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A Lattice Problem for Differential Forms in Euclidean Spaces A Lattice Problem for Dif
R. SCHUSTER

Im n-dimensionalen Euklidischen Raum Eⁿ wird in Verallgemeinerung eines Gitterpunktproblems ein Gitterproblem für (3-automorphe p-Differentialformen gelöst. (3 ist dabei eine eigentlich diskontinuierliche Gruppe von Isometrien des Eⁿ mit kompaktem Fundamentalbereich. Zur Behandlung werden Mittelwertoperatoren für Differentialformen und ein Landau-
sches Differenzenverfahren verwendet. R. SCHUSTER.

1

1

Im n-dimensionalen Euklidischen Raum E^a wird in Verallgemeinerung eines Gitt

problems ein Gitterproblem für G-automorphe p-Differentialformen gelöst. G ist de

eigentlich diskontinuierliche Gruppe v

Как обобщение одной сеточной проблемы в *п*-мерном евклидовом пространстве E^n решается сеточная проблема для \mathfrak{G} -автоморфных дифференциальных форм степени p . Π ри этом \mathfrak{G} .- впольне разрывная группа изометрий с компактной фундаментальной областью в Е^т. Для решения проблемы используются операторы среднего значения для дифференциальных форм и метод сеток Ландау.

Generalizing a lattice-point problem we solve a lattice problem for G-automorphic differential p-forms in the *n*-dimensional Euclidean space $Eⁿ$, where \mathfrak{G} is a properly discontinuous group of isometrics of $Eⁿ$ with compact fundamental domain. Our approach essentially uses meanvalue 'operators for differential forms and a Landau difference method. Hpu этом \mathfrak{B} - впольне разрывная группа изометрий с компактной фундамента

областью в Е°. Для решения проблемы используются операторы среднего зна

для дифференциальных форм и метод сегок Jangay.

Ceneralizing a lat

1. Introduction

Let G be a properly discontinuous group of isometries of the *n*-dimensional Euclidean space $Eⁿ$ with a compact fundamental domain $\mathcal F$. By generalizing the Landau ellipsoid problem, P. GÜNTHER [8] studied the estimation of a properly discontinuous group of
with a compact fundamental don
lem, P. GÜNTHER [8] studied the
 $A(t, x, y) = \sum_{\substack{b \in \mathcal{B} \\ r(x, by) < t}} 1$ for $t \to \infty$ *Y*
 Y is
 First of the Bytgend
 $\begin{aligned} \text{By gen} \end{aligned}$
 $\begin{aligned} \text{By gen} \end{aligned}$

with the Euclidean distance $r(x, y)$ of the points $x, y \in \mathbb{E}^n$. In [8] the elements of \mathfrak{G} (with the exception of the identity map id) were supposed to be without fixed points, but. instead of simply counting the lattice points, certain unimodular weights were used. The order of magnitude of the leading term and of the lattice remainder used there are the same as in the classical case treated by Landau. We refer to F. FRICKER, [3] 'and A WALFISZ [18] as basic references, see also the literature quoted there. Problems with weaker assumptions for the fundamental domain have recently been investigated by P. D. Lax and R. S. PHILLIPS $[14]$. In this paper we want to discuss a generalization involving alternating differential forms, and we will call it a latticeform problem. Every $b \in \mathcal{C}$ induces a mapping b^* for differential forms, see [11]. We call a differential form α on \mathbb{E}^n is automorphic if $b^*\alpha = \alpha$ is valid for all $b \in \mathbb{G}$. Following [7] we define components of differential forms. Let $(x¹, ..., xⁿ)$ be a Cartesian coordinate system of Eⁿ. The component of a p-form $\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

in the direction of the vector $v = (v^{i})$ then shall be defined by

$$
\alpha_{|v} = p ||v||^{-2} v_{\{i\}} v^i \alpha_{|i|i_1\cdots i_n|} dx^{i_1} \wedge \ldots \wedge dx^{i_p}
$$

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ction of the vector $v = (v^i)$ then shall
 $|v| = p ||v||^{-2} v_{[i_1} v^i \alpha_{[i] i_1 \cdots i_p]} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$

imponent $\alpha_{|v|}^{\perp}$ orthogonal to $\alpha_{|v}$ by $\alpha_{|v|}^{\perp}$ in the direction of the vector $v = (v^i)$ then shall be defined by
 $\alpha_{|v} = p ||v||^{-2} v_{\{i_1} v^i \alpha_{\{i \mid i_1, \dots, i_p\}} dx^{i_1} \wedge \dots \wedge dx^{i_p}$

and the component $\alpha_{|v|}^{\perp}$ orthogonal to $\alpha_{|v}$ by $\alpha_{|v|}^{\perp} = \alpha - \alpha_{|v|}$. | $[\cdot, \cdot, i] \cdots]$ shall denote the alternation without *i*. We lower and raise indices by the covariant and contravariant metric tensors δ_{ij} and δ^{ij} , respectively. Let $T_{x,y}$ be the parallel displacement of p-forms from the point $y \in \mathbb{E}^n$ to $x \in \mathbb{E}^n$ along the straight line joining these two points. We now define vection of the vector $v = (v^i)$ then shall be defined by
 $\alpha_{|v} = p ||v||^{-2} v_{(i_1} v^i \alpha_{|i_1, \ldots i_p)} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

component α_v^{\dagger} orthogonal to $\alpha_{|v}$ by $\alpha_v^{\dagger} = \alpha - \alpha_{|v}$. $||v||$ denotes the Euclides

the or $v = (v^i)$ then shall be define
 $\mathbf{v}_{[i|i_1\cdots i_p]} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$
 $\mathbf{v}_{[i|i_1\cdots i_p]} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$
 $\mathbf{v}_{[i]}$

$$
\mathsf{A}^{\mathsf{r}}[x]\,(t,\,x,\,y)=\sum_{\substack{b\in\mathfrak{B}\\0\leq t\mid x,bw\mid\leq t}}T_{x,by}\alpha_{|x-by}(by)\,,
$$

displacement of *p*-forms from the point
ing these two points. We now define

$$
A^{r}[x](t, x, y) = \sum_{\substack{b \in \mathbb{G} \\ 0 < r(x, by) < t}} T_{x, by} \alpha_{|x - by}(by),
$$

$$
A^{o}[\alpha](t, x, y) = \sum_{\substack{b \in \mathbb{G} \\ 0 < r(x, by) < t}} T_{x, by} \alpha_{|x - by}^{\perp}(by).
$$

We are interested to estimate $A^r[\alpha]$ and $A^q[\alpha]$ for a G-automorphic differential form α for $t \to \infty$. For $p = 0$, $\alpha = 1$ this lattice-form-problem for A^{σ} reduces to the problem for $A(t, x, y)$ mentioned above.

 (1)

 (2)

Our approach essentially uses kernels of mean value operators for differential forms which are defined by means of double differential forms σ_p ; τ_p introduced by P. GUNTHER [5]. The fact that the forms σ_p , τ_p are intimately related with the construction of the components of p -forms and their parallel displacement makes them well-suited. We will apply some standard arguments of the theory of Euler-Poisson-Darboux equations, but we will not make use of the approach by means of theta functions and Jacobi transformation laws, cf. [8]. We use the Fourier method, which also plays an important role in [8]. Mean value formulas turn out to be quite useful for this purpose. In the space of $\mathfrak{G}\text{-automorphic }p\text{-forms which are quadratically}$ integrable over $\mathcal F$ there exists in the sense of L^2 -norms over $\mathcal F$ a complete orthonormal system of $\mathcal{G}-a^2$ *s* $(\partial x^1)^2$ - ... - $\partial^2/(\partial x^n)^2$ with the corresponding eigenvalues μ_i^p : $\Delta \omega_i^p = \mu_i^p \omega_i^p$. To estimate A^{σ} , A^{τ} the harmonic forms ω_1^{σ} , ..., $\omega_{B_n}^{\sigma}$ turn out to be quite important. Thereby B_p denotes the multiplicity of the eigenvalue 0. If the elements of \otimes are without fixed points (with the exception of *id*), B_n is the pth Betti number of the Clifford-Klein space form corresponding to 03. Using the scalar product of the differential forms *a = dx'* A ... A *dx1 , =* Defined by a we will not make use of the approach by means of the norm \hat{I} and \hat{I}

$$
\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \qquad \beta = \beta_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}
$$

'following $\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \qquad \beta = \beta_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

by $\alpha \cdot \beta = p! \alpha_{i_1 \cdots i_p} \beta^{i_1 \cdots i_p}$ and the norm $||\alpha|| = (\alpha \cdot \alpha)^{1/2}$, we can

em: The lattice remainder defined by
 $\mathbf{P}^r[\alpha] (t, x, y) = \mathbf{A$ by $\alpha \cdot \beta =$
 $\text{em} \cdot \text{The}$
 $\text{Pr}[\alpha]$ (t, x)
 the relation *sational interior* $\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \qquad \beta = \beta_{i_1 \cdots i_p} c$ *

defined by* $\alpha \cdot \beta = p! \alpha_{i_1 \cdots i_p} \beta^{i_1 \cdots i_p}$ *and the norm ||

following

Theorem: The lattice remainder defined by
 P^r[\alpha](t, x, y) = A^r[\alpha](t, x, y)*

Theorem :The lattice remainder defined by

$$
\mathsf{P}^{\mathsf{r}}[\alpha](t, x, y) = \mathsf{A}^{\mathsf{r}}[\alpha](t, x, y)
$$

$$
f^{i_1} \wedge \ldots \wedge dx^{i_p}, \qquad \beta = \beta_{i_1 \cdots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}
$$

\n
$$
|\alpha_{i_1 \cdots i_p} \beta^{i_1 \cdots i_p}
$$
 and the norm $||\alpha|| = (\alpha \cdot \alpha)^{1/2}$, we can state t
\n*ice remainder defined by*
\n
$$
= \mathsf{A}^{\cdot}[\alpha] (t, x, y)
$$

\n
$$
- \sum_{i=1, \ldots, \beta_p} \frac{p}{n} \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})} (\alpha \cdot \omega_i^p) (y) \omega_i^p(x) t^n
$$

\n
$$
||\alpha|| = O\left(t^{n - \frac{2n}{n+1}}\right) ||\alpha(y)||.
$$

satisfies the relation

$$
\int_{\mathbb{R}^n} \|\mathsf{P}^{\ell}[\alpha](t,x,y)\| = O\left(t^{n-\frac{2n}{n+1}}\right)\|\alpha(y)\|.
$$

The O-term does not depend on α . We get an analogous result for σ instead of τ if we re*place the coefficient p/n of the leading term by* $(n - p)/n$ *.* α *is always supposed to be* α *continuous.*

A Lattice Problem for Differential Forms
 The O-term does not depend on α . We get an analogous result for σ instead of τ if v

place the coefficient $p|n$ of the leading term by $(n - p)|n$. α is always supp The property of $A(t, x, y)$ to be monotonic in *t* does not hold in general for $\|\mathbf{A}^{\mathsf{T}}[\alpha](\hat{t}, x, y)\|$. Nevertheless the lattice remainder is still estimable by $O(\hat{t}^{n-2n/(n+1)})$ in the case of p -forms. The order of magnitude of the leading terms of $A(t, x, y)$ and $A^{r}[\alpha]$ (*t, x, y*) for $t \to \infty$ are the same, too (if we suppose $B_p \neq 0$ and $p \neq 0$). The theorem points out the fact that the leading term of $||A^{r}[\alpha](t, x, y)||$ is essentially depending on the harmonic component
 $\sum_{i=1,...,B_p} (\alpha \cdot \omega_i^p)(y) \omega_i^p(x)$. A Lattice Problem for Differential Forms

The O-term does not depend on α . We get an analogous result for σ instead of τ if

place the coefficient $\eta |n \circ f$ the leading term by $(n - p)|n$. α is always suppose
 $\$ *v*roperty of $A(t, x, y)$ to be monotonic in *t* does x, y). Nevertheless the lattice remainder is still x, y for $t \to \infty$ are the same, too (if we suppose points out the fact that the leading term of $||A^r$ *g* on the ha *P*-forms. The order of magnitude of the leading term
for $t \to \infty$ are the same, too (if we suppose $B_p \neq 0$
ts out the fact that the leading term of $||A^{r}[x](t, x)$,
the harmonic component
 $\binom{\alpha \cdot \omega_i^p}{y} (y) \omega_i^p(x)$.
 \bin magnitude of the leading terms of $A(t, x, y)$ and
me, too (if we suppose $B_p \neq 0$ and $p \neq 0$). The
he leading term of $||A^{r}[x](t, x, y)||$ is essentially
ent
consequence of this theorem. Let $n = 2$ and write
bers. Let \emptyset b

$$
\sum_{i=1,\ldots,B_p} (\alpha \cdot \omega_i^p) (y) \omega_i^p(x).
$$

As an illustration we give a simple consequence of this theorem. Let $n=2$ and write the elements of E^2 as complex numbers. Let \emptyset be the translation group $\sum_{i=1,\ldots,B_p} (\alpha \cdot \omega_i^p) (y) \omega_i^p(x)$.
 *u*stration we give a simple consequence of this theorem. Lents of E² as complex numbers. Let \mathcal{Y} be the translation $u \to u + k_1v + k_2w =: u_{k_1,k_1}$ $(u, v, w \in \mathbb{C}; k_1, k_2 \in \mathbb{Z}),$

$$
u \to u + k_1 v + k_2 w =: u_{k_1,k_2} \qquad (u, v, w \in \mathbb{C}; k_1, k_2 \in \mathbb{Z}),
$$

Corollary: We have

N

g on the harmonic component
\n
$$
\sum_{i=1,\ldots,B_p} (\alpha \cdot \omega_i^p) (y) \omega_i^p(x).
$$
\n|
\nstraction we give a simple consequence of this there
\nents of E² as complex numbers. Let G¹ be the transl-
\n $u \to u + k_1v + k_2w =: u_{k_1,k_1}$ $(u, v, w \in \mathbb{C}; k_1, k_2)$
\n $vw \neq 0$, arg $(v/w) \neq 0$.
\n
$$
\sum_{k_1,k_1 \in \mathbb{Z}} \sin^2 (\arg u_{k_1,k_1}) = \frac{\pi}{2} \frac{1}{(\text{Im}(\overline{v}w))^2} t^2 + O(t^{2/3}).
$$

2. Mean value operators for differential forms

Our treatment of the mean value operators is based on the double differential forms

value operators for differential forms
\ntment of the mean value operators is based on the double di
\n
$$
\sigma_0(x, y) = 1
$$
, $\tau_0(x, y) = 0$,
\n $\sigma_1(x, y) = r(x, y) d\mathbf{d} r(x, y)$, $\tau_1(x, y) = dr(x, y) d r(x, y)$.
\n $\sigma_p = \frac{1}{p} \sigma_{p-1} \wedge \hat{\wedge} \sigma_1$, $\tau_p = \tau_1 \wedge \hat{\wedge} \sigma_{p-1}$,

introduced by P. GÜNTHER [5] for spaces of constant curvature $K = 0$. \hat{d} , $\hat{\lambda}$ shall denote that d , \wedge refer to the second variable *y*. As shown by P. GÜNTHER [6, 7], there is a geometric interpretation for these double differential forms:

$$
T_{x,y}\alpha_{|y-x} = (-1)^p \tau_p(x,y) \cdot \alpha(y),
$$

\n
$$
T_{x,y}\alpha_{|y-x}^{\perp} = (-1)^p \sigma_p(x,y) \cdot \alpha(y).
$$
\n(3)

4

Following G. DE RHAM [15] we can write the Laplace operator in the form $\Delta = d\delta$. $+ \delta d$, using the differential operator d and the codifferential operator $\delta = (-1)^{pn+n+1}$ $*d *$ for a p-form and the Hodge dualization $*$. The eigenforms ω_i^p we can suppose to be closed $(d\omega_i^p = 0)$ or coclosed $(\delta \omega_i^p = 0)$, cf. [1]. Let $K(x, t)$ be the ball and $S(x, t)$ the sphere around $x \in \mathbb{E}^n$ with radius *t.* P. GÜNTHER [6, 7] treated the spherical $T_{x,y} \alpha_{|y-x} = (-1)^p \tau_p(x, y) \cdot \alpha(y)$,
 $T_{x,y} \alpha_{|y-x}^1 = (-1)^p \sigma_p(x, y) \cdot \alpha(y)$.

Following G. DE RHAM [15] we can write the Laplace $\alpha + \delta d$, using the differential operator d and the codifferent
 $\ast d \ast$ for a p-form and the Hodg

ing the differential operator
$$
d
$$
 and the codifferential opera
\na p -form and the Hodge dualization \ast . The eigenforms
\nsed $(d\omega_i^p = 0)$ or colosed $(\delta\omega_i^p = 0)$, cf. [1]. Let $K(\alpha)$
\ne sphere around $x \in \mathbb{E}^n$ with radius t . P. GÜNTHER [6, 7]
\n]ues
\n
$$
M^{\sigma}[\alpha] (t, x) = (-1)^p \frac{c_0}{t^{n-1}} \int_{S(x,t)} \sigma_p(x, y) \cdot \alpha(y) d\sigma_y,
$$
\n
$$
M^{\sigma}[\alpha] (t, x) = (-1)^p \frac{c_0}{t^{n-1}} \int_{S(x,t)} \tau_p(x, y) \cdot \alpha(y) d\sigma_y
$$

 $\label{eq:2} \frac{1}{\sqrt{2\pi}}\int_{0}^{\pi} \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)^{2}dx$

with $c_0 = \Gamma(n/2)/2\pi^{n/2}$. Motivated by the Riemann-Liouville integrals used in the

444 R. SCHUSTER
\nwith
$$
c_0 = \Gamma(n/2)/2\pi^{n/2}
$$
. Motivated by the Riemann-Liouville integrals used in the
\npapers of A. WETISTEIN [19] and P. GÜNTIER [8] we take for $\lambda \ge n + 3$
\n
$$
M_2[\alpha] (t, x) = c_1 t^{3-1} \int_0^t (t^2 - r^2)^{\frac{1-n-3}{2}} r^{n-1} (M^{\sigma} + M^{\tau}) [\alpha] (r, x) dr,
$$
\n(4)
\n
$$
N_2[\alpha] (t, x) = c_1 t^{1-1} \int_0^t (t^2 - r^2)^{\frac{1-n-3}{2}} r^{n+1} (M^{\sigma} - M^{\tau}) [\alpha] (r, x) dr
$$
\nwith $c_1 = 2/B \left(\frac{n}{2}, \frac{\lambda - n - 1}{2}\right)$. Then it follows that
\n
$$
M_1[\alpha] (t, x) = c_2(-1)^p t^{3-1} \int_0^t (t^2 - r^2(x, y))^{\frac{1-n-3}{2}} (\sigma_p(x, y) + \tau_p(x, y)) \cdot \alpha_p(y) dv_y,
$$

$$
N_{\lambda}[\alpha](t, x) = c_1 t^{1-\lambda} \int\limits_0^1 (t^2 - r^2)^{-\frac{1}{2}} r^{n+1} (M^{\sigma} - M^{\tau}) [\alpha](r, x) dr
$$

¹). Then it follows that
 $\int_0^1 (t^2 - r^2(x, y))^{\frac{\lambda - n - 3}{2}} \left(\sigma_p(t) \right)$

$$
M_1[\alpha] (t,x) = c_2(-1)^p t^{3-\lambda} \int_{K(x,t)} (t^2 - r^2(x,y)) \frac{\lambda - n - 3}{2} \left(\sigma_p(x,y) + \tau_p(x,y) \right) \cdot \alpha_p(y) dv_y,
$$

$$
N_{\iota}[\alpha](t, x) = c_{2}(-1)^{p} t^{1-\iota} \int\limits_{K(x, t)} (t^{2} - r^{2}(x, y))^{1-\iota} \frac{1-n-3}{2} \big(\sigma_{p}(x, y) - \tau_{p}(x, y) \big) r^{2}(x, y) \cdot \alpha_{p}(y) \, dv_{y}
$$

with $c_2 = \frac{\Gamma(\lambda - 1)}{2} / \Gamma(\frac{\lambda - n - 1}{2}) \pi^{n/2}$. We want to use methods of Euler-Pois son-Darboux theory with respect to the parameter λ . For this reason, we define $z(t, \lambda, \mu)$ for $t \ge 0$ to be the unique solution of the differential equation *M*₁[*x*] $(t, x) = c_2(-1)^p t^{3-\lambda} \int_{K(x,t)} (t^2 - r^2(x, y))^{\frac{\lambda - n - 3}{2}} (\sigma_p(x, y) + \tau_p(x, y))$
 *N*₁[*x*] $(t, x) = c_2(-1)^p t^{1-\lambda} \int_{K(x,t)} (t^2 - r^2(x, y))^{\frac{\lambda - n - 3}{2}} (\sigma_p(x, y) - \tau_p(x, y)) r^2(x)$

with $c_2 = \frac{\lambda - 1}{2} \int_{K(x,t)}^{\lambda} \int_{T} (\frac{\lambda - n - 1}{2}) \pi^{n/2}$ *N*₁[*a*] $(t, x) = c_2(-1)^p t^{1-1} \int_{K(x,t)}^{x} (t) dt$
with $c_2 = \frac{\Gamma\left(\frac{\lambda-1}{2}\right)}{\Gamma\left(\frac{\lambda-1}{2}\right)} = \frac{\Gamma\left(\frac{\lambda-1}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)}$
son-Darboux theory with respectively $z(t, \lambda, \mu)$ for $t \geq 0$ to be the unital conditions $z(t)$
wit M₁[α] $(t, x) = c_2(-1)^p t^{3-\lambda} \int_{K(x,t)} (t^2 - r^2(x, y))$
 $N_1[\alpha] (t, x) = c_2(-1)^p t^{1-\lambda} \int_{K(x,t)} (t^2 - r^2(x, y)) \frac{d\alpha}{dt}$

with $c_2 = \Gamma\left(\frac{\lambda - 1}{2}\right) / \Gamma\left(\frac{\lambda - n - 1}{2}\right) \pi^{n/2}$. Where the parameters with respect to the parameters $z(t, \$

$$
\frac{d^2}{dt^2}z(t,\lambda,\mu)+\frac{\lambda}{t}\frac{d}{dt}z(t,\lambda,\mu)+\mu z(t,\lambda,\mu)=0
$$

with the initial conditions $z(0, \lambda, \mu) = 1$, $\frac{d}{dt} z(t, \lambda, \mu)_{|t=0} = 0$. It should be noted

son-Darboux theory with respect to the parameter
$$
\lambda
$$
. For this reason, we defi
\nz(l, λ , μ) for $t \ge 0$ to be the unique solution of the differential equation
\n
$$
\frac{d^2}{dt^2} z(l, \lambda, \mu) + \frac{\lambda}{t} \frac{d}{dt} z(l, \lambda, \mu) + \mu z(l, \lambda, \mu) = 0
$$
\nwith the initial conditions $z(0, \lambda, \mu) = 1$, $\frac{d}{dt} z(l, \lambda, \mu)_{|t=0} = 0$. It should be not
\nthat
\n
$$
z(l, \lambda, \mu) = \Gamma\left(\frac{\lambda + 1}{2}\right) \left(\frac{2}{\sqrt{\mu}}\right)^{\frac{\lambda - 1}{2}} J_{\frac{\lambda - 1}{2}}(\sqrt{\mu}t), \quad \mu > 0,
$$
\nusing the Bessel function J , with index ν . As a consequence of a correspondence pri-
\nciple of Euler-Poisson-Darboux theory we have the recursion formula
\n
$$
z(l, \lambda, \mu) = \left(\frac{1}{\lambda + 1} i \frac{d}{dt} + 1\right) z(l, \lambda + 2, \mu).
$$
\nWe set
\n
$$
u(l, \lambda, \mu) = 2 \frac{\lambda - 1 - q(\lambda)}{\lambda - 1} z(l, \lambda, \mu) - z(l, \lambda - 2, \mu),
$$

using the Bessel function *J_v* with index *v*. As a consequence of a correspondence prin-
ciple of Euler-Poisson-Darboux theory we have the recursion formula
 $z(t, \lambda, \mu) = \left(\frac{1}{\lambda + 1}t\frac{d}{dt} + 1\right)z(t, \lambda + 2, \mu).$
We set 0,
a corresp
formula

 \mathbb{R}^3

$$
z(t, \lambda, \mu) = \left(\frac{1}{\lambda+1}t\,\frac{d}{dt} + 1\right)z(t, \lambda+2, \mu).
$$

the Bessel function J, with index v. As a consequence of a correspondence pri
of Euler-Poisson-Darboux theory we have the recursion formula

$$
z(t, \lambda, \mu) = \left(\frac{1}{\lambda + 1}t\frac{d}{dt} + 1\right)z(t, \lambda + 2, \mu).
$$
et

$$
u(t, \lambda, \mu) = 2\frac{\lambda - 1 - q(\lambda)}{\lambda - 1}z(t, \lambda, \mu) - z(t, \lambda - 2, \mu),
$$

$$
v(t, \lambda, \mu) = -2\frac{q(\lambda)}{\lambda - 1}z(t, \lambda, \mu) + z(t, \lambda - 2, \mu)
$$

$$
q(\lambda) = p + (\lambda - n - 1)/2. \text{ By using the recursion formula for } z(t, \lambda, \mu) =
$$

$$
u(t, \lambda, \mu) = \left(\frac{1}{\lambda - 1}t\frac{d}{dt} + \frac{\lambda + 1}{\lambda - 1}\right)u(t, \lambda + 2, \mu),
$$

with $q(\lambda) = p + (\lambda - n - 1)/2$. By using the recursion formula for $z(t, \lambda, \mu)$ we obtain

$$
u(t, \lambda, \mu) = \left(\frac{1}{\lambda - 1} t \frac{d}{dt} + \frac{\lambda + 1}{\lambda - 1}\right) u(t, \lambda + 2, \mu),
$$

 \bullet

A Lattice Problem for Differential Forms
the same equation is valid for $v(\cdot, \cdot, \cdot)$ instead of $u(\cdot, \cdot, \cdot)$. One observes that closed
or coclosed eigenforms of the Laplace operator are at the same time eigenforms of
mean the same equation is valid for $v(\cdot, \cdot, \cdot)$ instead of $u(\cdot, \cdot, \cdot)$. One observes that closed or coclosed eigenforms of the Laplace operator are at the same time eigenforms of mean value operators. To make this more precise, we state the *A* Lattice Problem for Differential Forms
 e same equation is valid for $v(\cdot, \cdot, \cdot)$ instead of $u(\cdot, \cdot, \cdot)$. One observes that cl

coclosed eigenforms of the Laplace operator are at the same time eigenform

san value $\frac{1}{2}$ and $\frac{1}{2}$ a

- *Proposition: The following mean value formulas are true:*
-
- *(i) For* $\Delta \omega = \mu \omega$ *one has* $M_{\lambda}[\omega]$ $(t, x) = z(t, \lambda 2, \mu)$ $\omega(x)$.
 (ii) For $\Delta \omega = \mu \omega$, $d\omega = 0$ *one has* $N_{\lambda}[\omega]$ $(t, x) = u(t, \lambda, \mu) \omega(x)$.
- (iii) *For* $\Delta \omega = \mu \omega$, $\delta \omega = 0$ *one has* $N_i[\omega](t, x) = v(t, \lambda, \mu) \omega(x)$.

Proof: By referring to [6: Satz 2], it is quite easy to establish the following result: (i) For $\Delta \omega = \mu \omega$, $d\omega = 0$ one has $\begin{aligned} \n\frac{du}{d\omega} &= 0 \text{ on} \\ \n\frac{du}{d\omega} &= 0 \text{ on} \\ \n\frac{du}{d\omega} &= 0 \text{ on} \\ \n\frac{du}{d\omega} &= \frac{n - p}{n} \n\end{aligned}$

e equation is valid for
$$
v(\cdot, \cdot, \cdot)
$$
 instead of $u(\cdot, \cdot, \cdot)$. One observ-
sed eigenforms of the Laplace operator are at the same time
lue operators. To make this more precise, we state the
position: The following mean value formulas are true:
 $\cdot \Delta \omega = \mu \omega$ one has $M_1[\omega]$ $(t, x) = z(t, \lambda - 2, \mu) \cdot \omega(x)$.
or $\Delta \omega = \mu \omega$, $d\omega = 0$ one has $N_1[\omega]$ $(t, x) = u(t, \lambda, \mu) \omega(x)$.
or $\Delta \omega = \mu \omega$, $\delta \omega = 0$ one has $N_1[\omega]$ $(t, x) = v(t, \lambda, \mu) \omega(x)$.
 $[$: By referring to [6: Satz 2], it is quite easy to establish the fol-
 $\Delta \omega = \mu \omega$, $d\omega = 0$ one has
 $M^c[\omega]$ $(t, x) = \frac{n - p}{n} z(t, n + 1, \mu) \omega(x)$,
 $M^c[\omega]$ $(t, x) = \left(-\frac{n - p}{n} z(t, n + 1, \mu) + z(t, n - 1, \mu)\right) \omega(x)$.

(ii) For $\Delta \omega = \mu \omega$, $\delta \omega = 0$ one has

$$
M^{i}[\omega](t, x) = \left(-\frac{\kappa - \mu}{n} z(t, n + 1, \mu) + z(t, n - 1, \mu)\right) \omega
$$

\n
$$
4\omega = \mu\omega, \delta\omega = 0 \text{ one has}
$$

\n
$$
M^{o}[\omega](t, x) = \left(-\frac{\dot{p}}{n} z(t, n + 1, \mu) + z(t, n - 1, \mu)\right) \omega(x),
$$

\n
$$
M^{i}[\omega](t, x) = \frac{p}{n} z(t, n + 1, \mu) \omega(x).
$$

\nat $p = 0$ and $d\omega = 0$ as well as $p = n$ and $\delta\omega = 0$ implies
\n
$$
d\omega = \frac{1}{2} \omega + \frac{1}{2}
$$

Note that $p=0$ and $d\omega=0$ as well as $p=n$ and $\delta\omega=0$ implies $\mu=0$. Now the proof follows straightforward from (4) by applying the following integral equation $M'[\omega](t, x) = \frac{1}{n} z(t, n + 1, \mu)$
Note that $p = 0$ and $d\omega = 0$ as well as
proof follows straightforward from (4)
for Bessel functions $(\lambda_2 \ge \lambda_1 + 2 \ge 2)$:

$$
M^{i}[\omega](t, x) = \frac{p}{n} z(t, n + 1, \mu) \omega(x).
$$

\n
$$
M^{i}[\omega](t, x) = \frac{p}{n} z(t, n + 1, \mu) \omega(x).
$$

\n
$$
y(t, x) = 0 \text{ and } d\omega = 0 \text{ as well as } p' = n \text{ and } \delta\omega = 0 \text{ implies } \mu = 0. \text{ Now}
$$

\n
$$
\text{loss straightforward from (4) by applying the following integral equation}
$$

\n
$$
z(t, \lambda_2, \mu) = \frac{2t^{1-\lambda_1}}{B\left(\frac{\lambda_1 + 1}{2}, \frac{\lambda_2 - \lambda_1}{2}\right)} \int_0^t (t^2 - \tau^2)^{\frac{\lambda_1 - \lambda_1 - 2}{2}} \tau^{\lambda_1} z(\tau, \lambda_1, \mu) d\tau
$$

Let $\mathfrak G$ be a properly discontinuous group of isometries of $\mathbf E^n$. This shall mean that for every $x \in \mathbb{E}^n$ the set of *bx* for all $b \in \mathbb{G}$ has no accumulation point. Let $\mathcal F$ be a fundamental domain, that means first that the sets $b\mathcal{F}, b \in \mathcal{B}$, cover the space Eⁿ and. secondly that $b\mathcal{F} \cap c\mathcal{F}$ with $b, c \in \mathcal{G}, b \neq c$, has Lebesgue measure 0. We suppose $\mathcal F$ to be compact. Without loss of generality, we can suppose $\mathcal F$ to be the closure of an open, connected domain. For G-automorphic differential forms α we can rewrite the ntegration as an integration over a fundamental domain \mathcal{F} : **be a properly discontinuous group of isometries of** \mathbf{E}^n **. This s** $x \in \mathbf{E}^n$ **the set of** bx **for all** $b \in \mathbf{G}$ **has no accumulation points and its domain, that means first that the sets** $b\mathcal{F}, b \in \mathbf{G}$ **, cover t** $z(t, \lambda_2, \mu) = \frac{2t}{B\left(\frac{\lambda_1 + \mu_2}{2}\right)}$

Let \bigcirc be a properly discontifor every $x \in \mathbf{E}^n$ the set of bx fundamental domain, that mead secondly that $b\mathcal{F} \cap c\mathcal{F}$ with t \mathcal{F} to be compact. Without loss *roof* $c \in \mathcal{G}, b \neq c$, has Lebesgue measure 0. We suppose

of generality, we can suppose \mathcal{F} to be the closure of an
 F-automorphic differential forms α we can rewrite the

ver a fundamental domain \mathcal{F} :
 Note that $p = 0$ and $d\omega = 0$ as well as $p = n$ and $\delta\omega = 0$ implies $\mu = 0$. Now the

proof follows straightforward from (4) by applying the following integral equation

for Bessel functions $(\lambda_2 \ge \lambda_1 + 2 \ge 2)$:
 $z(t, \lambda$

$$
M_{\lambda}[\alpha](t,x)=t^{3-\lambda}\int\limits_{\mathcal{S}}\mathcal{M}_{\lambda}(t,x,y)\cdot\alpha(y)\,dv_{y}
$$

 $\label{eq:1} \begin{split} \Delta \Sigma_{\rm{eff}} & = \frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{2} \right) \right] \left[\frac{1$

 $\label{eq:1} \begin{split} \mathbf{A}^{(1)} & = \frac{1}{2} \mathbf{1} \mathbf$

$$
\mathcal{F} \text{ to be compact. Without loss of generality, we can suppose } \mathcal{F} \text{ to be the closure of} \\ \text{open, connected domain. For } \mathbb{G} \text{-automorphic differential forms } \alpha \text{ we can rewrite} \\ \text{integration as an integration over a fundamental domain } \mathcal{F}: \\ M_i[\alpha] \ (t, x) = t^{3-i} \int \mathcal{M}_i(t, x, y) \cdot \alpha(y) \, dv_y
$$
\nwith\n
$$
\mathcal{M}_i(t, x, y) = c_2(-1)^p \sum_{\substack{b \in \mathbb{G} \\ r(x, by) < t}} \left(t^2 - r^2(x, by) \right)^{\frac{1-n-3}{2}} b^* \big(\sigma_p(x, by) + \tau_p(x, by) \big)
$$
\nand\n
$$
N_i[\alpha] \ (t, x) = t^{1-i} \int \mathcal{N}_i(t, x, y) \cdot \alpha(y) \, dv_y
$$

$$
N_1[\alpha] (t, x) = t^{1-\lambda} \int_{x}^{\theta \in \mathcal{G}} \mathcal{N}_1(t, x, y) \cdot \alpha(y) dv_y
$$

with

446 R. SCHUSTER
with

$$
\mathcal{N}_1(t, x, y) = c_2(-1)^p \sum_{\substack{r \in \Theta \\ r(x, by) < t}} \left(t^2 - r^2(x, by)\right)^{\frac{1-n-3}{2}} r^2(x, by) b^* \left(\sigma_p(x, by) - \tau_p(x, by)\right).
$$

The induced mapping *b** is to be taken with respect to the second variable of the double differential forms. Since $\mathfrak G$ was supposed to be properly discontinuous only a finite number of terms in those integral kernels do not vanish. We immediately obtain *n*) = $c_2(-1)^p \sum_{b \in \mathfrak{S}} (t^2 - r^2(x, by))^{\frac{1-n-3}{2}} r^2(x, by) b^*(\sigma_p(x, by) - \tau_p(x, by))$.
 $\tau(x, by) \leq t$

ced mapping b^* is to be taken with respect to the second variable of the

fferential forms. Since \mathfrak{G} was supposed to b

number of terms in those integral kernels do not vanish. We immediately
\n
$$
(2 - 1) t M_1(t, x, y) = \frac{d}{dt} M_{\lambda+2}(t, x, y)
$$
\n(5)
\nmalogous equation for \mathcal{N} . Using $b^* \sigma_p(x, by) = (b^{-1})^* \sigma_p(y, b^{-1}x)$ and the ana-
\nquation for τ_p we find that
\n
$$
M_1(t, x, y) = M_1(t, y, x), \qquad \mathcal{N}_1(t, x, y) = \mathcal{N}_1(t, y, x).
$$

\nof the mean value formulas, it is possible to expand the integral kernels with

and an analogous equation for $\mathcal N$. Using $b^* \sigma_p(x, by) = (b^{-1})^* \sigma_p(y, b^{-1}x)$ and the analogous equation for τ_p we find that

$$
\mathscr{M}_\lambda(t,x,y)=\mathscr{M}_\lambda(t,y,x),\qquad \mathscr{N}_\lambda(t,x,y)=\mathscr{N}_\lambda(t,y,x).
$$

In view of the mean value formulas, it is possible to expand the integral kernels with respect to the complete eigenform system $\{\omega_i^p\}_{i\in \mathbb{N}}$:

obtain
\n
$$
(\lambda - 1) t \mathcal{M}_1(t, x, y) = \frac{d}{dt} \mathcal{M}_{\lambda+2}(t, x, y)
$$
\n(5)
\nand an analogous equation for \mathcal{N} . Using $b^* \sigma_p(x, by) = (b^{-1})^* \sigma_p(y, b^{-1}x)$ and the analogous equation for τ_p we find that
\n
$$
\mathcal{M}_1(t, x, y) = \mathcal{M}_1(t, y, x), \qquad \mathcal{N}_1(t, x, y) = \mathcal{N}_1(t, y, x).
$$
\nIn view of the mean value formulas, it is possible to expand the integral kernels with respect to the complete eigenform system $\{\omega_i^p\}_{i\in \mathbb{N}}$:
\n
$$
\mathcal{M}_1(t, x, y) = \sum_{i\in \mathbb{N}} z(t, \lambda - 2, \mu_i^p) t^{1-3} \omega_i^p(x) \omega_i^p(y),
$$
\n
$$
\mathcal{N}_1(t, x, y) = \sum_{i\in \mathbb{N}} u(t, \lambda, \mu_i^p) t^{1-1} \omega_i^p(x) \omega_i^p(y)
$$
\n
$$
+ \sum_{i\in \mathbb{N}} u(t, \lambda, \mu_i^p) t^{1-1} \omega_i^p(x) \omega_i^p(y).
$$
\nwhere the sum \sum is taken over eigenvalues of closed eigenforms of Λ (\sum " for co-
\nclosed eigenforms, respectively). First one has to understand the equations (6) in
\n2-3 are over \mathcal{F} with respect to x and y by standard continuity arguments if one uses the well-
\nvalue with respect to x and y by standard continuity arguments if one uses the well-
\nknown asymptotic behaviour of the eigenforms (see [4, 8, 10])
\n
$$
\sum_{0 \le \mu_i^p \leq \varepsilon} ||\omega_i^p(x)||^2 = O(\xi^{n/2}).
$$
\nThis implies by partial summation
\n
$$
\sum_{0 \le \mu_i^p \leq \varepsilon} ||\omega_i^p(x)||^2 (\mu_i^p)^{-e} = O(\xi^{n/2-e}) \quad \text{for } \rho < n/2,
$$
\n(7)

where the sum \sum' is taken over eigenvalues of closed eigenforms of $\Lambda(\sum''$ for coclosed 'eigenforms, respectively). First one has to understand the equations (6) in L²-sense over F with respect to y. But for $\lambda > 2n + 2$ one gets that (6) is pointwise valid with respect to x and y by standard continuity arguments if one uses the wellknown asymptotic behaviour of the eigenforms (see [4, 8, 10]) of closed eigenforms of Λ (Λ)
as to understand the equation
 $> 2n + 2$ one gets that (6) is
timuity arguments if one uses
ms (see [4, 8, 10])
for $\rho < n/2$,
for $\rho > n/2$. where the sum \sum' is taken over eigenvalues of closed eigenforms of $\Lambda(\sum'')$ for co-
closed eigenforms, respectively). First one has to understand the equations (6) in
 L^2 -sense over $\mathcal F$ with respect to y . But fo

$$
\sum_{\leq \mu_i P \leq \xi} ||\omega_i^p(x)||^2 = O(\xi^{n/2}).
$$

L using
$$
U(x)
$$
 and $U(x)$ and $U(x)$ and $U(x)$ are the same as the result known asymptotic behaviour of the eigenforms (see [4, 8, 10])\n
$$
\sum_{0 \leq \mu_i P \leq \xi} ||\omega_i^p(x)||^2 = O(\xi^{n/2}).
$$
\nThis implies by partial summation\n
$$
\sum_{0 \leq \mu_i P \leq \xi} ||\omega_i^p(x)||^2 (\mu_i^p)^{-e} = O(\xi^{n/2-e}) \quad \text{for } e < n/2,
$$
\n
$$
\sum_{\xi < \mu_i P} ||\omega_i^p(x)||^2 (\mu_i^p)^{-e} = O(\xi^{n/2-e}) \quad \text{for } e > n/2.
$$
\n(7\n
$$
\sum_{\xi < \mu_i P} ||\omega_i^p(x)||^2 (\mu_i^p)^{-e} = O(\xi^{n/2-e}) \quad \text{for } e > n/2.
$$
\n(8\nTurther on one has to use\n
$$
|z(t, \lambda, \mu)| \leq c_3 t^{-\lambda/2} \mu^{-\lambda/4} \quad \text{for } \mu > x > 0, \lambda \geq 1,
$$
\n(c₃ of coivre not depending on t and μ , see [8].\n(8\n3. Proof of the theorem

$$
\sum_{\xi < \mu_i p} \|\omega_i^p(x)\|^2 (\mu_i^p)^{-\varrho} = O(\xi^{n/2 - \varrho}) \quad \text{for } \varrho > n/2.
$$
\non one has to use

\n
$$
|z(t, \lambda, \mu)| \leq c_3 t^{-\lambda/2} \mu^{-\lambda/4} \quad \text{for } \mu > x > 0, \lambda \geq 1,
$$

1.

3. Proof of the theorem

 $|z(t, \lambda, \mu)| \le c_3 t^{-\lambda/2} \mu^{-\lambda/4}$ for $\mu > \kappa > 0, \lambda \ge 1$, (8)
 c_3 of course not depending on t and μ , see [8].

3. **Proof of the theorem**

On account of the kernel expansion above, the asymptotic behaviour of M_{λ} On account of the kernel expansion above, the asymptotic behaviour of \mathcal{M}_{λ} , \mathcal{N}_{λ} is quite clear for λ large enough. We now want to extract information about the case $|z(t, \lambda, \mu)| \leq c_3 t^{-\lambda/2} \mu^{-1/4}$ for $\mu > \kappa > 0, \lambda \geq 1$, (8)

course not depending on t and μ , see [8].

coof of the theorem

ccount of the kernel expansion above, the asymptotic behaviour of \mathcal{M}_λ , \mathcal{N}_λ is Further on one has to use
 $|z(t, \lambda, \mu)| \le c_3 t^{-\lambda/2} \mu^{-1/4}$ for $\mu > \varkappa > 0, \lambda \ge 1$,
 c_3 of course not depending on t and μ , see [8].

3. Proof of the theorem

On account of the kernel expansion above, the asymptotic

break up the kernels into two parts: ne k
.
....

A Lattice Problem for Differential Forms
\nbreak up the kernels into two parts:
\n
$$
\mathcal{H}_i(t, x, y) = t^{1-3} \sum_{i=1}^{B_p} \omega_i^p(x) \omega_i^p(y),
$$
\n
$$
\mathcal{R}_i(t, x, y) = \mathcal{A}_i(t, x, y) - \mathcal{H}_i(t, x, y),
$$
\n
$$
\mathcal{H}_i'(t, x, y) = \left(1 - 2 \frac{q(\lambda)}{\lambda - 1}\right) t^{\lambda - 1} \sum_{i=1}^{B_p} \omega_i^p(x) \omega_i^p(y),
$$
\n
$$
\mathcal{R}_i'(t, x, y) = \mathcal{N}_i(t, x, y) - \mathcal{H}_i'(t, x, y) \quad \text{with} \quad q(\lambda) = p + \frac{\lambda - n - 1}{2},
$$
\n
$$
\mathcal{H}_i \text{ and } \mathcal{H}_i' \text{ are the leading terms of } \mathcal{A}_i \text{ and } \mathcal{N}_i \text{, respectively. We will give estimates for } \mathcal{R}_i \text{ and } \mathcal{R}_i'. \text{ Next we define a difference operator for a mapping } f
$$

 \mathcal{X}_λ and \mathcal{X}_λ' are the leading terms of \mathcal{M}_λ and \mathcal{N}_λ , respectively. We will give error estimates for \mathcal{R}_1 and \mathcal{R}_1 . Next we define a difference operator for a mapping f from R into an arbitrary vector space by $\overline{\textbf{R}}$ into
 $\text{see} \overline{\textbf{R}}$ $\mathcal{R}_1'(t, x, y) = \mathcal{N}_1(t, x, y) - \mathcal{H}_1'(t, x, y)$ with $q(\lambda)$

od \mathcal{H}_1' are the leading terms of \mathcal{M}_1 and \mathcal{N}_1 , respect

tets for \mathcal{R}_1 and \mathcal{R}_1' . Next we define a difference opera

o an arbitrary vect

$$
\nabla_m f(\xi) = \sum_{\nu=0}^m \binom{m}{\nu} (-1)^{m-\nu} f(\xi + \nu \eta) \text{ with } \eta = \xi^a \text{ for } m \in \mathbb{N}, a \in (0, 1),
$$

see [9]. For convenience we transform $\xi = t^2/2$ and write this as

$$
\bar{\mathcal{M}}_1(\xi, x, y) = \mathcal{M}_1(t, x, y), \bar{\mathcal{N}}_1(\xi, x, y) = \mathcal{N}_1(t, x, y)
$$

and so on. Combining this with (5), we get

$$
\overline{\mathcal{M}}_{1,1}(s, x, y) = \overline{\mathcal{M}}_{1,1}(s, x, y), \overline{\mathcal{N}}_{1,1}(s, x, y) = \overline{\mathcal{N}}_{1,1}(s, x, y)
$$
\nand so on. Combining this with (5), we get\n
$$
\overline{\mathcal{M}}_{n+3+2m}(\xi) = c_4 \int_0^{\xi} \int_0^{n_m} \int_0^n \cdots \int_0^n \overline{\mathcal{M}}_{n+3}(\eta_1) d\eta_1 \cdots d\eta_m
$$
\nwith a constant c_4 depending on m and n , we have on
\nThis formula is also valid for $\mathcal{H}, \mathcal{R}, \mathcal{N}, \mathcal{H}', \mathcal{R}'$ instead\n
$$
\nabla_m \overline{\mathcal{M}}_{n+3+2m}(\xi) = c_4 \int_0^{\xi+n} \int_0^{n_m+n} \cdots \int_0^n \overline{\mathcal{M}}_{n+3}(\eta_1) d\eta_1 \cdots
$$
\nUsing the above decomposition, we find that

with a constant *c₄* depending on *m* and *n*, we have omitted the arguments *x* and *y*.
This formula is also valid for *H*, *R*, *H*, *H'*, *R'* instead of *M*. We deduce that
 $\sum_{i=1}^{k+n} \frac{n_i + n}{n_i + n_{i-1}}$ \cdots $\int_{\$ This formula is also valid for $\mathcal{H}, \mathcal{R}, \mathcal{N}, \mathcal{H}', \mathcal{N}'$ instead of \mathcal{M} . We deduce that

$$
\nabla_m \widetilde{\mathcal{M}}_{n+3+2m}(\xi) = c_4 \int\limits_{\xi}^{\xi+\eta} \int\limits_{\eta_m}^{\eta_m+\eta} \cdots \int\limits_{\eta_1}^{\eta_1+\eta} \widetilde{\mathcal{M}}_{n+3}(\eta_1) d\eta_1 \ldots d\eta_m.
$$

Using the above decomposition, we find that

$$
\bar{M}_{n+3+2m}(\xi) = c_4 \int_0^{\xi} \int_0^{n_m} \cdots \int_0^{n} \bar{M}_{n+3}(\eta_1) d\eta_1 \dots d\eta_m
$$
\nwith a constant c_4 depending on m and n , we have omitted the arguments x and y .
\nThis formula is also valid for $\mathcal{H}, \mathcal{R}, \mathcal{N}, \mathcal{H}', \mathcal{R}'$ instead of \mathcal{M} . We deduce that\n
$$
\nabla_m \bar{M}_{n+3+2m}(\xi) = c_4 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_m^{\eta_1} \bar{M}_{n+3}(\eta_1) d\eta_1 \dots d\eta_m.
$$
\nUsing the above decomposition, we find that\n
$$
c_5 \eta^m \bar{\mathcal{R}}_{n+3}(\xi) = \nabla_m \bar{\mathcal{R}}_{n+3+2m}(\xi)
$$
\n
$$
+ c_5 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_n^{\xi+n} \left(\bar{\mathcal{H}}_{n+3}(\eta_1) - \bar{\mathcal{H}}_{n+3}(\xi) \right) d\eta_1 \dots d\eta_m
$$
\n
$$
+ c_5 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_n^{\eta_1+n} \left(\bar{\mathcal{H}}_{n+3}(\eta_1) - \bar{\mathcal{H}}_{n+3}(\xi) \right) d\eta_1 \dots d\eta_m
$$
\n
$$
+ c_5 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_n^{\eta_1+n} \left(\bar{\mathcal{M}}_{n+3}(\eta_1) - \bar{\mathcal{M}}_{n+3}(\xi) \right) d\eta_1 \dots d\eta_m
$$
\n
$$
+ c_5 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_n^{\eta_1+n} \left(\bar{\mathcal{M}}_{n+3}(\eta_1) - \bar{\mathcal{M}}_{n+3}(\xi) \right) d\eta_1 \dots d\eta_m
$$
\n
$$
+ c_5 \int_0^{\xi+n} \int_0^{n_m+1} \cdots \int_n
$$

with a constant c_5 . This formula is also true if we replace M, \mathcal{H}, \mathcal{H} by N, $\mathcal{H}', \mathcal{H}', \mathcal{H}''$ respectively. The next point on the agenda is to obtain estimates for the right-hand side. To do this, we could use an a-priori estimate for the integrand of the last term. But an easier way is given by applying the known result for $p = 0$, see [8]. In [8] G was supposed to be without fixed points, with the exception of *id,* but it is obvious how to generalize the argumentation to our case. If we want to express the dependence of the kernel forms on the degree p , we write \mathcal{M}_{λ} ^p and so on. We recall that the norm $\|\varphi\|$ of a double differential form with a conservered in the separation of the kern and φ
and φ and φ and φ + c_s $\int_{\epsilon} \int_{\eta_m} \cdots \int_{\eta_n} \left(\overline{\mathscr{H}}_{n+3}(\eta_1) - \overline{\mathscr{H}}_{n+3}(\xi) \right) d\eta_1 \ldots d\eta_m$
 $\qquad \qquad \epsilon + \eta \frac{1}{\eta_m} + \eta \frac{1}{\eta_1} + \eta \frac{1}{\eta_2} + \eta \frac{1}{\eta_3} + \eta \frac{1}{\eta_4} + \eta \frac{1}{\eta_5} + \eta \frac{1}{\eta_6} + \eta \frac{1}{\eta_7} + \eta \frac{1}{\eta_8} + \eta \frac{1$

$$
\varphi = \varphi_{i_1 \cdots i_n j_1 \cdots j_n} dx^{i_1} \wedge \cdots \wedge dx^{i_p} dy^{j_1} \hat{\wedge} \cdots \hat{\wedge} dy^{j_p}
$$

$$
\begin{aligned} \text{is. scalars are} \\ \text{where} \\ \|\varphi\|^2 &= (p!)^2 \, \varphi_{i_1\cdots i_p j_1\cdots j_p} \, \varphi^{i_1\cdots i_p j_1\cdots j_p}, \end{aligned}
$$

448 R. SCHUSTER

can be given by
 $\|\varphi\|^2 = (p!)^2 \varphi_{i_1\cdots i_p j_1\cdots j_p} \varphi^{i_1\cdots i_p j_1\cdots j_p},$

see [4]. After a short computation, we see that the coefficients of σ_p see [4]. After a short computation, we see that the coefficients of $\sigma_p(x, y)$ and $\tau_p(x, y)'$ are bounded. From this we conclude that $\|\sigma_p(x, y)\| \leq c_6$, $\|\tau_p(x, y)\| \leq c_6$ with a constant c_6 . Using (9) twice (a second time for $p = 0$), taking the norms and combining this with the estimate ²¹
 A see that the coefficients of $\sigma_p(x, y)$

that $\|\sigma_p(x, y)\| \le c_6$, $\|\tau_p(x, y)\| \le c_6$

ae for $p = 0$), taking the norms and
 $\widehat{\mathcal{M}}_{n+3}^0(\eta_1) - \widehat{\mathcal{M}}_{n+3}^0(\xi)\|$,

$$
\|\tilde{\mathscr{M}}_{n+3}^p(\eta_1)-\tilde{\mathscr{M}}_{n+3}^p(\xi)\|\leqq c_6\,\|\tilde{\mathscr{M}}_{n+3}^0(\eta_1)-\tilde{\mathscr{M}}_{n+3}^0(\xi)\|.
$$

we get the inequality

II+3()II ' + IVm+3+2m()II ± IIVm+3+2m(4)II .+ *f I f* IP°+3(7i) **—** "fl43()II *di ¹dilm* E ,, **-+** *ff* •.. *I.* IP°+3(?li) — **0f3()** II *di 1 dim).* (10) we-have **fl 3()j** ^ for ^ analogous. - 3rd *summand*-*:* We break up the series- - - -

From [4], it is apparent how to bring $\|\cdot\|$ under the integral sign. According to [8] $\frac{n}{2} - \frac{n}{n+1}$

$$
\|\overline{\mathscr{R}}_{n+3}^0(\xi)\| \leq c_7 \xi^{\frac{n}{2} - \frac{n}{n+1}} \qquad \text{for } \xi \geq \xi_0,
$$
\n(11)

 $\xi_0 > 0$ arbitrary small. We consider the 3rd and 4th summand of the right-hand side, of (10). Choosing $p = 0$ we get the corresponding result for the 2^{nd} and 5^{th} summand. We take up the case of-even *n* and set $m = n/2$. The considerations for odd *n* are analogous. parent how to bring $\|\cdot\|$ under the integral sign. Acco

parent how to bring $\|\cdot\|$ under the integral sign. Acco
 $|\leq c_7 \xi^{\frac{n}{2} - \frac{n}{n+1}}$ for $\xi \geq \xi_0$,

mall. We consider the 3rd and 4th summand of the right of (10). Choosing $p =$
We take up the case
analogous.
 $3^{rd} summand:$ We
 $\overline{\mathscr{R}}_{2n+3}^{p}(\xi, x, \xi)$
into two parts
 $\mathscr{Q}_1 = \sum_{n=1}^{\infty}$ $\|\overline{\mathscr{R}}_{n+3}^{0}(\xi)\| \leq c_7 \xi^{\frac{n}{2} - \frac{n}{n}}$

(1) arbitrary small. We do the case of every small of the case of every goods.
 summand: We break using $\overline{\mathscr{R}}_{2n+3}^{p}(\xi, x, y) = \sum_{\mu_i > 0}$

wo parts
 $Q_1 = \sum_{\sigma \in \mu_i > c$ *2*_{*n*</sup> $\frac{n}{x+1}$ for $\xi \ge \xi_0$,
 2n consider the 3rd and 4th summand of the right-
 2nd and 5th *s* and 4th summand of the right-
 x n n and set $m = n/2$. The considerations for *c*
 p the series
 } $\|\overline{\mathcal{R}}_{n+3}^0(\xi)\| \leq c_7 \xi^{\frac{n}{2} - \frac{n}{n+1}}$ for $\xi \geq \xi_0$,
bitrary small. We consider the 3rd and 4th summand of the r
hoosing $p = 0$ we get the corresponding result for the 2nd and
up the case of even *n* and $\xi_0 > 0$ arbitrary small. We consider the 3rd
of (10). Choosing $p = 0$ we get the correspon
We take up the case of even *n* and set *m* =
analogous.
3rd summand: We break up the series
 $\overline{\mathscr{R}}_{2n+3}^n(\xi, x, y) = \sum_{\mu$

$$
\overline{\mathscr{R}}_{2n+3}^{p}(\xi,x,y)=\sum_{\mu,\nu>0}\xi^{n}2^{n}\overline{z}(\xi,2n+1,\mu_{i}\nu)\omega_{i}\nu(x)\omega_{i}\nu(y)
$$

$$
Q_1 = \sum_{0 \le \mu_i P \le \xi^b} \xi^{n} 2^{n} \bar{z}(\xi, 2n + 1, \mu_i P) \omega_i^p(x) \omega_i^p(y),
$$

\n
$$
Q_2 = \sum_{\xi^b \le \mu_i P} \xi^{n} 2^{n} \bar{z}(\xi, 2n + 1, \mu_i P) \omega_i^p(x) \omega_i^p(y)
$$

\nwith a constant $b > 0$ which we may choose later. We estimate Q_1 with the aid of
\n(8) with $\lambda = 2n + 1$:

rbitrary small. We consider the 3rd and 4th summand of the right-hand side
\nchoosing
$$
p = 0
$$
 we get the corresponding result for the 2nd and 5th summand.
\nup the case of even *n* and set $m = n/2$. The considerations for odd *n* are
\ns.
\n*mmand*: We break up the series
\n
$$
\overline{\mathscr{R}}_{2n+3}^{p}(\xi, x, y) = \sum_{\mu_i p > 0} \xi^{n} 2^{n} \overline{z}(\xi, 2n + 1, \mu_i p) \omega_i p(x) \omega_i p(y)
$$
\nparts
\n
$$
Q_1 = \sum_{\mu_i p > 0} \xi^{n} 2^{n} \overline{z}(\xi, 2n + 1, \mu_i p) \omega_i p(x) \omega_i p(y),
$$
\n
$$
Q_2 = \sum_{\ell_1, \ell_2, \ell_1} \xi^{n} 2^{n} \overline{z}(\xi, 2n + 1, \mu_i p) \omega_i p(x) \omega_i p(y).
$$
\nconstant $b > 0$ which we may choose later. We estimate Q_1 with the aid of
\n $\lambda = 2n + 1$:
\n
$$
|\nabla_m \{\xi^{n} \overline{z}(\xi, 2n + 1, \mu_i p)\}| \leq \sum_{r=0}^{m} {m \choose r} |(\xi + r\eta)^n \overline{z}(\xi + r\eta, 2n + 1, \mu_i p)|
$$
\n
$$
\leq c_8(\mu_i p)^{-\frac{2n+1}{4}} \xi^{\frac{2n-1}{4}}.
$$
\n(12)
\nthere hand, we will use (8) and the law of mean of the differential calculus in
\nprepare the estimation of Q_2 :

On the other hand, we will use (8) and the law of mean of the differential calculus in order to prepare the estimation of Q_2 :

$$
Q_1 = \sum_{0 \le \mu_i \ge \epsilon} \xi^n 2^n \bar{z}(\xi, 2n + 1, \mu_i^p) \omega_i^p(x) \omega_i^p(y),
$$
\n
$$
Q_2 = \sum_{\epsilon \nu_i \ge \mu_i^p} \xi^n 2^n \bar{z}(\xi, 2n + 1, \mu_i^p) \omega_i^p(x) \omega_i^p(y),
$$
\nwith a constant $b > 0$ which we may choose later. We estimate Q_1 with the aid of
\n(8) with $\lambda = 2n + 1$:
\n
$$
|\nabla_m \{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}| \le \sum_{r=0}^m \binom{m}{r} |(\xi + r\eta)^n \bar{z}(\xi + r\eta, 2n + 1, \mu_i^p)|
$$
\n
$$
\le c_8(\mu_i^p)^{-\frac{2n+1}{4}} \xi^{\frac{2n-1}{4}}.
$$
\n(12)
\nOn the other hand, we will use (8) and the law of mean of the differential calculus in order to prepare the estimation of Q_2 :
\n
$$
|\nabla_m \{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}| \le \eta^m \left| \frac{d^m}{d\xi^m} \{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}_{\xi = \bar{t}} \right|
$$
\n
$$
\le c_9 \eta^m |\xi^{n/2} \bar{z}(\xi, n + 1, \mu_i^p)_{\xi = \bar{t}}| \le c_{10} \eta^m (\mu_i^p)^{-\frac{n+1}{4}} \xi^{\frac{n-1}{4}}.
$$
\n(13)

A Lattice Problem for Differential Forms 449

A Lattice Problem for Differential Forms 449
 $h \xi \in (\xi, \xi + m\eta)$. Here we have used (8) with $\lambda = n + 1$. Using the estimates (7), with $\bar{\xi} \in (\xi, \xi + mn)$. Here we have used (8) with $\lambda = n + 1$. Using the estimates (7), (12) and (13) we see that

A Lattice Problem for Differential Forms
\nwith
$$
\xi \in (\xi, \xi + m\eta)
$$
. Here we have used (8) with $\lambda = n + 1$. Using the estimates (7),
\n(12) and (13) we see that
\n
$$
||\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)|| \leq c_{11} \left(\xi^{\frac{2n-1}{4} - \frac{b}{4}} + \xi^{\frac{n}{2}a + \frac{n-1}{4} + \frac{n-1}{4}b} \right).
$$
\n(14)
\nWe choose *b* optimally by minimizing the right-hand side of (14), we find that
\n $b = 1 - 2a$. Inserting this in (14) gives
\n
$$
||\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)|| \leq c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}.
$$
\n(15)
\n4th summand: Using the estimate $\|\sum_{n=0}^{\infty} \omega_i^p(x) \omega_i^p(y)\| \leq c_{13}$ we get

We choose *b* optimally by minimizing the right-hand side of (14), we find that $b = 1 - 2a$. Inserting this in (14) gives **4th summand:** Using the estimate $\| \cdot \|_{\mu}$ \leq $c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}$.
 4 th summand: Using the estimate $\| \sum_{\mu} c_{\mu} \varphi(\mu) \omega_{\mu} P(\mu) \| \leq c_{13}$ we get

$$
\|\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)\| \leq c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}.
$$
\n(15)

with
$$
\xi \in (\xi, \xi + m\eta)
$$
. Here we have used (8) with $\lambda = n + 1$. Using the estimates
\n(12) and (13) we see that
\n
$$
||\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)|| \leq c_{11} \left(\frac{2n-1}{\xi - 4} - \frac{b}{4} + \frac{n}{\xi} x + \frac{n-1}{4} + \frac{n-1}{4} b \right)
$$
\n(We choose *b* optimally by minimizing the right-hand side of (14), we find the
\n $b = 1 - 2a$. Inserting this in (14) gives
\n
$$
||\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)|| \leq c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}.
$$
\n(4th summand: Using the estimate $\|\sum_{\mu_i=0} \omega_i^p(x) \omega_i^p(y)\| \leq c_{13}$ we get
\n
$$
\sum_{i=1}^{i+n} \sum_{j=1}^{n-i} \cdots \int_{n} \|\overline{\mathcal{R}}_{n+3}^p(\eta_1) - \overline{\mathcal{R}}_{n+3}^p(\xi)\| d\eta_1 \cdots d\eta_m \leq c_{14} \xi^{\frac{n+2}{2} + \frac{n-2}{2}}.
$$
\nNext we choose *a* optimally by minimizing the sum of the 3rd and 4th summand
\n(10): we find that $a = 1/(n + 1)$. Combining this with the corresponding result
\n $p = 0$ and (11), we get
\n
$$
||\overline{\mathcal{R}}_{n+3}^p(\xi, x, y)|| \leq c_{14} \xi^{\frac{n}{2} - \frac{n}{n+1}}
$$
 for $\xi \geq \xi_0$.
\nWith small changes the arguments above give the estimate for $\overline{\mathcal{R}}_{n+3}^r$. Since \mathcal{R}_n^0 .

Next we choose a optimally by minimizing the sum of the 3rd and 4th summand of (10): we find that $a = 1/(n + 1)$. Combining this with the corresponding result for 4th summand: Using the estimate $\left\|\sum_{\mu_i p=0} \omega_i^p(x) \omega_i^p(y)\right\| \leq c_{13}$ we get
 $\int_{i}^{\frac{p+2}{\epsilon+n}} \int_{m}^{m+n} \int_{i}^{n+1} \left\|\overline{\mathscr{H}}_{n+3}^p(\eta_1) - \overline{\mathscr{H}}_{n+3}^p(\xi)\right\| d\eta_1 \cdots d\eta_m \leq c_{14} \xi^{\frac{n+2}{2}d + \frac{n-2}{2}}$.

Next we c We choose *b* optimally by minimizing the right
 $b = 1 - 2a$. Inserting this in (14) gives
 $\|\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)\| \leq c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}$.
 $4^{\text{th}} summand:$ Using the estimate $\left\|\sum_{\mu_i p=0} \omega_i^p(x)\right\|_2^{\xi + \eta} \sum$

$$
\|\overline{\mathscr{R}}_{n+3}^p(\xi,x,y)\| \leq c_{14} \xi^{\frac{n}{2}-\frac{n}{n+1}}, \quad \text{for } \xi \geq \xi_0.
$$
 (16)

With small changes the arguments above give the estimate for $\bar{\mathscr{R}}'_{n+3}$. Since $\mathscr{R}^0_{n+3}(t)$ with $\xi \in (\xi, \xi + m\eta)$. Here we have used (8) with $\lambda = n + 1$. Using the (12) and (13) we see that
 $||\nabla_m \overline{\mathcal{A}}_{2n+3}^p(\xi, x, y)|| \leq c_{11} \left(\frac{2^{n-1}}{\xi} - \frac{2}{4} + \xi^{\frac{n}{2} + \frac{1}{\xi} - \frac{1}{4} - \frac{1}{4}}\right)$.

We choose b op $t^{-2} \int r^2 d\mathcal{R}_{n+3}^0(r)$ we obtain from (11) *f* f^{n} π $\frac{1}{2}$
respond
r $\overline{\mathscr{R}}'_{n+3}$ $|t_3(\xi, x, y)| \leq c_{14} \xi^2 \frac{n+1}{n+1}$
hanges the arguments a
 $\frac{0}{n+3}(r)$ we obtain from (1
 $t_3(\xi)$ $\leq c_7 \xi^{\frac{n+2}{2} - \frac{n}{n+1}}$
with (12) we get imizing the sum of
mbining this with
for $\xi \geq \xi_0$.
bove give the esting
1)
for $\xi \geq \xi_0$. Next we choose *a* optimally by minimizing the (10): we find that $a = 1/(n + 1)$. Combining the $p = 0$ and (11), we get $\|\overline{\mathscr{R}}_{n+3}^p(\xi, x, y)\| \leq c_{14} \xi^{\frac{n}{2} - \frac{n}{n+1}}$ for $\xi \geq 0$
With small changes the arguments ab

$$
\overline{\mathscr{R}}'_{n+3}(\xi)\| \leq c_7 \overline{\zeta^{\frac{n+2}{2}-\frac{n}{n+1}}} \qquad \text{for } \xi \geq \xi_0. \tag{11}'
$$

--

By analogy with (12) we get

$$
= t^{-2} \int_{0}^{t} r^{2} d\mathcal{R}_{n+3}^{0}(r)
$$
 we obtain from (11)
\n
$$
\overline{\mathscr{R}}'_{n+3}(\xi) || \leq c_{7}^{\frac{n+2}{2} - \frac{n}{n+1}} \quad \text{for } \xi \geq \xi_{0}.
$$
\n(11)
\nBy analogy with (12) we get
\n
$$
|\nabla_{m} \{\xi^{n+1} \overline{u}(\xi, 2n+3, \mu)|
$$
\n
$$
\leq |\nabla_{m} \{\xi^{n+1} \overline{z}(\xi, 2n+3, \mu)|\} \geq \frac{|2n+2-q(2n+3)|}{2n+2}
$$
\n
$$
+ |\nabla_{m} \{\xi^{n+1} \overline{z}(\xi, 2n+1, \mu)|
$$
\n
$$
\leq c_{8}^{\prime} \left(\frac{2n+1}{\xi} - \frac{2n+3}{4} + \frac{2n+3}{4} - \frac{2n+1}{4}\right)
$$
\nand
\n
$$
|\nabla_{m} \{\xi^{n+1} \overline{v}(\xi, 2n+3, \mu)|\} \leq c_{8}^{\prime} \left(\frac{2n+1}{\xi} - \frac{2n+3}{4} + \frac{2n+3}{4} - \frac{2n+1}{4}\right).
$$
\n(12)[′]
\nFrom the recursion formulas for $z(\cdot, \cdot, \cdot)$ and $u(\cdot, \cdot, \cdot)$ we deduce, that $(2\xi)^{(2-3)/2}$
\n $\times \overline{z}(\xi, \lambda - 2, \mu)$ and $(2\xi)^{(4-1)/2} \overline{u}(\xi, \lambda, \mu)$ satisfy the same recursion formula

 $\begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}$

•

and
\n
$$
|\nabla_m{\xi^{n+1}\bar{v}(\xi, 2n+3, \mu)}| \leq c_8'' \left(\frac{2n+1}{\xi} \mu^{-\frac{2n+3}{4}} + \frac{2n+3}{\xi} \mu^{-\frac{2n+1}{4}}\right).
$$
\n(12)''
\nFrom the recursion formulas for $z(\cdot, \cdot, \cdot)$ and $u(\cdot, \cdot, \cdot)$ we deduce, that $(2\xi)^{(2-3)/2}$
\n $\times \bar{z}(\xi, \lambda - 2, \mu)$ and $(2\xi)^{(1-1)/2} \bar{u}(\xi, \lambda, \mu)$ satisfy the same recursion formula
\n
$$
\frac{d}{d\xi} \{(2\xi)^{(1+1)/2} \bar{u}(\xi, \lambda + 2, \mu)\} = (\lambda - 1) \{(2\xi)^{(1-1)/2} \bar{u}(\xi, \lambda, \mu)\}.
$$
\n29 Analysis Bd. 7, Hett 3 (1988)

 λ , μ) satisfy the same recursion formula

$$
e_1 \cos \theta + \cos \theta + \cos \theta + \cos \theta + \cos \theta
$$

\n
$$
f(x, y, z) = \cos \theta + \cos \theta + \cos \theta
$$

\n
$$
f(x, y, z) = \cos \theta + \cos \theta
$$

\n
$$
f(x, y, z) = \sin \theta + \cos \theta
$$

\n
$$
f(x, y, z) = \sin \theta + \cos \theta
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$$
f(x, y, z) = \sin \theta + \cos \theta
$$

/
*t*ely connec This property is intimately connected with the equations (6). So by analogy with (13) we get 450
This
we go
 $\frac{1}{2}$

450 R. SCHUSTER
\nThis property is intimately connected with the equations (6). So by analogy with (13)
\nwe get
\n
$$
|\nabla_m \{\xi^{n+1} \overline{u}(\xi, 2n+3, \mu)\}| \leq \eta^m \left| \frac{d^m}{d\xi^m} \left\{ \xi^{n+1} \overline{u}(\xi, 2n+3, \mu) \right\} \right|_{|\xi = \overline{\xi}}
$$
\n
$$
\leq c_9' \eta^m \left| \xi^{\frac{n+2}{2}} \overline{u}(\xi, n+3, \mu) \right|_{|\xi = \overline{\xi}}
$$
\nand an analogous equation for \overline{v} . These equations imply by analogy with (14)
\n
$$
||\nabla_m \overline{\mathcal{R}}'_{2n+3}(\xi, x, y)|| \leq c_{11} \left(\xi^{\frac{2n+3}{4}} + \xi^{\frac{2n+1}{4}} + \xi^{\frac{2n+1}{4}} \right)
$$
\n
$$
+ \xi^{\frac{n}{2}a + \frac{n+1}{4} + b\frac{n-3}{4}} + \xi^{\frac{n}{2}a + \frac{n+3}{4} + b\frac{n-1}{4}} \right)
$$
\n(13)

and an analogous equation for \bar{v} . These equations imply by analogy with (14)

and an analogous equation for
$$
\bar{v}
$$
. These equations imply by analogy
\nand an analogous equation for \bar{v} . These equations imply by analogy
\n
$$
\|\nabla_m \overline{\mathscr{R}}_{2n+3}^{\prime p}(\xi, x, y)\| \leq c_{11} \left(\xi^{\frac{2n+3}{4} - \frac{b}{4}} + \xi^{\frac{2n+1}{4} - \frac{3b}{4}} \right)
$$
\nIf we set $b = 1 - 2a$, we obtain $(0 < a < 1)$
\n
$$
\|\nabla_m \overline{\mathscr{R}}_{2n+3}^{\prime p}(\xi, x, y)\| \leq O\left(\xi^{\frac{n-1}{2} + \frac{3}{2}a} \right) + \xi^{\frac{n}{2}a + \frac{n+3}{4} + b\frac{n-1}{4}}.
$$
\nIf we set $b = 1 - 2a$, we obtain $(0 < a < 1)$
\n
$$
\|\nabla_m \overline{\mathscr{R}}_{2n+3}^{\prime p}(\xi, x, y)\| = O\left(\xi^{\frac{n-1}{2} + \frac{3}{2}a} \right) + O\left(\xi^{\frac{n+1}{2} + \frac{a}{2}} \right) = O\left(\xi^{\frac{n+1}{2} + \frac{a}{2}} \right)
$$
\nFor the 4th summand we get
\n
$$
\int_{\xi}^{\xi + \eta} \int_{\eta_m}^{\eta_m + \eta} \int_{\eta}^{\eta_1 + \eta} |\overline{\mathscr{K}}_{\eta+3}^{\prime p}(\eta_1) - \overline{\mathscr{K}}_{\eta+3}^{\prime p}(\xi)| d\eta_1 \cdots d\eta_m = O\left(\xi^{\frac{n+2}{2}a + \frac{a}{2}} \right).
$$
\nIf we use $a = 1$:\n $\int_{\eta}^{\eta} \int_{\eta}^{\eta} \cdots \int_{\eta}^{\eta} |\overline{\mathscr{K}}_{\eta+3}^{\prime p}(\eta_1) - \overline{\mathscr{K}}_{\eta+3}^{\prime p}(\xi)| d\eta_1 \cdots d\eta_m = O\left(\xi^{\frac{n+2}{2}a + \frac{a}{2}} \right).$

If we set
$$
b = 1 - 2a
$$
, we obtain $(0 < a < 1)$
\n
$$
\|\nabla_m \overline{\mathcal{R}}_{2n+3}^{\prime p}(\xi, x, y)\| = O\left(\xi^{\frac{n-1}{2} + \frac{3}{2}a}\right) + O\left(\xi^{\frac{n+1}{2} + \frac{a}{2}}\right) = O\left(\xi^{\frac{n+1}{2} + \frac{a}{2}}\right)
$$
\nFor the 4th summand we get

$$
+\frac{n}{2}a+\frac{n+1}{4}+b\frac{n-3}{4}+\frac{n}{5}a+\frac{n+3}{4}+b\frac{n-1}{4})
$$

If we set $b = 1 - 2a$, we obtain $(0 < a < 1)$

$$
\|\nabla_m \overline{\mathcal{R}}_{2n+3}^p(\xi, x, y)\| = O\left(\frac{n-1}{2} + \frac{3}{2}a\right) + O\left(\frac{n+1}{2} + \frac{a}{2}\right) = O\left(\frac{n+1}{2} + \frac{a}{2}\right).
$$

For the 4th summand we get

$$
\int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \int_{\eta_3}^{\eta_2+\eta} \|\overline{\mathcal{H}}_{n+3}^p(\eta_1) - \overline{\mathcal{H}}_{n+3}^p(\xi)\| d\eta_1 \cdots d\eta_m = O\left(\frac{n+2}{2}a+\frac{n}{2}\right).
$$

If we use $a = 1/(n + 1)$, we get

$$
\|\mathcal{R}_{n+3}^p(\xi, x, y)\| = O\left(\frac{n}{2} - \frac{n}{n+1}\right) \quad \text{for } \xi \ge \xi_0
$$

$$
\text{by analogy with (16).}
$$

We are interested in

$$
\Gamma\left(\frac{n+2}{2}\right)
$$

If we use $a = 1/(n + 1)$, we get

$$
\|\mathcal{R}_{n+3}^p(\xi, x, y)\| = O\left(\zeta^{\frac{n}{2} - \frac{n}{n+1}}\right) \quad \text{for } \xi \ge \xi
$$

$$
\|\nabla_m \overline{\mathscr{R}}_{2n+3}^{p}(s, x, y)\| = O\left(\frac{n-1}{2} + \frac{3}{2}a\right) + O\left(\frac{n+1}{2} + \frac{a}{2}\right) = O\left(\frac{n+1}{2} + \frac{a}{2}\right).
$$

For the 4th summand we get

$$
\int_{\xi}^{t+\eta} \int_{\eta_m + \eta}^{\eta_m + \eta} \frac{n+\eta}{\eta} \frac{1}{\|\mathscr{H}_{n+3}^{p}(n, 1) - \mathscr{H}_{n+3}^{p}(s)\|} d\eta_1 \cdots d\eta_m = O\left(\frac{n+2}{2} + \frac{n}{2}\right).
$$
If we use $a = 1/(n + 1)$, we get

$$
\|\mathscr{H}_{n+3}^{p}(s, x, y)\| = O\left(\frac{n}{2} - \frac{n}{n+1}\right) \quad \text{for } \xi \geq \xi_0
$$

by analogy with (16).
We are interested in

$$
\mathscr{K}_{n+3}^{p}(t, x, y) = (-1)^p \frac{\left(\frac{n+2}{2}\right)}{\frac{n^{n/2}}{n^{n/2}} \sum_{\substack{b \in \mathbb{Q} \\ b \in \mathbb{Q} \\ b}} b^*(\sigma_p(x, by) - \tau_p(x, by)).
$$

Using $\mathscr{N}_{n+3}(t, x, y) = \mathscr{K}_{n+3}^{p}(t_0, x, y) + \int_{t_1}^{t} r^{-2} d\mathscr{N}_{n+3}^{p}(t_1, x, y) (r)$
for a small $t_0 > 0$. We split \mathscr{K}_{n+3}^{p} into

$$
\mathscr{K}_{n+3}^{p}(t, x, y) = \frac{n+2}{n!^2} \mathscr{K}_{n+3}^{p}(t, x, y)
$$

and

$$
\mathscr{H}_{n+3}^{p}(t, x, y) = \mathscr{K}_{n+3}^{p}(t, x, y) - \mathscr{K}_{n+3}^{p}(t, x, y)
$$

and get

$$
n = \frac{2n}{n!^2} \qquad \mathscr{K}_{n+3}^{p}(t, x, y) = \mathscr{K}_{n+3}^{p}(t, x
$$

Using \mathcal{N}_{n+3} we can rewrite \mathcal{K}_{n+3} as a Stieltjes integral

$$
\mathcal{K}_{n+3}^{p}(t, x, y) = (-1)^{p} \frac{2}{\pi^{n/2}} \sum_{\substack{b \in \mathbb{G} \\ 0 < r(x, by) < t}} b^{*}(\sigma_{p}(x, by) - \tau_{p}(x, by)).
$$
\nUsing \mathcal{N}_{n+3} we can rewrite \mathcal{K}_{n+3} as a Stieltjes integral

\n
$$
\mathcal{H}_{n+3}^{p}(t, x, y) = \mathcal{K}_{n+3}^{p}(t_0, x, y) + \int_{t_0}^{t} r^{-2} d\mathcal{N}_{n+3}^{p}(\cdot, x, y) \ (r)
$$
\nfor a small $t_0 > 0$. We split \mathcal{K}_{n+3}^{p} into

\n
$$
\mathcal{H}_{n+3}^{\prime p}(t, x, y) = \frac{n+2}{nt^{2}} \mathcal{H}_{n+3}^{\prime p}(t, x, y)
$$
\nand

\n
$$
\mathcal{H}_{n+3}^{\prime p}(t, x, y) = \mathcal{K}_{n+3}^{p}(t, x, y) - \mathcal{H}_{n+3}^{\prime p}(t, x, y)
$$
\nand get

\n
$$
\|\mathcal{R}_{n+3}^{\prime p}(t, x, y)\| \leq c_{16}t^{n-\frac{2n}{n+1}} \quad \text{for } t \geq t_{0} \leq \sqrt{2\xi_{0}}.
$$
\n(17)

$$
\mathscr{E}_{n+3}^{\prime\prime p}(t,x,y)=\frac{n+2}{n t^2}\,\mathscr{H}_{n+3}^{\prime p}(t,x,y)
$$

$$
\mathcal{R}_{n+3}^{\prime\prime p}(t,x,y)=\mathcal{K}_{n+3}^p(t,x,y)-\mathcal{K}_{n+3}^{\prime\prime p}(t,x,y)
$$

$$
\|\mathcal{R}_{n+3}^{\prime\, p}(t,x,y)\| \leq c_{16} t^{n-\frac{2n}{n+1}} \quad \text{for } t \geq t_0 \leq \sqrt{2\xi_0}.
$$

A Lattice Problem for Differential Forms 451

Combining this with (16) , we get the asymptotic behaviour of

A Lattice Problem for Differential Forms 451\nCombining this with (16), we get the 'asymptotic behaviour of\n
$$
\mathcal{S}^e(t, x, y) = c_1, \sum_{b \in \mathbb{Q}} b^b \sigma_p(x, by),
$$
\n
$$
\mathcal{S}^e(t, x, y) = c_1, \sum_{b \in \mathbb{Q}} b^b \sigma_p(x, by),
$$
\n
$$
\mathcal{S}^e(t, x, y) = c_1, \sum_{b \in \mathbb{Q}} b^b \sigma_p(x, by)
$$
\nwith
$$
c_{17} = (-1)^p \Gamma((n+2)/2) \pi^{-n/2}. \text{ In fact, setting}
$$
\n
$$
\mathcal{R}^e(t, x, y) = \mathcal{S}^e(t, x, y) - \frac{n- p}{n} t^n \sum_{\mu, \nu = 0} \omega_{\nu}^p(x) \omega_{\nu}^p(y),
$$
\n(19)\nwe have\n
$$
||\mathcal{R}^e(t, x, y)|| \leq c_{18} t^{n-2n}.
$$
\n
$$
||\mathcal{R}^e(t, x, y)|| \leq c_{19} t^{n-2n}.
$$
\n(20)\nand a similar estimate if we take τ and $n - p$ instead of σ, p , respectively. Recalling $\mathbb{N}^2(a, 1)$, (18) and (19) we see that $\mathbb{N}^2(a, t, y) = c_{19} \mathcal{R}^e(t, x, y) \alpha(y),$ \nwith
$$
c_{19} = \pi^{n/2} \Gamma((x, x) \log n) \alpha(y),
$$
\nwith
$$
c_{19} = \pi^{n/2} \Gamma((x + 2)/2). \text{ From (19) and (20) we deduce the theorem \mathbb{I} To prove our corollary we use
$$
dx_{1a} = |u|^{-2} (\text{Re } u)^2 dx + |u|^{-2} \text{ Re } u \text{ Im } u \text{ dy}
$$
\nfor the \mathbb{G} -autometric 1-form dx and get thereby\n
$$
\mathbf{A}^r[dx](t, 0, 0) = \sum_{b \in \mathbb{N} \setminus \mathbb{Z}} \left(\frac{(\text{Re } u_{k,k})^2}{|u_{k
$$
$$

with $c_{17} = (-1)^p \Gamma((n+2)/2) \pi^{-n/2}$. In fact, setting

$$
\mathcal{F}(t, x, y) = \mathcal{F}(t, x, y)
$$
\n
$$
= (-1)^p \Gamma((n+2)/2) \pi^{-n/2}.
$$
 In fact, setting\n
$$
\mathcal{R}^{\sigma}(t, x, y) = \mathcal{F}^{\sigma}(t, x, y) - \frac{n-2}{n} t^n \sum_{\mu_i p = 0} \omega_i^p(x) \omega_i^p(y),
$$
\n
$$
\|\mathcal{R}^{\sigma}(t, x, y)\| \leq c_{18} t^{n-2n}
$$
\n
$$
\|\mathcal{R}^{\sigma}(t, x, y)\| \leq c_{18} t^{n-2n}
$$
\n
$$
\|\mathcal{R}^{\sigma}(t, x, y)\| \leq c_{18} t^{n-2n}
$$
\n
$$
\|\mathcal{R}^{\sigma}(t, x, y)\| \leq c_{18} t^{n-2n}
$$
\n
$$
\text{for } t \geq t_0
$$
\n
$$
\text{inialar estimate if we take } \tau \text{ and } n-p \text{ instead of } \sigma, p \text{, respectively. Recalling} \text{tions (1)–(3), (18) and (19) we see that\n
$$
\mathsf{A}^{\tau}[\alpha](t, x, y) = c_{19} \mathcal{F}^{\tau}(t, x, y) \alpha(y),
$$
\n
$$
\mathsf{P}^{\tau}[\alpha](t, x, y) = c_{19} \mathcal{R}^{\tau}(t, x, y) \alpha(y)
$$
\n
$$
= \pi^{n/2} \Gamma((n+2)/2).
$$
\nFrom (19) and (20) we deduce the theorem\n
$$
\mathsf{Prove} \text{ our corollary we use } d x_{1u} = |u|^{-2} (\mathbb{R}e u)^2 dx + |u|^{-2} \mathbb{R}e u \mathbb{I}m u \, du
$$
$$

we have

 Δ

$$
\|\mathcal{R}^{\sigma}(t, x, y)\| \leq c_{18}t^{n-\frac{2n}{n+1}} \qquad \text{for } t \geq t_0 \tag{2}
$$

and a similar estimate if we take τ and $n - p$ instead of σ , p , respectively. Recalling the equations (1) - (3) , (18) and (19) we see that

$$
A^{r}[\alpha] (t, x, y) = c_{19} \mathcal{F}^{r}(t, x, y) \alpha(y),
$$

\n
$$
P^{r}[\alpha] (t, x, y) = c_{19} \mathcal{F}^{r}(t, x, y) \alpha(y)
$$

with $c_{19} = \pi^{n/2} \Gamma((n+2)/2)$. From (19) and (20) we deduce the theorem

To prove our corollary we use $dx_{|u} = |u|^{-2}(\text{Re }u)^2 dx + |u|^{-2} \text{Re }u \text{ Im }u dy$

$$
\|\mathcal{R}^{c}(t, x, y)\| \leq c_{18}t^{n-n+1} \quad \text{for } t \geq t_{0}
$$

and a similar estimate if we take τ and $n - p$ instead of σ , p , respectively. Rec
the equations (1)–(3), (18) and (19) we see that

$$
A^{r}[\alpha](t, x, y) = c_{19}\mathcal{F}^{c}(t, x, y) \alpha(y),
$$

$$
P^{r}[\alpha](t, x, y) = c_{19}\mathcal{R}^{r}(t, x, y) \alpha(y)
$$
with $c_{19} = \pi^{n/2}|\Gamma((n + 2)/2)$. From (19) and (20) we deduce the theorem
To prove our corollary we use $dx_{|u|} = |u|^{-2}(\text{Re } u)^{2} dx + |u|^{-2} \text{ Re } u$ In
for the G-automorphic 1-form dx and get thereby

$$
A^{r}[dx](t, 0, 0) = \sum_{k_{1},k_{2} \in \mathbb{Z}} \left(\frac{(\text{Re } u_{k_{1},k_{2}})^{2}}{|u_{k_{1},k_{1}}|^{2}} dx + \frac{(\text{Re } u_{k_{1},k_{2}})(\text{Im } u_{k_{1},k_{2}})}{|u_{k_{1},k_{2}}|^{2}} dy \right).
$$

We set $v = (v_{1}, v_{2}), w = (w_{1}, w_{2}), D = v_{1}w_{2} - v_{2}w_{1}$ and get $D = \text{Im } (\overline{v}w)$. $dx/dy/D$ form an orthonormal basis of the G-automorphic harmonic 1-forms. As a

We set $v = (v_1, v_2), w = (w_1, w_2), D = v_1w_2 - v_2w_1$ and get $D = \text{Im} (\bar{v}w)$. dz/D and dy/D form an orthonormal basis of the G-automorphic harmonic 1-forms. As a consequence of the theorem above we get $\left\| \mathbf{A}^r[dx] (t, 0, 0$ du/D form an orthonormal basis of the $\mathfrak G$ -automorphic harmonic 1-forms. As a consequence of the theorem above we get e set $v = (v_1, v_2), w = (w_1, w_2), D = v_1w_2 - v_2w_1$ and get $D = \text{Im} (\bar{v}w)$. $dz/D = \text{Im} (\bar{v}w)$. $dz/D = \text{Im} (\bar{v}w)$. $dz/D = \text{Im} (\bar{v}w)$. As a ence of the theorem above we get
 $\left\| \mathbf{A}^r[dx] (t, 0, 0) - \frac{\pi}{2} \frac{dx}{D^2} \right\| = O(t^{2/3$

$$
\left|\mathbf{A}^{\mathrm{r}}[dx]\left(t,0,0\right)-\frac{\pi}{2}\frac{dx}{D^2}\right|=O(t^{2/3})
$$

We set
$$
v = (v_1, v_2)
$$
, $w = (w_1, w_2)$, $D = v_1w_2 - v_2w_1$ and get $D = \text{Im}(\overline{v}w)$. dx/D and dy/D form an orthonormal basis of the (9-automorphic harmonic 1-forms. As a consequence of the theorem above we get\n
$$
\left\| \mathbf{A}^r[dx] (t, 0, 0) - \frac{\pi}{2} \frac{dx}{D^2} \right\| = O(t^{2/3})
$$
\nand thereby our conclusion is proved \mathbf{I} \nWe remark, that we also could write the conclusion in the form\n
$$
\sum_{\substack{k_1, k_2 \in \mathbb{Z} \\ |u_{k_1, k_2}| < t}} \frac{(u_1 + k_1v_1 + k_2w_1)^2}{(u_1 + k_1v_1 + k_2w_1)^2 + (u_2 + k_1v_2 + k_2w_2)^2}
$$
\n
$$
= \frac{\pi}{2} \frac{t^2}{(v_1w_2 - v_2w_1)^2} + O(t^{2/3}).
$$
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452 R. Scruster

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