

On Nonlocal Problems for Pseudoparabolic Equations

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Es werden drei nichtlokale Aufgaben für pseudoparabolische Gleichungen betrachtet. Ihre Lösungsmethode basiert auf der Konstruktion der Riemannschen Funktion für einen gewissen Differentialoperator dritter Ordnung. Damit werden die Aufgaben auf die Lösung von Integralgleichungen geführt. Es werden hinreichende Existenzbedingungen für die Lösung der nichtlokalen Aufgaben angegeben.

Рассматриваются три нелокальные задачи для псевдопараболических уравнений. Метод их решения основан на конструкции функции Римана для некоторого дифференциального оператора третьего порядка. Этим задачи сводятся к интегральным уравнениям. Сформулированы достаточные условия для существования решения нелокальных задач.

Three nonlocal problems for pseudoparabolic equations are considered. The method of solving of these problems is based on constructing the Riemann function for some differential operator of third order. The problems lead to respective integral equations. Sufficient conditions for existence of the solution of the nonlocal problems are given.

1. Introduction. Partial differential equations of the 3rd order, e.g. pseudoparabolic equations, possess many physical applications. These equations describe for example diffusion in a fissured medium with absorption or partial saturation, congelation of glue, they appear in the weak formulation of the 2-phase Stefan problem, in fluid mechanics and in other problems (cf. [2]). Nonlocal problems for pseudoparabolic equations appear also during the numerical solving of some special kind of boundary value problems [9]. Nonlocal problems were at first considered for partial differential equations of the 2nd order of elliptic type [1] and later for parabolic equations of the 2nd order and parabolic systems (cf. [4–8, 10]).

In the present paper three nonlocal problems for pseudoparabolic equations are considered. The method of solving of these problems is based on a construction of the Riemann function for some differential operator of 3rd order. D. COLTON proved in 1972 (c.f. [3]) the existence of this Riemann function by a fixed point method. In 1982 the existence of the Riemann function was proved by V. A. VODACHOVA by means of integral equations [11]. In this paper we will use properties of the Riemann function constructed by D. Colton. For pseudoparabolic equations, a nonlocal problem was considered only in [11]. The nonlocal problems for pseudoparabolic equations considered in this paper correspond to nonlocal problems for parabolic systems of equations of 2nd order in [6–8].

Let $\Omega = \{(x, t) : x \in (0, 1) \text{ and } t \in (0, T)\}$, for $0 < l, T < \infty$. Now, we can define the operator L by

$$Lu \equiv u_{xxt} + Au_{xx} + a(x)u_x + b(x, t)u_t + c(x)u \quad (1)$$

for $u = u(x, t)$, $(x, t) \in \Omega$. We assume that $A = \text{const}$, $a \in C^1[0, l]$, $c \in C^1[0, l]$, $b \in C^1(\Omega)$ and $b(x, t) < 0$ for $(x, t) \in \Omega$. In the case $b(x, t) \geq 0$ the solution for some boundary value problems for the operator (1) may be not unique (c.f. [3]).

2. Nonlocal problems. For the operator L defined by (1) we consider the following nonlocal problems.

Problem (N-I): Find a function $u = u(x, t)$ that is regular in Ω and satisfies the equation

$$Lu(x, t) = f(x, t) \quad (2)$$

subject to the conditions

$$u(x, 0) = \tau(x) \quad \text{for } x \in [0, l], \quad (3a)$$

$$u_x(l, t) = \psi(t) \quad \text{for } t \in [0, T], \quad (3c)$$

$$u(l, t) = \sum_{j=1}^n \alpha_j(t) u(x_j, t) + \vartheta(t) \quad \text{for } t \in [0, T], x_j \in [0, l], \quad (4)$$

where $\tau, \psi, \alpha_j, \vartheta, f$ are of class C^1 in the respective domains. Some additional conditions for α_j will be given later. A. M. NACHUSCHEV [9] proved that some special problems with conditions of integral type lead to conditions of form (4).

Problem (N-II): Find a function $u = u(x, t)$ that is regular in Ω and satisfies the equation $Lu(x, t) = f(x, t)$ subject to the conditions

$$u(x, 0) = \tau(x) \quad \text{for } x \in [0, l],$$

$$u(l, t) = \varphi(t) \quad \text{for } t \in [0, T], \quad (3b)$$

$$u_x(l, t) = \sum_{j=1}^n \alpha_j(t) u(x_j, t) + \vartheta(t) \quad \text{for } t \in [0, T], x_j \in [0, l], \quad (5)$$

where $\tau, \varphi, \alpha_j, \vartheta, f$ are of class C^1 in the respective domains. Some additional conditions for α_j will be given later. A nonlocal condition of form (5) was considered in [6, 7] for a parabolic system of 2nd order.

Problem (N-III): Find a function $u = u(x, t)$ that is regular in Ω and satisfies the equation $Lu(x, t) = f(x, t)$ subject to the boundary conditions

$$u(l, t) = \varphi(t) \quad \text{for } t \in [0, T],$$

$$u_x(l, t) = \psi(t) \quad \text{for } t \in [0, T],$$

and the nonlocal condition

$$u(x, 0) = \sum_{j=1}^n \alpha_j(x) u(x_j, t) + \vartheta(x) \quad \text{for } x \in [0, l], t_j \in [0, T], \quad (6)$$

where $\varphi, \psi, \vartheta, \alpha_j, f$ are of class C^1 in the respective domains. Some additional conditions for α_j will be given later. A nonlocal condition of the form (6) was considered in [6, 7] for a parabolic system of 2nd order.

3. Auxiliary problems. To solve the nonlocal problems (N-I) – (N-III) we need the solutions of the following auxiliary

Problem (P): Find a function $u = u(x, t)$ that is regular in Ω and satisfies the equation $Lu(x, t) = f(x, t)$ subject to the boundary conditions

$$u(x, 0) = \tau(x) \quad \text{for } x \in [0, l], \quad (3a)$$

$$u(l, t) = \varphi(t) \quad \text{for } t \in [0, T], \quad (3b)$$

$$u_x(l, t) = \psi(t) \quad \text{for } t \in [0, T]; \quad (3c)$$

where τ, φ, ψ, f are of class C^1 in the respective domains and satisfy $\varphi(0) = \tau(l)$, $\tau'(l) = \psi(0)$. To solve problem (P) we shall need the following operator M , which was defined by D. COLTON (cf. [3]) as

$$Mv \equiv v_{xxt} - Av_{xx} + (a(x)v)_x + b(x, t)v_t - c(x)v. \tag{7}$$

Let $v = v(x, t, x_0, t_0)$ be a solution of the problem

$$Mv = 0 \text{ in } \Omega, \tag{8}$$

$$v(x_0, t, x_0, t_0) = 0 \text{ for } t \in [0, T], \tag{9}$$

$$v(x, t_0, x_0, t_0) = 0 \text{ for } x \in [0, l],$$

$$v_x(x_0, t, x_0, t_0) = \frac{1}{A} [e^{A(t-t_0)} - 1] \text{ for } t \in [0, T], \tag{10}$$

where (x_0, t_0) is an arbitrary but fixed point from Ω . The function $v(x, t) \equiv v(x, t, x_0, t_0)$ is called a *Riemann function* for the problem (P). D. COLTON showed (cf. [3]) that the Riemann function v for (P) exists and is sufficiently smooth. His proof results from the Banach fixed point theorem.

Now we apply Green's formula in the rectangle $x < \xi < l, 0 < \eta < t$, to the identity

$$\begin{aligned} v_t Lu - u_t Mv &= \frac{\partial}{\partial x} [u_x v_t - u_t v_{xt} - au_x v + Au_x v_t + Au_t v_x] \\ &+ \frac{\partial}{\partial t} [au_x v + cuv - Au_x v_x] \end{aligned} \tag{11}$$

and from conditions (2), (3), (8), (10) we obtain the following integral representation for the solution of problem (P):

$$\begin{aligned} v(x, t) &= \tau(x) + \int_x^l [a(\xi) \tau'(\xi) v(\xi, 0, x, t) + c(\xi) \tau(\xi) v(\xi, 0, x, t) \\ &- A\tau'(\xi) v_\xi'(\xi, 0, x, t)] d\xi \\ &- \int_0^t [\psi'(\eta) v_\eta'(l, \eta, x, t) - \varphi'(\eta) v_{\xi\eta}''(l, \eta, x, t) - a(l) \varphi'(\eta) v(l, \eta, x, t) \\ &+ A\psi(\eta) v_\eta'(l, \eta, x, t) + A\varphi_\xi'(\eta) v_\xi'(l, \eta, x, t)] d\eta \\ &+ \int_0^t \int_x^l v_\eta'(\xi, \eta, x, t) f(\xi, \eta) d\xi d\eta \end{aligned} \tag{12}$$

(for details see [3]).

4. Solutions of the nonlocal problems. We can formulate the following theorems.

Theorem 1: *If the given functions $\tau, \psi, \delta, f, \alpha_j$ ($j = 1, \dots, n$) are of class C^1 in the respective domains and*

$$1 - \sum_{j=1}^n \alpha_j(t) [v_{\xi\eta}''(l, t, x_j, t) - Av_\xi'(l, t, x_j, t)] \neq 0$$

for every $t \in [0, T]$, then a solution of problem (N-I) exists.

Proof: Let $\varphi(t) = u(l, t)$ be such a function that the solution of the auxiliary problem (P) defined by (12) satisfies the nonlocal condition (4). We substitute (12) into (4), which leads to the equation

$$\varphi(t) = \int_0^t \varphi'(\eta) \left\{ \sum_{j=1}^n \alpha_j(t) [v''_{\xi\eta}(l, \eta, x_j, t) - Av'_{\xi}(l, \eta, x_j, t) + a(l)v(l, \eta, x_j, t)] \right\} d\eta + \sum_{j=1}^n \alpha_j(t) \Phi(x_j, t) + \vartheta(t), \quad (13)$$

where

$$\begin{aligned} \Phi(x, t) = & \tau(x) + \int_x^l [a(\xi)\tau'(\xi)v(\xi, 0, x, t) + c(\xi)\tau(\xi)v(\xi, 0, x, t) \\ & - A\tau'(\xi)v'_{\xi}(\xi, 0, x, t)] d\xi + \int_0^t [A\psi'(\eta)v'_{\eta}(l, \eta, x, t) \\ & - \psi'(\eta)v_{\eta}(l, \eta, x, t)] d\eta + \int_0^t \int_x^l v_{\eta}(\xi, \eta, x, t) f(\xi, \eta) d\xi d\eta. \end{aligned}$$

Integrating by parts in (13) leads to the following Volterra integral equation of the 2nd kind for φ :

$$\begin{aligned} \varphi(t)w(t) - \int_0^t \varphi(\eta) \frac{d}{d\eta} \left\{ \sum_{j=1}^n \alpha_j(t) [Av'_{\xi}(l, \eta, x_j, t) \right. \\ \left. - v''_{\xi\eta}(l, \eta, x_j, t) - a(l)v(l, \eta, x_j, t)] \right\} d\eta = F_1(t), \quad (14) \end{aligned}$$

where

$$w(t) = 1 - \sum_{j=1}^n \alpha_j(t) [v''_{\xi\eta}(l, t, x_j, t) - Av'_{\xi}(l, t, x_j, t)]$$

and

$$\begin{aligned} F_1(t) = & \sum_{j=1}^n \alpha_j(t) \{ \Phi(x_j, t) \\ & - \tau(l) [v''_{\xi\eta}(l, 0, x_j, t) - Av'_{\xi}(l, 0, x_j, t) + a(l)v(l, 0, x_j, t)] \} + \delta(t). \end{aligned}$$

It follows immediately from the assumption of Theorem 1 that there exists a solution of equation (14). The solution of the Volterra equation (14) is of class $C^1[0, T]$. This follows from the assumption of Theorem 1 and from properties of the Riemann function v (for details see [3, 11]). The function φ defines the solution of the problem (N-I) by formula (12) ■

Theorem 2: *If the given functions $\tau, \varphi, \vartheta, f$ and α_j ($j = 1, \dots, n$) are of class C^1 in the respective domains and*

$$1 + \sum_{j=1}^n \alpha_j(t) v_{\eta}(l, t, x_j, t) \neq 0 \quad \text{for every } t \in [0, T],$$

then a solution of problem (N-II) exists.

Proof: Let $\psi(t) = u_x(l, t)$ be such a function that the solution of problem (P) defined by (12) satisfies the nonlocal condition (5). Now, we obtain the equation

$$\begin{aligned} \psi(t) = & - \int_0^t \sum_{j=1}^n \alpha_j(t) [A\psi(\eta) v_\eta'(l, \eta, x_j, t) + \psi'(\eta) v_\eta'(\bar{l}, \eta, x_j, t)] d\eta \\ & + \sum_{j=1}^n \alpha_j(t) \bar{\Phi}(x_j, t) + \vartheta(t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \bar{\Phi}(x, t) = & \tau(x) \\ & + \int_x^l [a(\xi) \tau'(\xi) v(\xi, 0, x, t) + c(\xi) \tau(\xi) v(\xi, 0, x, t) \\ & - A\tau'(\xi) v_\xi'(\xi, 0, x, t)] d\xi + \int_0^t [\varphi'(\eta) v''_{\xi\eta}(l, \eta, x, t) \\ & + a(l) \varphi'(\eta) v(l, \eta, x, t) - A\varphi'(\eta) v_\xi'(l, \eta, x, t)] d\eta \\ & + \int_0^t \int_x^l v_\eta'(\xi, \eta, x, t) f(\xi, \eta) d\xi d\eta. \end{aligned}$$

Hence,

$$\dot{\tilde{w}}(t) \psi(t) + \int_0^t \psi(\eta) \left\{ \sum_{j=1}^n \alpha_j(t) [Av_\eta'(l, \eta, x_j, t) - v''_{\eta\eta}(l, \eta, x_j, t)] \right\} d\eta = F_2(t), \tag{16}$$

where

$$\tilde{w}(t) = 1 + \sum_{j=1}^n \alpha_j(t) v_\eta'(l, t, x_j, t)$$

and

$$F_2(t) = \sum_{j=1}^n \alpha_j(t) [\bar{\Phi}(x_j, t) + \tau'(l) v_\eta'(l, 0, x_j, t)] + \vartheta(t).$$

It follows immediately from the assumption of Theorem 2 that a solution of equation (16) exists. Owing to properties of the Riemann function v (cf. [3]) this solution of the integral equation (16) is of class $C^1[0, T]$. The function ψ defines the solution of the problem (N-II) by formula (12) ■

Theorem 3: *If the given functions $\varphi, \psi, \vartheta, f$ and α_j ($j = 1, \dots, n$) are of class C^1 in the respective domains and*

$$1 - \sum_{j=1}^n \alpha_j(x) e^{-Ax} \neq 0 \text{ for every } x \in [0, l],$$

then a solution of the problem (N-III) exists.

Proof: Let $\tau(x) = u(x, 0)$ be such a function that the solution of the problem (P) defined by (12) satisfies the nonlocal condition (6). It leads to the equation

$$\begin{aligned} \tau(x) = & \tau(x) \sum_{j=1}^n \alpha_j(x) + \sum_{j=1}^n \alpha_j(x) \int_x^l c(\xi) v(\xi, 0, x, t_j) \tau(\xi) d\xi \\ & + \sum_{j=1}^n \alpha_j(x) \int_x^l [a(\xi) v(\xi, 0, x, t_j) - Av_\xi'(\xi, 0, x, t_j)] \tau'(\xi) d\xi + F(x), \end{aligned} \tag{17}$$

where

$$\begin{aligned}
 F(x) = & - \sum_{j=1}^n \alpha_j(x) \int_0^{t_j} [\psi'(\eta) v_\eta'(l, \eta, x, t_j) - \varphi'(\eta) v_{\xi n}''(l, \eta, x, t_j) \\
 & - a(l) \varphi'(\eta) v(l, \eta, x, t_j) + A\psi(\eta) v_\eta'(l, \eta, x, t_j) + A\varphi'(\eta) v_\xi'(l, \eta, x, t_j)] d\eta \\
 & + \sum_{j=1}^n \alpha_j(x) \int_0^{t_j} \int_0^l v_\eta'(\xi, \eta, x, t_j) f(\xi, \eta) d\xi d\eta + \vartheta(x).
 \end{aligned}$$

From integrating by parts in (17), we obtain

$$\tau(x) \hat{w}(x) = \int_x^l K(x, \xi) \tau(\xi) d\xi + \bar{F}(x), \quad (18)$$

where

$$\begin{aligned}
 \hat{w}(x) &= 1 - \sum_{j=1}^n \alpha_j(x) e^{-At_j}, \\
 \bar{F}(x) &= F(x) + \sum_{j=1}^n \alpha_j(x) [a(l) v(l, 0, x, t_j) - Av_\xi'(l, 0, x, t_j)] \varphi(0), \\
 K(x, \xi) &= \sum_{j=1}^n \alpha_j(x) [c(\xi) v(\xi, 0, x, t_j) - \frac{d}{d\xi} [a(\xi) v(\xi, 0, x, t_j) \\
 &\quad - Av_\xi'(\xi, 0, x, t_j)]].
 \end{aligned}$$

It follows immediately from the assumption of Theorem 3 that a solution of equation (18) exists. This solution is of class $C^1[0, l]$. The function τ defines the solution of the problem (N-III) by formula (12) ■

Remark: The situation in the problem (N-III) with a nonlocal condition of the form (6) is more "simple" than in the corresponding problem for a parabolic system of equations of 2nd order. Problem (N-III) leads to a Volterra-type integral equation but the corresponding nonlocal problem for a parabolic system of equations of 2nd order leads to a Fredholm-type integral equation that is always solvable.

Nonlinear case. It is possible to consider the following nonlinearities:

$$\begin{aligned}
 \vartheta(t) &= \vartheta(t, u(l, t)) && \text{in problem (N-I),} \\
 \vartheta(t) &= \vartheta(t, u_x(l, t)) && \text{in problem (N-II),} \\
 \vartheta(x) &= \vartheta(x, u(x, 0)) && \text{in problem (N-III)}
 \end{aligned}$$

and

$$f(x, t) = f(x, t, u(x, t)) \quad \text{in (N-I), (N-II), (N-III).}$$

In this case the integral equations (14), (16), (18) become nonlinear equations. One can show that if the Lipschitz constant for the functions δ and f as functions of the last argument are sufficiently small, then there exists a solution of the corresponding integral equation. The solution of the nonlocal problem is defined by (12).

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