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On the Identification of the Adiabatic Equation of State in Compressible Gas Flow

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Es wird das inverse Problem der Bestimmung des Koeffizienten in der Differentialgleichung $\nabla(\varrho \nabla\varphi) = 0$, $\varrho = \varrho(|\nabla\varphi|)$ betrachtet. Die zugrunde liegende Idee wird am Beispiel der Gasdynamik erläutert. Bei geeigneter Formulierung des Problems findet sich eine explizite Lösung.

Рассматривается обратная задача определения коэффициента в дифференциальном уравнении $\nabla(\varrho\nabla\varphi) = 0$, $\varrho = \varrho(|\nabla\varphi|)$. Основная идея объясняется на прикладном примере из газовой динамики. Показано, как при подходящей формулировке проблемы можно найти явное решение.

The inverse problem of determining the coefficient in the differential equation $\nabla(\varrho \nabla\varphi) = 0$, $\varrho = \varrho(|\nabla\varphi|)$ is considered. The underlying idea is demonstrated by a consideration of gas dynamics. An appropriate formulation of the problem yields an explicit solution.

1. Introduction. Consider a compressible gas in steady (i.e. time-independent) two-dimensional irrotational homentropic flow. See e.g. [1, 3]. Let u, v be the x, y velocity components. Set, as usual, $q = (u^2 + v^2)^{1/2}$ and denote pressure and density by p and ϱ, respectively. The equations governing the above flow are

$$\frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{1}$$

the equation of continuity and that for the absence of vorticity, respectively. For homentropic flows the equation of state, see [3], reads

$$p = F(\varrho). \tag{2}$$

The function F (increasing) characterizes the behaviour of compressibility of the considered gas. Next, for homentropic steady irrotational flow the Bernoulli equation

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} = \text{const} \tag{3}$$

holds throughout the region of flow [3], provided the gas is subject to no extraneous force, e.g. gravity. By (2), (3) a relation between speed and density

$$\varrho = G(q) \tag{4}$$

is established, where G is decreasing.

Frequently the equation of state is assumed to be that of an ideal gas, namely (with a proper choice of units)

$$p = \varrho^\gamma \tag{5}$$

where γ is a constant larger than unity, usually $\gamma = 1.4$ for air. Then (4) becomes, see [3],

$$\varrho = \frac{\gamma - 1}{2\gamma} (q_{\max}^2 - q^2)^{1/(\gamma-1)} \tag{6}$$

where $q_{\max}$ is the maximum possible speed, the so-called *escape velocity*, which is attained when the gas flows into vacuum. In this paper (5), (6) are dropped. Instead, the consideration aims at the identification of (4) by the use of appropriate observation, i.e. measurement of the gas flow on hand.

The forthcoming consideration deals with subsonic gas flows only. The gas equation (4), although yet unknown in detail, is mildly decreasing. The mass flux ϱq as function of the speed q is increasing for small q, and decreasing for large q. (Eventually it approaches zero at the escape velocity $q_{\max}$.) There is a critical speed for which the mass flux attains a maximum value. The flow is said to be *subsonic* if the gas speed is below this critical value everywhere in the flow region.

Consider again, for example, state equation (6). If $\varrho = 1$ for $q = 0$, then $q_{\max}^2 = 2\gamma/(\gamma - 1)$ and (6) becomes $\varrho = (1 - (\gamma - 1)/2\gamma\, q^2)^{1/(\gamma-1)}$. As may be easily checked, for the critical speed we have $q_{\mathrm{crit}}^2 = 2\gamma/(1 + \gamma)$.

2. Statement of the problem. Equations (1) give rise to the stream function Ψ and the potential φ, respectively:

$$\varrho u = \frac{\partial \Psi}{\partial y}, \quad \varrho v = -\frac{\partial \Psi}{\partial x} \quad \text{and} \quad u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}.$$

Thus (1) give the system

$$\varrho \varphi_x = \Psi_y, \quad \varrho \varphi_y = -\Psi_x. \tag{7}$$

Now consider a particular flow, i.e. boundary value problem for (6). We investigate the efflux of a plane jet from an aperture in a vessel bounded by semi-infinite straight vertical walls, as in Fig. 1, assuming the flow to be symmetric with respect to the x-axis.

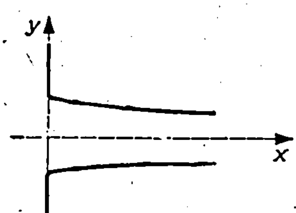

Fig. 1

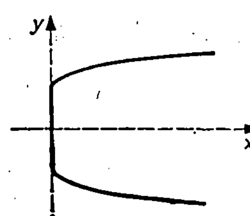

Fig. 2

To characterize the unknown shape of the free flow boundary we need additional information. Following Helmholtz and Kirchhoff, the gas speed has constant value along the free boundary. Assume the flux of matter (per unit of time) to be π. Thus the boundary values are, see e.g. [2],

$$\left.\begin{aligned} &\Psi = \frac{\pi}{2} && \text{on the upper line segment and on the upper free stream line,} \\ &\Psi = -\frac{\pi}{2} && \text{on the lower line segment and on the lower free stream line,} \\ &q = \mathrm{const} =: q_0 && \text{on the free stream lines.} \end{aligned}\right\} \tag{8}$$

This free boundary value problem is a problem of mapping the region of flow (bounded by the impermeable walls and the free stream boundaries) onto the parallel strip $-\infty < \varphi < +\infty$, $-\pi/2 < \Psi < \pi/2$. The determination of the free flow boundary is, of course, part of the problem. For existence and uniqueness see [3]. We want to use this particular solution to determine (4). To this end additional data, e.g. from measurement, are needed. In particular it is sensible to observe the gas speed on the vertical walls inside the vessel. This will become obvious in the next section.

Apart from the boundary value **Problem 1** made up by Fig. 1 and (8), different flow problems may be useful as well. Consider the infinite cavity occuring when a steady plane infinite stream in the horizontal direction impinges on a fixed vertical line segment. We again imagine the flow to be symmetric with respect to the x-axis. See Figure 2.
At the obstacle the stream line on the negative x-axis, say $\Psi = 0$, is split into an upper and lower branch. Hence the boundary values are, see [2],

$$\left.\begin{array}{ll} \Psi = 0 & \text{everywhere on the boundary of the flow region enclosing the cavity behind the obstacle, i.e. on the vertical wall and on the free stream boundaries;} \\ q = \text{const} =: q_0 & \text{on the free stream boundaries.} \end{array}\right\} \quad (9)$$

At the origin of coordinates q vanishes. Let there $\varphi = 0$. For the corresponding mapping problem the image of the domain of flux is the entire potential plane cut along the non-negative φ-axis $-\infty < \varphi, \Psi < +\infty$, $|\Psi| + |\varphi| - \varphi > 0$. For the treatment of the free boundary value **Problem 2**, which consists of Fig. 2 and (9), see [2]. To identify the relation (4) we again may observe the gas speed on the surface of the vertical obstacle.

As pointed out in [2], the presented model performs quite well as a description of cavity phenomena, whereas it is a poor account of wakes. Hence the observations needed should be done for cavities rather than wakes.

3. Treatment of the free boundary value problems. The appropriate method to treat Problems 1 and 2 is hodograph transformation, since for both 1 and 2 the hodograph, i.e. the domain covered by u, $-v$ in the velocity plane, is a semidisk, see [1—3].

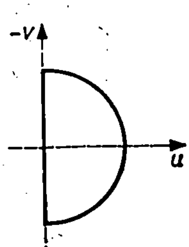

Fig. 3

The vertical segment of the hodograph-semidisk corresponds to the rigid wall, and the circular arc corresponds to the free boundaries in the plane of flow. The radius of the semicircle is the maximum gas speed attained at the free stream boundaries. Assume that these values q_0 for Experiment 1 and 2 coincide. The density is constant on circular arcs around the origin. Its maximum value occurs at the origin $q = 0$, its minimum value for $q = q_0$. It is useful to introduce polar coordinates in the velocity plane: $q = (u^2 + v^2)^{1/2}$ and $\vartheta = \arctan (v/u)$. Transforming (7) to hodograph coordinates q, ϑ gives [1, 3]

$$\varphi_q = q \frac{d}{dq}\left(\frac{1}{q\varrho}\right) \Psi_\vartheta, \qquad \varphi_\vartheta = \frac{q}{\varrho} \Psi_q. \tag{10}$$

For subsonic gas flows this system is elliptic. To get a system more suitable for investigation we substitute q by r according to

$$\frac{dr}{r} = \sqrt{\frac{1}{\varrho} \frac{d(q\varrho)}{dq}} \frac{dq}{q}. \tag{11}$$

That is $r = \exp \int\limits_{q_0}^{q} \sqrt{\frac{1}{\varrho} \frac{d(q\varrho)}{dq}} \frac{dq}{q}$, $r(q_0) = 1$. So we arrive at

$$\tau\varphi_r = -\frac{1}{r} \Psi_\vartheta, \qquad \tau\varphi_\vartheta = r\Psi_r, \tag{12}$$

where

$$\tau(r) = \varrho \Big/ \sqrt{\frac{1}{\varrho} \frac{d(q\varrho)}{dq}}. \tag{13}$$

Now the hodograph in the plane with polar coordinates r, ϑ is a unit-semidisk. The transformation q to r is a distortion of the semidisk on polar rays only, whereas the polar angles remain the same. The boundary conditions for the system (12) are for

Problem 1: $\Psi = \frac{\pi}{2}$ on the upper boundary $-\vartheta > 0$,

$\Psi = -\frac{\pi}{2}$ on the lower boundary $-\vartheta < 0$,

Problem 2: $\Psi = 0$ on all the boundary.

The point is that the system (12) is invariant with respect to conformal transformations of the independent coordinates, as may easily be checked. Introduce the complex notation $w = r\,e^{-i\vartheta}$ and use in particular the conformal transformation $z = 2/(1/w - w)$. By this the hodograph unit-semidisk is mapped onto the right half-plane $\operatorname{Re} z > 0$. Let $z = Re^{-i\alpha}$. The upper half of the semidisk $0 < r < 1$, $0 < -\vartheta < \pi/2$ is mapped onto the quadrant $0 < R < \infty$, $0 < -\alpha < \pi/2$. By $z = 2/(1/w - w)$ the system (12) is transformed to

$$\tau\varphi_R = -\frac{1}{R} \Psi_\alpha, \tag{14}$$

$$\tau\varphi_\alpha = R\Psi_R. \tag{15}$$

Now it is a simple matter to construct the solutions to Problems 1 and 2. For an idea consider the case of incompressibility. That is $\varrho = \text{const}$, $\tau = \text{const}$. Then

$$\tau\varphi_1 + i\Psi_1 = \log z,$$
$$\tau\varphi_2 + i\Psi_2 = -z^2.$$

Accordingly, in the case of a compressible fluid

$$\Psi_1 = \arg z = -\alpha, \tag{16}$$
$$\Psi_2 = -\operatorname{im} z^2 = R^2 \sin 2\alpha. \tag{17}$$

4. Treatment of the inverse problem. For Experiment 1 the upper impermeable wall is mapped onto $0 < R < 1$, $\alpha = -\pi/2$, as is true for the lower half of the obstacle for Experiment 2. Each value of gas speed inside the flow region is also attained on the wall. Hence it should be possible to reconstruct (4) from 0 to q_0 by measurements on the rigid wall. In both cases we observe $y = y(q)$ on just these segments. Since at these boundaries $u = 0$, $q = -v > 0$ we have $q = -d\varphi/dy$, whence

$$\frac{d\varphi}{dq} = \frac{d\varphi}{dy}\frac{dy}{dq} = -q\,\frac{dy}{dq}. \tag{18}$$

We now consider the inverse problem of determining (4) on the line segment $0 < R < 1$, $\alpha = -\pi/2$. From (14) we get $d\varphi/dR = -(1/\tau R)\,\Psi_\alpha$. Together with (18) we have

$$\frac{dR}{dq} = \tau\,\frac{Rq}{\Psi_\alpha}\,\frac{dy}{dq}. \tag{19}$$

Further, $z = 2/(1/w - w)$ gives on the interval of observation

$$R = 2r/(1 + r^2). \tag{20}$$

Thus, to identify (4), the system (11), (13), (19), (20) must be solved. But this turns out to be fairly involved, no matter whether it is based on (16) or (17). Instead, we use both Experiment 1 and 2 to derive

$$\frac{d\varphi_1}{d\varphi_2} = \frac{dy_1}{dy_2} =: \frac{1}{2}\,f(q) \tag{21}$$

where f is known from $y_1 = y_1(q)$ and $y_2 = y_2(q)$. On the other hand (14), (16), (17) yield $d\varphi_1/d\varphi_2 = 1/2R^2$. Thus, with (20), (21) we get

$$f(q) = \frac{1}{4}\left(r + \frac{1}{r}\right)^2, \tag{22}$$

$$r = f^{1/2} - (f - 1)^{1/2}. \tag{23}$$

Necessarily $f > 1$. In (23) the sign of the second square root has to be minus because $0 < r < 1$. From (23) it follows that $dr/r = -df/\left(2f^{1/2}(f-1)^{1/2}\right)$. By (11) one concludes that

$$-\frac{df}{2f^{1/2}(f-1)^{1/2}} = \sqrt{\frac{1}{\varrho}\,\frac{d(q\varrho)}{dq}}\,\frac{dq}{q},$$

$$\left(\frac{df}{dq}\right)^2 \frac{q^2}{4f(f-1)} = \frac{1}{\varrho}\,\frac{d(q\varrho)}{dq} = 1 + \frac{d(\ln \varrho)}{d(\ln q)},$$

$$\varrho = \frac{1}{q}\exp\int^q \left(\frac{df}{dq}\right)^2 \frac{q}{4f(f-1)}\,dq.$$

Thus the material relation (4) can be explicitly determined if both experiments are carried out, instead of one of them alone. The consideration above is based on the fact that the material property (4) is one and the same, and therefore involved in both φ_1 and φ_2. Although the system (11), (13), (19), (20) is much more difficult to solve for (4) than (22) is, the inverse problem might be completely solvable by one experiment only.

The material property (4), even if unknown yet, controls both φ_1 and φ_2. Hence the data φ_1, φ_2 depend on each other. They fulfil some relation of compatibility which is derived as follows. Assume we know $f = f(q)$. From (11) we get

$$\frac{1}{\varrho} \frac{d\varrho}{dq} = \left(\frac{dr}{dq}\right)^2 \frac{q}{r^2} - \frac{1}{q} =: g(q), \tag{24}$$

which can be evaluated by (23). Further, (14), (16) give $\frac{d\varphi_1}{dq} = \frac{1}{\tau} \frac{1}{R} \frac{dR}{dr} \frac{dr}{dq}$. Together with (11), (13) we obtain

$$\varrho \frac{d\varphi_1}{dq} = \frac{q}{rR} \frac{dR}{dr} \left(\frac{dr}{dq}\right)^2 =: h(q), \tag{25}$$

where $h = h(q)$ can be determined by (20), (23). Now, with (24), logarithmic differentiation of (25) yields

$$\frac{1}{q} + \frac{d}{dq} \ln\left(-\frac{dy_1}{dq}\right) = \frac{d}{dq} (\ln(h)) - g. \tag{26}$$

This is the relation of compatibility between $\varphi_1(q)$ and $\varphi_2(q)$. Note that by (20), (21), (23) the quantities g and h involve the observed relation $f = f(q)$.

To reconstruct (4) from the first observation $y_1 = y_1(q)$ only means to solve the system (11), (20), (24)–(26), where in this case g, h, r, ϱ are unknown functions of q.

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