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Regularized Parameter Identification in Elliptic Boundary Value Problems¹)

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In der Arbeit wird die Methode der Regularisierung zur numerischen Identifikation des Diffusionskoeffizienten in elliptischen Gleichungen zweiter Ordnung untersucht. Zur Lösung der entsprechenden regularisierten diskreten und stetigen Minimierungsaufgaben wird das Gauß-Newton-Verfahren analysiert. Es wird ein effektiver Weg beschrieben, der pro Iterationsschritt das Lösen einer Gleichung zweiter Ordnung und einer Gleichung vierter Ordnung erfordert.

В работе исследуются методы регуляризации численной идентификации коэффициента диффузии эллиптического уравнения второго порядка. Обсуждается применение метода Гаусса-Ньютона для решения соответствующих регуляризованных дискретных и непрерывных экстремальных задач. Опивывается эффективный алгоритм, требующий решения одного уравнения второго порядка и одного уравнения четвертого порядка на каждой итерации.

In this paper the regularization method applied to the numerical identification of the diffusion coefficient in second order elliptic equations is investigated. For solving the corresponding regularized discrete and continuous nonlinear minimization problems the Gauss-Newton method is analyzed. We describe an effective way for performing one iteration step which requires to solve one problem of 2nd order and one problem of 4th order.

1. Introduction

In this paper we investigate parameter identification problems and their approximations for a special class of elliptic boundary value problems. Although our ideas are applicable to a wide class of identification problems, our work here is devoted to a special inverse problem of aquifer transmissivity identification (cf. [1, 27]). As our fundamental state equation we consider the elliptic equation

$$
-\text{div}(a \text{ grad } u) = f, \qquad x \in \Omega
$$

 (1.1)

This equation, for example, describes the steady flow in a confined nonuniform aquifer Q, where $a = a(x)$ denotes the spatially varying transmissivity coefficient of the aquifer, $u = u(x)$ the (observed) piezometric head and $f \stackrel{\sim}{=} f(x)$ the source term. In the direct problem for given a , f and appropriate boundary conditions the unique solution u is to be determined. This direct problem, in general, is a well-posed one.

In the parameter identification problem a is to be found for given u and f . In this problem we are faced with the identifiability problem and with a noncontinuous dependence of a on the observation data $z = u + \eta$ (η denotes the observation error) with respect to any meaningful observation topology. We say a is *identifiable* if, for given u and f and given boundary conditions, a is unique. Several authors (cf. [4, 6, 22]) study the identifiability of a in one- and higher-dimensional problems (1.1) and

¹) Vortrag auf der gemeinsamen Tagung vom 9. - 13. 11: 1987 in Berga (DDR) der HFR Analysis und HFR Numerische Mathematik.

the paper $[18]$ investigates the identifiability of a in different discretized versions of (1.1) for one space dimension. The noncontinuous dependence of a from z is caused by the differentiation of *z* in order to determine grad *u* and can lead to serious difficulties if the infinite-dimensional identification problem is approximated by finitedimensional problems. The numerical solution may 'show undesirable oscillations or may not converge at all. Therefore, special identification techniques are needed to overcome such difficulties. The accompanying is a partial list of numerical methods for parameter identification: the differentiation of z in order to determine grad u and can lead to serious difficulties if the infinite-dimensional identification problem is approximated by finite-
dimensional problems. The numerical solution may 'sh

1. Least Squares Methods $\{3, 5, 6, 18\}$, 2. Tichonov's Regularization Methods $[9, 16, 21, 24]$,

3. Adaptive (or Asymptotic) Regularization Methods [2, 12].

All these methods use further a priori information about the parameter to be determined in order'to overcome the ill-posedness of the identification problem.

The present paper is organized-as follows, in Section 2 we describe the Tichonov's regularization method for parameter identification problems and its.Galerkin approximations. Section 3 contains results to the effective numerical solution of the corresponding discrete nonlinear minimization problems by the Gauss-Newton method. In Section 4 we generalize the results of Section 3 to the continuous case and show that one iteration step of the Gauss-Newton method requires only to solve one problem of 2nd order and one problem of 4th order. *x* et al. **m** exercises one functionary or entertionary interesting in an ized as follows. In Section 2 we describe the Tichonov's
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2. Parameter identification by Tichonov's regularization

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with any of t As our fundamental state equation which is used in the sequel we consider the following prototype of a two point boundary value problem */ . —(au*

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-(au_x)_x = f, \qquad x \in (0, 1), \tag{2.1}
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underer identification by Tichonov's regularization

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-(au_x)_x = f
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 $u(0) = g_0$, $u(1) = g_1$,
 $a(0) u_x(0) = g_0$, $a(1) u_x(1) = g_1$,
 $a(0) u_x(0) + s_0 u(0) = g_0$, $a(1) u_x(1) + s_1 u(1) = g_1$.
(2.3)
 $a(0) u_x(0) + s_0 u(0) = g_0$, $a(1) u_x(1) + s_1 u(1) = g_1$.
(2.4)
ation a is supposed to be' unknown so we shall consider the problem of identity-
junction from observations of the state variable u. This problem is ill-posed.
consider a specific case in which continuity of a with respect to u is violated.

$$
a(0) u_x(0) + s_0 u(0) = g_0, a(1) u_x(1) + s_1 u(1) = g_1.
$$
\n(2.4)

The function a is supposed to be unknown so we shall consider the problem of identifying the function from observations of the state variable *u.* This problemis ill-posed. Let us consider a specific case in which continuity of a with respect to *u* is violated. Consider (2.1), (2.2) with $f = 0$, $g_0 = 1$, $g_1 = 2$ and $a(0) = 1$. Given the data $u(x)$ $= 1 + x$, then (2.4) has the unique solution $a(x) = 1$. Given the *data'u_k*(x) = $u(x)$ $+ \sin 2k\pi x/(2k\pi)$, then (2.1) has the unique solution $a_k(x) = 3/(2 + \cos 2k\pi x)$. Then $(1) + s_0 u(0) = g_0$, $a(1) u_x(1) + s_1 u(1)$
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the unique solution $a(x) = 1$. Given bservations of the state variable *u*. This posservations of the state variable *u*. This pic case in which continuity of *a* with resp th $f = 0$, $g_0 = 1$, $g_1 = 2$ and $a(0) = 1$. Given th (2.1) has the unique solution 1. Given the data $u_k(x) = 3/(2 + \cos 2k\pi x)$. Thereas for all k,
the data $u_k(x) = 3/(2 + \cos 2k\pi x)$. Thereas for all k,

obviously,
$$
u_k
$$
 converges to u uniformly with increasing k , whereas for all k ,
\n
$$
||a - a_k||_{L^{1}(0,1)}^2 = \int_0^1 \frac{(1 - \cos 2k\pi x)^2}{(2 + \cos 2k\pi x)^2} dx
$$
\n
$$
= \left[x - \frac{3}{2k\pi} \frac{\sin 2k\pi x}{2 + \cos 2k\pi x}\right]_0^1 = 1.
$$

 $=\left[x - \frac{1}{2k\pi} \frac{1}{2 + \cos 2k\pi x}\right]_0^{\pi} = 1.$
In summary, $u_k \to u$, yet $a_k \to \hat{a}$ and a is therefore not a continuous function of the data. data.

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The state equations (2.1) with any of the three boundary conditions $(2.2) - (2.4)$ (in case (2.2) for simplicity we suppose homogeneous boundary conditions) can, be R

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 $\langle au_x, v_x \rangle = \langle f, v \rangle + g_1 v(1) - g_0 v(0)$

Regularized Parameter Identification
\nThe state equations (2.1) with any of the three boundary conditions (2.2)–(2.4)
\n(in case (2.2) for simplicity we suppose homogeneous boundary conditions) can be
\nwritten in weak formulation:
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$$
\langle au_x, v_x \rangle = \langle f, v \rangle
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$$
\langle au_x, v_x \rangle = \langle f, v \rangle + g_1 v(1) - g_0 v(0) \qquad | \qquad \forall v \in U,
$$
\n
$$
\langle au_x, v_x \rangle + s_1 u(1) v(1) - s_0 u(0) v(0) = \langle f, v \rangle + g_1 v(1) - g_0 v(0)
$$
\n
$$
\forall v \in U
$$
\n
$$
\langle \langle \cdot, \cdot \rangle \text{ denotes the scalar product of } L^2(0, 1) \rangle. Choosing $A = H^1(0, 1), f \in L^2(0, 1),$
$$

 $A_c = {a \in A : a(x) \ge a \text{ a.e. } x \in (0, 1), \text{ for some } a > 0}$, then the following results with respect to solvability of the variational equations (2.5) – (2.7) are valid (cf. The state equations (2)

(in case (2.2) for simplic

written in weak formulat
 $\langle au_x, v_x \rangle = \langle f, v \rangle$
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with respect to solv

Lemma 1: For any $a \in A_c$, the following is true:

'(i). there exists a unique solution $u \in U = H_0^1(0, 1)$ to (2.5) ;

(ii) under the condition $\int f dx + g_1 - g_0 = 0$ *there exists a unique solution* $u \in U_c$

$$
= \left\{ u \in U = H^{1}(0, 1) : \int_{0}^{1} (u - z) dx = 0 \right\} to (2.6);
$$

(iii) *under the conditions* $s_0 < 0 < s_1$ *there exists a unique solution* $u \in U = H^1(0, 1)$ *to(2.7).*

Remark: In fact, a stronger result is possible, i.e. Lemma 1 remains true for any $a \in A_{c}$, ${a \in L^{\infty}(0, 1): a(x) \geq a \text{ a.e. } x \in (0, 1); \text{ for some } a > 0}.$

Let us introduce the set $A_{ad} \subset A_c$ of physically admissible parameters a and let $b \in A_{ad}$ a suitable estimate of the unknown parameter *a*. Furthermore, for a brief notation let us rewrite the problems $(2.5) - (2.7)$ in form of an operator equation

$$
f(a, u) = \dot{\phi}, \tag{2.8}
$$

where the bilinear operator $\mathfrak{F}: A_{c} \times U \to U^{*}$ induced by the left-hand-sides of where the bilinear operator $\mathfrak{F}: A_c \times U \to U^*$ induced by the left-hand sides of (2.5)—(2.7) is bounded. Applying Tichonov's regularization method (cf. [25]) to the identification problem we are led to the nonlinear prog the identification problem we are led to the nonlinear programming problem $b \in A_{ad}$ a suitable estimate of the unotation let us rewrite the problems (
 $\mathfrak{F}(a, u) = \phi$,

where the bilinear operator $\mathfrak{F} : A_c$
 $(2.5) - (2.7)$ is bounded. Applying

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$$
\inf_{a \in A_{n}a}{J_{a}(a); J_{a}(a)} = \frac{1}{2} ||u - z||_{L^{2}(0, 1)}^{2} + \frac{\alpha}{2} ||a - b||_{L^{2}(0, 1)}^{2},
$$

where $u(x; a)$ satisfies (2.8) .

Here $\alpha > 0$ denotes the regularization parameter to be chosen appropriately (cf. [13]). Roughly speaking, the term $\|a - b\|_{L^{1}(0,1)}^2$ in the functional J_a prevents the solution *a* from the divergence. We now consider the approximation of problem (P). In the following we describe a Galerkin-based parameter/state approximation scheme in the same spirit of the ideas found in [18] and divide our considerations into two stages. In the first stage we carry out the parameter approximation. That is, we seek the parameters \dot{a} in the form $a^m = a_1 \varphi_1 + \ldots + a_m \varphi_m$, where $A_m = \mathfrak{L}(\varphi_1)$, $\varphi_2, \ldots, \varphi_m$) denotes an *m*-dimensional subspace of $A := \overline{\mathfrak{X}}(\varphi_1, \varphi_2, \ldots, \varphi_m, \varphi_{m+1}, \ldots)$ is bounded. Applying Tichonov's regularization method (cf. [25]
aation problem we are led to the nonlinear programming problem
 $\dot{f} J_s(a); J_s(a) = \frac{1}{2} ||u - z||_{L^1(0, 1)}^2 + \frac{\alpha}{2} ||a - b||_{L^1(0, 1)}^2$,
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i.e. $u(x; a)$ and describe this approximation by the bounded linear projection operator \mathfrak{P}^m : *A* $\rightarrow \mathbb{R}^m$, $\mathfrak{P}^m a:= \tilde{a}$ if $a'= a^m$, $\mathfrak{P}^m \varphi_i := 0$ if $i > m$ with $\tilde{a} = (a_1, a_2, \ldots, a_m)^T \in \mathbb{R}^m$. Introducing the finite-dimensional set $A_{ad}^m = \mathfrak{P}^m A_{ad}$, we are led to the parameter approxi-

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mation problem

$$
(\mathbf{P}_m) \qquad \min_{\tilde{a} \in \mathcal{A}^m} J_{\mathfrak{a}}(\tilde{a}); J_{\mathfrak{a}}(\tilde{a}) = \frac{1}{2} ||u - z||_{L^2(0, 1)}^2 + \frac{\alpha}{2} ||\tilde{a} - \mathfrak{P}^m b||_{\mathbf{R}^m}^2,
$$

where
$$
u(x; a^m)
$$
 satisfies (2.8).

In the second stage we consider the approximation of the state equations $(2.5) - (2.7)$. That is, we seek approximate solutions to $(2.5) - (2.7)$ in the form $u^n = u_1 y_1 + \dots$ $u_1 + u_n \psi_n$, where $U_n = \mathfrak{L}(\psi_1, \psi_2, \ldots, \psi_n)$ denotes an *n*-dimensional subspace of $U = \overline{\mathfrak{L}}(\psi_1, \psi_2, \ldots, \psi_n)$ $\psi_2, \ldots, \psi_n, \psi_{n+1}, \ldots$ and describe this approximation by the bounded linear projection operator \mathfrak{D}^n : $U \to \mathbf{R}^n$, $\mathfrak{D}^n u := \tilde{u}$ if $u = u^n$, $\mathfrak{D}^n \psi_i := 0$ if $i > n$ with $\tilde{u} = (u_1, u_2, \dots, u_n)$ $u_2, ..., u_n$ ^T $\in \mathbb{R}^n$. Introducing the differentiation operator $D = d/dx$ the corresponding state equations $(2.5) - (2.7)$ can be approximated by $(i = 1, 2, ..., n)$

$$
a^m D u^n, D \psi_i \rangle = \langle f, \psi_i \rangle, \tag{2.9}
$$

$$
\langle a^m D u^n, D \psi_i \rangle = \langle f, \psi_i \rangle + g_1 \psi_i(1) - g_0 \psi_i(0), \qquad (2.10)
$$

$$
\langle a^m D u^n, D \psi_i \rangle + s_1 u^n(1) \psi_i(1) - s_0 u^n(0) \psi_i(0) = \langle f, \psi_i \rangle + g_1 \psi_i(1) - g_0 \psi_i(0).
$$
\n(2.11)

Note that these approximated state equations can be written in the compact form

$$
F(\tilde{a},\tilde{u}) = \tilde{f}, \qquad F: (\tilde{a},\tilde{u}) \in \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n. \tag{2.12}
$$

Let us regard a special example. Let $(0, 1)$ be partitioned into N subintervals of length $h = 1/N$. For $i = 1, ..., N$ we define the 0th order spline basis functions

 $\varphi_i(x) = \begin{cases} 1, & (i-1) \ h \leq x \leq ih, \ 0, & \text{otherwise} \end{cases}$

and for $j = 1, ..., N - 1$ the linear spline basis functions

 $\psi_j(x) = \begin{cases} Nx + 1 - j, & (j - 1) \ h \leq x \leq jh, \\ -Nx + 1 + j, & jh \leq x \leq (j + 1) \ h, \\ 0, & \text{otherwise.} \end{cases}$

Thus we have chosen $m = N$ and $n = N - 1$. Let us regard the approximate state equation (2.9). Then it is straightforward to compute analytically the integrals of the system (2.9) and hence to obtain the corresponding parametric finite-dimensional operator equation (2.12) in the form

$$
-a_{i+1}(u_{i+1}-u_i)+a_i(u_i-u_{i-1})=h^2\langle f,\psi_i\rangle, \qquad i=1,\ldots,N-1, \qquad (2.13)
$$

where $u_0 = u_N = 0$.

Now we are able to formulate the parameter/state approximation problem:

 $(P_{m,n})$ $\min_{\tilde{a}\in A_m^m} J_a(\tilde{a}); J_a(\tilde{a}) = \frac{1}{2} ||\tilde{u} - \mathfrak{D}^n z||_{\mathbf{R}^n}^2 + \frac{\alpha}{2} ||\tilde{a} - \mathfrak{P}^m b||_{\mathbf{R}^m}^2$

where \tilde{u} satisfies (2.12).

3. Solution of the discrete minimization problems

In this chapter we examine the numerical solution of problem $(P_{m,n})$, which is a finitedimensional nonlinear programming problem to obtain the coefficients a_1, a_2, \ldots, a_m of the unknown function $a^m(x) = a_1 \varphi_1(x) + \ldots + a_m \varphi_m(x)$. One possibility for

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solving this nonlinear programming problem is to use a gradient projection method. The main'work required at each iteration is to evaluate the gradient $(\partial J_a(\bar{a})/\partial a_1, \ldots, \partial J_a(a)/\partial a_m)$. This can be done after introducing of an appropriate adjoint variable **a**
a
a Regularized Parameter Identification
a The main work required at each iteration is to use a gradient projection method
The main work required at each iteration is to evaluate the gradient $(\partial J_a(\bar{a})/\partial a_1, \$ by the solution of only two linear systems of equations with a sparse matrix (cf. [3, 24]). Since problems $(P_{m,n})$ in general have a flat global minimum in deep banana shaped valleys, they should better be treated by (projecied) Gauss-Newton methods. which make it possible to proceed in great steps along the deep valleys and assures fast convergence. Let $\Delta u[\Delta a]$ be the solution of the linearized finite-dimensional state equation solving this nonlinear programming proble

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 $F_{\tilde{a}}'\$ *a*: α ² work required at each iteratio α ³. This can be done after intimulation of only two linear system Since problems $(P_{m,n})$ in general alleys, they should better be treake it possible to proceed in gree erge

$$
F_{\tilde{\mathbf{a}}'} \widetilde{\triangle u} + F_{\tilde{\mathbf{a}}'} \widetilde{\triangle a} = 0 \tag{3.1}
$$

where $F_{\tilde{a}}' = F_{\tilde{a}}'(\tilde{a}, \tilde{u})$ and $F_{\tilde{a}}' = F_{\tilde{a}}'(\tilde{a}, \tilde{u})$ denote the Frechet derivatives of the mapping F with respect to \tilde{u} and \tilde{a} . Then in the (projected) Gauss-Newton method a given approximation $\tilde{a} \in \mathbb{R}^m$ is improved by

where $\widehat{\triangle a}$ is the solution of the linearized problem

fast convergence. Let
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\widehat{\Delta u}[\Delta a]
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 be the solution of the linearized finite-dir
equation
 $F_{\tilde{u}}'\widehat{\Delta u} + F_{\tilde{a}}'\widehat{\Delta a} = 0$
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mapping F with respect to \tilde{u} and \tilde{a} . Then in the (projected) Gauss-N
a given approximation $\tilde{a} \in \mathbb{R}^m$ is improved by
 $\tilde{a} := P_{ad}^m[\tilde{a} + \gamma \widehat{\Delta a}],$
where $\widehat{\Delta a}$ is the solution of the linearized problem
 $(P_{m,n}^1)$ inf $J_a^1(\widehat{\Delta a})$; $J_a^1(\widehat{\Delta a})' = \frac{1}{2} ||\tilde{u} + \widehat{\Delta u}[\widehat{\Delta a}] - \mathfrak{D}^n z||_{\mathbb{R}^n}^2 + \frac{\alpha}{2} ||\tilde{a} + \widehat{\Delta a} - \mathfrak{P}^m b||_{\mathbb{R}^m}^2$,
where $\widehat{\Delta u}[\Delta a]$ satisfies (3.1),
 P_{ad}^m is the projection onto A_{ad}^m and γ is the steplength parameter to be

where $\widetilde{\Delta u}(\widetilde{\triangle a})$ satisfies (3.1),

 P_{ad}^{m} is the projection onto A_{ad}^{m} and γ is the steplength parameter to be chosen appropriately (cf. [23]). At first glance, method (3.2) seems to be rather expensive compared with gradient projection methods. Fortunately we have found a route how the amount of computational work in solving $(P_{m,n}^1)$ can be reduced substantially $\tilde{a} := P_{\text{ad}}^m[\tilde{a} + \gamma \widehat{\triangle} a],$

where $\widehat{\triangle a}$ is the solution of the linearized problem
 $(P_{m,n}^{\parallel})$ inf $J_s^1(\widehat{\triangle} a)$; $J_s^1(\widehat{\triangle} a) = \frac{1}{2} ||\tilde{a} + \widehat{\triangle} a||\widehat{\triangle} a] - \mathfrak{D}^n z||_{\mathbf{R}^n}^2$,
 $+ \frac{\alpha}{2} ||\tilde{a} + \widehat$ Int $J_a^*(\triangle a)$; $J_a^*(\triangle a) = \frac{1}{2} \|b - \frac{a}{2}\|$

where $\widehat{\triangle u}[\widehat{\triangle a}]$ satisfies (3.1),

a projection onto A_{aa}^{π} and γ is

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th gradient projection methods. Fortunately we have found a route how red with gradient projection methods. Fortunately we have found a route how

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mpared with traditional strategies.

Theorem 1: Let F_{\math *Factors* $B = \frac{1}{2}$
 Fa $P_{m,n}^1$ and be rather expected and $(P_{m,n}^1)$ can be reduced so
 Fhen problem $(P_{m,n}^1)$ *is uniqual* $P_{m,n}^1$ *is uniqual* $\tilde{q} \in \mathbb{R}^n$ *is the unique solut*
 $\mathbb{R}^m b) + F_a'(\tilde{u} - \mathbb{$ pared with gradient projection methods. Fortunately we have found a route how
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compared with traditional strategies.
Theorem 1: Let

compared with traditional strategies.

Theorem 1: Let $F_{\tilde{u}}'$ be a regular matrix. Then problem $(P_{m,n}^1)$ is uniquely solvable. *The unique solution* $\widetilde{\Lambda}a \in \mathbb{R}^m$ *is given by*

$$
\widetilde{\triangle a} = \mathfrak{P}^{\mathfrak{m}} b - \tilde{a} + F_{\tilde{a}}^{\mathfrak{m}} \tilde{q}, \qquad (3.3)
$$

where $\tilde{u} \in \mathbb{R}^n$ *is the unique solution of* (2.12) *and* $\tilde{q} \in \mathbb{R}^n$ *is the unique solution of*

$$
(F_{\tilde{\mathbf{a}}}'F_{\tilde{\mathbf{a}}}^{\prime\bullet} + \alpha F_{\tilde{\mathbf{a}}}^{\prime}F_{\tilde{\mathbf{a}}}^{\prime\bullet})\tilde{q} = F_{\tilde{\mathbf{a}}}^{\prime}(\tilde{a} - \mathfrak{P}^{\mathbf{m}}b) + F_{\tilde{\mathbf{a}}}^{\prime}(\tilde{u} - \mathfrak{D}^{\mathbf{n}}z). \tag{3.4}
$$

$$
\frac{\partial^2 J_a{}^1(\widetilde{\triangle a})}{\partial \widetilde{\triangle a}^2} = F_{\vec{a}}{}^{\prime\bullet}(F_{\vec{a}}{}^{\prime}F_{\vec{a}}{}^{\prime\bullet})^{-1}F_{\vec{a}}{}^{\prime} + \alpha I
$$

is positive definite. Now let us regard

compared with traditional strategies.
\nTheorem 1: Let
$$
F_{\vec{u}}
$$
 be a regular matrix. Then problem $(P_{m,n}^1)$ i
\nThe unique solution $\tilde{\Delta a} \in \mathbb{R}^m$ is given by
\n
$$
\tilde{\Delta a} = \mathfrak{P}^m b - \tilde{a} + F_{\vec{a}}^* \tilde{\Phi} \tilde{\Phi}
$$
\nwhere $\tilde{u} \in \mathbb{R}^n$ is the unique solution of (2.12) and $\tilde{q} \in \mathbb{R}^n$ is the unique
\n($F_{\vec{a}}' F_{\vec{a}}'^* + \alpha F_{\vec{a}}' F_{\vec{a}}'^* \tilde{\Phi} \tilde{\Phi} = F_{\vec{a}}' (\tilde{a} - \mathfrak{P}^m b) + F_{\vec{a}}' (\tilde{u} - \mathfrak{D}^n z)$
\nProof: The existence and uniqueness of $\tilde{\Delta a}$ follows from the fact
\n $\partial^2 J_a^{-1}(\tilde{\Delta a})/\partial \tilde{\Delta a}^2 = F_{\vec{a}}'^*(F_{\vec{a}}' F_{\vec{a}}'^*)^{-1} F_{\vec{a}}' + \alpha I$
\nis positive definite. Now let us regard
\n
$$
J_a^{-1}(\tilde{\Delta a} + t\tilde{v}) = \frac{1}{2} \|\tilde{\Delta u}(\tilde{\Delta a} + t\tilde{v}) + \tilde{u} - \tilde{\Delta}^n z\|_{\mathbb{R}^n}^2 + \frac{\alpha}{2} \|\tilde{a} + \tilde{\Delta a} + t\tilde{v} - \mathfrak{P}^m b\|_{\mathbb{R}^m}^2
$$
\n
$$
+ \frac{\alpha}{2} \|\tilde{a} + \tilde{\Delta a} + t\tilde{v} - \mathfrak{P}^m b\|_{\mathbb{R}^m}^2.
$$
\nDifferentiation with respect to t yields for all $\tilde{v} \in \mathbb{R}^m$
\n
$$
\frac{\partial}{\partial t} J_a^{-1}(\tilde{\Delta a} + t\tilde{v})|_{t=0} = (\tilde{\Delta u}(\tilde{\Delta a}) + \tilde{u} - \tilde{\Delta}^n z, \tilde{\Delta u}(\tilde{v}) + \alpha(\tilde{a} +
$$

$$
\frac{\partial}{\partial t} J_{\alpha}^{-1}(\widetilde{\triangle a}+t\tilde{v})|_{t=0}=(\widetilde{\triangle u}(\widetilde{\triangle a})+\tilde{u}-\mathfrak{Q}^n z,\widetilde{\triangle u}(\tilde{v}))+\alpha(\tilde{a}+\widetilde{\triangle a}-\mathfrak{P}^m b,\tilde{v})=0
$$
\n(3.5)

(3.2)

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as necessary and sufficient condition for $\widetilde{\triangle{a}}$ being the unique solution of as necessary and sufficient condition for $\widetilde{\Delta a}$ being the unique solution of $(P_{m,n}^1)$.
Substitution of (3.3) into (3.5) yields $(\widetilde{\Delta u}[\mathfrak{P}^m b - \bar{a} + F_{\bar{a}}^{\prime *} \bar{q}] + \bar{u} - \mathfrak{L}^n z, \widetilde{\Delta u}[\bar{v}] + \alpha(\bar{q}, F_{\bar{a$

$$
(\widetilde{\triangle u}[\mathfrak{P}^{\mathfrak{m}}b - \tilde{a} + F_{\tilde{a}}^{\prime\bullet}\tilde{q}] + \tilde{u} - \mathfrak{Q}^{\mathfrak{m}}z, \widetilde{\triangle u}[\tilde{v}]) + \alpha(\tilde{q}, F_{\tilde{a}}^{\prime}\tilde{v}) = 0 \qquad \forall v \in \mathbb{R}^m.
$$

Using (3.1) we find

$$
(\widetilde{\triangle u}[\mathfrak{P}^{\mathfrak{m}}b - \bar{a} + F_{\tilde{a}}^{\prime\bullet}\tilde{q}] + \tilde{u} - \mathfrak{Q}^{\mathfrak{m}}z, \widetilde{\triangle u}[\tilde{v}]) + \alpha(\tilde{q}, F_{\tilde{a}}^{\prime}\tilde{v}) = 0 \quad \forall v \in \mathbb{R}^m.
$$

Using (3.1) we find

$$
\left(-F_{\tilde{a}}^{\prime-1}F_{\tilde{a}}^{\prime}(\mathfrak{P}^{\mathfrak{m}}b - \bar{a} + F_{\tilde{a}}^{\prime\bullet}\tilde{q}) + \tilde{u} - \mathfrak{Q}^{\mathfrak{m}}z, \widetilde{\triangle u}[\tilde{v}]\right) - \alpha(F_{\tilde{a}}^{\prime\bullet}\tilde{q}, \widetilde{\triangle u}[\tilde{v}]) = 0
$$

or what is the same,

$$
\left(-F_{\tilde{a}}^{\prime}F_{\tilde{a}}^{\prime\bullet} + \alpha F_{\tilde{a}}^{\prime}F_{\tilde{a}}^{\prime\bullet} + \beta F_{\tilde{a}}^{\prime\prime}(\tilde{a} - \mathfrak{R}^{\mathfrak{m}}b) + F_{\tilde{a}}^{\prime}(\tilde{u} - \mathfrak{Q}^{\mathfrak{m}}a) \cdot (F_{\tilde{a}}^{\prime\bullet}) - \widetilde{\triangle u}(\tilde{u})\right) = 0
$$

$$
\left(-(F_{\tilde{a}}'F_{\tilde{a}}^{\prime\bullet}+\alpha F_{\tilde{u}}'F_{\tilde{u}}^{\prime\bullet})\tilde{q}+F_{\tilde{a}}'(\tilde{a}-\mathfrak{B}^{m}b)+F_{\tilde{u}}'(\tilde{u}-\mathfrak{D}^{n}z),(F_{\tilde{u}}^{\prime\bullet})^{-1}\widetilde{\triangle u}[\tilde{v}]\right)=0
$$

for all $\tilde{v} \in \mathbb{R}^m$, which yields the expected result (3.4) \blacksquare

The advantage of our approach from the computational standpoint is that the two systems to be solved in each iteration step are independent of the size *m;* the number of unknowns, to be determined. This enables us to increase the size of the parameter space and therefore to handle a larger number of parameter degrees of freedom without increasing the computational work. A related theory for twodimensional problems can be developed in a similar way. $\tilde{a}'F_{\tilde{a}}'{}^* + \alpha F_{\tilde{a}}'F_{\tilde{a}}'{}^* \tilde{q} + F_{\tilde{a}}'(\tilde{c})$
 l $\tilde{v} \in \mathbf{R}^m$, which yields the ex

e advantage of our approach

get a set of unknowns, to be determeter space and therefore to

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Let us discuss example (2.13) in more detail. Here we have

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, which yields the expected result (3.4)
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, which yields the expected result (3.4)
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where $e_i = u_i - u_{i-1}$ and $d_i = d_i + d_{i+1}$. In this case the computation of the Gauss-Newton correction $\widetilde{\triangle}a$ requires only

(1) the solution of the 3-diagonal system (2.13) in order to find *u,*

(2) the solution of the 5-diagonal system (3.4) in order to find q and

(3) the computation of Δa according to (3.3).

.4. The continuous analogue **of the problem (3.4)**

To discuss the problems which may arise in (3.4) for decreasing stepsizes we consider the continuous problem (P) in more detail for the spatially one-dimensional case. The linearization of *u* in (P) at the function $a \in L^{\infty}(0, 1)$ with $a(x) \ge a > 0$ leads to the minimization problem where $e_i = u_i - u_{i-1}$ and $d_i = a_i + a_{i+1}$.
correction Δa requires only
(1) the solution of the 3-diagonal system
(2) the solution of the 5-diagonal system
(3) the computation of Δa according to (
4. The continuous a To discuss the problems which may arise in (3.4) for decreasing sto
the continuous problem (P) in more detail for the spatially one
The linearization of u in (P) at the function $a \in L^{\infty}(0, 1)$ with $a(x)$
the minimizatio **4.** The continuous analog

To discuss the problems

the continuous problem

The linearization of u in

the minimization problem
 (\mathbf{P}^1) inf $J_a^{-1}(\triangle a)$
 $\triangle a \in L^{\infty}(0, 1)$ 1.3).

1.3).

1.3.1.

1.3.4.

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1.4.4.

1.4.4.

1.4.4.

1.4.1.

$$
\inf_{\triangle a\in L^{\infty}(0;\;1)}J_{a}{}^{1}(\triangle a);J_{a}{}^{1}(\triangle a)=\frac{1}{2}\;\|u\>+ \triangle u[\triangle a]-\zeta\|_{L^{1}(0,\;1)}^{2}
$$

$$
+ \frac{\alpha}{2} ||a + \triangle a^{i} - b||^{2}_{L^{4}(0,1)},
$$

$$
\mathbf{8}^{-}
$$

external regularized Parameter Identification **9**

where $\triangle u[w]$, $w \in L^{\infty}(0, 1)$, solves one of the linear variational problems

Required Parameter Identification
\nwhere
$$
\triangle u[w]
$$
, $w \in L^{\infty}(0, 1)$, solves one of the linear variational problems
\n $\langle a \triangle u_x, v_x \rangle = -\langle w u_x, v_x \rangle$
\n $\langle \alpha \triangle u_x, v_x \rangle = -\langle w u_x, v_x \rangle$
\n $\langle \alpha \triangle u_x, v_x \rangle = -\langle w u_x, v_x \rangle$
\n $\langle \alpha \triangle u_x, v_x \rangle + s_1 \triangle u(1) v(1) - s_0 \triangle u(0) v(0) = -\langle w u_x, v_x \rangle$
\n $\forall v \in U = H^1(0, 1),$
\naccording to the corresponding boundary conditions (2.2)–(2.4). These second
\norder equations we consider in an H^{-1} -weak formulation. That means the set of
\ntest functions v will be reduced to allow a further partial integration on the left-
\nhand side. After defining the space of test functions
\n
$$
V = \begin{cases} v(0) = 0, v(1) = 0 \text{ or} \\ v_x(0) = 0, v_x(1) = 0 \text{ or} \\ a(0) v_x(0) + s_0 v(0) = 0, a(1) v_x(1) + s_1 v(1) = 0 \end{cases}
$$
\n(4.3)
\nwe get the H^{-1} -weak formulation for the problem (4.2)

according to the corresponding boundary conditions *(2.2)—(?.4).* These second order equations we consider in an H^{-1} -weak formulation. That means the set of test functions *v* will be reduced to allow a further partial integration on the lefthand side. After defining the space of test functions

$$
\forall v \in U - \exists w_x, v_x \land \forall v_y \in U - \exists w_y, v_z \land \forall v_z \in U - \exists P(0, 1),
$$
\n
$$
\forall v \in U = H^1(0, 1)
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\forall v \in U = H^1(0, 1)
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\forall v \in U = H^1(1, 1)
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\forall v \in
$$

we get the H^{-1} -weak formulation for the problem (4.2)

$$
\langle \triangle u, (av_x)_x \rangle = \langle w u_x, v_x \rangle = \langle w, u_x v_x \rangle \qquad \forall v \in V \qquad (4.4)
$$

for all the three cases of boundary conditions. Note that in this formulation the supposition $w \in L^{\infty}(0, 1)$ can be weakened to $w \in L^{2}(0, 1)$. Analogously to Theorem 1

Theorem 2: Let be $a \in L^{\infty}(0, 1)$ with $a(x) \geq \underline{a} > 0$. There exists one and only one *element* $\Delta a^* \in L^2(0, 1)$, *which minimizes the linearized functional* (4.1). This element $\triangle a^*$ can be characterized by the weak solution $q \in V$ of the 4th order equation **(b)** ω (c), 1) can be included to $\omega \in L(0, 1)$. Interesting to Theorem 1
 $\alpha^* \in L^2(0, 1)$, which minimizes the linearized functional (4.1). This element

be characterized by the weak solution $q \in V$ of the 4th orde *A* $A^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
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 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$
 $\Delta u^* = u_x q_x - (a - b).$

$$
\langle u_x q_x - (a - b), u_x v_x \rangle + \langle u - z + \alpha (a q_x)_x, (a v_x)_x \rangle = 0 \quad \forall v \in V \quad (4.5)
$$

\n
$$
\triangle a^* = u_x q_x - (a - b).
$$

\nthat the equation (4.5) is the continuous analogue to the discrete problem (3.4).
\n
$$
\text{sof: The minimizing element } \triangle a^* \text{ has to fulfill the necessary and sufficient}
$$

\n
$$
\text{from condition for the quadratic functional (P^1)}
$$

\n
$$
\langle \triangle u[\triangle a^*] + u - z, \triangle u[r] \rangle + \alpha \langle \triangle a^* + a - b, r \rangle = 0 \quad \forall r \in L^2(0, 1).
$$

\n(4.7)
\n
$$
q \text{ be a solution of the equation (4.5). The equation (4.5), (4.6) together with}
$$

\ngives us an H^{-1} -weak solution
\n
$$
\triangle u[\triangle a^*] = -\{u - z + \alpha (a q_x)_x\}.
$$

\n
$$
\triangle u[\triangle a^*] = -\{u - z + \alpha (a q_x)_x\}.
$$

\n
$$
\therefore u \in V, \text{ we get } \langle \triangle u[r], (a q_x)_x \rangle = \langle r, u_x q_x \rangle. \text{ This relation together with (4.8) and}
$$

\nshows, that the minimum condition (4.7) holds for all $r \in L^2(0, 1)$

by

Note that the equation (4.5) is the continuous analogue to the discrete problem (3.4).

Proof: The minimizing element Δa^* has to fulfil the necessary and sufficient minimum condition for the quadratic functional $(P¹)$

$$
\langle \triangle u[\triangle a^*]+u-z, \triangle u[r]\rangle + \alpha \langle \triangle a^*+a-b, r\rangle = 0 \qquad \forall r \in L^2(0,1).
$$
\n(4.7)

Let q be a solution of the equation (4.5) . The equation (4.5) , (4.6) together with (4.4) gives us an $H⁻¹$ -weak solution

$$
\triangle u[\triangle a^*] = -\{u, -z + \alpha(aq_z)_z\}.
$$
\n(4.8)

Rewriting the H^{-1} -weak formulation (4.4) for $\Delta u(r)$ with $q \in V$ instead of an arbitrary $v \in V$, we get $\langle \triangle u(r), (aq_x)_x \rangle = \langle r, u_x q_x \rangle$. This relation together with (4.8) and (4.6) shows, that the minimum condition (4.7) holds for all $r \in \overline{L}^2(0, 1)$ \blacksquare

Due to the well-posedness of (4.5) for fixed $\alpha > 0$ there is no fear of ill-conditioning in the discrete problem (3.4) for decreasing stepsizes.

Remarks: 1. Theorem 2 deals only with the case $\Delta a^* \in L^2(0, 1)$. The representation (4.6) shows that u_x is the decisive quantity for Δa^* being an element of $L^{\infty}(0, 1)$. In order to get $u_x \in L^{\infty}(0, 1)$ we need some not too hard additional assumptions on $f \in H^{-1}(0, 1)$ for the problem (2.1) (for instance, $f \in L^2(0, 1)$ gives $u_x \in L^{\infty}(0, 1)$). **2.** If we formally perform (4.5)

THE V. FRIEDRICH and U. TAUTENHAHN

by partial integration we get the 4th order equation

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by partial integration we get the 4th order equation

$$
\alpha(a(aq_x)_{xx})_x - (u_x^2 q_x)_x = (a(z - u)_x)_x - ((a - b) u_x)_x
$$
with the boundary conditions at $x = 0$ and $x = 1$
 $q = 0$, $-\alpha(aq_x)_x = u - z$
for the problem (2.1–2.2) and

$$
aq_x + sq = 0
$$
,

with the boundary conditions at $x=0$ and $x=1$

$$
q=0, \qquad, -\alpha(aq_x)_x=u-z
$$

for the problem $(2.1-2.2)$ and

aq±sq=0, S - - *—u 2q ± c[a(aqZ) XX \± s(aq)] = —(a - b) U^x + a(z— u) + s(z - u)* for the problems,(2.1,2.3/4). REFERENCES .

 $\frac{\int_0^1 f(x) dx}{\int_0^1}$

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