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On the Nonlinear Boltzmann Equation of the Carrier Transport in Semiconductors. II: Numerical Approximation of Solutions[1])

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Zur numerischen Lösung einer speziellen Form der stationären Boltzmann-Gleichung, wie man sie in der Transporttheorie in Halbleitern benutzt, wird ein Kollokationsverfahren vorgeschlagen und dessen Konvergenz bewiesen.

Предлагается метод коллокации для численного решения стационарного уравнения Больцмана специального вида, возникающего в теории транспорта носителей заряда в полупроводников, и доказывается сходимость этого метода.

A collocation method for the numerical solution of a special kind of the steady-state nonlinear Boltzmann equation used in the transport theory in semiconductors is proposed and the convergence of the method is proved.

This paper is concerned with numerical methods for solving a special kind of a Boltzmann equation, which is used in the carrier transport theory in semiconductors. In Part I of this paper [12] we formulated this equation as an operator equation in suitable anisotropic Sobolev spaces and proved the existence and uniqueness of solution under rather general suppositions. Two cases were distinguished: in a first case we assumed that no carrier sources or sinks exist, in a second case we took sources and sinks into account.

In case of small carrier concentrations it is convenient to use a linearized form of the Boltzmann equation in order to describe the transport phenomena adequately [16]. There are already some works which are devoted to the solution of the linearized equation. The methods used are: Expansion of the solution into a Fourier series with respect to Legendre polynomials and truncating after a finite number of members [7], Monte Carlo methods [5, 14], transformation of the equation into an integral equation and computation of an eigensolution [3, 16, 18, 19], finite-difference methods [1, 15, 25–27]. If the solution is expanded with respect to Legendre polynomials, the coefficients satisfy a system of ordinary differential equations. This system has a satisfactory solution only for small numbers of coefficients. The use of more coefficients leads to inaccurate results. The Monte Carlo technique is used for investigating many substances. But for computing a sufficiently accurate solution a high-computational expense is necessary. The finite-difference method and the integral-equation method are closely related. Essentially they differ in the derivation and representation of the difference equations.

The finite-difference methods turned out to be favourable. In our paper we introduce a special collocation method. This method is constructed in such a way that close relations exist to finite-difference methods. As basis functions we use tensor

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products of piecewise linear spline functions. The set of collocation conditions consists of pointwise conditions and averaging conditions (subregion method [17, 24]). We prove the convergence of the method by means of a discretization theory developed by STUMMEL [21] and VAINIKKO [23, 24]. For the solution of the discretized equations it is appropriate to use multigrid methods. A convergence proof can be found in [11]. Numerical results using models of p-type germanium and p-type tellur are contained in [8–10]. Section 1 summarizes the suppositions and essential results of Part I of this paper [12], which will be needed in the following. In Section 2 we construct the discrete approximations of the Banach spaces used. We prove a general principle for constructing discrete approximations of spaces $C(K, Z)$, where K is a compact set and Z is a Banach space, supposed that we have discrete approximations of $C(K)$ and Z. Section 3 contains the convergence proof of the collocation method if carrier sources and sinks are taken into account (Case (II)), whereas we prove the convergence for the equation free of sources and sinks in Section 4 (Case (I)). All proofs will be done using a 2-dimensional phase space. This is due to the expensive notations. But a thorough analysis of the proofs shows that the results are valid in higher-dimensional phase spaces, too.

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1. Summary of results from Part I

In this Section we summarize the suppositions and essential results of Part I of this paper [12] in order to facilitate understanding. For a detailed discussion of these facts we refer to [12]. Finally, we quote some results of a discretization theory [21, 23, 24], which will be used extensively in the following.

We investigate the equation

$$F \frac{\partial}{\partial x} u + c_1 u = g + \int_G \{W(\cdot, k') (1 - u) u(k') - W(k', \cdot) (1 - u(k')) u\} z(k') \, dk' \tag{1.1}$$

subject to the boundary conditions

$$u(-l, t) = u(l, t) \quad \text{for all} \quad t \in \bar{G}_2. \tag{1.2}$$

Let $G = I \times G_2$, $I = (-l, l) \subset \mathbf{R}$ $(l > 0)$ and $G_2 \subseteqq \mathbf{R}^{n-1}$ open and bounded. Furthermore, let $k = (x, t) \in G$ where $x \in I$ and $t \in G_2$. Suppose $z \in C(\bar{G})$, $z(x, t) \equiv z(t) \geqq 0$ almost everywhere. Let $F \in \mathbf{R}$, $F > 0$ be fixed. The integral kernel W has the form

$$W(k, k') = \sum_{s=-r}^{r} K_s(k, k') \, \delta\big(E(k) - E(k') + w_0^s\big). \tag{1.3}$$

$E \in C^1(\bar{G})$ is a given function (level structure), and $w_0^s \in \mathbf{R}$. Examples are found in [12]. A definition of the integrals with the kernel W is also given in [12]. We distinguish between two cases:

(I) $c_1 \equiv 0, g \equiv 0$.

(II) $c_1(k) \geqq 0$ for all $k \in \bar{G}$, $c_1 \not\equiv 0$.

We introduce the Banach spaces X and Y:

$$H_p^1(I) := \{v \in W^{1,2}(I) \mid v(-l) = v(l)\},$$

$$X = C(\bar{G}_2, H_p^1(I)), \qquad Y = C(\bar{G}_2, L^2(I)).$$

Here, for a compact set K and a Banach space Z, $C(K, Z)$ denotes the Banach space, equipped with the supremum norm, of all continuous mappings defined on K and mapping into Z. The norms in $W^{1,2}(I)$ and $L^2(I)$ are denoted by $\|\cdot\|_1$ and $\|\cdot\|_0$, respectively. The following continuous imbeddings are true:

$$X \to C(\bar{G}) \to Y \to L^2(G).$$

We denote the Banach space of all continuous linear operators defined on X and mapping into Y by $B(X, Y)$. Let $B_0(X, Y) \subseteq B(X, Y)$ be the subspace of all compact linear operators. If Z is a Banach space, let I_Z be the identity mapping of Z. For $A \in B(X, Y)$, $N(A)$ and $R(A)$ denote the kernel and the range of A, respectively. Let $c_1 \in Y$. We define the following operators for all $u, v \in X$ ($k, k' \in \bar{G}$):

$$Au(k) = F \frac{\partial}{\partial x} u(k) + c(k)\, u(k), \tag{1.4}$$

$$c = c_0 + c_1, \qquad c_0(k) = \int_G W(k', k)\, z(k')\, dk',$$

$$Bu(k) = \int_G W(k, k')\, u(k')\, z(k')\, dk', \tag{1.5}$$

$$\bar{B}u(k) = \int_G \big(W(k', k) - W(k, k')\big)\, u(k')\, z(k')\, dk', \tag{1.6}$$

$$Cu(k) = u(k)\, \bar{B}u(k), \tag{1.7}$$

$$Tu = Au - Bu - Cu, \tag{1.8}$$

$$W_u(k, k') = W(k, k') + \big(W(k', k) - W(k, k')\big)\, u(k), \tag{1.9}$$

$$A_u v = F \frac{\partial}{\partial x} v + c_u v,$$

$$c_u = c_1 + c_{0,u}, \qquad c_{0,u}(k) = \int_G W_u(k', k)\, z(k')\, dk',$$

$$B_u v(k) = \int_G W_u(k, k')\, v(k')\, z(k')\, dk'. \tag{1.10}$$

Equation (1.1)–(1.2) is equivalent to

$$Tu = g, \qquad u \in X. \tag{1.11}$$

Moreover, T is analytical and it holds that $T'(u) = A_u - B_u$. Suppose that there is a $d \in \mathbf{R}$ such that

$$\int_I c(x, t)\, dx \geqq d > 0 \qquad \text{for all } t \in \bar{G}_2.$$

In the following, an *eigenvalue* of (A, B) is a $\lambda \in \mathbf{C}$ such that the complexified operator $A - \lambda B$ has a nontrivial null-space. Under additional assumptions on G and E given in Part I the following theorems are true.

Theorem 1.1: *There are constants* $\sigma < 0$ *and* $\tau > 1$ *such that for all* $u \in D := \{u \in X \mid \sigma < u(k) < \tau \text{ for all } k \in \bar{G}\}$, *it holds:*

(i) $A_u \in B(X, Y)$ *is bijective and* $B_u \in B(X, Y)$ *is compact.*

(ii) *For all* $z \in \mathbf{C}$, *the complexified operator* $A_u - zB_u$ *is of Fredholm type with index zero. The eigenvalues have no finite point of accumulation.*

(iii) *There exists an eigenvalue* $\lambda_0 \in \mathbf{R}$ *possessing the properties:*

a) $\lambda_0 > 0$ *and* $|\lambda| > \lambda_0$ *for all eigenvalues* $\lambda \neq \lambda_0$ *of* (A_u, B_u).

b) λ_0 *is algebraically simple. The eigenvector* $e \in X$ *belonging to* λ_0 *can be chosen to be strictly positive, i.e.* $e(k) > 0$ *for all* $k \in \bar{G}$.

(iv) *In Case* (I) $\lambda_0 = 1$ *holds, whereas* $\lambda_0 > 1$ *in Case* (II).

Theorem 1.2: *In Case* (II) *we have:*

(i) *Let* $u \in D$ *and* $Tu = g$. *Then there exist open neighbourhoods* $U \subseteq X$, $V \subseteq Y$ *of* u, g, *respectively, such that* $\tilde{T} = T|_U : U \to V$ *is bijective and* $\tilde{T}^{-1}$ *is continuously differentiable.*

(ii) *There exists* $\delta > 0$ *such that the equation* $Tu = g$ *has a solution* $u^* \in D$ *for all* $g \in Y$, $\|g\|_Y < \delta$.

In Case (I) the derivative $T'(u) = A_u - B_u$ is singular for every $u \in D$ because of Theorem 1.1. But it holds

Lemma 1.3: *Let*

$$Y' = \left\{ v \in Y \mid \int_G v(k)\, z(k)\, dk = 0 \right\}. \tag{1.12}$$

Then, in Case (I), $T'(u)\, X = Y'$ *for all* $u \in D$ *and* $TX \subseteq Y'$.

Therefore, we introduce the following notations:

$$e^* \in X^*, \qquad \langle e^*, u \rangle := \int_G u(k)\, z(k)\, dk,$$

$$H: X \times \mathbf{R} \to Y' \times \mathbf{R}, \qquad H(u, p) = \begin{pmatrix} Tu \\ \langle e^*, u \rangle - p \end{pmatrix}.$$

Instead of (1.1)–(1.2) we consider the equation $H(u, p) = 0$.

Theorem 1.4: *Let* $p_{\max} = \langle e^*, 1 \rangle$ *and* $D' = \{u \in X \mid 0 \leqq u(k) \leqq 1 \text{ for all } k \in \bar{G}\}$. *In Case* (I) *it holds:*

(i) $\frac{\partial}{\partial u} H(u, p)$ *is bijective for all* $(u, p) \in D \times \mathbf{R}$.

(ii) *There is exactly one analytic solution path* $u^*: [0, p_{\max}] \to D'$ *of* $H(u, p) = 0$ *with* $u^*(0) = 0$ *and* $u^*(p_{\max}) = 1$. *Moreover, the equation* $H(u, p) = 0$ *has no further solutions in* $D' \times [0, p_{\max}]$.

Regarding Theorem 1.4, the following representation of the Boltzmann equation is appropriate in Case (I):

$$\begin{aligned} &\mathcal{X} = C([0, p_{\max}], X), \qquad \mathcal{Y} = C([0, p_{\max}], Y' \times \mathbf{R}), \\ &\mathcal{T}: \mathcal{X} \to \mathcal{Y}, \qquad \mathcal{T}u(p) = H(u(p), p). \end{aligned} \tag{1.13}$$

(1.1)–(1.2) is replaced by $\mathcal{T}u = 0$.

Corollary: *The equation* $\mathcal{T}u = 0$ *possesses exactly one solution* $u^* \in \mathcal{D} := \{u \in \mathcal{X} \mid u(p) \in D \text{ for all } p \in [0, p_{\max}]\}$. *Furthermore, the derivative* $\mathcal{T}'(u)$ *is bijective for all* $u \in \mathcal{D}$.

Let Im be a directed infinite index set. Let Z_1, Z_2, Z_{1h}, Z_{2h} be Banach spaces for $h \in Im$. Let $\mathcal{A}(Z_1, \prod Z_{1h}, P)$ and $\mathcal{A}(Z_2, \prod Z_{2h}, Q)$ be discrete approximations of Z_1 and Z_2, respectively. The discrete convergence shall be defined by restriction operators $P = (P_h)_{h \in Im}$ and $Q = (Q_h)_{h \in Im}$. Let $A \in B(Z_1, Z_2)$ and $A_h \in B(Z_{1h}, Z_{2h})$ be continuous linear operators. The sequence (A_h) converges discretely to A if and only if this sequence is stable and consistent with A. The sequence (A_h) is called *inversely stable* if there are $h_0 > 0$ and $\beta > 0$ such that, for all $|h| < h_0$, $A_h^{-1} \in B(Z_{2h}, Z_{1h})$ exists and $\|A_h^{-1}\| \leqq \beta$. Obviously, a sequence (A_h) of Fredholm operators with index zero is inversely stable if and only if there are $h_0 > 0$ and $\gamma > 0$ such that $\gamma \|u_h\|_{Z_{1h}} \leqq \|A_h u_h\|_{Z_{2h}}$ for all $|h| < h_0$ and all $u_h \in Z_{1h}$. The sequence (A_h) is called *regularly convergent* to A if (A_h) converges discretely to A and every bounded sequence $(u_h) \in \prod Z_{1h}$ is discretely compact if $(A_h u_h)$ is discretely compact. In the following we denote the discrete convergence and the convergence in norm by the same symbol "$\to$" since there is no fear of ambiguity. The notation "$u_h \to u$ $(h \in Im')$" denotes the convergence of the subsequence $(u_h)_{h \in Im'}$ for $Im' \subseteqq Im$.

Theorem 1.5 [24]: *The following propositions are equivalent:*

(i) (A_h) *converges to* A *regularly,* A_h $(h \in Im)$ *are of Fredholm type with index zero,* $N(A) = \{0\}$.

(ii) (A_h) *converges discretely to* A, (A_h) *is inversely stable,* $R(A) = Z_2$.

Theorem 1.6 [24]: *Let* $T\colon D \subseteqq Z_1 \to Z_2$, $T_h\colon D_h \subseteqq Z_{1h} \to Z_{2h}$, $g \in Z_2$ *and* $(g_h) \in \prod Z_{2h}$. *Let the following be fulfilled:*

(i) *The equation* $Tu = g$ *has a solution* $u^* \in D$ *and* T *is Frechet differentiable at* u^*.

(ii) *There is a* $\delta > 0$ *such that the operators* T_h $(h \in Im)$ *are Frechet differentiable in the corresponding balls* $\|u_h - P_h u^*\|_{Z_{1h}} \leqq \delta$ *of* Z_{1h}, *and for any* $\varepsilon > 0$ *there is a* $\delta_\varepsilon \in (0, \delta)$ *such that, for every* $h \in Im$, $\|T_h'(u_h) - T_h'(P_h u^*)\| \leqq \varepsilon$ *whenever* $\|u_h - P_h u^*\|_{Z_{1h}} \leqq \delta_\varepsilon$.

(iii) $\|T_h P_h u^* - g_h\|_{Z_{2h}} \to 0$.

(iv) $(T_h'(P_h u^*))_{h \in Im}$ *converges to* $T'(u^*)$ *regularly,* $T_h'(P_h u^*)$ *are Fredholm operators of index zero,* $N(T'(u^*)) = \{0\}$.

Then there exist $h_0 > 0$ *and* $\delta_0 \in (0, \delta)$ *such that the equation* $T_h u_h = g_h$ *has, for* $|h| < h_0$, *a unique solution* u_h^* *in the ball* $\|u_h - P_h u^*\|_{Z_{1h}} < \delta_0$. *Besides* $u_h^* \to u^*$ *with an error estimate* $(c_1, c_2 > 0)$:

$$c_1 \|T_h P_h u^* - g_h\|_{Z_{2h}} \leqq \|u_h^* - P_h u^*\|_{Z_{1h}} \leqq c_2 \|T_h P_h u^* - g_h\|_{Z_{2h}}.$$

Theorem 1.7 [28]: *Let* $K \subset \mathbf{R}^m$ *be compact. Let* $(A_h)_{h \in Im}$ *be a sequence of operators* $A_h\colon K \to B(Z_{1h}, Z_{2h})$ *with the properties:*

(i) $(A_h(t))_{h \in Im}$ *is inversely stable for all* $t \in K$.

(ii) *For all* $t \in K$, $(t^h)_{h \in Im} \subseteqq K$, *and every bounded sequence* $(u_h) \in \prod Z_{1h}$, $(A_h(t^h) - A_h(t))\, u_h \to 0$ *if* $t^h \to t$.

Then there are $h_0 > 0$ *and* $\gamma > 0$ *such that* $\gamma \|u_h\|_{Z_{1h}} \leqq \|A_h(t) u_h\|_{Z_{1h}}$ *for all* $u_h \in Z_{1h}$, $t \in K$, *and* $|h| < h_0$.

2. Discrete approximation of anisotropic Sobolev spaces

Now we introduce the discrete approximations of the Banach spaces X, Y and $\mathcal{X}$, $\mathcal{Y}$, respectively. At a first stage, we prove a general method for constructing approximations of spaces of the kind $C(K, Z)$ by means of tensor products. Then, using spaces of piecewise linear resp. constant spline functions we obtain the desired discrete approximations.

Tensor products of Banach spaces are well known (cf. e. g. [6, 20, 22]). We use the notation of [20]. Let C and Z be Banach spaces. First, we define the product space $C \odot Z$: Consider the set

$$M = \left\{ \sum_{j=1}^{m} x_j \otimes y_j \mid m \in \mathbf{N}, x_j \in C, y_j \in Z \right\}$$

in which for two elements the addition and multiplication by a scalar $\alpha \in \mathbf{R}$ are defined according to

$$\left(\sum_{j=1}^{m} x_j \otimes y_j\right) + \left(\sum_{j=1}^{m'} x_j' \otimes y_j'\right) = \sum_{j=1}^{m} x_j \otimes y_j + \sum_{j=m+1}^{m+m'} x_{j-m}' \otimes y_{j-m}',$$

$$\alpha \left(\sum_{j=1}^{m} x_j \otimes y_j\right) = \sum_{j=1}^{m} (\alpha x_j) \otimes y_j.$$

We introduce an equivalence relation in M calling the two elements above equivalent if

a) $m' = m$, and $x_j = \alpha x_j'$, $\alpha y_j = y_j'$ for some $\alpha \in \mathbf{R}$ *or* $x_j' = x_{\pi(j)}$, $y_j' = y_{\pi(j)}$ for some permutation π of $\{1, \ldots, n\}$ $(j = 1, \ldots, m)$, or

b) $m' = m + 1$ and $x_j' = x_j$, $y_j' = y_j$ $(j = 1, \ldots, m - 1)$, and $y_m' = y_{m'}' = y_m$, $x_m = x_m' + x_{m'}'$ *or* $x_m' = x_{m'}' = x_m$, $y_m = y_m' + y_{m'}'$,

or if one element can be transformed into the other by a finite number of successive applications of these rules. Then, by definition, $C \odot Z$ is the related factor space of M in which we introduce the following norm (*λ-norm* in the notations of [20]):

$$\left\| \sum_{j=1}^{m} x_j \otimes y_j \right\|_{C \otimes Z} = \sup \left\{ \left| \sum_{j=1}^{m} f(x_j)\, g(y_j) \right| \,\middle|\, f \in C^*, g \in Z^*, \|f\| = \|g\| = 1 \right\}.$$

Finally, the *tensor product* $C \otimes Z$ of C and Z is defined by the completion of $C \odot Z$ in the norm $\|\cdot\|_{C \otimes Z}$.

The following lemma is well known (cf. [6, 9, 22]).

Lemma 2.1: *Let $K \subset \mathbf{R}^n$ be a compact set and Z a Banach space. Then $C(K) \otimes Z = C(K, Z)$.*

Let $\bar{C}$ and $\bar{Z}$ be further Banach spaces, $R: C \to \bar{C}$ and $S: Z \to \bar{Z}$ be given linear operators. The product $R \odot S: C \odot Z \to \bar{C} \odot \bar{Z}$ defined by

$$(R \odot S) \left(\sum_{j=1}^{m} x_j \otimes y_j\right) = \sum_{j=1}^{m} (Rx_j) \otimes (Sy_j)$$

is a linear operator. If R and S are bounded, then $R \odot S$ is so, moreover, $\|R \odot S\| = \|R\|\,\|S\|$, and its unique extension $R \otimes S$ onto $C \otimes Z$ is called the *tensor product* of R and S.

Theorem 2.2: *Let $K \subset \mathbf{R}^n$ be a compact set and Z a separable Banach space. Let $\mathcal{A}(C(K), \prod C_h, R)$ and $\mathcal{A}(Z, \prod Z_h, S)$ be discrete approximations of the Banach spaces $C(K)$ and Z, respectively. Every operator R_h and S_h be linear. Then, $\mathcal{A}(C(K) \odot Z, \prod C_h \otimes Z_h, (R_h \odot S_h))$ is a discrete approximation of the set $C(K) \odot Z$, which is dense in $C(K, Z)$.*

Proof: It is sufficient to show that, for every $x \in C(K) \odot Z$,

$$\|(R_h \odot S_h)\, x\|_{C_h \otimes Z_h} \to \|x\|_{C(K,Z)}. \tag{2.1}$$

Let $x = \sum_{j=1}^{m} x_j \otimes y_j$. For every $\varepsilon > 0$, there are functionals $f \in C(K)^*$ and $g \in Z^*$ such that $\|f\| = \|g\| = 1$ and

$$\left| \|x\|_{C(K,Z)} - \left| \sum_{j=1}^{m} f(x_j)\, g(y_j) \right| \right| < \varepsilon.$$

since $C(K)$ and Z are separable, there exist functionals $f_h \in C_h^*$ and $g_h \in Z_h^*$ such that $f_h \rightharpoonup f$, $\|f_h\| \to \|f\|$, $g_h \rightharpoonup g$, $\|g_h\| \to \|g\|$ $(h \in Im)$ ([23: Theorem 1 (37)]; Here

„$\rightharpoonup$" denotes weak discrete convergence). Hence,

$$\varliminf_{h\in Im} \|(R_h \odot S_h)\, x\|_{C_h \otimes Z_h} \geqq \varliminf_{h\in Im} \frac{\left|\sum_{j=1}^{m} f_h(R_h x_j)\, g_h(S_h y_j)\right|}{\|f_h\|\,\|g_h\|}$$

$$\geqq \left|\sum_{j=1}^{m} f(x_j)\, g(y_j)\right| \geqq \|x\|_{C(K,Z)} - \varepsilon.$$

Therefore, $\varliminf_{h\in Im} \|(R_h \odot S_h)\, x\|_{C_h\otimes Z_h} \geqq \|x\|_{C(K,Z)}$. Suppose now that $\varlimsup_{h\in Im} \|(R_h \odot S_h)\, x\|_{C_h\otimes Z_h} > \|x\|_{C(K,Z)}$. Then there would be an $\varepsilon > 0$, a subsequence $Im' \subseteqq Im$ and functionals $f_h \in C_h^*$, $g_h \in Z_h^*$ $(h \in Im')$ which $\|f_h\| = \|g_h\| = 1$,

$$\left|\sum_{j=1}^{m} f_h(R_h x_j)\, g_h(S_h y_j)\right| \geqq \|x\|_{C(K,Z)} + \varepsilon \ (h \in Im').$$

As $C(K)$ and Z are separable, there are functionals $f \in C(K)^*$, $g \in Z^*$, as well as a subsequence $Im'' \subseteqq Im'$ such that $f_h \rightharpoonup f$, $g_h \rightharpoonup g$ $(h \in Im'')$ [23: Theorem 1 (47)]. Using $\|f\| \leqq \varliminf_{h\in Im''} \|f_h\| = 1$, $\|g\| \leqq \varliminf_{h\in Im''} \|g_h\| = 1$ we obtain

$$\|x\|_{C(K,Z)} + \varepsilon \leqq \lim_{h\in Im''} \left|\sum_{j=1}^{m} f_h(R_h x_j)\, g_h(S_h y_j)\right|$$

$$= \left|\sum_{j=1}^{m} f(x_j)\, g(y_j)\right| \leqq \|x\|_{C(K,Z)}.$$

This contradiction proves (2.1) ∎

When constructing concrete approximations of a Banach space E it is possible, in most cases, to define the restriction operators S_h on a dense linear manifold $E_d \subseteqq E$. But there exists an extension of S_h onto E such that $\mathcal{A}(E, \prod E_h, S)$ is a discrete approximation of E. This extension is uniquely determined up to equivalence. Usually, the operators S_h are linear such that the extensions can be chosen to be linear (not necessarily bounded) [23: Theorem 1 (18)]. Hence, the supposition on the linearity of the restriction operators in Theorem 2.2 is often fulfilled.

If $C_h \subseteqq C$ and $Z_h \subseteqq Z$ are subspaces, $C_h \odot Z_h$ can be viewed as a linear subset of $C \otimes Z$. Let $x = \sum_{j=1}^{m} x_j \otimes y_j \in C_h \odot Z_h$. If $f \in C^*$, then $f|_{C_h} \in C_h^*$ and $\|f|_{C_h}\| \leqq \|f\|$. Therefore, $\|x\|_{C_h\otimes Z_h} \geqq \|x\|_{C\otimes Z}$. Let $f \in C_h^*$, $g \in Z_h^*$ such that $\|x\|_{C_h\otimes Z_h} - \left|\sum_{j=1}^{m} f(x_j) \times g(y_j)\right| \big/ (\|f\|\,\|g\|) < \varepsilon$. By the Hahn-Banach Theorem, there exist extensions $\bar{f} \in C^*$, $\bar{g} \in Z^*$ such that $\|\bar{f}\| = \|f\|$, $\|\bar{g}\| = \|g\|$. Hence, $\|x\|_{C_h\otimes Z_h} \leqq \|x\|_{C\otimes Z}$. Consequently, $C_h \otimes Z_h$ is a subspace of $C \otimes Z$.

In connection with projection methods the following theorem is useful.

Theorem 2.3: *Let $K \subset \mathbf{R}^n$ be a compact subset and Z a Banach space. Let $\mathcal{A}\big(C(K), \prod C_h, R\big)$ and $\mathcal{A}(Z, \prod Z_h, S)$ be discrete approximations of $C(K)$ and Z, respectively, such that, for all $h \in Im$, $C_h \subseteqq C(K)$ and $Z_h \subseteqq Z$ are subspaces and*

a) *$R_h \in B\big(C(K), C_h\big)$, $\|R_h x - x\|_{C(K)} \to 0$ for all $x \in C(K)$,*

b) *$S_h \in B(Z, Z_h)$, $\|S_h y - y\|_Z \to 0$ for all $y \in Z$.*

Then, $\mathcal{A}\left(C(K, Z), \prod C_h \otimes Z_h, (R_h \otimes S_h)\right)$ *is a discrete approximation of* $C(K, Z)$ *with the properties*

(i) $R_h \otimes S_h \in B\left(C(K, Z), C_h \otimes Z_h\right)$, $h \in Im$.

(ii) $\|(R_h \otimes S_h)\, x - x\|_{C(K,Z)} \to 0$ *for all* $x \in C(K, Z)$.

Proof: It is sufficient to show (ii) for $x \in C(K) \odot Z$ since the sequence $(\|R_h \otimes S_h\|)$ is bounded because of a), b). Let $x = \sum_{j=1}^{m} x_j \otimes y_j$. Then

$$\|(R_h \otimes S_h)\, x - x\|_{C(K,Z)}$$

$$\leq \sum_{j=1}^{m} \|(R_h \otimes S_h)\,(x_j \otimes y_j) - x_j \otimes y_j\|_{C(K,Z)}$$

$$\leq \sum_{j=1}^{m} \left(\|R_h x_j - x_j\|_{C(K)}\, \|S_h y_j\|_Z + \|x_j\|_{C(K)}\, \|S_h y_j - y_j\|_Z\right) \to 0 \quad \blacksquare$$

Let us remark that the above construction can be applied recursively. If $I, J \subset \mathbf{R}$ are compact intervals, then $C(I) \otimes C(J) = C(I \times J)$ [6].

For our purposes we use the construction principle according to Theorem 2.3. We confine ourselves to the case of the two-dimensional Boltzmann equation, i.e. $G_2 = (a, b) \subset \mathbf{R}$. This is mostly due to the complicate notation. The results are easily extended to the higher dimensional case. The Banach spaces $C[a, b]$, $H_p^1(a, b)$, $L^2(a, b)$ are approximated by spaces of spline functions. Let π, ϱ be given grids where

$$\begin{aligned} &\pi\colon -l = x_0 < x_1 < \dots < x_m = l, \\ &\varrho\colon a = t_1 < t_2 < \dots < t_n = b. \end{aligned} \tag{2.2}$$

We define the following B-splines:

$$B_{i,1}(x) = \begin{cases} 1, & x \in [x_{i-1}, x_i) \\ 0, & \text{otherwise} \end{cases} \qquad \begin{pmatrix} i = 1, \dots, m \\ x \in I \end{pmatrix},$$

$$B_{i,2}(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases} \qquad \begin{pmatrix} i = 0, \dots, m \\ x \in I \end{pmatrix},$$

$$B_{j,3}(t) = \begin{cases} \dfrac{t - t_{j-1}}{t_j - t_{j-1}}, & t \in [t_{j-1}, t_j] \\ \dfrac{t_{j+1} - t}{t_{j+1} - t_j}, & t \in [t_j, t_{j+1}] \\ 0, & \text{otherwise} \end{cases} \qquad \begin{pmatrix} j = 1, \dots, n \\ t \in [a, b] \end{pmatrix}.$$

Now, let

$$\mathrm{Sp}(\pi, 1) = \mathrm{lin}\,\{B_{i,1} \mid i = 1, \dots, m\},$$

$$\mathrm{Sp}(\pi, 2) = \left\{\sum_{i=0}^{m} \alpha_i B_{i,2} \mid \alpha_i \in \mathbf{R}\ (i = 0, \dots, m),\ \alpha_0 = \alpha_m\right\},$$

$$\mathrm{Sp}(\varrho, 3) = \mathrm{lin}\,\{B_{j,3} \mid j = 1, \dots, n\}.$$

Denote by Im a directed infinite index set. Let, for every $h \in Im$, grids $\pi(h)$, $\varrho(h)$ be given according to (2.2). We denote

$$\Delta\pi(h) = \max_{1\leq i\leq m} \{x_i - x_{i-1}\}, \qquad \nabla\pi(h) = \min_{1\leq i\leq m} \{x_i - x_{i-1}\},$$

$$\Delta\varrho(h) = \max_{2\leq j\leq n} \{t_j - t_{j-1}\}.$$

Suppose that there is an $\alpha \in \mathbf{R}$ such that $\Delta\pi(h)/\nabla\pi(h) \leqq \alpha$ for all $h \in Im$. If $h \in Im$, let $|h| = \max\{\Delta\pi(h), \Delta\varrho(h)\}$. We assume $h \to 0$ $(h \in Im)$ to hold. The dependence of the knot sequences $\pi(h)$ and $\varrho(h)$ on h will not be noted for the sake of simplifying the notations. Let

$$C_h = \big(\mathrm{Sp}(\varrho(h), 3), \|\cdot\|_{C[a,b]}\big),$$
$$L_h = \big(\mathrm{Sp}(\pi(h), 1), \|\cdot\|_0\big),$$
$$W_h = \big(\mathrm{Sp}(\pi(h), 2), \|\cdot\|_1\big).$$

We choose the restriction operators as follow:

$$R_h\colon C[a,b] \to C_h, \qquad R_h u = \sum_{j=1}^{m} u(t_j)\, B_{j,3},$$

$$S_h\colon L^2(I) \to L_h, \qquad S_h v = \sum_{i=1}^{m} (x_i - x_{i-1})^{-1} \int_{x_{i-1}}^{x_i} v(\xi)\, d\xi B_{i,1},$$

$$S_{h,1}\colon H_p^1(I) \to W_h, \qquad S_{h,1} u = \sum_{i=1}^{m} u(x_i)\, B_{i,2}.$$

The following lemma is well known (cf. [2]).

Lemma 2.4: *We have:*

(i) $R_h \in B(C[a,b], C_h)$, $\|R_h u - u\|_{C[a,b]} \to 0$ *for all* $u \in C[a,b]$.

(ii) $S_h \in B\big(L^2(I), L_h\big)$, $\|S_h v - v\|_0 \to 0$ *for all* $v \in L^2(I)$.

(iii) $S_{h,1} \in B\big(H_p^1(I), W_h\big)$, $\|S_{h,1} u - u\|_1 \to 0$ *for all* $u \in H_p^1(I)$.

Let

$$X_h = C_h \otimes W_h, \qquad P_h = R_h \otimes S_{h,1}, \qquad Y_h = C_h \otimes L_h, \qquad Q_h = R_h \otimes S_h.$$

Corollary 1: *$\mathcal{A}\big(X, \prod X_h, (P_h)\big)$ and $\mathcal{A}\big(Y, \prod Y_h, (Q_h)\big)$ are discrete approximations of the Banach spaces X and Y, respectively, by subspaces. Moreover, $P_h \in B(X, X_h)$, $Q_h \in B(Y, Y_h)$, $\|P_h u - u\|_X \to 0$ $(h \in Im)$, $\|Q_h v - v\|_Y \to 0$.*

The operator Q_h has for $v \in Y$ the representation

$$Q_h v = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - x_{i-1})^{-1} \int_{x_{i-1}}^{x_i} v(\xi, t_j)\, d\xi B_{j,3} B_{i,1}. \tag{2.3}$$

In order to solve the Boltzmann equation in Case (I) it is necessary to approximate the Banach space $Y' \times \mathbf{R}$. If we want to preserve the bijectivity of $\partial H(u, p)/\partial u$ (Theorem 1.4(i)) while dicretizing, we have to ensure the condition $\dim X_h = \dim (Y'_h \times \mathbf{R})$. Therefore it is appropriate to approximate Y' by spaces $Y_h' \subseteqq Y_h$ with codimension 1. We choose $Y_h' = Y_h \cap Y'$. Let a sequence $(e_h)_{h\in Im} \subset Y_h$, having the properties

$$e_h(k) \geqq 0 \; (k \in \bar{G}), \qquad \int_G e_h(k)\, z(k)\, dk = 1, \qquad e_h \to e \in Y$$

be fixed. Now, set

$$V_h \in B(Y_h, Y_h'), \quad V_h v = v - \int_G v(k)\, z(k)\, dk\, e_h,$$

$$\underline{Q}_h = V_h Q_h.$$

Corollary 2: *$\mathcal{A}\big(Y' \times \mathbf{R}, \prod(Y_h' \times \mathbf{R}), (\underline{Q}_h \times I_{\mathbf{R}})\big)$ is a discrete approximation of $Y' \times \mathbf{R}$. Moreover, $\underline{Q}_h \times I_{\mathbf{R}} \in B(Y' \times \mathbf{R}, Y_h' \times \mathbf{R})$ and $\|(\underline{Q}_h \times I_{\mathbf{R}})(v, \alpha) - (v, \alpha)\|_{Y' \times \mathbf{R}} \to 0$ for all $v \in Y'$ and $\alpha \in \mathbf{R}$.*

Finally, we define

$$\mathcal{X}_h = C[0, p_{\max}] \otimes X_h, \qquad \mathcal{P}_h = I_{C[0,p_{\max}]} \otimes P_h,$$

$$\mathcal{Y}_h = C[0, p_{\max}] \otimes (Y_h' \times \mathbf{R}), \qquad \mathcal{Q}_h = I_{C[0,p_{\max}]} \otimes (\underline{Q}_h \times I_{\mathbf{R}}).$$

Corollary 3: *$\mathcal{A}\big(\mathcal{X}, \prod \mathcal{X}_h, (\mathcal{Q}_h)\big)$ and $\mathcal{A}\big(\mathcal{Y}, \prod \mathcal{Y}_h, (\mathcal{Q}_h)\big)$ are discrete approximation of the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Moreover, $\mathcal{P}_h \in B(\mathcal{X}, \mathcal{X}_h)$, $\mathcal{Q}_h \in B(\mathcal{Y}, \mathcal{Y}_h)$ $\|\mathcal{P}_h u - u\|_{\mathcal{X}} \to 0$ and $\|\mathcal{Q}_h v - v\|_{\mathcal{Y}} \to 0$ for all $u \in \mathcal{X}$ and all $v \in \mathcal{Y}$.*

3. Discrete approximation of the Boltzmann equation in Case (II)

In this section the collocation method is introduced and its convergence will be proved.

The operator $T_h := Q_h T|_{X_h}$, $h \in Im$, describes the discretization of the Boltzmann equation (1.1)—(1.2) in Case (II). Namely, if $g_h \in Y_h$ is an approximation of $g \in Y$, e.g. $g_h = Q_h g$, (1.1)—(1.2) is approximated by the equation

$$T_h u_h = g_h. \tag{3.1}$$

If $g_h = Q_h g$, (3.1) is equivalent to $Q_h(Tu_h - g) = 0$, $u_h \in X_h$. Regarding the representation (2.3) of Q_h, this is equivalently given by

$$\int_{x_{i-1}}^{x_i} \big(Tu_h(\xi, t_j) - g(\xi, t_j)\big)\, d\xi = 0 \quad \left(u_h \in X_h;\ \begin{matrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix}\right).$$

These are the collocation equations, where averaging conditions (in the x-direction) and pointwise conditions (in the t-direction) are used.

We recall that the operators A_u and B_u are defined by (1.9)—(1.10) for $u \in X$.

Lemma 3.1: *Let $u, u^h \in X$ $(h \in Im)$ and $\|u^h - u\|_X \to 0$. Define $B_{h,u} = Q_h B_{u^h}|_{X_h}$. Then, the sequence $(B_{h,u})_{h \in Im}$ is discretely compact and consistent with B_u.*

Proof: For all $v \in X$ it holds

$$\|(B_u - B_{u^h})\, v\|_Y = \|(u - u^h)\, \overline{B}v\|_Y \leqq \alpha\, \|u - u^h\|_X\, \|v\|_X.$$

Hence, $\|B_u - B_{u^h}\|_{B(X,Y)} \to 0$. By Corollary 1 of Lemma 2.4, the sequence (Q_h) is bounded. Using $B_{h,u}v = Q_h B_u v + Q_h(B_{u^h} - B_u)\, v$ for all $v \in X_h$ and Theorem 1.1(i) we obtain the assertions ∎

Lemma 3.2: *Let $u \in D$, $u^h \in X$ $(h \in Im)$ and $\|u^h - u\|_X \to 0$. Define $A_{h,u} = Q_h A_{u^h}|_{X_h}$. Then $(A_{h,u})$ converges regularly to A_u, $N(A_u) = \{0\}$, and every operator $A_{h,u}$ is of Fredholm type with index zero.*

Proof: (i) $N(A_u) = \{0\}$ is a consequence of Theorem 1.1. Because of $\dim X_h = \dim Y_h$, for all h, every operator $A_{h,u}$ is of Fredholm type with index zero. It holds

$$\|(A_{u^h} - A_u)\, v\|_Y = \|(c_{0,u} - c_{0,u^h})\, v\|_Y$$
$$\leqq \gamma\, \|c_{0,u} - c_{0,u^h}\|_Y\, \|v\|_X \leqq \gamma\, \|\bar{B}(u^h - u)\|_Y\, \|v\|_X$$
$$\leqq \alpha\gamma\, \|u^h - u\|_X\, \|v\|_X$$

for all $v \in X$. Therefore, $\|A_{u^h} - A_u\|_{B(X,Y)} \to 0$. By Corollary 1 of Lemma 2.4, the sequence $(A_{h,u})$ is stable and consistent with A_u. Hence, $A_{h,u} \to A_u$.

(ii) It remains to show that the sequence $(A_{h,u})$ converges regularly to A_u. For $t \in [a, b]$, let $A_u(t) \in B(H_p{}^1(I), L^2(I))$ and $A_{h,u}(t) \in B(W_h, L_h)$ $(h \in Im)$ be defined by

$$A_u(t)\, v = A_u w(\cdot, t), \qquad w(x, t) \equiv v(t)$$
$$A_{h,u}(t)\, v_h = A_{h,u} w_h(\cdot, t), \qquad w_h(x, t) \equiv v_h(t) \quad (x \in I).$$

Since $A_u(t)$ is bijective, $N(A_u(t)) = \{0\}$. Because of $\dim W_h = \dim L_h$, every operator $A_{h,u}(t)$ is of Fredholm type with index zero. Now, let $v_h \in W_h$ and $v \in H_p{}^1(I)$ such that $v_h \to v$. Therefore, the elements w_h and w fulfil $w_h \to w$. Hence,

$$\|A_{h,u}(t)\, v_h - A_u(t)\, v\|_0 \leqq \sup_{s\in[a,b]} \|A_{h,u}(s)\, v_h - A_u(s)\, v\|_0$$
$$= \|A_{h,u} w_h - A_u w\|_Y \to 0\,,$$

since by (i) $A_{h,u} \to A_u$. But this implies $A_{h,u}(t) \to A_u(t)$ for every $t \in [a, b]$.

(iii) Now let $v_h \in W_h$, $\|v_h\|_1 \leqq \varkappa < \infty$ such that the sequence $(A_{h,u}(t)\, v_h)$ is discretely compact. Since the sequence (v_h) is bounded, it is precompact in $C(\bar{I})$. Therefore, $\{c_u(\cdot, s)\, v_h \mid h \in Im\}$ is precompact in $L^2(I)$ for every $s \in [a, b]$. Because of

$$\|c_{u^h}(\cdot, s)\, v_h - v\|_0 \leqq \|(c_{u^h}(\cdot, s) - c_u(\cdot, s))\, v_h\|_0 + \|c_u(\cdot, s)\, v_h - v\|_0$$
$$\leqq \gamma\varkappa\, \|c_{u^h}(\cdot, s) - c_u(\cdot, s)\|_0 + \|c_u(\cdot, s) - v\|_0$$

for every $v \in L^2(I)$,

$$\{c_{u^h}(\cdot, s)\, v_h \mid h \in Im\} \text{ is precompact for every } s \in [a, b]. \tag{3.2}$$

(c_{u^h}) is a convergent sequence, therefore, it is also precompact. By the Theorem of Arzela-Ascoli [4: Theorem 7.5.7], the set (c_{u^h}) is equicontinuous with respect to s. Now, the set $M = \{c_{u^h} w_h \mid h \in Im\}$ is equicontinuous with respect to s. Together with (3.2) we obtain by the Theorem of Arzela-Ascoli that M is precompact in Y. Since the sequence (Q_h) is stable, $Q_h(M)$ is precompact. Moreover, the set $\{Q_h c_{u^h} w_h(t) \mid h \in Im\}$ is precompact. Using the discrete compactness of $(A_{h,u}(t)\, v_h)$ we conclude that the sequence $(Fv_h')_{h\in Im}$ is discretely compact in $\mathcal{A}(L^2(I), \prod L_h, (S_h))$ (cf. (1.4)). The boundedness of (v_h) in $H_p{}^1(I)$ implies the precompactness of (v_h) in $L^2(I)$. Hence, (v_h) is discretely compact in $\mathcal{A}(L^2(I), \prod L_h, (S_h))$. Summarizing we obtain that (v_h) is discretely compact in $\mathcal{A}(H_p{}^1(I), \prod W_h, (S_{h,1}))$. Step (ii) of the proof and Theorem 1.5 provide the inverse stability of the sequence $(A_{h,u}(t))$.

(iv) Now, let $t \in [a, b]$ be fixed and $(t^h)_{h\in Im} \subseteqq [a, b]$ a sequence such that $t^h \to t$. Moreover, let $(v_h) \in \prod W_h$ be a bounded sequence. We will investigate the sequence $((A_{h,u}(t^h) - A_{h,u}(t))\, v_h)_{h\in Im}$. Let

$$\alpha_{il} = \int_{x_{i-1}}^{x_i} c_{u^h}(\zeta, t_l)\, v_h(\zeta)\, d\zeta \qquad \begin{pmatrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{pmatrix}.$$

We fix $j, j(h) \in \{2, 3, \ldots, n\}$, such that $t \in [t_{j-1}, t_j]$, $t^h \in [t_{j(h)-1}, t_{j(h)}]$. Using (2.5) we obtain

$$\|(A_{h,u}(t) - A_{h,u}(t^h))\, v_h\|_0^2$$
$$\leqq \sum_{i=1}^{m} (x_i - x_{i-1})^{-1} (2 \max \{|\alpha_{ij} - \alpha_{i,j(h)}|, |\alpha_{ij} - \alpha_{i,j(h)-1}|,$$
$$|\alpha_{i,j-1} - \alpha_{i,j(h)}|, |\alpha_{i,j-1} - \alpha_{i,j(h)-1}|\})^2.$$

(c_{u^h}) is a sequence converging in Y, therefore, it is equicontinuous in particular. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $\|c_{u^h}(\cdot, s_1) - c_{u^h}(\cdot, s_2)\|_0 < \varepsilon$ for all $s_1, s_2 \in [a, b]$ such that $|s_1 - s_2| < \delta$ and all $h \in Im$. Choose $h_0 > 0$ such that, for all $h \in Im$ with $|h| < h_0$, it holds that $\max \{|t_j - t_{j(h)}|, |t_{j-1} - t_{j(h)}|, |t_j - t_{j(h)-1}|, |t_{j-1} - t_{j(h)-1}|\} < \delta$. This is possible because of $t^h \to t$ and $|h| \to 0$. Hence, for all $l, l' \in \{j, j-1, j(h), j(h)-1\}$,

$$|\alpha_{il} - \alpha_{il'}| \leqq \gamma \|v_h\|_1 (x_i - x_{i-1})^{1/2} \left(\int_{x_{i-1}}^{x_i} \big(c_{u^h}(\zeta, t_l) - c_{u^h}(\zeta, t_{l'})\big)^2 d\zeta \right)^{1/2}.$$

If $\|v_h\|_1 \leqq \varkappa$, we have

$$\|(A_{h,u}(t) - A_{h,u}(t^h))\, v_h\|_0^2$$
$$\leqq \sum_{i=1}^{m} 4\gamma^2 \|v_h\|_1^2 \int_{x_{i-1}}^{x_i} \big(c_{u^h}(\zeta, t_l) - c_{u^h}(\zeta, t_{l'})\big)^2 d\zeta \leqq (2\gamma\varkappa\varepsilon)^2.$$

Therefore, $\big(A_{h,u}(t) - A_{h,u}(t^h)\big)\, v_h \to 0$.

(v) Because of $\dim X_h = \dim Y_h$, every operator $A_{h,u}$ is Fredholm type with index zero. Using (iii), (iv), and Theorem 1.7 we obtain the inverse stability of the sequence $(A_{h,u})$. Now, $A_{h,u}$ converges regularly to A_u by (i) and Theorem 1.5 ▮

After this preparation we are in the position to prove the main theorem of this section on the convergence of the solutions of the discrete equations (3.1) to the solution of (1.11).

Theorem 3.3: *Let the suppositions of the Case* (II) *be fulfilled. Let $u^* \in D$ and $g \in Y$ such that $Tu^* = g$. Moreover, let $(g_h) \in \prod Y_h$ be a sequence such that $g_h \to g$. Then there exist $h_0 > 0$ and $\delta_0 > 0$ such that, for every $h \in Im$ with $|h| < h_0$, we have*:

(i) *The equations $T_h u_h = g_h$ have unique solutions u_h^* in the balls defined by $\|P_h u - u_h\|_X < \delta_0$.*

(ii) *These solutions fulfil $u_h^* \to u^*$. Furthermore, the following two-sided error estimate is true $(c_1, c_2 > 0)$:*

$$c_1 \|T_h P_h u^* - g_h\|_Y \leqq \|u_h^* - P_h u^*\|_X \leqq c_2 \|T_h P_h u^* - g_h\|_Y.$$

Proof: The statements are consequences of Theorem 1.6. Obviously, assumptions (i) and (iii) of Theorem 1.6 are fulfilled. Assumption (ii) of Theorem 1.6 holds, since $T_h'(u_h) = Q_h T'(u_h)|_{X_h} (u_h \in X_h; h \in Im)$ and the sequence $(Q_h)_{h \in Im}$ is bounded. By Lemma 3.1 and Lemma 3.2, $T_h'(P_h u^*)$ converges regularly to $T'(u^*)$ [23: Theorem 2 (55)]. $N\big(T'(u^*)\big) = \{0\}$ because of Theorem 1.1. Obviously, every operator $T_h'(P_h u^*)$ is of Fredholm type with index zero. Hence assumption (iv) of Theorem 1.6 is fulfilled, too ▮

It remains to solve the discrete equations (3.1). Under the hypotheses of Theorem 3.3 the Theorem of Kantorovič [13: Theorem XVIII. 1.6] applies such that the method of Newton-Kantorovič is appropriate for solving (3.1). Since sufficiently fine grids lead to high-dimensional discrete problems, the expense for solving the arising linear systems of equations becomes considerable. The special structure (1.3) of the integral kernel W does not lead to full matrices certainly, nevertheless, there are relatively many non-zero entries. Therefore it is advisable to use iterative methods. In [9] the convergence of special block-Jacobi and block-Gauss-Seidel procedures is investigated. But in practical cases sufficient efficiency is not obtained unless multi-grid methods are applied. A convergence proof is given in [11], whereas [10] contains some numerical results.

4. Discrete approximation of the Boltzmann equation in Case (I)

In Case (I) the collocation method is slightly modified. The reason is the special structure of the image space Y' (1.12). Therefore, the convergence proof is considerably more complicated.

The operator $\mathcal{T}_h := Q_h\mathcal{T}|_{\mathcal{X}_h}$, $h \in Im$, defines the discretization of the equation $\mathcal{T}u = 0$, where $\mathcal{T}$ is given by (1.13):

$$\mathcal{T}_h u_h = 0, \qquad u_h \in \mathcal{X}_h. \tag{4.1}$$

Let α_{ij} denote the expansion coefficients of e_h of (2.4):

$$e_h = \sum_{i=1}^{m} \sum_{j=2}^{n} \alpha_{ij} B_{i,1} B_{j,3}.$$

Now, (4.1) is equivalent to

$$\int_{x_{i-1}}^{x_i} \mathcal{T}u(\zeta, t_j)\, d\zeta = (x_i - x_{i-1})\, d(u_h)\, \alpha_{ij} \quad \begin{pmatrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{pmatrix},$$

$$\langle e^*, u_h\rangle = p, \qquad p \in [0, p_{\max}], \qquad d(u_h) := \langle e^*, \mathcal{T}_h u_h\rangle.$$

In order to show the convergence of the solutions u_h^* of the equations (4.1) to the solution u^* of $\mathcal{T}u = 0$ we will use Theorem 1.6 again. The most expensive part of the proof consists in establishing its assumption (iv). For this, we need resolvent integrals [21]. Let Z^1, Z^2 be Banach spaces over the field $\mathbf{C}$ of the complex numbers and $U, V \in B(Z^1, Z^2)$, V compact. Assume that the spectrum $\sigma(U, V)$ consists of at most countably infinitely many values which possess no finite point of accumulation. Let $\Delta \subseteqq \mathbf{C}$ be a bounded domain such that its boundary $\Gamma = \partial\Delta$ is a rectifiable Jordan curve and $\Gamma \cap \sigma(U, V) = \emptyset$. Denote $S(\lambda) = U - \lambda V$ for $\lambda \in \mathbf{C}$. Then the integrals

$$r = \frac{-1}{2\pi i} \int_\Gamma S(\lambda)^{-1} V\, d\lambda, \qquad q = \frac{-1}{2\pi i} \int_\Gamma V S(\lambda)^{-1}\, d\lambda \tag{4.2}$$

exist (i — imaginary unit). $r \in B(Z^1)$ and $q \in B(Z^2)$ are projections which induce the decompositions

$$\begin{aligned} Z^1 &= Z^{11} \oplus Z^{12}, \quad Z^{11} = r(Z^1), \quad Z^{12} = (I - r)(Z^1), \\ Z^2 &= Z^{21} \oplus Z^{22}, \quad Z^{21} = q(Z^2), \quad Z^{22} = (I - q)(Z^2). \end{aligned} \tag{4.3}$$

Denote

$$U^j = U|_{Z^{1j}}, \qquad V^j = V|_{Z^{1j}}, \qquad j = 1, 2. \tag{4.4}$$

Then $U^j, V^j \in B(Z^{1j}, Z^{2j})$. For the spectra it holds

$$\sigma(U^1, V^1) = \sigma(U, V) \cap \Delta, \qquad \sigma(U^2, V^2) = \sigma(U, V) \setminus \bar{\Delta}. \tag{4.5}$$

Lemma 4.1: *Let $u \in \mathcal{D}$. Then the sequence $(\mathcal{T}_h'(\mathcal{P}_h u))_{h \in Im}$ converges regularly to $\mathcal{T}'(u)$, the operators $\mathcal{T}_h'(\mathcal{P}_h u)$ are of Fredholm type with index zero, and $N(\mathcal{T}'(u)) = \{0\}$.*

Proof: (i) By the corollary of Theorem 1.4, $\mathcal{T}'(u)$ is bijective, that means in particular also $N(\mathcal{T}'(u)) = \{0\}$.

(ii) In the sequel we assume all spaces and operators to be complexified. For the time being let $p \in [0, p_{\max}]$ be fixed. We only write u instead of $u(p)$. Let A_u and B_u be defined by (1.9)–(1.10) and set $A_{h,u} = Q_h A_{P_h u}|_{X_h}$, $B_{h,u} = Q_h B_{P_h u}|_{X_h}$, and $J_{h,u} = (I_{Y_h} - V_h)(A_{h,u} - B_{h,u})$ for $h \in Im$. Then $T'(u) = A_u - B_u$ and

$$Q_h T'(P_h u) = A_{h,u} - B_{h,u} - J_{h,u}. \tag{4.6}$$

(iii) By Lemma 3.2 and Theorem 1.5, the sequence $(A_{h,u})$ is stable and inversely stable, and consistent with A_u.

(iv) By Lemma 3.1, the sequence $(B_{h,u})$ is discretely compact and consistent with B_u.

(v) Let $V \in B(Y, Y')$ be defined by

$$Vv = v - \int_G v(k)\, z(k)\, dk\, e, \qquad v \in Y$$

with e given by (2.4). Obviously, the sequence (V_h) converges discretely to V. Hence, $(I_{Y_h} - V_h)$ converges to $I_Y - V$. Let $(v_h)_{h \in Im'}$, $Im' \subseteqq Im$, be a bounded sequence with $v_h \in Y_h$. This implies that the sequence $\left\{\int_G v_h(k)\, z(k)\, dk \mid h \in Im'\right\} \subseteqq \mathbf{C}$ is bounded. Therefore, there exist an $\alpha \in \mathbf{C}$ and a subsequence $Im'' \subseteqq Im'$ such that $\int_G v_h(k) z(k)\, dk \to \alpha$ $(h \in Im'')$. Since $e_h \to e$ $(h \in Im)$, the sequence $((I_{Y_h} - V_h) v_h)_{h \in Im''}$ converges. Hence, the sequence $(I_{Y_h} - V_h)$ is discretely compact. Because of the steps (iii) and (iv), the sequence $(J_{h,u})$ is so, too. By Lemma 1.3, $(I_Y - V)(A_u - B_u) = 0$. This implies $J_{h,u} \to 0$ $(h \in Im)$. Moreover, $R(I_{Y_h} - V_h) = \operatorname{lin}\{e_h\}$ holds such that every operator $J_{h,u}$ is compact.

(vi) The number 1 is an algebraically simple eigenvalue of the pair (A_u, B_u) which has the smallest modulus among all eigenvalues (Theorem 1.1). Let $\Delta = \{z \in \mathbf{C} \mid |z| < 1 + \varepsilon\}$ such that $\Delta \cap \sigma(A_u, B_u) = \{1\}$ and $\varepsilon > 0$. By [21: Theorem II-3.2(8)] and steps (iii)–(v), for every $h \in Im$, $|h|$ sufficiently small, there exists exactly one eigenvalue $\lambda_h \in \sigma(A_{h,u}, B_{h,u} + J_{h,u})$ (which is algebraically simple) such that $\Delta \cap \sigma(A_{h,u}, B_{h,u} + J_{h,u}) = \{\lambda_h\}$. Moreover, $\lambda_h \to 1$. Furthermore, for every λ_h there is an eigenvector $w_h \in X_h$ such that $w_h \to w$. Here, w denotes a strictly positive (real) eigenvector of the eigenvalue 1 of (A_u, B_u). On the other hand, since $\langle e^*, (A_{h,u} - B_{h,u} - J_{h,u}) v\rangle = 0$ for all $v \in X_h$, $h \in Im$, by the definition of $J_{h,u}$, $1 \in \sigma(A_{h,u}, B_{h,u} + J_{h,u})$ for all $h \in Im$. Hence, $\lambda_h = 1$. But this implies that the real part of w_h is also an eigenvector. Because of $w_h \to w$, this real part is strictly positive for sufficiently small $|h|$, say for $|h| < h_0$. We assume without loss of generality that w_h has a vanishing imaginary part. Hence, we obtain

$$w_h(k) > 0\ (k \in \bar{G}), \quad \langle e^*, w_h \rangle \geqq \alpha > 0 \quad (h \in Im, |h| < h_0). \tag{4.7}$$

(vii) Let $h_0 > 0$ be chosen such that (4.7) holds. Let $\Gamma = \partial\Delta$. We define for $h \in Im$, $|h| < h_0$ (cf. (4.2)–(4.5)):

$$r = \frac{-1}{2\pi i} \int_\Gamma (A_u - \lambda B_u)^{-1} B_u\, d\lambda, \qquad q = \frac{-1}{2\pi i} \int_\Gamma B_u (A_u - \lambda B_u)^{-1}\, d\lambda,$$

$$r_h = \frac{-1}{2\pi i} \int_\Gamma \left(A_{h,u} - \lambda(B_{h,u} + J_{h,u})\right)^{-1} (B_{h,u} + J_{h,u})\, d\lambda,$$

$$q_h = \frac{-1}{2\pi i} \int_\Gamma (B_{h,u} + J_{h,u}) \left(A_{h,u} - \lambda(B_{h,u} + J_{h,u})\right)^{-1} d\lambda,$$

$$X^2 = (I_X - r)(X), \qquad Y^2 = (I_Y - q)(Y),$$

$$X_h^2 = (I_{X_h} - r_h)(X_h), \qquad Y_h^2 = (I_{Y_h} - q_h)(Y_h),$$

$$A_u^2 = A_u|_{X^2}, \qquad B_u^2 = B_u|_{X^2},$$

$$A_{h,u}^2 = A_{h,u}|_{X_h^2}, \qquad B_{h,u}^2 = (B_{h,u} + J_{h,u})|_{X_h^2}.$$

It holds that A_u^2, $B_u^2 \in B(X^2, Y^2)$ and $A_{h,u}^2$, $B_{h,u}^2 \in B(X_h^2, Y_h^2)$. $\mathcal{A}(X^2, \prod X_h^2, ((I_{X_h} - r_h) P_h))$ and $\mathcal{A}(Y^2, \prod Y_h, ((I_{Y_h} - q_h) Q_h))$ are discrete approximations of X^2 and Y^2, respectively [21: Theorem II-3.1(8)]. For them, the following is true:

$(A_{h,u}^2)$ is consistent with A_u^2 as well as inversely stable (cf. step (iii)).

$(B_{h,u}^2)$ is consistent with B_u^2 as well as discretely compact (cf. step (iv)).

Using [21: Theorem II-2.2(1)] we obtain that, for all $\lambda \in \varrho(A_u^2, B_u^2)$, there is an $h_0(\lambda)$ such that $A_{h,u}^2 - B_{h,u}^2 \in B(X_h^2, Y_h^2)$ is bijective for every $|h| < h_0(\lambda)$ and $(A_{h,u}^2 - \lambda B_{h,u}^2)^{-1} \to (A_u^2 - \lambda B_u^2)^{-1}$ $(h \in Im, |h| < h_0(\lambda))$. By (4.6), this is true in particular for $\lambda = 1$. Hence, there is a $\beta \in \mathbf{R}$ such that

$$\|(A_{h,u}^2 - B_{h,u}^2)^{-1}\| \leqq \beta \qquad (h \in Im, |h| < h_0(1)). \tag{4.8}$$

(viii) Next we show that $Y^2 = Y'$. Let $y \in Y' = R(A_u - B_u)$. Let $v \in X$ such that $(A_u - B_u) v = y$. Assume $v = v_1 + v_2$ with $v_1 \in R(r)$ and $v_2 \in R(I_X - r) = X^2$. Since the eigenvalue 1 is algebraically simple, $N(A_u - B_u) = R(r)$ [21: Theorem II-1. 3 (14)—(16)]. Hence, $(A_u - B_u) v = (A_u - B_u) v_2 = (A_u^2 - B_u^2) v_2 = y \in Y^2$. Thus, we obtain the relations $R(A_u^2 - B_u^2) \subseteqq R(A_u - B_u) \subseteqq Y^2 = R(A_u^2 - B_u^2)$ which prove $Y^2 = Y'$. Analogously, one shows $Y_h' = Y_h^2$ for $h \in Im$, $|h| < h_0(1)$, using step (vii).

(ix) Now we return to the real spaces. If $y \in Y_h'$, y can be viewed as an element of the respective complexified space with vanishing imaginary part. $(A_{h,u}^2 - B_{h,u}^2)^{-1} y$ is well defined for $|h| < h_0(1)$ and has vanishing imaginary part. Therefore, $(A_{h,u}^2 - B_{h,u}^2)^{-1} \in B(Y_h', X_h)$ is well defined as an operator acting in the real spaces.

(x) For fixed $p \in [0, p_{\max}]$ we have $\mathcal{T}_h'(\mathcal{P}_h u)(p) = \begin{pmatrix} Q_h T'(P_h u(p)) \\ \langle e^*, \cdot \rangle \end{pmatrix}$. We define, for $(y_h, \mu) \in Y_h' \times \mathbf{R}$, $|h| < h_0(1)$,

$$v_h = \left[\mu - \left\langle e^*, \frac{A_{h,u}^2 - B_{h,u}^2)^{-1} y_h}{\langle e^*, w_h \rangle} \right\rangle\right] w_h + (A_{h,u}^2 - B_{h,u}^2)^{-1} y_h,$$

with w_h defined according to step (vi) with (4.7). We will show that the operator G_h: $(y_h, \mu) \mapsto v_h$ is the inverse of $\mathcal{T}_h'(\mathcal{P}_h u)(p)$. Let $v \in X_h$ such that $\mathcal{T}_h'(\mathcal{P}_h u)(p) v = 0$. By (4.6) and step (vi), $v = t w_h$ for some $t \in \mathbf{R}$ and $\langle e^*, v \rangle = 0$. This implies $v = 0$ because of (4.7). Hence, $\mathcal{T}_h'(\mathcal{P}_h u)(p)$ is injective. Next we prove $\mathcal{T}_h'(\mathcal{P}_h u)(p) G_h = I_{Y_h' \times \mathbf{R}}$. Then, together with the injectivity, the assertion will follow. Let $(y_h, \mu) \in Y_h' \times \mathbf{R}$. Because of $w_h \in N(A_{h,u} - B_{h,u} - J_{h,u})$ it holds

$$\begin{aligned}(A_{h,u} - B_{h,u} - J_{h,u}) v_h &= (A_{h,u} - B_{h,u} - J_{h,u}) (A_{h,u}^2 - B_{h,u}^2)^{-1} y_h \\ &= (A_{h,u}^2 - B_{h,u}^2) (A_{h,u}^2 - B_{h,u}^2)^{-1} y_h = y_h.\end{aligned}$$

Finally, one easily computes $\langle e^*, v_h \rangle = 1$.

(xi) Let α, β be chosen according to (4.7) and (4.8). For all $h \in Im$, $|h| < h_0(1)$, and all $(y_h, \mu) \in Y_h' \times \mathbf{R}$,

$$\begin{aligned}
\|v_h\|_X &= \|G_h(y_h, \mu)\|_X \\
&\leqq \left\| \left[\mu - \left\langle e^*, \frac{(A_{h,u}^2 - B_{h,u}^2)^{-1} y_h}{\langle e^*, w_h \rangle} \right\rangle \right] w_h \right\|_X + \|(A_{h,u}^2 - B_{h,u}^2)^{-1} y_h\|_X \\
&\leqq \frac{\|w_h\|_X}{\alpha} \left(|\mu| + \langle e^*, 1 \rangle \|(A_{h,u}^2 - B_{h,u}^2)^{-1} y_h\|_{C(\bar{G})} \right) + \|(A_{h,u}^2 - B_{h,u}^2)^{-1} y_h\|_X \\
&\leqq \alpha'(|\mu| + \langle e^*, 1 \rangle \gamma\beta \|y_h\|_Y) + \beta \|y_h\|_Y \\
&\leqq \text{const } \|(y_h, \mu)\|_{Y \times \mathbf{R}}.
\end{aligned}$$

Since $\dim X_h = \dim Y_h' \times \mathbf{R}$, every operator $\mathcal{T}_h'(\mathcal{P}_h u)(p)$ is of Fredholm type with index zero. Hence, the sequence $(\mathcal{T}_h'(\mathcal{P}_h u)(p))$ is inversely stable.

(xii) For every $p_1, p_2 \in [0, p_{\max}]$ (cf. (1.4)—(1.8))

$$\mathcal{T}_h'(\mathcal{P}_h u)(p_1) - \mathcal{T}_h'(\mathcal{P}_h u)(p_2) = \begin{pmatrix} Q_h(C'(P_h u(p_1)) - C'(P_h u(p_2))) \\ 0 \end{pmatrix}.$$

Using (1.7) we obtain

$$\|\mathcal{T}_h'(\mathcal{P}_h u)(p_1) - \mathcal{T}_h'(\mathcal{P}_h u)(p_2)\| \leqq \text{const } \|u(p_1) - u(p_2)\|_X.$$

Because of $u \in \mathcal{X}$ and step (xi) the assumptions of Theorem 1.7 are fulfilled. Hence, there is a $\gamma > 0$ and an $h_1 > 0$ such that, for all $v_h \in \mathcal{X}_h$ and $|h| < h_1$, $\gamma \|v_h\|_{\mathcal{X}_h} \leqq \|\mathcal{T}_h'(\mathcal{P}_h u) v_h\|_{\mathcal{Y}_h}$.

(xiii) In order to obtain the inverse stability of the sequence $(\mathcal{T}_h'(\mathcal{P}_h u))$ it remains to show that, for $|h| < h_1$, $\mathcal{T}_h'(\mathcal{P}_h u)$ is surjective. Let $(y_h, \mu) \in \mathcal{Y}_h$. By step (xi), for every $p \in [0, p_{\max}]$, there is exactly one solution $v_h(p)$ of the equation $\mathcal{T}_h'(\mathcal{P}_h u) v_h(p) = (y_h(p), \mu(p))$. But

$$\begin{aligned}
&\|v_h(p_1) - v_h(p_2)\|_X \\
&= \|[\mathcal{T}_h'(\mathcal{P}_h u)(p_1)]^{-1} (y_h(p_1), \mu(p_1)) - [\mathcal{T}_h'(\mathcal{P}_h u)(p_2)]^{-1} (y_h(p_2), \mu(p_2))\|_X \\
&\leqq \gamma^{-1} \|(y_h(p_1), \mu(p_1)) - (y_h(p_2), \mu(p_2))\|_{Y \times \mathbf{R}} \\
&\quad + \gamma^{-2} \|\mathcal{T}_h'(\mathcal{P}_h u)(p_1) - \mathcal{T}_h'(\mathcal{P}_h u)(p_2)\| \|y(_h(p_1), \mu(p_1))\|_{Y \times \mathbf{R}}
\end{aligned}$$

on account of step (xii). Hence, $v_h \colon [0, p_{\max}] \to X_h$ is continuous. This gives finally $v_h \in \mathcal{X}_h$ and $\mathcal{T}_h'(\mathcal{P}_h u) v_h = (y_h, \mu)$.

(xiv) The assertions of the lemma follow from steps (i), (xii), (xiii), and Theorem 1.5 ∎.

Theorem 4.2: *Let $u^* \in \mathcal{D}$ be a solution of the equation $\mathcal{T} u = 0$. Then there are $h_0 > 0$ and $\delta_0 > 0$ such that, for all $h \in Im$ with $|h| < h_0$, the equations $\mathcal{T}_h u_h = 0$ have unique solutions u_h^* in the balls $\{u_h \mid \|u_h - \mathcal{P}_h u\|_{\mathcal{X}} < \delta_0\}$. Besides, $u_h^* \to u^*$ $(h \in Im)$ with the error estimate $(c_1, c_2 > 0)$*

$$c_1 \|\mathcal{T}_h \mathcal{P}_h u^*\|_{\mathcal{Y}} \leqq \|u_h^* - \mathcal{P}_h u^*\|_{\mathcal{X}} \leqq c_2 \|\mathcal{T}_h \mathcal{P}_h u^*\|_{\mathcal{Y}}.$$

Proof: The assertions are an immediate consequence of Theorem 1.6 and Lemma 4.1 ∎

For sufficiently small $|h|$, there is a solution path $u_h^* \in C([0, p_{\max}], X_h)$ of the discrete equations which is unique in a neighbourhood of $\mathcal{P}_h u^*$. Since, for every $h \in Im$, the functions $u = 0$ and $u = 1$ belong to $\mathcal{X}_h$, $u_h^*(0) = 0$ and $u_h^*(p_{\max}) = 1$. For computing the solution path usual homotopy methods can be applied if locally convergent iteration methods for solving equations of the kind $\mathcal{T}_h u_h(p) = 0$ are known [29: Theorem 2.3]. In [9] the Newton method is used. In every iteration step there occur overdetermined linear systems of equations which possess a unique solution however. For solving these systems a least-squares method and a special iteration method, respectively, are applied. But it is more favourable to use the multigrid method proposed in [11] instead of the Newton method. As a smoothing procedure, e.g., a simple nonlinear Jacobi method in connection with the power method can be used.

The construction of the discrete operators $\mathcal{T}_h$ depends on a sequence $(e_h)_{h \in Im}$, which can be chosen arbitrarily in certain limits. By this sequence the properties of $\mathcal{T}_h$ can be slightly modified. But for all $h \in Im$ and $u_h \in X_h$, the estimate

$$\|Q_h T u_h\|_Y \leqq \|V_h\| \, \|Q_h T u_h\|_Y \leqq (1 + \|e^*\| \, \|e_h\|_Y) \, \|Q_h T u_h\|_Y$$

holds. Since the sequence (e_h) converges, there exists a bound independent of h for the factors $1 + \|e^*\| \, \|e_h\|_Y$. Hence, the sequence $(Q_h T)_{h \in Im}$ is not worse than the sequence $(Q_h T)_{h \in Im}$ with respect to the approximation quality. The concrete choice of (e_h) should be made in dependence on the semiconductor model under consideration. As a rule of thumb, $e_h(k)$ should be large where a solution peak is expected. But in the same measure as $|h|$ tends to zero, $\langle e^*, Q_h T u_h \rangle$ decreases such that the influence of the sequence (e_h) vanishes.

REFERENCES

[1] Beer, M.: Die numerische Lösung der zweidimensionalen Boltzmanngleichung der Halbleiterphysik. In: 3. Fachtagung „Numer. Realisierung Math. Modelle", Zingst (DDR), Okt. 1979. Akad. Wiss. DDR (Berlin), Zentrum f. Rechentechnik: ZfR-Berichte 79. 12 (1979), S. 135–155.

[2] de Boor, C.: A practical guide to splines (Appl. Math. Sci. 27). New York–Heidelberg–Berlin: Springer-Verlag 1978.

[3] Budd, H.: Path variable formulation of the hot carrier problem. Phys. Rev. 158 (1967), 798–804.

[4] Dieudonné, J.: Grundzüge der modernen Analysis I. Berlin: Dt. Verlag Wiss. 1971.

[5] Fawcett, W., Boardman, A. D., and S. Swain: Monte Carlo determination of electron transport properties in gallium arsenide. J. Phys. Chem. Solids 31 (1970), 1963–1990.

[6] Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaire. Mem. Amer. Math. Soc. 16 (1955), 1–140.

[7] Hammar, C.: Iterative method for calculating hot carrier distributions in semiconductors. J. of Phys. C: Solid State Phys. 6 (1973), 70–78.

[8] Hanke, M.: Zur numerischen Lösung der nichtlinearen Boltzmanngleichung der Halbleiterphysik. In: 4. Fachtagung „Numer. Realisierung Math. Modelle", Zingst (DDR), Sept. 1981. Akad. Wiss. DDR (Berlin), Zentrum f. Rechentechnik: ZfR-Berichte 81. 16 (1981), S. 55–63.

[9] Hanke, M.: Zur numerischen Lösung der nichtlinearen Boltzmanngleichung der Halbleiterphysik. Dissertation A. Berlin: Humboldt-Univ., Math-Nat. Fak. 1982.

[10] Hanke, M.: Eine Anwendung der Mehrgittermethode auf eine Boltzmanngleichung. In: 5. Fachtagung „Numer. Realisierung Math. Modelle", Zingst (DDR), Sept. 1983. Akad. Wiss. DDR (Berlin), Zentrum f. Rechentechnik: ZfR-Berichte 83. 12 (1983), S. 239–246.

[11] Hanke, M.: On the convergence of a multigrid method and its application to a nonlinear Boltzmann equation. In: Numer. Beh. von Dgl. II. Wiss. Beiträge Univ. Jena 1984, S. 69–84.

[12] Hanke, M.: On the nonlinear Boltzmann equation of the carrier transport in semiconductors I. Existence and uniqueness of solutions. Berlin: Humboldt-Univ., Sektion Math., Preprint Nr. 109 (1986).
[13] Канторович, Л. В., и Г. П. Акилов: Функциональный анализ. Изд. второе. Москва: Изд-во Наука 1977.
[14] Kurosawa, T.: Monte Carlo calculation of hot electron problems. Proc. Intern. Conf. Phys. Semiconductors, Kyoto 1966. J. Phys. Soc. Japan 21 (1966) Suppl., 424—426.
[15] Lebwohl, P. A., and P. M. Markus: Hot electron distributions by direct integration of the Boltzmann equation. Solid State Comm. 9 (1971), 1671—1674.
[16] Mocker, M.: Zur Theorie heißer Ladungsträger in Halbleitern. Dissertation A. Berlin: Humboldt-Univ., Math.-Nat. Fak. 1972.
[17] Петерсен, И.: О сходимости приближённых методов интерполяционного типа для обыкновенных дифференциальных уравнений. Изв. Акад. Наук Эст. ССР, Сер. физ.-мат. и техн. наук 10 (1961), 3—12.
[18] Primatarowa, M.: Zur Berechnung der Verteilerfunktion von Ladungsträgern im Halbleiter, Ergebnisse für p-Tellur. Dissertation A. Berlin: Humboldt-Univ., Math.-Nat. Fak. 1976.
[19] Primatarowa, M., und M. Wille: Zur numerischen Lösung der Boltzmanngleichung für heiße Ladungsträger in p-Tellur. Diplomarbeit. Berlin: Humboldt-Univ., Sektion Physik 1972.
[20] Schatten, R.: A theory of cross-spaces. Princeton (N. J.): Univ. Press 1950.
[21] Stummel, F.: Diskrete Konvergenz linearer Operatoren, I—III. Math. Ann. 190 (1970), 45—92; Math. Z. 120 (1971), 231—264 und Proc. Oberwolfach Conf. Lin. Oper. Approx. 1971 (ISNM 20), Basel: Birkhäuser Verlag 1972, S. 196—216.
[22] Umegaki, H., and A. T. Bharucha-Reid: Banach spaces valued random variables and tensor products of Banach spaces. J. Math. Anal. Appl. 31 (1970), 49—67.
[23] Vainikko, G.: Funktionalanalysis der Diskretisierungsmethoden (Teubner-Texte zur Mathematik: Bd. 5). Leipzig: B. G. Teubner Verlagsges. 1976.
[24] Vainikko, G.: Approximative methods for nonlinear equations. Nonlinear Anal. — Theory, Methods & Appl. 2 (1978), 647—687.
[25] Wendt, W.: Über eine eindimensionale lineare Integro-Differentialgleichung und deren approximative Lösung mit dem Differenzenverfahren. Wiss. Z. Humboldt-Univ., Math.-Nat. R., 28 (1979), 535—548.
[26] Wendt, W.: Über eine Klasse von parameterabhängigen Funktional-Differentialgleichungen und ihre diskrete Approximation mit einer Anwendung auf stationäre Boltzmanngleichungen der Halbleiterphysik. Berlin: Humboldt-Univ., Sektion Math., Preprint Nr. 52 (1984).
[27] Wendt, W.: Eine Klasse von linearen Funktional-Differentialgleichungen und deren diskrete Approximation — eindimensionale Probleme. Berlin: Humboldt-Univ., Sektion Math., Seminarber. Nr. 54 (1984).
[28] Wendt, W.: Personal communication.
[29] Winkler, R.: Kurvenverfolgung bei Zwei-Punkt-Randwertaufgaben. Berlin: Humboldt-Univ., Sektion Math., Seminarber. Nr. 78 (1986).

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