saturation or inverse theorems (cf. [3, p. 131, 264], [4]). Note that if $T_n p_0 = p_0$, then conversely $T_n((u-x)^2; x)^2 \leqq T_n((u-x)^4; x)$ by the Cauchy-Schwarz inequality. On the other hand, without condition (9) the best result, known so far, is

$$\int |T_n f - f| \leqq C\tau_2\big(f, (\nu_n\, |\log \nu_n|)^{1/2}\big), \qquad \nu_n := \|T_n((u-x)^2; x)\|_B,$$

given in [14, p. 45].

3. The sharpness

To derive the sharpness of Theorem 3, a quantitative extension of the uniform boundedness principle will be applied. To this end, let $\omega(\delta)$ be an abstract modulus of continuity, i.e., $\omega(\delta)$ is a positive, continuous, subadditive, and increasing function of $\delta > 0$ satisfying

$$\omega(\delta) = o(1), \qquad \delta = o\big(\omega(\delta)\big) \qquad (\delta \to 0+). \tag{14}$$

Furthermore, let X be a Banach space with norm $\|\cdot\|_X$ and X^* be the space of nonnegative, sublinear, bounded functionals T on X, i.e., for $f, g \in X$ and scalars α

$$0 \leqq T(f+g) \leqq Tf + Tg, \qquad T(\alpha f) = |\alpha|\, Tf,$$

$$\sup\{Tf : f \in X, \|f\|_X \leqq 1\} < \infty.$$

The following resonance principle holds true (cf. [7] and the literature cited there).

Theorem 4: *Let $\{\varphi_n\}$ be a decreasing nullsequence, let $\sigma(\delta) > 0$ and $\omega(\delta)$ be subject to (14). Suppose that for $U_\delta, R_n \in X^*$ there are elements $h_n \in X$ satisfying*

$$\|h_n\|_X = \mathcal{O}(1), \tag{15}$$

$$U_\delta h_n \leqq M \min\{1, \sigma(\delta)/\varphi_n\} \qquad (\delta > 0, n \in \mathbb{N}), \tag{16}$$

$$R_n h_n \neq o(1). \tag{17}$$

Then there exists a counterexample $f_\omega \in X$ with

$$U_\delta f_\omega = \mathcal{O}\big(\omega(\sigma(\delta))\big), \qquad R_n f_\omega \neq o\big(\omega(\varphi_n)\big). \qquad (18)$$

In the present context X will be the Banach space R with norm $\|\cdot\|_B$, $U_\delta f = \tau_2(f, \delta)$ with $\varphi_n = \mu_n$ and $\sigma(\delta) = \delta^2$, and $R_n f = \overline{\int} \sup_{k \geq n} |T_k f - f|$. The sharpness of the estimate (10) is then established by

Theorem 5: *Let $\{T_n\}$ be a sequence of positive linear operators of R into itself, satisfying (8), (9) with $\mu_n = o(1)$ (cf. (1)). Moreover, for each $n \in \mathbf{N}$ let I_n be the union of j_n disjoint subintervals $I_{jn} := (a_{jn}, b_{jn})$ of $[a, b]$ such that*

$$0 < c_1 \leq j_n \mu_n^{1/2} \leq c_2; \qquad b_{jn} - a_{jn} \geq c_3 \mu_n^{1/2} > 0, \qquad (19)$$

$$0 < c_4 \mu_n \leq T_n\big((u - x)^2; x\big) \qquad (x \in I_n). \qquad (20)$$

Then for each ω subject to (14) there exists a counterexample $f_\omega \in R$ such that

$$\tau_2(f_\omega, \delta) = \mathcal{O}\big(\omega(\delta^2)\big), \qquad \overline{\int} \sup_{k \geq n} |T_k f_\omega - f_\omega| \neq o\big(\omega(\mu_n)\big).$$

Proof: To construct the elements h_n, following [5], let H be four times continuously differentiable on $\mathbf{R}$ with

$$0 \leq H(u) \leq 1, \qquad H(u) = \begin{cases} u^2, & |u| \leq 1/2, \\ 0, & |u| \geq 1. \end{cases} \qquad (21)$$

Choose a constant $c_5 > 0$ such that (cf. (9)) $K_0 := 2Kc_5^2 \leq c_4 c_5 - \|H^{(4)}\|_B Kc_5^2/24$ and define

$$d_n = (\mu_n/c_5)^{1/2}, \qquad h_{zn}(u) = H\big((u - x)/d_n\big), \qquad u, x \in [a, b].$$

Without loss of generality let $x_{jn} := (a_{jn} + b_{jn})/2$ satisfy $|x_{jn} - x_{j\pm 1,n}| \geq 3d_n$ (otherwise omit some intervals, namely each time at most m ones, where m, independent of n, is determined by $m c_3 \geq 3c_5^{-1/2}$). Now set

$$h_n = \sum_{j=1}^{j_n} h_{jn}, \qquad h_{jn} = h_{x_{jn}, n}.$$

Let $u \in [a, b]$ with $h_{jn}(u) \neq 0$. Then $|u - x_{jn}| \leq d_n$ which implies $|u - x_{in}| > d_n$, thus $h_{in}(u) = 0$ for $i \neq j$. Therefore one has (15) since $\|h_n\|_B = \max_j \|h_{jn}\|_B = \|H\|_B \leq 1$. Moreover (cf. [15, p. 24]), one obtains (16) in view of (19) and

$$\tau_2(h_n, \delta) \leq C\delta^2 \|h_n''\|_1 \leq C\delta^2 d_n^{-1} j_n \int_{-1}^{1} |H''(u)| \, du \leq C\delta^2/\mu_n. \qquad (22)$$

Now let $x, y \in I_n$ with $|x - y| \leq d_n/2$. By (21) one has $H''\big((x - y)/d_n\big) = 2$, $H^{(3)}\big((x - y)/d_n\big) = 0$ so that for $u \in [a, b]$ there exists ξ between x and u with

$$h_{yn}(u) = h_{yn}(x) + (u - x) H'\big((x - y)/d_n\big)/d_n + (u - x)^2/d_n^2$$
$$+ (u - x)^4 H^{(4)}\big((\xi - y)/d_n\big)/24 d_n^4.$$

In view of (8), (9), (20) this gives

$$\begin{aligned} |T_n(h_{yn}; x) - h_{yn}(x)| &\geq T_n\big((u - x)^2; x\big)/d_n^2 - T_n\big((u - x)^4; x\big) \|H^{(4)}\|_B/24 d_n^4 \\ &\geq c_4 \mu_n/d_n^2 - K\mu_n^2 \|H^{(4)}\|_B/24 d_n^4 \\ &= c_4 c_5 - \|H^{(4)}\|_B Kc_5^2/24 \geq K_0, \end{aligned}$$

i.e., with $y = x_{jn}$ one has

$$|T_n(h_{jn}; x) - h_{jn}(x)| \geqq K_0 \qquad (x \in I_n, |x - x_{jn}| < d_n/2). \tag{23}$$

Setting

$$\psi_n = \min\{d_n/2, c_3\mu_n^{1/2}/2\} =: c_6\mu_n^{1/2}, \quad B_n = \bigcup_{j=1}^{j_n} [x_{jn} - \psi_n, x_{jn} + \psi_n],$$

B_n is of total length $j_n 2\psi_n \geqq 2c_6c_1 > 0$ by (19).

Now consider $x \in B_n$ which implies $|x - x_{kn}| \leqq d_n/2$ for some $1 \leqq k \leqq j_n$. Since $|x - x_{jn}| > d_n$ for $j \neq k$, one has $h_{jn}(x) = 0$, thus in view of (23)

$$|T_n(h_n; x) - h_n(x)| \geqq K_0 - T_n\left(\sum_{j \neq k} h_{jn}(u); x\right). \tag{24}$$

I f $u \in [a, b]$ satisfies $|x - u| \leqq d_n$, then

$$|x_{k \pm 1, n} - u| \geqq |x_{k \pm 1, n} - x_{kn}| - |x_{kn} - x| - |x - u| \geqq d_n,$$

thus $h_{jn}(u) = 0$ for $j \neq k$ so that $\sum_{j \neq k} h_{jn}(u) = 0$ for $|x - u| \leqq d_n$. On the other hand, if $|x - u| > d_n$, then $\sum_{j \neq k} h_{jn}(u) \leqq 1 \leqq ((x - u)/d_n)^4$. Altogether this delivers $\sum_{j \neq k} h_{jn}(u) \leqq ((x - u)/d_n)^4$ for $u \in [a, b]$ which gives in connection with (9), (24)

$$|T_n(h_n; x) - h_n(x)| \geqq K_0 - K\mu_n^2/d_n^4 = K_0 - Kc_5^2 = K_0/2 \qquad (x \in B_n).$$

This establishes (17) since

$$R_n h_n \geqq \int_{B_n} |T_n h_n - h_n| \geqq c_6 c_1 K_0 > 0 \quad \blacksquare \tag{25}$$

For $U_\delta f = \tau_1(f, \delta)$ one may analogously deduce the following contribution concerning the sharpness of Theorem 1 (cf. (22)).

Corollary 6: *Under the assumptions of Theorem 5 there exists a counterexample $f_\omega \in R$ such that*

$$\tau_1(f_\omega, \delta) = \mathcal{O}(\omega(\delta)), \qquad \int \sup_{k \geqq n} |T_k f_\omega - f_\omega| \neq o(\omega(\mu_n^{1/2})).$$

Note that the proof still works (cf. (25)) if one considers $R_n f := \int |T_n f - f|$ so that one also obtains the sharpness of estimates of type

$$\int |T_n f - f| \leqq C\tau_2(f, \mu_n^{1/2}) \qquad (f \in R), \tag{26}$$

thus even for the L^1-error (cf. (10)).

4. Applications

Let us discuss the preceding Bohman-Korovkin theorem and its sharpness in connection with Bernstein polynomials and spline approximation.

The *Bernstein polynomials* are defined for $f \in R[0, 1]$ by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

One has $B_n p_i = p_i$ for $i = 0, 1$ and $B_n((u - x)^2; x) = \varphi(x)/n$, $\varphi(x) := x(1 - x)$, thus (8), (19), (20) with $\mu_n = 1/4n$ are fulfilled if one chooses

$$I_{jn} = \left(1/4 + (j - 1) n^{-1/2}, 1/4 + jn^{-1/2}\right), \qquad n^{1/2}/2 \leqq j_n < n^{1/2}/2 + 1.$$

Concerning (9) only note that $B_n((u - x)^4; x) = 3\varphi^2(x)/n^2 + \varphi(x)(1 - 6\varphi(x))/n^3$. Hence our theorems deliver

Corollary 7: *For* $f \in R[0, 1]$ *there holds true*

$$\overline{\int} \sup_{k \geq n} |B_k f - f| \leq C\tau_2(f, n^{-1/2}). \tag{27}$$

Moreover, for each ω *subject to* (14) *there exists an* $f_\omega \in R[0, 1]$ *such that*

$$\tau_2(f_\omega, \delta) = \mathcal{O}(\omega(\delta^2)), \quad \overline{\int} \sup_{k \geq n} |B_k f_\omega - f_\omega| \neq o(\omega(1/n)). \tag{28}$$

Concerning the L^1-error $\int |B_n f - f|$, results like (27) (cf. (26)) are well-known (cf. [1], [15, p. 107]). In fact, instead of (9), convexity arguments are available for $B_n'' g_\delta - g_\delta''$ (cf. (12)). Though (27) is sharp in the sense of (28), no inverse theorem can be established. Therefore, in [10] the relevant L^1-estimate was strengthened by substituting the (unweighted) τ_2-modul by

$$\tau_2^\varphi(f, \delta) := \overline{\int} \sup \{|\Delta_h^2 f(t)| : t \pm h \in U_{\delta\varphi^{1/2}(x)+\delta^2}(x)\} \, dx.$$

Partial inverse theorems were then established in [11] for the L^p-error, $p > 1$. Concerning the Riemann error $\overline{\int} \sup_{k \geq n} |B_k f - f|$ discussed here, an improved direct as well as inverse result can be found in [18], namely

$$\left\| \sup_{k \geq n} |B_k f - f| \right\|_{1/n}^\varphi \leq M\tau_2^\varphi(f, n^{-1/2}), \quad \tau_2^\varphi(f, n^{-1/2}) \leq \frac{M}{n} \sum_{k=1}^n \left\| \sup_{m \geq k} |B_m f - f| \right\|_{1/k}^\varphi$$

in terms of the weighted norm

$$\|f\|_{\delta^2}^\varphi = \int \sup \{|f(y)| : y \in U_{\delta\varphi^{1/2}(x)+\delta^2}(x)\} \, dx.$$

Let us also mention that the sharpness of the latter results is discussed in [17].

Finally consider linear interpolating splines on the equidistant nodes $x_{kn} := k/n$ in the interval [0, 1]. With the continuous functions

$$g_{kn}(x) = \begin{cases} 1 - n |x_{kn} - x|, & |x_{kn} - x| \leq 1/n, \\ 0 & \text{else} \end{cases}$$

the *spline operator* S_n is defined for $f \in R[0, 1]$ by

$$S_n(f; x) = \sum_{k=0}^n f(x_{kn}) g_{kn}(x) \qquad (x \in [0, 1]).$$

Obviously, $S_n p_i = p_i$ for $i = 0, 1$, thus (8). Furthermore, one has ($x \in [0, 1], r = 2, 4$)

$$S_n((u - x)^r; x) = \sum_{k=0}^n (x_{kn} - x)^r g_{kn}(x) = \sum_{g_{kn}(x) \neq 0} (x_{kn} - x)^r g_{kn}(x) \leq \frac{1}{n^r}, \tag{29}$$

since $|x_{kn} - x| \leq 1/n$ if $g_{kn}(x) \neq 0$. Set $I_n = \bigcup_{k=1}^n ((4k - 3)/4n, (4k - 1)/4n)$. Then $x \in I_n$ implies $|x_{kn} - x| \geq 1/4n$ which gives

$$S_n((u - x)^2; x) = \sum_{k=0}^n (x_{kn} - x)^2 g_{kn}(x) \geq \frac{1}{16n^2} \sum_{k=0}^n g_{kn}(x) = \frac{1}{16n^2}. \tag{30}$$

Therefore $1/16n^2 \leq \mu_n \leq 1/n^2$ in view of (29), and (9), (19), (20) with $j_n = n$ follow by (29), (30).

Corollary 8: *For* $f \in R[0, 1]$ *there holds true*

$$\overline{\int} \sup_{k \geq n} |S_k f - f| \leq C\tau_2(f, 1/n), \tag{31}$$

and for each ω *subject to* (14) *there exists an* $f_\omega \in R[0, 1]$ *such that*

$$\tau_2(f_\omega, \delta) = \mathcal{O}(\omega(\delta^2)), \qquad \overline{\int} \sup_{k \geq n} |S_k f_\omega - f_\omega| \neq o(\omega(1/n^2)).$$

The result, corresponding to (31) for the L^1-error, is again well-known (cf. [15, p. 123]).

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