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How to Measure Smoothness of Distributions on Riemannian Symmetric Manifolds and Lie Groups?

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Wir diskutieren verschiedene Arten von Mittelbildungen auf Riemannschen (global symmetrischen) Mannigfaltigkeiten, hyperbolischen Mannigfaltigkeiten und Lie-Gruppen, ihre gegenseitigen Beziehungen und damit zusammenhängende Funktionenräume.

Мы обсуждаем разного рода средних на римановых (глобально симметричных) многообразиях, гиперболических многообразиях и групп Ли, их всаимные отношения и связанные пространства функций.

We discuss several types of means on Riemannian (globally symmetric) manifolds, hyperbolic manifolds and Lie groups, their mutual interrelations and related function spaces.

1. Introduction

Let $\mathbf{R}^n$ be the euclidean n-space and let $B = \{y \mid |y| < 1\}$ be the unit ball in $\mathbf{R}^n$. Let k be a function defined on the real line such that $y \to k(|y|)$ is a C^∞ function on $\mathbf{R}^n$, supported by B. We introduce the means

$$k(t, f)(x) = \int_{\mathbf{R}^n} k(|y|)\, f(x + ty)\, dy, \qquad x \in \mathbf{R}^n, t > 0, \tag{1}$$

which make sense for any $f \in D'(\mathbf{R}^n)$ (appropriately interpreted). Let $L_p(\mathbf{R}^n)$ with $0 < p \leqq \infty$ be the usual spaces quasi-normed via

$$\|f \mid L_p(\mathbf{R}^n)\| = \Big(\int_{\mathbf{R}^n} |f(x)|^p\, dx\Big)^{1/p}, \qquad 0 < p < \infty, \tag{2}$$

with the usual modification if $p = \infty$. Let k_0 be a second function defined on the real line such that $y \to k_0(|y|)$ is a C^∞ function on $\mathbf{R}^n$, supported by B. Let $k_0(t, f)(x)$ be the corresponding means. Let $-\infty < s < \infty$, $0 < \varepsilon < \infty$ and $0 < r < \infty$. Let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Under additional assumptions for k_0 and k, which will be described in detail in Subsection 3.1, we introduce the spaces

$$F^s_{pq}(\mathbf{R}^n) = \Big\{ f \in D'(\mathbf{R}^n) \mid \|f \mid F^s_{pq}(\mathbf{R}^n)\|^{k_0,k}_{\varepsilon,r} = \|k_0(\varepsilon, f) \mid L_p(\mathbf{R}^n)\| + \Big\| \Big(\int_0^r t^{-sq}\, |k(t, f)(\cdot)|^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| L_p(\mathbf{R}^n) \Big\| < \infty \Big\} \tag{3}$$

(with the usual modification if $q = \infty$), which are independent of k_0, k, ε and r. Let $-\infty < s < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Let $-\infty < s_0 < s < s_1 < \infty$ and

$s = (1 - \theta) s_0 + \theta s_1$. Then we introduce the spaces

$$B^s_{pq}(\mathbf{R}^n) = \left(F^{s_0}_{pp}(\mathbf{R}^n), F^{s_1}_{pp}(\mathbf{R}^n)\right)_{\theta,q} \tag{4}$$

via the real interpolation method $(\cdot, \cdot)_{\theta,q}$, which are independent of the chosen numbers s_0 and s_1. These two scales $F^s_{pq}(\mathbf{R}^n)$ and $B^s_{pq}(\mathbf{R}^n)$ coincide with the spaces extensively treated in [13]. The above local-global approach goes essentially back to [14], see also the recent surveys [19] and [7], where the latter paper describes what has been done in this direction by the Russian school. We mention that the two above scales cover many well-known classical function spaces:

the Hölder-Zygmund spaces $\mathscr{C}^s = F^s_{\infty\infty}$ with $s > 0$;
the Besov spaces $\Lambda^s_{pq} = B^s_{pq}$ with $s > 0$, $1 \le p < \infty$, $1 \leqq q \leqq \infty$;
the fractional Sobolev spaces $H_p{}^s = F^s_{p2}$ with $-\infty < s < \infty$, $1 < p < \infty$;
the Sobolev spaces $W_p{}^s = F^s_{p2}$ with $1 < p < \infty$, $s = 0, 1, 2, \ldots$;
the (inhomogeneous) Hardy spaces $h_p = F^0_{p2}$, $0 < p \leqq 1$.

The characteristic feature of (3) reads as follows: Smoothness is measured locally via means, in particular via the behaviour of $k(t, f)(x)$ from (1) for $t \to 0$ (x fixed). Afterwards some global growth restrictions with respect to $x \in \mathbf{R}^n$ are required. The advantage of (1) compared with the Fourier-analytical approach preferred in [13] is its local nature, which is the basis to extend (1), (3) to more general structures: Riemannian manifolds and Lie groups. In this context it is more natural to introduce spaces of type F^s_{pq} first and then the spaces of type B^s_{pq} via real interpolation. We followed this path in a series of papers, see [15—18]. The first task is to find an appropriate counterpart of (1). As far as connected complete Riemannian manifolds M (with positive injectivity radius and bounded geometry) are concerned we interpreted the integration over $\mathbf{R}^n$ in (1) as an integration over the tangent space T_xM and $x + ty$ as the Riemannian geodesic $c(x, y, t)$ with $c(x, y, 0) = x$ and $dc(x, y, 0)/dt = y$. In the case of a (connected) Lie group G we replaced T_xG by the Lie algebra $\mathfrak{g}$ and we preferred the Lie geodesics $x \cdot \exp(ty)$ instead of the (left-invariant) Riemannian geodesics. However, beside these two interpretations of the euclidean means (1) on more general structures (Riemannian manifolds and Lie groups) there are few other possibilities, for example as spectral means via the Laplace operator or as convolutions. The aim of this paper is to study these diverse possibilities and their mutual interrelations. However, there is little hope (by the restricted knowledge of the author) to handle effectively Fourier analytical tools, spectral means and convolutions on general complete Riemannian manifolds with bounded geometry and positive injectivity radius. The situation improves essentially if one restricts the considerations to Riemannian globally symmetric manifolds or, even more restrictive, to hyperbolic manifolds. Then tools from the theory of Lie groups are available.

The paper is organized as follows. In Section 2 we discuss several types of means on manifolds. Section 3 deals with spaces on Riemannian (globally symmetric) manifolds and Lie groups. Spectacular new results should not be expected. It is our aim to discuss the diverse possibilities and to link our approach developed in [15—18] with Fourier analytical techniques in symmetric manifolds and Lie groups.

2. Several types of means

2.1. Manifolds. Let M be a connected n-dimensional C^∞ Riemannian manifold with bounded geometry and positive injectivity radius r_0. For details and references about this definition we refer to [15, 16]. Let T_PM be the tangent space at the point

$P \in M$ and let $|X| = \sqrt{g_P(X, X)}$ where $g_P(X, Y)$ with $P \in M$, $X \in T_PM$, $Y \in T_PM$, is a positive definite bilinear symmetric form standing for the Riemannian metric g on M. Let $P \in M$ and $X \in T_PM$, and let $c(P, X, t)$ be the geodesic with $c(P, X, 0) = P$ and $dc(P, X, 0)/dt = X$ where $t\,|X|$ is the arc length. Let (Ω, φ) be a local chart with $P \in M$ and let $U = \varphi(\Omega)$. Then one possible replacement of the euclidean means (1) is given by

$$k^R(t, f)\,(P) = \int_{T_PM} k(|X|)\, f\big(c(P, X, t)\big)\, dX$$

$$= \int_{T_{\varphi(P)}U} k(|\varphi_* X|)\,(f \circ \varphi^{-1})\,\big(c(\varphi(P), \varphi_* X, t)\big)\sqrt{|\det g_{\varphi(P)}|}\, d\varphi_* X \tag{5}$$

where the latter expression is the definition of the former one, $0 < t$ small. In particular, $k^R(t, f)\,(P)$ is independent of the chosen local chart (Ω, φ). For details we refer again to [15, 16]. Next we specialize the above local chart by $(\Omega, \exp_P^{-1})$, where $\exp_P$ stands for the exponential map. We identify T_0U with $\mathbf{R}^n$ (recall $\exp_P 0 = P$) and $\exp_P^{-1} X$ with X. Then we have

$$k^R(t, f)\,(P) = \int_{\mathbf{R}^n} k(|X|)\, f \circ \exp_P(tX)\, dX, \tag{6}$$

$0 < t \leqq r_0$, where r_0 has the above meaning.

2.2. Symmetric manifolds. Next we specialize the above manifold M. We assume that M is a connected analytic Riemannian globally symmetric manifold, see [3; IV, §3, p. 205] or [8; 2.2] or [9; XI] for a definition and properties (usually "symmetric manifolds" are called "symmetric spaces", but we prefer here the word "manifold", in order to avoid confusions with the "spaces" F^s_{pq} and B^s_{pq}). In particular, the Lie group $I(M)$ of all isometries on M acts transitively on M. Let $Q \in M$ be a fixed point and let $U_{QP} \in I(M)$ be an isometry which maps Q in P, where P is an arbitrary point. Let $\Omega_P(r)$ be the geodesic ball centered at $P \in M$ with radius r, where $r < r_0$. Then we have the distinguished local charts $\big(\Omega_P(r), \exp_Q^{-1} \circ U_{PQ}\big)$ with $U_{PQ} = U_{QP}^{-1}$ and (6) looks as

$$k^R(t, f)\,(P) = \int_{\mathbf{R}^n} k(|X|)\, f \circ U_{QP} \circ \exp_Q(tX)\, dX, \tag{7}$$

$0 < t \leqq r$ (all functions are extended outside of the unit ball by zero). In other words, first we shift f from $\Omega_P(r)$ into $\Omega_Q(r)$ via $U_{PQ}\Omega_P(r) = \Omega_Q(r)$, and then we use (6) where $\mathbf{R}^n$ is identified with T_QM. In particular, (7) has to be understood as an integral on the tangent space T_QM. We modify (7) somewhat. In (7) we equipped the tangent space $T_QM = \mathbf{R}^n$ with the euclidean metric $\sqrt{|\det g_O|}\, dX$. A second possibility is to equip $\mathbf{R}^n$ (in a neighbourhood of the origin) with the Riemannian metric $\sqrt{|\det g_X|}\, dX$ connected with the local chart $\big(\Omega_P(r), \exp_Q^{-1} \circ U_{PQ}\big)$. First (7) can be rewritten as

$$k^R(t, f)\,(P) = \int_{\mathbf{R}^n} k^t(|X|)\, f \circ U_{QP} \circ \exp_Q(X)\, dX, \tag{8}$$

$0 < t \leqq r$, with

$$k^t(|X|) = t^{-n} k(t^{-1}|X|). \tag{9}$$

Now the modification we have in mind reads as

$$k^{\bar{R}}(t, f)\,(P) \doteq \int_{\mathbf{R}^n} k^t(|X|)\, f \circ U_{QP} \circ \exp_Q(X)\sqrt{|\det g_X|}\, dX, \tag{10}$$

$0 < t \leqq r$ (again functions are extended outside of a ball of radius t by zero). The retransformation in the sense of (9) yields

$$k^{\bar{R}}(t, f)(P) = \int_{\mathbf{R}^n} k(|X|)\, f \circ U_{QP} \circ \exp_Q(tX) \sqrt{|\det g_{tX}|}\, dX, \qquad (11)$$

$0 < t \leqq r$. However, (10) can be interpreted as an integral on M itself. Let

$$k^{t,Q}(y) = \begin{cases} k^t(|\exp_Q^{-1} y|) & \text{if } y \in \Omega_Q(r), \\ 0 & \text{if } y \in M \setminus \Omega_Q(r) \end{cases} \qquad (12)$$

and let dy be the Riemannian volume element on M. Then (10) can be rewritten as

$$k^{\bar{R}}(t, f)(P) = \int_M k^{t,Q}(y)\, f \circ U_{QP}(y)\, dy. \qquad (13)$$

Finally one can retransform (11) (and hence also (13)) in the sense of (5) with $T_P M$ as basis. We have

$$k^{\bar{R}}(t, f)(P) = \int_{T_P M} k(|X|)\, f(c(P, X, t)) \sqrt{|\det g_{c(P,X,t)}|}\, dX, \qquad (14)$$

$0 < t \leqq r$ (again functions are extended outside of the unit ball by zero). This follows from the mentioned independence of the means in (5) of their concrete realization in local coordinates: for this purpose one has to replace f in (5) by $f \sqrt{|\det g|}$ and to interpret $\mathbf{R}^n$ in (11) as $T_{\varphi(P)}U$ with $\varphi = \exp_Q^{-1} \circ U_{PQ}$. Now (5) and (14) show the difference between these two means.

2.3. Convolutions. Let again M be the Riemannian globally symmetric manifold from Subsection 2.2. It is our aim to rewrite the means $k^{\bar{R}}(t, f)(P)$ as convolutions. For this purpose we have to recall some known facts for Riemannian globally symmetric manifolds. Let G be the identity component of the group of all isometries on M. Then G is a connected Lie group. Let K be the subgroup of G which leaves a given point $Q \in M$ fixed. Then K is compact and G/K is analytically diffeomorphic to M under the map $\gamma K \to \gamma \circ Q$ with $\gamma \in G$, see [3: IV, 3, p. 208] or [8: 2.2]. We assume in addition that G is unimodular (this holds also for K because K is compact). We put $M = G/K$. Let dy be the Riemannian volume element (invariant under G) and let $d\mu$ be the (right- and left-invariant) Haar measure on G. Let f be an integrable function on M. Then we may assume that dy and $d\mu$ are normed in such a way that

$$\int_M f(y)\, dy = \int_G f(\gamma \circ Q)\, d\mu \qquad (15)$$

holds. As far as (15) is concerned we refer to [2: V, 3, in particular Proposition 5, Theorem 9, Proposition 16 and the first example on p. 267] or [10: III, § 1, Theorem 1]. A special case may also be found in [4: p. 77] where $M = \mathrm{SU}(1,1)/\mathrm{SO}(2)$ is the unit circle in the complex plane equipped with the Poincaré metric. We wish to reformulate (13) in the sense of (15). The necessary extension of (15) to compactly supported distributions $h(y) \in D'(M)$ and $h(\gamma \circ Q) \in D'(G)$ causes no problem. We may assume $U_{QP} \in G$ in (13). Further we remark

$$k^{t,Q}(\gamma \circ Q) = k^{t,Q}(\gamma^{-1} \circ Q). \qquad (16)$$

This is a consequence of (12). Then (13) and (15), (16) yield

$$\begin{aligned} k^{\bar{R}}(t, f)(P) &= \int_G k^{t,Q}(\gamma^{-1} \circ Q)\, f(U_{QP} \circ \gamma \circ Q)\, d\mu \\ &= \int_G k^{t,Q}(\gamma^{-1} \circ U_{QP} \circ Q)\, f(\gamma \circ Q)\, d\mu = f * k^{t,Q}(U_{QP}K). \end{aligned} \qquad (17)$$

This is the convolution on Lie groups G and on manifolds $M = G/K$. We refer to [6: § 20, (20.10)] and [4: p. 77], the latter as far as the last formulation is concerned. Cf. also [2: V, 4].

2.4. Hyperbolic manifolds. A further specialization of the Riemannian globally symmetric manifolds trated in the Subsections 2.2 and 2.3 are the hyperbolic manifolds (again we speak about "hyperbolic manifolds" instead of "hyperbolic spaces", where the latter is the usual notation). We restrict ourselves to $n = 2$ mostly for sake of convenience and in order to have a quick reference to [4]. The general case may be found in [5], we refer also to [8: 1.11, 3.8]. Let $n = 2$. Then Poincaré's model of the non-euclidean plane is given by the open unit disk D in the plane $\mathbf{R}^2$ furnished with the usual Riemannian metric

$$ds^2 = (1 - x^2 - y^2)^{-2} (dx^2 + dy^2). \tag{18}$$

The Riemannian volume element and the Laplace-Beltrami operator are given by

$$dz = \frac{dx\, dy}{(1 - x^2 - y^2)^2} \quad \text{and} \quad \Delta = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{19}$$

respectively. As usual in this case we prefer the complex notation $z = x + iy$. Now D becomes a Riemannian globally symmetric manifold via the interpretation

$$D = \mathrm{SU}(1, 1)/\mathrm{SO}(2). \tag{20}$$

Here SU(1, 1) is the Lie group

$$\mathrm{SU}(1, 1) = \left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \,\middle|\, a \text{ and } b \text{ complex}, |a|^2 - |b|^2 = 1\right\}. \tag{21}$$

The action

$$\gamma: z \to \frac{az + b}{\bar{b}z + \bar{a}}, \qquad z \in D, \tag{22}$$

is isometric and transitiv. SO(2) stands for the subgroup of all rotations with the origin O as the fixed point ($b = 0$ in (21)). We refer to [4: pp. 48–52] for more details. SU(1, 1) (and of course also SO(2)) is unimodular: This follows from the fact that SU(1, 1) is isomorphic to SL(2, **R**) (which is well known and stated explicitly in [12: p. 80], including the Iwasawa decomposition of SU(1, 1)) and the known assertion that SL(2, **R**) is unimodular (which is stated explicitly in [10: p. 4]). Hence the considerations developed in the Subsections 2.2 and 2.3 are applicable. It is our aim to connect the means $k^{\mathbf{R}}(t, f)$ from (10), (13), (17) with some spectral means, i.e. with $\varphi(-\Delta)$ where φ is an appropriate function and Δ is the above Laplace-Beltrami operator from (19). Let $\langle z, b\rangle$ be the distance of the horocycle ξ (with sign) form the origin O with $z \in \xi$ and $b \in \xi \cap \partial D$ (i.e. ξ is the circle in $\overline{D}$ tangential to ∂D in the point b and with $z \in \xi$), cf. [4: p. 53]. The Fourier transform in D is given by

$$\tilde{f}(\lambda, b) = \int_D f(z)\, e^{(-i\lambda+1)\langle z,b\rangle}\, dz, \tag{23}$$

cf. (19), where λ is complex and $b \in \partial D$. The inverse Fourier transform is given by

$$f(z) = \frac{1}{4\pi} \int_{\mathbf{R}} \int_{\partial D} \tilde{f}(\lambda, b)\, e^{(i\lambda+1)\langle z,b\rangle}\, \lambda \tanh\left(\frac{\pi\lambda}{2}\right) db\, d\lambda \tag{24}$$

with $\int_{\partial D} db = 1$. We refer again to [4: pp. 55/56]. Let f be a radial C^∞ function in D with compact support. Then $\tilde{f}(\lambda, b)$ from (23) is independent of $b \in \partial D$ and coincides with the spherical transform $\tilde{f}(\lambda)$ in [4: pp. 66/67]. Let f_1 and f_2 be two such radial functions and let $f_1 * f_2$ be the convolution, see (17) and [4: p. 77]. Then we have

$$(f_1 * f_2)^{\sim} (\lambda, b) = \tilde{f}_1(\lambda, b)\, \tilde{f}_2(\lambda, b), \qquad b \in \partial D. \tag{25}$$

This follows from [4; pp. 77/78]. Next we use the fact that $e^{(i\lambda+1)\langle z,b\rangle}$ is an eigenfunction of Δ with respect to z, where λ complex and b with $|b| = 1$ are fixed,

$$\Delta(e^{(i\lambda+1)\langle z,b\rangle}) = -(\lambda^2 + 1)\, e^{(i\lambda+1)\langle z,b\rangle}, \tag{26}$$

cf. [4; p. 55]. Furthermore Δ is formally self-adjoint. At least for compactly supported C^∞ functions in D it follows from (23) and (26)

$$(\Delta f)^{\sim} (\lambda, b) = -(\lambda^2 + 1)\, \tilde{f}(\lambda, b), \qquad \lambda \text{ complex}, b \in \partial D. \tag{27}$$

Let φ be an appropriate function on the real line $\mathbf{R}$. Then it follows in the sense of the spectral theory of the positive definite self-adjoint Laplace-Beltrami operator $-\Delta$ in $L_2(D)$

$$\big(\varphi(-\Delta)\, f\big)^{\sim} (\lambda, b) = \varphi(1 + \lambda^2)\, \tilde{f}(\lambda, b), \qquad \lambda \in \mathbf{R}, b \in \partial D, \tag{28}$$

cf. also the mapping properties of $f \to \tilde{f}$ described in [4: p. 57, (iii)] (Plancherel's formula). Now we return to the means $k^{\bar{R}}(t, f)(P)$ specialized to the hyperbolic manifold $M = D$ under consideration. We put $Q = 0$, the origin, in (12) and (17). Then $k^{t,0}(y)$ with $y \in D$ is a radial compactly supported C^∞ function in D. Let us assume that f is also a radial compactly supported C^∞ function in D. Then (17) and (25) yield

$$k^{\bar{R}}(t, f)^{\sim} (\lambda, b) = \widetilde{k^{t,0}}(\lambda, b)\, \tilde{f}(\lambda, b), \tag{29}$$

where all the functions are independent of b. If one compares (28) and (29), then

$$\varphi^t(1 + \lambda^2) = \widetilde{k^{t,0}}(\lambda), \qquad \lambda \in \mathbf{R}, \tag{30}$$

(where we omitted b) is desirable. First we remark that $\widetilde{k^{t,0}}(\lambda)$ is an even entire function of exponential type, see [4; Theorem 4.7, p. 68]. Then we find a C^∞ function φ on $\mathbf{R}$ with (30). Then (28)—(30) yield

$$k^{\bar{R}}(t, f)(y) = \varphi^t(-\Delta)\, f(y), \qquad y \in D, \tag{31}$$

under the above restrictions. The moral of the story: For hyperbolic manifolds there is a connection between the Riemannian means $k^{\bar{R}}(t, f)$ and some spectral means $\varphi^t(-\Delta) f$.

3. Spaces

3.1. The euclidean case. All notations have the same meaning as in the introduction. We must clarify what conditions the functions k and k_0 have to satisfy such that the definition of $F^s_{pq}(\mathbf{R}^n)$ in (3) makes sense. We assume that there exists a rotation-symmetric C^∞ function $\varkappa$ in $\mathbf{R}^n$ supported by the unit ball B, such that

$$k(|y|) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\right)^N \varkappa(y), \qquad y \in \mathbf{R}^n, \tag{32}$$

where the natural number N will be determined later on. Let $\varkappa_0(y) = k_0(|y|)$, and let $\hat{\varkappa}_0$ and $\hat{\varkappa}$ be the Fourier transform of $\varkappa_0$ and $\varkappa$, respectively. We assume

$$\hat{\varkappa}(0) \neq 0 \qquad \text{and} \qquad \hat{\varkappa}_0(y) \neq 0 \quad \text{for all } y \in \mathbf{R}^n. \tag{33}$$

Now we are in the position to give a formal definition of the spaces $F^s_{pq}(\mathbf{R}^n)$ from (3):

Let $-\infty < s < \infty$, $0 < \varepsilon < \infty$ and $0 < r < \infty$. Let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Let k_0 and k be the above functions satisfying (33) and (32) with $2N > \max(s, n(1/p - 1), 0)$. Then $F^s_{pq}(\mathbf{R}^n)$ is given by (3).

Then one defines $B^s_{pq}(\mathbf{R}^n)$ by (4). Comments and references may be found in the Introduction.

3.2. Riemannian manifolds. We assume that M is the manifold from Subsection 2.1 with the positive injectivity radius $r_0 > 0$. Let again $\Omega_P(r)$ be the geodesic ball centered at $P \in M$ with radius $r < r_0$. If $\delta > 0$ is sufficiently small, then there exist a uniformly locally finite covering of M by a sequence of balls $\Omega_{P_j}(\delta)$ and a corresponding C^∞ resolution of unity $\psi = \{\psi_j\}$ with supp $\psi_j \subset \Omega_{P_j}(\delta)$. We refer for details and necessary explanations to [15, 16]. Coverings of this type have been used first by Calabi and Aubin, see [1]. Now we define the spaces $F^s_{pq}(M)$ as follows: Let $-\infty < s < \infty$ and let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Then

$$F^s_{pq}(M) = \left\{ f \in D'(M) \mid \|f \mid F^s_{pq}(M)\|^p = \left(\sum_{j=1}^{\infty} \|\psi_j f \circ \exp_{P_j} \mid F^s_{pq}(\mathbf{R}^n)\|^p \right)^{1/p} < \infty \right\} \tag{34}$$

(modification if $p = \infty$). Of course in (34) we extend $\psi_j f \circ \exp_{P_j}$ outside of $\exp^{-1}_{P_j} \Omega_{P_j}(\delta)$ by zero. Let $-\infty < s_0 < s < s_1 < \infty$, $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s = (1 - \theta) s_0 + \theta s_1$. Then

$$B^s_{pq}(M) = \left(F^{s_0}_{pp}(M), F^{s_1}_{pp}(M)\right)_{\theta,q} \tag{35}$$

where again $(\cdot, \cdot)_{\theta,q}$ stands for the real interpolation method. Again we refer for necessary explanations to [15, 16]. One of the main aims of the just cited papers is to give intrinsic descriptions. This is possible for both the spaces $F^s_{pq}(M)$ and $B^s_{pq}(M)$. But for sake of brevity we restrict ourselves to $F^s_{pq}(M)$. In comparison with [15, 16] it is clear how the corresponding formulations for the spaces $B^s_{pq}(M)$ look like. We assume that the means $k^R(t, f)(P)$ and $k^{\overline{R}}(t, f)(P)$ are given by (5) and (14), respectively, where the latter makes also sense for the manifold under consideration, which need not be globally symmetric. Furthermore k and k_0 have the same meaning as in Subsection 3.1, in particular we have (32), (33). Then it is clear what is meant by $k_0{}^R(t, f)(P)$ and $k_0{}^{\overline{R}}(t, f)(P)$. Let $L_p(M)$ with $0 < p \leqq \infty$ be the counterpart of $L_p(\mathbf{R}^n)$, cf. (2), now with respect to the Riemannian volume element on M.

Theorem 1: *Let $-\infty < s < \infty$ and let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Let $0 < r < r_0$, $\varepsilon > 0$ be small and*

$$N > \max(s, 5 + 2n/p) + \max\left(0, n(1/p - 1)\right). \tag{36}$$

Then

$$\|k_0{}^R(\varepsilon, f) \mid L_p(M)\| + \left\| \left(\int_0^r t^{-sq} \mid k^R(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right| L_p(M) \Bigg\| \tag{37}$$

and

$$\|k_0{}^{\overline{R}}(\varepsilon, f) \mid L_p(M)\| + \left\| \left(\int_0^r t^{-sq} |k^{\overline{R}}(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right| L_p(M) \Bigg\| \tag{38}$$

(modification if $q = \infty$) are equivalent quasi-norms in $F^s_{pq}(M)$.

Remark 1: It was one of the main aims of [15, 16] to prove this assertion as far as the quasi-norm in (37) is concerned. The corresponding claim for the quasi-norm (38) is new. But one can follow the arguments in [15, 16] with few technical changes. Then one obtains the desired assertion with respect to (38).

Remark 2: If M is the above manifold, then one would prefer the means $k^{\mathrm{R}}(t, f)$, i.e. the quasi-norms (37), because they look simpler and the use of the tangent space seems to be quite natural. But if one knows in addition that M is globally symmetric, then the means $k^{\overline{\mathrm{R}}}(t, f)$ are more attractive, because we have now the reformulations (13) and (17). In the case of hyperbolic manifolds one has also the (somewhat vague) connection with spectral means described in (31).

3.3. Lie groups. Let G be a n-dimensional connected Lie group (it is sufficient to assume that G consists of a finite number of connected components). Let e be the unit element of G and let $\mathfrak{g} = T_eG$ be the corresponding Lie algebra. Let g be a real positive definite symmetric bilinear form on $\mathfrak{g}$. Let $L_a: x \to ax$ be the left translation on G, where $a \in G$ and $x \in G$. Then the pull back operation $g_a = (L_{a^{-1}})_a^* g$ with $a \in G$ generates a left-invariant analytic Riemannian metric. This n-dimensional manifold is connected and complete, it has a positive injectivity radius and a bounded geometry. Hence we can apply the above theory, in particular we can introduce the spaces $F^s_{pq}(G)$ and $B^s_{pq}(G)$ via (34), (35) (with $M = G$). It is easy to see that these spaces are independent of the chosen bilinear form g on $\mathfrak{g}$. Hence we have Theorem 1 with respect to this left-invarinant Riemannian metric. However, it seems to be reasonable to replace the Riemannian means $k^{\mathrm{R}}(t, f)$ and $k^{\overline{\mathrm{R}}}(t, f)$ by the Lie means

$$k^L(t, f)(x) = \int_{\mathfrak{g}} k(|X|)\, f(x \cdot \exp(tX))\, dX, \quad x \in G, \tag{39}$$

$0 < t \leqq r$, where the Lie algebra $\mathfrak{g}$ is equipped with an euclidean metric, and exp stands for the usual exponential map on G. We assume again that k and k_0 have the same meaning as in Subsection 3.1, in particular we have (32), (33). Then it is clear what is meant by $k_0{}^L(t, f)(x)$. Let $L_p(G)$ with $0 < p \leqq \infty$ be the counterpart of $L_p(\mathbf{R}^n)$, see (2), now with respect to a fixed left-invariant Haar measure on G (which may be identified with the Riemannian volume element of the above left-invariant Riemannian metric).

Theorem 2: *Let G be the above Lie group. Let $-\infty < s < \infty$ and let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Let $\varepsilon > 0$ and $r > 0$ be sufficiently small and let N be given by* (36). *Then*

$$\|k_0{}^L(\varepsilon, f) \mid L_p(G)\| + \left\| \left(\int_0^r t^{-sq}\, |k^L(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \middle| L_p(G) \right\| \tag{40}$$

is an equivalent quasi-norm in $F^s_{pq}(G)$ (modification if $q = \infty$).

Remark 3: In [17, 18] we sketched two different proofs of this theorem (and its obvious B^s_{pq}-counterpart). The first proof in [17] is baed on the Campbell-Baker-Hausdorff formula. The second proof in [18] used the fact that $x \cdot \exp(tX)$ coincides with the geodesics of a left-invariant covariant derivation in the sense of [8; 1.7.7, 1.7.10, 1.7.13], see also [3; pp. 102 and 104]. Below we sketch a third proof which uses on the one hand the just cited left-invariant covariant derivation on G and on the other hand the above introduced means $k^{\overline{\mathrm{R}}}(t, f)$.

Proof of Theorem 2 (outline): Similarly as in the case of Riemannian globally symmetric manifolds we have also for the above Lie group G the group $\{L_a\}_{a \in G}$ of isometries (with respect to the introduced left-invariant Riemannian metric) which acts transitively on G. Let $k^{\bar{R}}(t, f)(x)$ with $t > 0$ small and $x \in G$ be the Riemannian means given by (10) or (11) with respect to the above left-invariant Riemannian metric. We put $P = x$, $Q = e$ and $U_{QP} = L_x$. Then (13) yields in our case

$$k^{\bar{R}}(t, f)(x) = \int_G k^{t,e}(y) f(x \cdot y) \, dy, \tag{41}$$

with the counterpart of (12), i.e.

$$k^{t,e}(y) = t^{-n} k\big(t^{-1} |\exp_e^{-1}(y)|\big) \tag{42}$$

in a neighbourhood of e. Here dy is the left-invariant Haar measure on G and $\exp_e$ is the exponential map from $\mathfrak{g} = T_e G$ into G in the Riemannian sense. We transform the result with the help of the Lie exponential map exp to $\mathfrak{g}$:

$$\begin{aligned} & k^{\bar{R}}(t, f)(x) \\ & = \int_{\mathfrak{g}} f(x \cdot \exp Y) \, t^{-n} k(t^{-1} |\exp_e^{-1} \circ \exp Y|) \, |d \exp Y| \, dY \\ & = \int_{\mathfrak{g}} f\big(x \cdot \exp (tX)\big) \, k(t^{-1} |\exp_e^{-1} \circ \exp (tX)|) \, |d \exp (tX)| \, dX. \end{aligned} \tag{43}$$

We wish to compare (43) with (39). For this purpose we remark that $X \to \exp_e^{-1} \circ \exp X$ is a diffeomorphic map near the origin of $\mathbf{R}^n = \mathfrak{g}$. However both maps $X \to \exp X$ and $X \to \exp_e X$ are governed by systems of ordinary differential equations for geodesic lines. This is obvious for $X \to \exp_e X$ and it follows for $X \to \exp X$ from the above remarks about left-invariant covariant derivation, see the cited references. Then we have

$$\exp_e^{-1} \circ \exp (tX) = tX + O(t^2), \tag{44}$$

where $O(t^2)$ stands for an analytic expression in X and t of the indicated order. Furthermore

$$d \exp (tX) = 1 + O(t), \tag{45}$$

where $O(t)$ is also an analytic expression of the indicated order. We put (44), (45) in (43). Then (39) and (43) yield

$$k^{\bar{R}}(t, f)(x) = k^{L}(t, f)(x) + \int_{\mathfrak{g}} f\big(x \cdot \exp (tX)\big) \, O(t) \, dX, \tag{46}$$

what means that $k^{\bar{R}}(t, f)$ and $k^{L}(t, f)$ coincide beside a harmless perturbation. Let s be large. Then we developed in [15: 4.2] in detail a machinery how to handle such perturbation terms, see also [17]. Then Theorem 1, in particular (38), (46) and this technique prove that (40) is an equivalent quasi-norm in $F^s_{pq}(G)$. The extension of this assertion to arbitrary values of s can be done by a lifting procedure described in [15, 16] ∎

Remark 4: We have $|\exp_e^{-1} y| = |\exp_e^{-1} y^{-1}|$ because L_y is an isometry on G: the Riemannian distance between y^{-1} and e is the same as between $e = L_y y^{-1}$ and $y = L_y e$. Hence we can replace $k^{t,e}(y)$ in (41) by $k^{t,e}(y^{-1})$. In other words, $k^{\bar{R}}(t, f)(x) = f * k^{t,e}(x)$, where the latter stands for the convolution on G, see [6: (20.10)].

Remark 5: In general exp and $\exp_e$ do not coincide. But if G is compact (or abelian), then we have $\exp = \exp_e$. Then the Lie geodesics coincide with the Riemannian geodesics and the above constructed left-invariant Riemannian metric is bi-invariant. We refer to [11: 4.2, 4.3], see also [3: IV, § 6].

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