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A Lattice Problem for Differential Forms in Euclidean Spaces

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Im n-dimensionalen Euklidischen Raum $\mathbf{E}^n$ wird in Verallgemeinerung eines Gitterpunktproblems ein Gitterproblem für $\mathfrak{G}$-automorphe p-Differentialformen gelöst. $\mathfrak{G}$ ist dabei eine eigentlich diskontinuierliche Gruppe von Isometrien des $\mathbf{E}^n$ mit kompaktem Fundamentalbereich. Zur Behandlung werden Mittelwertoperatoren für Differentialformen und ein Landausches Differenzenverfahren verwendet.

Как обобщение одной сеточной проблемы в n-мерном евклидовом пространстве $\mathbf{E}^n$ решается сеточная проблема для $\mathfrak{G}$-автоморфных дифференциальных форм степени p. При этом $\mathfrak{G}$ — впольне разрывная группа изометрий с компактной фундаментальной областью в $\mathbf{E}^n$. Для решения проблемы используются операторы среднего значения для дифференциальных форм и метод сеток Ландау.

Generalizing a lattice-point problem we solve a lattice problem for $\mathfrak{G}$-automorphic differential p-forms in the n-dimensional Euclidean space $\mathbf{E}^n$, where $\mathfrak{G}$ is a properly discontinuous group of isometries of $\mathbf{E}^n$ with compact fundamental domain. Our approach essentially uses mean value operators for differential forms and a Landau difference method.

1. Introduction

Let $\mathfrak{G}$ be a properly discontinuous group of isometries of the n-dimensional Euclidean space $\mathbf{E}^n$ with a compact fundamental domain $\mathcal{F}$. By generalizing the Landau ellipsoid problem, P. GÜNTHER [8] studied the estimation of

$$\mathsf{A}(t, x, y) = \sum_{\substack{b \in \mathfrak{G} \\ r(x, by) < t}} 1 \qquad \text{for } t \to \infty$$

with the Euclidean distance $r(x, y)$ of the points $x, y \in \mathbf{E}^n$. In [8] the elements of $\mathfrak{G}$ (with the exception of the identity map id) were supposed to be without fixed points, but instead of simply counting the lattice points, certain unimodular weights were used. The order of magnitude of the leading term and of the lattice remainder used there are the same as in the classical case treated by Landau. We refer to F. FRICKER [3] and A. WALFISZ [18] as basic references, see also the literature quoted there. Problems with weaker assumptions for the fundamental domain have recently been investigated by P. D. LAX and R. S. PHILLIPS [14]. In this paper we want to discuss a generalization involving alternating differential forms, and we will call it a lattice-form problem. Every $b \in \mathfrak{G}$ induces a mapping b^* for differential forms, see [11]. We call a differential form α on $\mathbf{E}^n$ $\mathfrak{G}$-automorphic if $b^*\alpha = \alpha$ is valid for all $b \in \mathfrak{G}$. Following [7] we define components of differential forms. Let $(x^1, \ldots, x^n)$ be a Cartesian coordinate system of $\mathbf{E}^n$. The component of a p-form $\alpha = \alpha_{i_1 \ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

in the direction of the vector $v = (v^i)$ then shall be defined by

$$\alpha_{|v} = p\, \|v\|^{-2}\, v_{[i_1} v^i \alpha_{|i|i_2 \cdots i_p]}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

and the component $\alpha_{|v}^{\perp}$ orthogonal to $\alpha_{|v}$ by $\alpha_{|v}^{\perp} = \alpha - \alpha_{|v}$. $\|v\|$ denotes the Euclidean norm of the vector v. We adopt the convention of summing over repeated indices. $[\cdots |i| \cdots]$ shall denote the alternation without i. We lower and raise indices by the covariant and contravariant metric tensors δ_{ij} and δ^{ij}, respectively. Let $T_{x,y}$ be the parallel displacement of p-forms from the point $y \in \mathbf{E}^n$ to $x \in \mathbf{E}^n$ along the straight line joining these two points. We now define

$$\begin{aligned} \mathsf{A}^{\tau}[\alpha]\,(t, x, y) &= \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} T_{x,by} \alpha_{|x-by}(by), \\ \mathsf{A}^{\sigma}[\alpha]\,(t, x, y) &= \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} T_{x,by} \alpha_{|x-by}^{\perp}(by). \end{aligned} \tag{1}$$

We are interested to estimate $\mathsf{A}^{\tau}[\alpha]$ and $\mathsf{A}^{\sigma}[\alpha]$ for a $\mathfrak{G}$-automorphic differential form α for $t \to \infty$. For $p = 0$, $\alpha = 1$ this lattice-form-problem for A^{σ} reduces to the problem for $\mathsf{A}(t, x, y)$ mentioned above.

Our approach essentially uses kernels of mean value operators for differential forms which are defined by means of double differential forms σ_p; τ_p introduced by P. GÜNTHER [5]. The fact that the forms σ_p, τ_p are intimately related with the construction of the components of p-forms and their parallel displacement makes them well-suited. We will apply some standard arguments of the theory of Euler-Poisson-Darboux equations, but we will not make use of the approach by means of theta functions and Jacobi transformation laws, cf. [8]. We use the Fourier method, which also plays an important role in [8]. Mean value formulas turn out to be quite useful for this purpose. In the space of $\mathfrak{G}$-automorphic p-forms which are quadratically integrable over $\mathcal{F}$ there exists in the sense of L^2-norms over $\mathcal{F}$ a complete orthonormal system of $\mathfrak{G}$-automorphic eigenforms $\{\omega_i^p\}_{i \in \mathbf{N}}$ of the Laplace operator $-\partial^2/(\partial x^1)^2 - \ldots - \partial^2/(\partial x^n)^2$ with the corresponding eigenvalues μ_i^p: $\Delta \omega_i^p = \mu_i^p \omega_i^p$. To estimate A^{σ}, A^{τ} the harmonic forms $\omega_1^p, \ldots, \omega_{B_p}^p$ turn out to be quite important. Thereby B_p denotes the multiplicity of the eigenvalue 0. If the elements of $\mathfrak{G}$ are without fixed points (with the exception of id), B_p is the p^{th} Betti number of the Clifford-Klein space form corresponding to $\mathfrak{G}$. Using the scalar product of the differential forms

$$\alpha = \alpha_{i_1 \cdots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \qquad \beta = \beta_{i_1 \cdots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

defined by $\alpha \cdot \beta = p!\,\alpha_{i_1 \cdots i_p} \beta^{i_1 \cdots i_p}$ and the norm $\|\alpha\| = (\alpha \cdot \alpha)^{1/2}$, we can state the following

Theorem: *The lattice remainder defined by*

$$\mathsf{P}^{\tau}[\alpha]\,(t, x, y) = \mathsf{A}^{\tau}[\alpha]\,(t, x, y) - \sum_{i=1,\ldots,B_p} \frac{p}{n} \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} (\alpha \cdot \omega_i^p)(y)\, \omega_i^p(x)\, t^n \tag{2}$$

satisfies the relation

$$\|\mathsf{P}^{\tau}[\alpha]\,(t, x, y)\| = O\left(t^{n - \frac{2n}{n+1}}\right) \|\alpha(y)\|.$$

The O-term does not depend on α. We get an analogous result for σ instead of τ if we replace the coefficient p/n of the leading term by $(n-p)/n$. α is always supposed to be $\mathfrak{G}$-automorphic and continuous.

The property of $\mathsf{A}(t, x, y)$ to be monotonic in t does not hold in general for $\|\mathsf{A}^\tau[\alpha](t, x, y)\|$. Nevertheless the lattice remainder is still estimable by $O(t^{n-2n/(n+1)})$ in the case of p-forms. The order of magnitude of the leading terms of $\mathsf{A}(t, x, y)$ and $\mathsf{A}^\tau[\alpha](t, x, y)$ for $t \to \infty$ are the same, too (if we suppose $B_p \neq 0$ and $p \neq 0$). The theorem points out the fact that the leading term of $\|\mathsf{A}^\tau[\alpha](t, x, y)\|$ is essentially depending on the harmonic component

$$\sum_{i=1,\dots,B_p} (\alpha \cdot \omega_i^p)(y)\, \omega_i^p(x).$$

As an illustration we give a simple consequence of this theorem. Let $n = 2$ and write the elements of $\mathbf{E}^2$ as complex numbers. Let $\mathfrak{G}$ be the translation group

$$u \to u + k_1 v + k_2 w =: u_{k_1,k_2} \qquad (u, v, w \in \mathbf{C};\ k_1, k_2 \in \mathbf{Z}),$$
$$vw \neq 0,\ \arg(v/w) \neq 0.$$

Corollary: *We have*

$$\sum_{\substack{k_1,k_2 \in \mathbf{Z} \\ |u_{k_1,k_2}| < t}} \sin^2(\arg u_{k_1,k_2}) = \frac{\pi}{2} \frac{1}{\left(\operatorname{Im}(\bar{v}w)\right)^2}\, t^2 + O(t^{2/3}).$$

2. Mean value operators for differential forms

Our treatment of the mean value operators is based on the double differential forms

$$\sigma_0(x, y) = 1, \qquad \tau_0(x, y) = 0,$$
$$\sigma_1(x, y) = r(x, y)\, d\hat{d}\, r(x, y), \qquad \tau_1(x, y) = dr(x, y)\, \hat{d}r(x, y),$$
$$\sigma_p = \frac{1}{p}\, \sigma_{p-1} \wedge\hat{\wedge}\, \sigma_1, \qquad \tau_p = \tau_1 \wedge\hat{\wedge}\, \sigma_{p-1},$$

introduced by P. Günther [5] for spaces of constant curvature $K \neq 0$. $\hat{d}$, $\hat{\wedge}$ shall denote that d, $\wedge$ refer to the second variable y. As shown by P. Günther [6, 7], there is a geometric interpretation for these double differential forms:

$$\begin{aligned} T_{x,y}\alpha_{|y-x} &= (-1)^p\, \tau_p(x, y) \cdot \alpha(y), \\ T_{x,y}\alpha^{\perp}_{|y-x} &= (-1)^p\, \sigma_p(x, y) \cdot \alpha(y). \end{aligned} \tag{3}$$

Following G. de Rham [15] we can write the Laplace operator in the form $\Delta = d\delta + \delta d$, using the differential operator d and the codifferential operator $\delta = (-1)^{pn+n+1} * d *$ for a p-form and the Hodge dualization $*$. The eigenforms ω_i^p we can suppose to be closed ($d\omega_i^p = 0$) or coclosed ($\delta\omega_i^p = 0$), cf. [1]. Let $K(x, t)$ be the ball and $S(x, t)$ the sphere around $x \in \mathbf{E}^n$ with radius t. P. Günther [6, 7] treated the spherical mean values

$$M^\sigma[\alpha](t, x) = (-1)^p \frac{c_0}{t^{n-1}} \int\limits_{S(x,t)} \sigma_p(x, y) \cdot \alpha(y)\, do_y,$$

$$M^\tau[\alpha](t, x) = (-1)^p \frac{c_0}{t^{n-1}} \int\limits_{S(x,t)} \tau_p(x, y) \cdot \alpha(y)\, do_y$$

with $c_0 = \Gamma(n/2)/2\pi^{n/2}$. Motivated by the Riemann-Liouville integrals used in the papers of A. WEINSTEIN [19] and P. GÜNTHER [8] we take for $\lambda \geqq n + 3$

$$\begin{aligned} M_\lambda[\alpha](t, x) &= c_1 t^{3-\lambda} \int_0^t (t^2 - r^2)^{\frac{\lambda-n-3}{2}} r^{n-1}(M^\sigma + M^\tau)[\alpha](r, x)\, dr, \\ N_\lambda[\alpha](t, x) &= c_1 t^{1-\lambda} \int_0^t (t^2 - r^2)^{\frac{\lambda-n-3}{2}} r^{n+1}(M^\sigma - M^\tau)[\alpha](r, x)\, dr \end{aligned} \tag{4}$$

with $c_1 = 2/B\left(\frac{n}{2}, \frac{\lambda - n - 1}{2}\right)$. Then it follows that

$$M_\lambda[\alpha](t, x) = c_2(-1)^p\, t^{3-\lambda} \int_{K(x,t)} \left(t^2 - r^2(x, y)\right)^{\frac{\lambda-n-3}{2}} \left(\sigma_p(x, y) + \tau_p(x, y)\right) \cdot \alpha_p(y)\, dv_y,$$

$$N_\lambda[\alpha](t, x) = c_2(-1)^p\, t^{1-\lambda} \int_{K(x,t)} \left(t^2 - r^2(x, y)\right)^{\frac{\lambda-n-3}{2}} \left(\sigma_p(x,y) - \tau_p(x, y)\right) r^2(x, y) \cdot \alpha_p(y)\, dv_y$$

with $c_2 = \Gamma\left(\frac{\lambda - 1}{2}\right) \Big/ \Gamma\left(\frac{\lambda - n - 1}{2}\right) \pi^{n/2}$. We want to use methods of Euler-Poisson-Darboux theory with respect to the parameter λ. For this reason, we define $z(t, \lambda, \mu)$ for $t \geqq 0$ to be the unique solution of the differential equation

$$\frac{d^2}{dt^2} z(t, \lambda, \mu) + \frac{\lambda}{t} \frac{d}{dt} z(t, \lambda, \mu) + \mu z(t, \lambda, \mu) = 0\;,$$

with the initial conditions $z(0, \lambda, \mu) = 1$, $\frac{d}{dt} z(t, \lambda, \mu)_{|t=0} = 0$. It should be noted that

$$z(t, \lambda, \mu) = \Gamma\left(\frac{\lambda + 1}{2}\right) \left(\frac{2}{\sqrt{\mu}\, t}\right)^{\frac{\lambda-1}{2}} J_{\frac{\lambda-1}{2}}\left(\sqrt{\mu}\, t\right), \qquad \mu > 0,$$

using the Bessel function J_ν with index ν. As a consequence of a correspondence principle of Euler-Poisson-Darboux theory we have the recursion formula

$$z(t, \lambda, \mu) = \left(\frac{1}{\lambda + 1}\, t\, \frac{d}{dt} + 1\right) z(t, \lambda + 2, \mu).$$

We set

$$u(t, \lambda, \mu) = 2\, \frac{\lambda - 1 - q(\lambda)}{\lambda - 1}\, z(t, \lambda, \mu) - z(t, \lambda - 2, \mu),$$

$$v(t, \lambda, \mu) = -2\, \frac{q(\lambda)}{\lambda - 1}\, z(t, \lambda, \mu) + z(t, \lambda - 2, \mu)$$

with $q(\lambda) = p + (\lambda - n - 1)/2$. By using the recursion formula for $z(t, \lambda, \mu)$ we obtain

$$u(t, \lambda, \mu) = \left(\frac{1}{\lambda - 1}\, t\, \frac{d}{dt} + \frac{\lambda + 1}{\lambda - 1}\right) u(t, \lambda + 2, \mu),$$

the same equation is valid for $v(\cdot, \cdot, \cdot)$ instead of $u(\cdot, \cdot, \cdot)$. One observes that closed or coclosed eigenforms of the Laplace operator are at the same time eigenforms of mean value operators. To make this more precise, we state the

Proposition: *The following mean value formulas are true:*
(i) *For* $\Delta\omega = \mu\omega$ *one has* $M_\lambda[\omega]\,(t, x) = z(t, \lambda - 2, \mu)\,\omega(x)$.
(ii) *For* $\Delta\omega = \mu\omega$, $d\omega = 0$ *one has* $N_\lambda[\omega]\,(t, x) = u(t, \lambda, \mu)\,\omega(x)$.
(iii) *For* $\Delta\omega = \mu\omega$, $\delta\omega = 0$ *one has* $N_\lambda[\omega]\,(t, x) = v(t, \lambda, \mu)\,\omega(x)$.

Proof: By referring to [6: Satz 2], it is quite easy to establish the following result:
(i) For $\Delta\omega = \mu\omega$, $d\omega = 0$ one has

$$M^\sigma[\omega]\,(t, x) = \frac{n-p}{n}\, z(t, n+1, \mu)\,\omega(x),$$

$$M^\tau[\omega]\,(t, x) = \left(-\frac{n-p}{n}\, z(t, n+1, \mu) + z(t, n-1, \mu)\right)\omega(x).$$

(ii) For $\Delta\omega = \mu\omega$, $\delta\omega = 0$ one has

$$M^\sigma[\omega]\,(t, x) = \left(-\frac{p}{n}\, z(t, n+1, \mu) + z(t, n-1, \mu)\right)\omega(x),$$

$$M^\tau[\omega]\,(t, x) = \frac{p}{n}\, z(t, n+1, \mu)\,\omega(x).$$

Note that $p = 0$ and $d\omega = 0$ as well as $p = n$ and $\delta\omega = 0$ implies $\mu = 0$. Now the proof follows straightforward from (4) by applying the following integral equation for Bessel functions ($\lambda_2 \geqq \lambda_1 + 2 \geqq 2$):

$$z(t, \lambda_2, \mu) = \frac{2t^{1-\lambda_2}}{B\left(\frac{\lambda_1+1}{2}, \frac{\lambda_2-\lambda_1}{2}\right)} \int_0^t (t^2 - \tau^2)^{\frac{\lambda_2-\lambda_1-2}{2}}\, \tau^{\lambda_1} z(\tau, \lambda_1, \mu)\, d\tau \quad \blacksquare$$

Let $\mathfrak{G}$ be a properly discontinuous group of isometries of $\mathbf{E}^n$. This shall mean that for every $x \in \mathbf{E}^n$ the set of bx for all $b \in \mathfrak{G}$ has no accumulation point. Let $\mathcal{F}$ be a fundamental domain, that means first that the sets $b\mathcal{F}$, $b \in \mathfrak{G}$, cover the space $\mathbf{E}^n$ and secondly that $b\mathcal{F} \cap c\mathcal{F}$ with $b, c \in \mathfrak{G}$, $b \neq c$, has Lebesgue measure 0. We suppose $\mathcal{F}$ to be compact. Without loss of generality we can suppose $\mathcal{F}$ to be the closure of an open, connected domain. For $\mathfrak{G}$-automorphic differential forms α we can rewrite the integration as an integration over a fundamental domain $\mathcal{F}$:

$$M_\lambda[\alpha]\,(t, x) = t^{3-\lambda} \int_{\mathcal{F}} \mathcal{M}_\lambda(t, x, y) \cdot \alpha(y)\, dv_y$$

with

$$\mathcal{M}_\lambda(t, x, y) = c_2(-1)^p \sum_{\substack{b \in \mathfrak{G} \\ r(x, by) < t}} \left(t^2 - r^2(x, by)\right)^{\frac{\lambda-n-3}{2}} b^*\left(\sigma_p(x, by) + \tau_p(x, by)\right)$$

and

$$N_\lambda[\alpha]\,(t, x) = t^{1-\lambda} \int_{\mathcal{F}} \mathcal{N}_\lambda(t, x, y) \cdot \alpha(y)\, dv_y$$

with

$$\mathcal{N}_\lambda(t, x, y) = c_2(-1)^p \sum_{\substack{b\in\mathfrak{G}\\ r(x,by)<t}} \left(t^2 - r^2(x, by)\right)^{\frac{\lambda-n-3}{2}} r^2(x, by)\, b^*\left(\sigma_p(x, by) - \tau_p(x, by)\right).$$

The induced mapping b^* is to be taken with respect to the second variable of the double differential forms. Since $\mathfrak{G}$ was supposed to be properly discontinuous only a finite number of terms in those integral kernels do not vanish. We immediately obtain

$$(\lambda - 1)\, t\mathcal{M}_\lambda(t, x, y) = \frac{d}{dt}\, \mathcal{M}_{\lambda+2}(t, x, y) \tag{5}$$

and an analogous equation for $\mathcal{N}$. Using $b^*\sigma_p(x, by) = (b^{-1})^*\, \sigma_p(y, b^{-1}x)$ and the analogous equation for τ_p we find that

$$\mathcal{M}_\lambda(t, x, y) = \mathcal{M}_\lambda(t, y, x), \qquad \mathcal{N}_\lambda(t, x, y) = \mathcal{N}_\lambda(t, y, x).$$

In view of the mean value formulas, it is possible to expand the integral kernels with respect to the complete eigenform system $\{\omega_i{}^p\}_{i\in\mathbf{N}}$:

$$\begin{aligned} \mathcal{M}_\lambda(t, x, y) &= \sum_{i\in\mathbf{N}} z(t, \lambda - 2, \mu_i{}^p)\, t^{\lambda-3}\, \omega_i{}^p(x)\, \omega_i{}^p(y), \\ \mathcal{N}_\lambda(t, x, y) &= \sum_{i\in\mathbf{N}, \mu_i{}^p>0}{}' \; u(t, \lambda, \mu_i{}^p)\, t^{\lambda-1} \omega_i{}^p(x)\, \omega_i{}^p(y) \\ &\quad + \sum_{i\in\mathbf{N}}{}'' \; v(t, \lambda, \mu_i{}^p)\, t^{\lambda-1} \omega_i{}^p(x)\, \omega_i{}^p(y), \end{aligned} \tag{6}$$

where the sum $\sum'$ is taken over eigenvalues of closed eigenforms of Δ ($\sum''$ for coclosed eigenforms, respectively). First one has to understand the equations (6) in L^2-sense over $\mathcal{F}$ with respect to y. But for $\lambda > 2n + 2$ one gets that (6) is pointwise valid with respect to x and y by standard continuity arguments if one uses the well-known asymptotic behaviour of the eigenforms (see [4, 8, 10])

$$\sum_{0\leq\mu_i{}^p\leqq\xi} \|\omega_i{}^p(x)\|^2 = O(\xi^{n/2}).$$

This implies by partial summation

$$\begin{aligned} \sum_{0\leq\mu_i{}^p\leqq\xi} \|\omega_i{}^p(x)\|^2\, (\mu_i{}^p)^{-\varrho} &= O(\xi^{n/2-\varrho}) \qquad \text{for } \varrho < n/2, \\ \sum_{\xi<\mu_i{}^p} \|\omega_i{}^p(x)\|^2\, (\mu_i{}^p)^{-\varrho} &= O(\xi^{n/2-\varrho}) \qquad \text{for } \varrho > n/2. \end{aligned} \tag{7}$$

Further on one has to use

$$|z(t, \lambda, \mu)| \leqq c_3 t^{-\lambda/2} \mu^{-\lambda/4} \qquad \text{for } \mu > \varkappa > 0,\ \lambda \geqq 1, \tag{8}$$

c_3 of course not depending on t and μ, see [8].

3. Proof of the theorem

On account of the kernel expansion above, the asymptotic behaviour of $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ is quite clear for λ large enough. We now want to extract information about the case $\lambda = n + 3$. This interest is motivated by the geometric interpretation of $\mathcal{M}_{n+3}$, $\mathcal{N}_{n+3}$. To go further, we shall use a variant of a Landau difference method. We now break up the kernels into two parts:

$$\mathcal{H}_\lambda(t, x, y) = t^{\lambda-3} \sum_{i=1}^{B_p} \omega_i^p(x)\, \omega_i^p(y),$$

$$\mathcal{R}_\lambda(t, x, y) = \mathcal{M}_\lambda(t, x, y) - \mathcal{H}_\lambda(t, x, y),$$

$$\mathcal{H}_\lambda'(t, x, y) = \left(1 - 2\,\frac{q(\lambda)}{\lambda - 1}\right) t^{\lambda-1} \sum_{i=1}^{B_p} \omega_i^p(x)\, \omega_i^p(y),$$

$$\mathcal{R}_\lambda'(t, x, y) = \mathcal{N}_\lambda(t, x, y) - \mathcal{H}_\lambda'(t, x, y) \quad \text{with} \quad q(\lambda) = p + \frac{\lambda - n - 1}{2},$$

$\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda'$ are the leading terms of $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$, respectively. We will give error estimates for $\mathcal{R}_\lambda$ and $\mathcal{R}_\lambda'$. Next we define a difference operator for a mapping f from $\mathbf{R}$ into an arbitrary vector space by

$$\nabla_m f(\xi) = \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^{m-\nu} f(\xi + \nu\eta) \text{ with } \eta = \xi^a \text{ for } m \in \mathbf{N}, a \in (0, 1),$$

see [9]. For convenience we transform $\xi = t^2/2$ and write this as

$$\bar{\mathcal{M}}_\lambda(\xi, x, y) = \mathcal{M}_\lambda(t, x, y),\ \bar{\mathcal{N}}_\lambda(\xi, x, y) = \mathcal{N}_\lambda(t, x, y)$$

and so on. Combining this with (5), we get

$$\bar{\mathcal{M}}_{n+3+2m}(\xi) = c_4 \int_0^{\xi} \int_0^{\eta_m} \cdots \int_0^{\eta_2} \bar{\mathcal{M}}_{n+3}(\eta_1)\, d\eta_1 \ldots d\eta_m$$

with a constant c_4 depending on m and n, we have omitted the arguments x and y. This formula is also valid for $\mathcal{H}$, $\mathcal{R}$, $\mathcal{N}$, $\mathcal{H}'$, $\mathcal{R}'$ instead of $\mathcal{M}$. We deduce that

$$\nabla_m \bar{\mathcal{M}}_{n+3+2m}(\xi) = c_4 \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \bar{\mathcal{M}}_{n+3}(\eta_1)\, d\eta_1 \ldots d\eta_m .$$

Using the above decomposition, we find that

$$\begin{aligned} c_5 \eta^m \bar{\mathcal{R}}_{n+3}(\xi) = {} & \nabla_m \bar{\mathcal{R}}_{n+3+2m}(\xi) \\ & + c_5 \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \left(\bar{\mathcal{H}}_{n+3}(\eta_1) - \bar{\mathcal{H}}_{n+3}(\xi)\right) d\eta_1 \ldots d\eta_m \\ & - c_5 \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \left(\bar{\mathcal{M}}_{n+3}(\eta_1) - \bar{\mathcal{M}}_{n+3}(\xi)\right) d\eta_1 \ldots d\eta_m \end{aligned} \tag{9}$$

with a constant c_5. This formula is also true if we replace $\mathcal{M}$, $\mathcal{H}$, $\mathcal{R}$ by $\mathcal{N}$, $\mathcal{H}'$, $\mathcal{R}'$, respectively. The next point on the agenda is to obtain estimates for the right-hand side. To do this, we could use an a-priori estimate for the integrand of the last term. But an easier way is given by applying the known result for $p = 0$, see [8]. In [8] $\mathfrak{G}$ was supposed to be without fixed points, with the exception of id, but it is obvious how to generalize the argumentation to our case. If we want to express the dependence of the kernel forms on the degree p, we write $\mathcal{M}_\lambda^p$ and so on. We recall that the norm $\|\varphi\|$ of a double differential form

$$\varphi = \varphi_{i_1 \cdots i_p j_1 \cdots j_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\, dy^{j_1} \hat{\wedge} \cdots \hat{\wedge}\, dy^{j_p}$$

can be given by

$$\|\varphi\|^2 = (p!)^2\, \varphi_{i_1\cdots i_p j_1\cdots j_p}\, \varphi^{i_1\cdots i_p j_1\cdots j_p},$$

see [4]. After a short computation, we see that the coefficients of $\sigma_p(x, y)$ and $\tau_p(x, y)$ are bounded. From this we conclude that $\|\sigma_p(x, y)\| \leqq c_6$, $\|\tau_p(x, y)\| \leqq c_6$ with a constant c_6. Using (9) twice (a second time for $p = 0$), taking the norms and combining this with the estimate

$$\|\bar{\mathcal{M}}^p_{n+3}(\eta_1) - \bar{\mathcal{M}}^p_{n+3}(\xi)\| \leqq c_6\, \|\bar{\mathcal{M}}^0_{n+3}(\eta_1) - \bar{\mathcal{M}}^0_{n+3}(\xi)\|,$$

we get the inequality

$$\begin{aligned} \|\eta^m \bar{\mathscr{R}}^p_{n+3}(\xi)\| \leqq c_6'(&\|\bar{\mathscr{R}}^0_{n+3}(\xi)\,\eta^m\| \\ &+ \|\nabla_m \bar{\mathscr{R}}^0_{n+3+2m}(\xi)\| + \|\nabla_m \bar{\mathscr{R}}^p_{n+3+2m}(\xi)\| \\ &+ \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \|\bar{\mathscr{H}}^p_{n+3}(\eta_1) - \bar{\mathscr{H}}^p_{n+3}(\xi)\|\, d\eta_1 \cdots d\eta_m \\ &+ \int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \|\bar{\mathscr{H}}^0_{n+3}(\eta_1) - \bar{\mathscr{H}}^0_{n+3}(\xi)\|\, d\eta_1 \cdots d\eta_m). \end{aligned} \tag{10}$$

From [4], it is apparent how to bring $\|\cdot\|$ under the integral sign. According to [8] we have

$$\|\bar{\mathscr{R}}^0_{n+3}(\xi)\| \leqq c_7 \xi^{\frac{n}{2} - \frac{n}{n+1}} \qquad \text{for } \xi \geqq \xi_0, \tag{11}$$

$\xi_0 > 0$ arbitrary small. We consider the 3rd and 4th summand of the right-hand side of (10). Choosing $p = 0$ we get the corresponding result for the 2nd and 5th summand. We take up the case of even n and set $m = n/2$. The considerations for odd n are analogous.

3rd *summand*: We break up the series

$$\bar{\mathscr{R}}^p_{2n+3}(\xi, x, y) = \sum_{\mu_i^p > 0} \xi^n 2^n \bar{z}(\xi, 2n + 1, \mu_i^p)\, \omega_i^p(x)\, \omega_i^p(y)$$

into two parts

$$Q_1 = \sum_{0 < \mu_i^p < \xi^b} \xi^n 2^n \bar{z}(\xi, 2n + 1, \mu_i^p)\, \omega_i^p(x)\, \omega_i^p(y),$$

$$Q_2 = \sum_{\xi^b \leqq \mu_i^p} \xi^n 2^n \bar{z}(\xi, 2n + 1, \mu_i^p)\, \omega_i^p(x)\, \omega_i^p(y)$$

with a constant $b > 0$ which we may choose later. We estimate Q_1 with the aid of (8) with $\lambda = 2n + 1$:

$$\begin{aligned} |\nabla_m\{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}| &\leqq \sum_{\nu=0}^{m} \binom{m}{\nu} |(\xi + \nu\eta)^n\, \bar{z}(\xi + \nu\eta, 2n + 1, \mu_i^p)| \\ &\leqq c_8(\mu_i^p)^{-\frac{2n+1}{4}}\, \xi^{\frac{2n-1}{4}}. \end{aligned} \tag{12}$$

On the other hand, we will use (8) and the law of mean of the differential calculus in order to prepare the estimation of Q_2:

$$\begin{aligned} |\nabla_m\{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}| &\leqq \eta^m \left| \frac{d^m}{d\xi^m} \{\xi^n \bar{z}(\xi, 2n + 1, \mu_i^p)\}_{\xi=\tilde{\xi}} \right| \\ &\leqq c_9 \eta^m\, |\xi^{n/2} \bar{z}(\xi, n + 1, \mu_i^p)_{\xi=\tilde{\xi}}| \leqq c_{10} \eta^m (\mu_i^p)^{-\frac{n+1}{4}}\, \xi^{\frac{n-1}{4}} \end{aligned} \tag{13}$$

with $\bar{\xi} \in (\xi, \xi + m\eta)$. Here we have used (8) with $\lambda = n + 1$. Using the estimates (7), (12) and (13) we see that

$$\|\nabla_m \overline{\mathscr{R}}^p_{2n+3}(\xi, x, y)\| \leqq c_{11} \left(\xi^{\frac{2n-1}{4} - \frac{b}{4}} + \xi^{\frac{n}{2}a + \frac{n-1}{4} + \frac{n-1}{4}b}\right). \tag{14}$$

We choose b optimally by minimizing the right-hand side of (14), we find that $b = 1 - 2a$. Inserting this in (14) gives

$$\|\nabla_m \overline{\mathscr{R}}^p_{2n+3}(\xi, x, y)\| \leqq c_{12} \xi^{\frac{n-1}{2} + \frac{a}{2}}. \tag{15}$$

4th *summand*: Using the estimate $\left\| \sum_{\mu_i^p = 0} \omega_i^p(x)\, \omega_i^p(y) \right\| \leqq c_{13}$ we get

$$\int_{\xi}^{\xi+\eta} \int_{\eta_m}^{\eta_m+\eta} \cdots \int_{\eta_2}^{\eta_2+\eta} \|\mathscr{H}^p_{n+3}(\eta_1) - \mathscr{H}^p_{n+3}(\xi)\|\, d\eta_1 \cdots d\eta_m \leqq c_{14} \xi^{\frac{n+2}{2}a + \frac{n-2}{2}}.$$

Next we choose a optimally by minimizing the sum of the 3rd and 4th summand of (10): we find that $a = 1/(n + 1)$. Combining this with the corresponding result for $p = 0$ and (11), we get

$$\|\overline{\mathscr{R}}^p_{n+3}(\xi, x, y)\| \leqq c_{14} \xi^{\frac{n}{2} - \frac{n}{n+1}} \qquad \text{for } \xi \geqq \xi_0. \tag{16}$$

With small changes the arguments above give the estimate for $\overline{\mathscr{R}}'_{n+3}$. Since $\mathscr{R}'^0_{n+3}(t) = t^{-2} \int_0^t r^2\, d\mathscr{R}^0_{n+3}(r)$ we obtain from (11)

$$\|\overline{\mathscr{R}}'_{n+3}(\xi)\| \leqq c_7' \xi^{\frac{n+2}{2} - \frac{n}{n+1}} \qquad \text{for } \xi \geqq \xi_0. \tag{11$'$}$$

By analogy with (12) we get

$$\begin{aligned} &|\nabla_m \{\xi^{n+1} \bar{u}(\xi, 2n + 3, \mu)\}| \\ &\leqq |\nabla_m \{\xi^{n+1} \bar{z}(\xi, 2n + 3, \mu)\}|\, 2\, \frac{|2n + 2 - q(2n + 3)|}{2n + 2} \\ &\quad + |\nabla_m \{\xi^{n+1} \bar{z}(\xi, 2n + 1, \mu)\}| \\ &\leqq c_8' \left(\xi^{\frac{2n+1}{4}} \mu^{-\frac{2n+3}{4}} + \xi^{\frac{2n+3}{4}} \mu^{-\frac{2n+1}{4}}\right) \end{aligned} \tag{12$'$}$$

and

$$|\nabla_m \{\xi^{n+1} \bar{v}(\xi, 2n + 3, \mu)\}| \leqq c_8'' \left(\xi^{\frac{2n+1}{4}} \mu^{-\frac{2n+3}{4}} + \xi^{\frac{2n+3}{4}} \mu^{-\frac{2n+1}{4}}\right). \tag{12$''$}$$

From the recursion formulas for $z(\cdot, \cdot, \cdot)$ and $u(\cdot, \cdot, \cdot)$ we deduce, that $(2\xi)^{(\lambda-3)/2} \times \bar{z}(\xi, \lambda - 2, \mu)$ and $(2\xi)^{(\lambda-1)/2)}\, \bar{u}(\xi, \lambda, \mu)$ satisfy the same recursion formula

$$\frac{d}{d\xi} \{(2\xi)^{(\lambda+1)/2}\, \bar{u}(\xi; \lambda + 2, \mu)\} = (\lambda - 1)\, \{(2\xi)^{(\lambda-1)/2}\, \bar{u}(\xi, \lambda, \mu)\}.$$

This property is intimately connected with the equations (6). So by analogy with (13) we get

$$|\nabla_m\{\xi^{n+1}\,\bar{u}(\xi, 2n+3, \mu)\}| \leqq \eta^m \left|\frac{d^m}{d\xi^m}\{\xi^{n+1}\bar{u}(\xi, 2n+3, \mu)\}\right|_{|\xi=\tilde{\xi}}$$
$$\leqq c_9'\eta^m \left|\xi^{\frac{n+2}{2}}\bar{u}(\xi, n+3, \mu)\right|_{|\xi=\tilde{\xi}}$$
$$\leqq c_{10}' \left\{\eta^m\xi^{\frac{n+1}{4}}\mu^{-\frac{n+3}{4}} + \eta^m\xi^{\frac{n+3}{4}}\mu^{-\frac{n+1}{4}}\right\} \tag{13$'$}$$

and an analogous equation for $\bar{v}$. These equations imply by analogy with (14)

$$\|\nabla_m\bar{\mathscr{R}}'^p_{2n+3}(\xi, x, y)\| \leqq c_{11}\Big(\xi^{\frac{2n+3}{4}-\frac{b}{4}} + \xi^{\frac{2n+1}{4}-\frac{3b}{4}} + \xi^{\frac{n}{2}a+\frac{n+1}{4}+b\frac{n-3}{4}} + \xi^{\frac{n}{2}a+\frac{n+3}{4}+b\frac{n-1}{4}}\Big).$$

If we set $b = 1 - 2a$, we obtain $(0 < a < 1)$

$$\|\nabla_m\bar{\mathscr{R}}'^p_{2n+3}(\xi, x, y)\| = O\left(\xi^{\frac{n-1}{2}+\frac{3}{2}a}\right) + O\left(\xi^{\frac{n+1}{2}+\frac{a}{2}}\right) = O\left(\xi^{\frac{n+1}{2}+\frac{a}{2}}\right).$$

For the 4th summand we get

$$\int_{\xi}^{\xi+\eta}\int_{\eta_m}^{\eta_m+\eta}\cdots\int_{\eta_2}^{\eta_2+\eta}\|\bar{\mathscr{H}}'^p_{n+3}(\eta_1) - \bar{\mathscr{H}}'^p_{n+3}(\xi)\|\,d\eta_1\cdots d\eta_m = O\left(\xi^{\frac{n+2}{2}a+\frac{n}{2}}\right).$$

If we use $a = 1/(n+1)$, we get

$$\|\mathscr{R}'^p_{n+3}(\xi, x, y)\| = O\left(\xi^{\frac{n}{2}-\frac{n}{n+1}}\right) \qquad \text{for } \xi \geqq \xi_0 \tag{16}$$

by analogy with (16).

We are interested in

$$\mathscr{K}^p_{n+3}(t, x, y) = (-1)^p \frac{\Gamma\left(\frac{n+2}{2}\right)}{\pi^{n/2}} \sum_{\substack{b\in\mathfrak{G}\\ 0<r(x,by)<t}} b^*\big(\sigma_p(x, by) - \tau_p(x, by)\big).$$

Using $\mathscr{N}_{n+3}$ we can rewrite $\mathscr{K}_{n+3}$ as a Stieltjes integral

$$\mathscr{K}^p_{n+3}(t, x, y) = \mathscr{K}^p_{n+3}(t_0, x, y) + \int_{t_0}^{t} r^{-2}\,d\mathscr{N}^p_{n+3}(\cdot, x, y)\,(r)$$

for a small $t_0 > 0$. We split $\mathscr{K}^p_{n+3}$ into

$$\mathscr{H}''^p_{n+3}(t, x, y) = \frac{n+2}{nt^2}\,\mathscr{H}'^p_{n+3}(t, x, y)$$

and

$$\mathscr{R}''^p_{n+3}(t, x, y) = \mathscr{K}^p_{n+3}(t, x, y) - \mathscr{H}''^p_{n+3}(t, x, y)$$

and get

$$\|\mathscr{R}''^p_{n+3}(t, x, y)\| \leqq c_{16}t^{n-\frac{2n}{n+1}} \qquad \text{for } t \geqq t_0 = \sqrt{2\xi_0}. \tag{17}$$

Combining this with (16), we get the asymptotic behaviour of

$$\begin{aligned}\mathcal{S}^\sigma(t, x, y) &= c_{17} \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} b^* \sigma_p(x, by), \\ \mathcal{S}^\tau(t, x, y) &= c_{17} \sum_{\substack{b \in \mathfrak{G} \\ 0 < r(x, by) < t}} b^* \tau_p(x, by)\end{aligned} \tag{18}$$

with $c_{17} = (-1)^p \Gamma\big((n+2)/2\big) \pi^{-n/2}$. In fact, setting

$$\mathcal{R}^\sigma(t, x, y) = \mathcal{S}^\sigma(t, x, y) - \frac{n-p}{n} t^n \sum_{\mu_i^p = 0} \omega_i^p(x) \, \omega_i^p(y), \tag{19}$$

we have

$$\|\mathcal{R}^\sigma(t, x, y)\| \leqq c_{18} t^{n - \frac{2n}{n+1}} \quad \text{for } t \geqq t_0 \tag{20}$$

and a similar estimate if we take τ and $n - p$ instead of σ, p, respectively. Recalling the equations (1)—(3), (18) and (19) we see that

$$\begin{aligned}\mathsf{A}^\tau[\alpha] \, (t, x, y) &= c_{19} \mathcal{S}^\tau(t, x, y) \, \alpha(y), \\ \mathsf{P}^\tau[\alpha] \, (t, x, y) &= c_{19} \mathcal{R}^\tau(t, x, y) \, \alpha(y)\end{aligned}$$

with $c_{19} = \pi^{n/2}/\Gamma\big((n+2)/2\big)$. From (19) and (20) we deduce the theorem ∎

To prove our corollary we use $dx_{|u} = |u|^{-2} (\operatorname{Re} u)^2 \, dx + |u|^{-2} \operatorname{Re} u \operatorname{Im} u \, dy$ for the $\mathfrak{G}$-automorphic 1-form dx and get thereby

$$\mathbf{A}^\tau[dx] \, (t, 0, 0) = \sum_{\substack{k_1, k_2 \in \mathbf{Z} \\ 0 < |u_{k_1, k_2}| < t}} \left(\frac{(\operatorname{Re} u_{k_1, k_2})^2}{|u_{k_1, k_2}|^2} \, dx + \frac{(\operatorname{Re} u_{k_1, k_2}) \, (\operatorname{Im} u_{k_1, k_2})}{|u_{k_1, k_2}|^2} \, dy \right).$$

We set $v = (v_1, v_2)$, $w = (w_1, w_2)$, $D = v_1 w_2 - v_2 w_1$ and get $D = \operatorname{Im} (\bar{v} w)$. dx/D and dy/D form an orthonormal basis of the $\mathfrak{G}$-automorphic harmonic 1-forms. As a consequence of the theorem above we get

$$\left\| \mathbf{A}^\tau[dx] \, (t, 0, 0) - \frac{\pi}{2} \frac{dx}{D^2} \right\| = O(t^{2/3})$$

and thereby our conclusion is proved ∎

We remark, that we also could write the conclusion in the form

$$\begin{aligned}&\sum_{\substack{k_1, k_2 \in \mathbf{Z} \\ |u_{k_1, k_2}| < t}} \frac{(u_1 + k_1 v_1 + k_2 w_1)^2}{(u_1 + k_1 v_1 + k_2 w_1)^2 + (u_2 + k_1 v_2 + k_2 w_2)^2} \\ &= \frac{\pi}{2} \frac{t^2}{(v_1 w_2 - v_2 w_1)^2} + O(t^{2/3}).\end{aligned}$$

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