Zeitschrift für Analysis
und ihre Anwendungen
Bd. 8 (1) 1989, S. 25–40

Basic Properties of Some Differential-Algebraic Equations[1])

E. GRIEPENTROG and R. MÄRZ

Es wird die Lösbarkeit der Gleichung $Ax' + Bx = q$ mit regulärem modifizierten lokalen Matrixbüschel von höherem Index untersucht. Dabei wird gezeigt, wie Anfangsbedingungen geeignet formuliert werden können und daß die Anfangswertprobleme zu Operatoren mit unbeschränkten Inversen führen. Eine enge Verwandtschaft der für die Untersuchung aufgebauten Matrix-Ketten mit den entsprechenden Ketten bei den Reduktionsmethoden wird herausgestellt.

Рассматривается вопрос расрешимости уравнения $Ax' + Bx = q$ с неособенным модифицированным локальным матричным пучком высшего индекса. Показывается, как надлежащим образом могут быть сформулированы начальные условия и что задачи с начальными значениями ведут к операторам с неограниченными обратными. Выявляется тесное родство построенных для исследования матричных цепочек с соответствующими цепочками методов редукции.

The solvability of the equation $Ax' + Bx = q$ with a regular modified local matrix pencil of higher index is considered. It is shown how to formulate initial conditions properly and that the initial value problems lead to operators with unbounded inverses. The close relationship of the matrix pencils needed in the investigation and of the related chains of the reduction methods is pointed out.

Introduction

Differential-algebraic equations are certain uniformly singular ordinary differential equations

$$f(x'(t), x(t), t) = 0, \tag{0.1}$$

where the partial Jacobian $f_y'(y, x, t)$ is everywhere singular but has constant rank. Those equations originate from different applications (descriptor systems for electric networks, Euler-Lagrange equations for systems of rigid bodies etc.). Further, e.g. reduced systems in singular perturbation theory, certain semi-discretized Navier-Stokes systems, also control problems represent differential-algebraic equations. Often differential-algebraic equations are given in semi-explicit form

$$\left.\begin{aligned} u'(t) - \varphi(u(t), v(t), t) &= 0; \\ \psi(u(t), v(t), t) &= 0 \end{aligned}\right\} \tag{0.2}$$

(e.g. dynamical systems subjected to constraints, reduced equations of singularly perturbed systems with separated fast and slow components) which may be considered to be differential equations on manifold (cf. [2, 17]), supposed φ, ψ are smooth enough.

[1]) Vortrag auf der gemeinsamen Tagung vom 9.–13. 11. 1987 in Berga (DDR) der HFR Analysis und HFR Numerische Mathematik.

Notice that, in view of the application, we should accept $u \in C^1$, $v \in C$ satisfying the above semi-explicit system to be solutions but we should not demand $v \in C^1$. Clearly, if $\psi(u, v, t) = 0$ may be solved uniquely with respect to v, $v = h(u, t)$, the system (0.2) is well-understood. It is not surprising that numerical methods aproved for regular ordinary differential equations can be modified to work well also for (0.2). The class of *transferable* differential-algebraic equations (0.1) considered in [8] is the related generalization of this case.

However, what happens when $\psi_v'(u, v, t)$ becomes singular? It should be mentioned that, for instance, $\psi_v'(u, v, t) \equiv 0$ holds when only holonomic constraints are given for a system of rigid bodies. Integration methods working well when applied to transferable differential-algebraic equations fail or work unreliably in case of nontransferable equations (cf. [8, 14]). Only for very restricted classes of nontransferable differential-algebraic equations the usual integration methods can be managed to work well by special error controls (e.g. [11, 14]).

The behaviour of discretizations as integration methods is closely related to the underlying functional-analytic characterization of the original problem. In particular unbounded inverses give rise to the instability of the related discretizations.

In the present paper, we consider linear nontransferable differential-algebraic equations

$$A(t)\, x'(t) + B(t)\, x(t) = q(t) \tag{0.3}$$

with continuous coefficients A, B. Note that in view of possible linearizations of (0.1) we are interested even in continuous coefficients (but not in analytic ones). The first questions we should deal with is whether the solutions of the homogeneous equation form a finite-dimensional function space, and how to formulate initial conditions appropriately. It is also important to have solvability assertions. In particular, for the numerical treatment we should know whether the related maps are Fredholm, have bounded inverses etc.

To answer these questions requires a lot of matrix analysis. In § 1, basic properties of certain matrix chains are proved. By this, some open questions of [15] are also answered. In particular, Theorem 4.7 formulated in [15] for $\mu \leq 3$ becomes true also for $\mu > 3$. In § 2, initial conditions are formulated appropriately, and solvability statements are given for a new class of equations (0.3). This class is characterized by an everywhere regular modified local matrix pencil (cf. [8]). Via the modified local matrix pencil, or, equivalently, via the matrix chaines proposed, an *index* k of (0.3) is defined. So-called higher index equations ($k > 1$) lead to ill-posed initial value problems (unbounded inverses) even in the case the initial conditions are stated properly (injective maps). In § 3, so-called reduction methods are considered. They allow to reduce the index k of (0.3) supposed some differentiations may be carried out exactly, and supposed the solution of (0.3) is $x \in C^1$. It is also the purpose of the present paper to point out the close relations of the inner decoupling of the differential-algebraic equation in § 2 and the reduction method in § 3 by means of the common matrix chain approach. Notice that the matrix chain results proposed in § 1 generalize resp. simplify both [7] and [15].

§ 1. Matrix-theoretical relations

We consider sequences of $n \times n$-matrices $\{J_i\}_{i=0}^k$ and $\{\hat{J}_i\}_{i=0}^k$ defined by a given matrix $J_0 := \hat{J}_0 := J$ and the relations $J_1 := \hat{J}_1 := J_0 + Q_0$,

$$J_l := J_{l-1} + P_0 P_1 \dots P_{l-2} Q_{l-1}, \quad \hat{J}_l := \hat{J}_{l-1} + \hat{Q}_{l-1} \qquad (l = 2, 3, \dots, k),$$

where the $Q_i = I - P_i$ and $\hat{Q}_i = I - \hat{P}_i$ are arbitrary projectors onto $\ker(\mathcal{J}_i)$ and $\ker(\hat{J}_i)$, respectively. Furthermore, we define index sets

$$\mathcal{J}_m^{(r,s)} = \{(i_1, \ldots, i_m): i_j \in \mathbb{N} \cup \{0\}, r \leqq i_1 < \cdots < i_m \leqq s\}$$

and for $l = 1, \ldots, k$ the matrices

$$G_l = \sum_{\mathcal{J}_l^{(0,k-1)}} Q_{i_1} Q_{i_2} \ldots Q_{i_l}, \qquad \hat{G}_l = \sum_{\mathcal{J}_l^{(0,k-1)}} \hat{Q}_{i_1} \hat{Q}_{i_2} \ldots \hat{Q}_{i_l}.$$

Then the following theorem holds.

Theorem 1: *For* $l = 0, 1, \ldots, k-1$ *the relations* $J^l J_k = J^{l+1} + \sum_{i=l+1}^{k} (-1)^{i-1} \times \binom{i-1}{l} G_i$ *and* $J^l \hat{J}_k = J^{l+1} + (-1)^l G_{l+1}$ *hold.*

Proof: Obviously, both the assertions are true for $l = 0$, because of $J_k = J + Q_0 + (I - Q_0) Q_1 + \cdots + (I - Q_0)(I - Q_1) \ldots (I - Q_{k-2}) Q_{k-1} = J + \sum_{i=1}^{k} (-1)^{i-1} G_i$ and $\hat{J}_k = J + \sum_{i=0}^{k-1} \hat{Q}_i = J + \hat{G}_1$. Now we verify the assertion concerning $J^l \hat{J}_k$ with $l \geqq 1$. The assumption $J^{l-1} \hat{J}_k = J^l + (-1)^{l-1} \hat{G}_l = J^l + (-1)^{l-1} \sum_{\mathcal{J}_l^{(0,k-1)}} \hat{Q}_{i_1} \hat{Q}_{i_2} \ldots \hat{Q}_{i_l}$ implies

$$\begin{aligned} J^l \hat{J}_k &= J^{l+1} + (-1)^{l-1} \sum_{\mathcal{J}_l^{(0,k-1)}} \hat{J}_0 \hat{Q}_{i_1} \hat{Q}_{i_2} \ldots \hat{Q}_{i_l} \\ &= J^{l+1} + (-1)^{l-1} \sum_{\mathcal{J}_l^{(0,k-1)}} \Big(\hat{J}_{i_1} - \sum_{0 \leqq j \leqq i_1 - 1} \hat{Q}_j \Big) \hat{Q}_{i_1} \hat{Q}_{i_2} \ldots \hat{Q}_{i_l} \\ &= J^{l+1} + (-1)^l \sum_{0 \leqq j < i_1 < \cdots < i_l \leqq k-1} \hat{Q}_j \hat{Q}_{i_1} \ldots \hat{Q}_{i_l} = J^{l+1} + (-1)^l \hat{G}_{l+1}. \end{aligned}$$

The proof of the assertion for $J^l J_k$ is a little more complicated. Defining

$$G_m^{(r,s)} = \sum_{\mathcal{J}_m^{(r,s)}} Q_{i_1} Q_{i_2} \ldots Q_{i_m} \qquad (1 \leqq m \leqq s - r + 1), \quad G_0^{(r,s)} = I,$$

we obtain immediately $G_m = G_m^{(0,k-1)} = \sum_{r=1}^{k-m+1} Q_{r-1} G_{m-1}^{(r,k-1)} = \sum_{r=m+1-i}^{k+1-i} G_{m-i}^{(0,r-2)} Q_{r-1} G_{i-1}^{(r,k-1)}$ $(1 \leqq i \leqq m)$. Now from the assumption $J^{l-1} J_k = J^l + \sum_{i=l}^{k} (-1)^{i-1} \binom{i-1}{l-1} G_i^{(0,k-1)} = J^l + \sum_{i=l}^{k} (-1)^{i-1} \binom{i-1}{l-1} \sum_{r=1}^{k-i+1} Q_{r-1} G_{i-1}^{(r,k-1)}$ follows the equation

$$\begin{aligned} J^l J_k &= J^{l+1} + \sum_{i=l}^{k} (-1)^{i-1} \binom{i-1}{l-1} \sum_{r=1}^{k+1-i} \Big\{ J_{r-1} - \sum_{j=1}^{r-1} (-1)^{j-1} G_j^{(0,r-2)} \Big\} Q_{r-1} G_{i-1}^{(r,k-1)} \\ &= J^{l+1} + \sum_{i=l}^{k} (-1)^i \binom{i-1}{l-1} \sum_{r=1}^{k+1-i} \sum_{j=1}^{r-1} (-1)^{j-1} G_j^{(0,r-2)} Q_{r-1} G_{i-1}^{(r,k-1)}, \end{aligned}$$

and by convenient exchanges of the summing-up processes $(s := i + j)$

$$\begin{aligned} J^l J_k &= J^{l+1} + \sum_{i=l}^{k} \sum_{j=1}^{k-i} (-1)^{i+j-1} \binom{i-1}{l-1} \sum_{r=j+1}^{k+1-i} G_j^{(0,r-2)} Q_{r-1} G_{i-1}^{(r,k-1)} \\ &= J^{l+1} + \sum_{s=l+1}^{k} (-1)^{s-1} \sum_{i=l}^{s-1} \binom{i-1}{l-1} \sum_{r=s+1-i}^{k+1-i} G_{s-i}^{(0,r-2)} Q_{r-1} G_{i-1}^{(r,k-1)} \\ &= J^{l+1} + \sum_{s=l+1}^{k} (-1)^{s-1} \sum_{i=l}^{s-1} \binom{i-1}{l-1} G_s^{(0,k-1)}. \end{aligned}$$

Finally, the well-known identity $\sum_{i=l}^{s-1} \binom{i-1}{l-1} = \sum_{j=0}^{s-1-l} \binom{l-1+j}{l-1} = \binom{s-1}{l}$ yields the assertion $J^l J_k = J^{l+1} + \sum_{s=l+1}^{k} (-1)^{s-1} \binom{s-1}{l} G_s$. ∎

For nilpotent matrices J Theorem 1 has far-reaching consequences.

Theorem 2: *If the matrix J is nilpotent of the order k, i.e. $J^k = 0$ and $J^{k-1} \neq 0$ holds, then all J_l and $\hat{J}_l$ with $l < k$ are singular matrices, whereas J_k and $\hat{J}_k$ are non-singular.*

Proof: From Theorem 1 and $J^k = 0$ there immediately follows $J^{k-1}J_k = (-1)^{k-1} \times Q_0Q_1 \dots Q_{k-1}$ and $J^{k-1}\hat{J}_k = (-1)^{k-1} \hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}$. In order to prove the non-singularity of J_k and $\hat{J}_k$ we have to confirm $\ker(J_k) = \ker(\hat{J}_k) = \{0\}$. The assumption $\hat{J}_k v = 0$ implies $0 = \hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}v = (\hat{J}_1 - \hat{J}_0)\hat{Q}_1 \dots \hat{Q}_{k-1}v = -J\hat{Q}_1 \dots \hat{Q}_{k-1}v$, that means $\hat{Q}_1 \dots \hat{Q}_{k-1}v \in \ker(J)$ and hence $\hat{Q}_1 \dots \hat{Q}_{k-1}v = \hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}v = 0$. Now $(\hat{J}_2 - \hat{J}_1)\hat{Q}_2 \dots \hat{Q}_{k-1}v = -\hat{J}_1\hat{Q}_2 \dots \hat{Q}_{k-1}v$ yields $\hat{Q}_2 \dots \hat{Q}_{k-1}v \in \ker(\hat{J}_1)$, i.e. $\hat{Q}_2 \dots \hat{Q}_{k-1}v = \hat{Q}_1\hat{Q}_2 \dots \hat{Q}_{k-1}v = 0$ and so on. Finally, we obtain $0 = \hat{Q}_{k-1}v = (\hat{J}_{k-1} - \hat{J}_k)v = -\hat{J}_{k-1}v$ and therefore $v \in \ker(\hat{J}_{k-1})$, i.e. $v = \hat{Q}_{k-1}v = 0$ verifying $\ker(\hat{J}_k) = \{0\}$.

Analogously the assumption $J_k v = 0$ leads to $Q_0Q_1 \dots Q_{k-1}v = 0$ and $Q_1 \dots Q_{k-1}v = 0$, that is $P_0Q_1 \dots Q_{k-1}v = 0$. Hence, we obtain $0 = Q_2 \dots Q_{k-1}v = P_0P_1Q_2 \dots Q_{k-1}v$ using $(J_2 - J_1)Q_2 \dots Q_{k-1}v = 0$, i.e. $Q_2 \dots Q_{k-1}v \in \ker(J_1)$. Continueing this process at last we receive $0 = Q_{k-1}v = P_0P_1 \dots P_{k-2}Q_{k-1}v$, that means $0 = (J_k - J_{k-1})v = -J_{k-1}v$, and hence $v = Q_{k-1}v = 0$ proving $\ker(J_k) = \{0\}$.

To complete the proof we have to verify the singularity of the matrices J_l, $\hat{J}_l$ with $l < k$. $J^{k-1}\hat{J}_k = J^k + (-1)^{k-1}\hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}$ implies $J^{k-1} = (-1)^{k-1}\hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}\hat{J}_k^{-1}$. Since $J^{k-1} \neq 0$ we may choose a vector w with $J^{k-1}w \neq 0$; then we obtain $\hat{Q}_0\hat{Q}_1 \dots \hat{Q}_{k-1}\hat{J}_k^{-1}w \neq 0$ and therefore $w_l := \hat{Q}_l \dots \hat{Q}_{k-1}\hat{J}_k^{-1}w \neq 0$. On the other hand, $\hat{J}_l\hat{Q}_l = 0$ implies $\hat{J}_l w_l = 0$ demonstrating the singularity of $\hat{J}_l$. The proof concerning J_l is completely analogous. ∎

Now we try to generalize our results to pairs of matrices. A pair (A, B) of $m \times m$-matrices is called a *regular pencil*, if $p(z) = \det(zA + B)$ does not vanish identically. For each regular pencil a decomposition

$$E^{-1}AF^{-1} = \operatorname{diag}(I_s, J),\ E^{-1}BF^{-1} = \operatorname{diag}(W, I_{m-s}) \tag{1.1}$$

with $\det(E) \neq 0$, $\det(F) \neq 0$ exists, where J is nilpotent (cf. [3, 8]). The order k of nilpotency is called *index* of the pencil: $k = \operatorname{ind}(A, B)$. If A is regular, then per definition $\operatorname{ind}(A, B) = 0$. The integers k and s do not depend on the special choice of E and F.

Starting with $A_0 := \hat{A}_0 := A$, $B_0 := \hat{B}_0 := B$ we construct sequences $\{(A_l, B_l)\}_{l=0}^k$, $\{(\hat{A}_l, \hat{B}_l)\}_{l=0}^k$ of matrix pairs by the following rules $(l = 0, \dots, k-1)$:

$$A_{l+1} = A_l + B_lQ_l,\quad B_{l+1} = B_lP_l,\quad \hat{A}_{l+1} = \hat{A}_l + \hat{B}_l\hat{Q}_l,\quad \hat{B}_{l+1} = \hat{B}_l. \tag{1.2}$$

Here the matrices $Q_l = I - P_l$ and $\hat{Q}_l = I - \hat{P}_l$ are projectors onto $\ker(A_l)$ and $\ker(\hat{A}_l)$, respectively.

Theorem 3: *If (A, B) is a regular pencil with the index k, then the matrices A_l, $\hat{A}_l$ with $l < k$ are singular, whereas A_k and $\hat{A}_k$ are non-singular. Conversely, if A_k or $\hat{A}_k$ is non-singular, then (A, B) is a regular pencil.*

Proof: First we prove the second assertion. For this purpose we simply demonstrate, that $\det(zA_l + B_l) \equiv 0$ implies $\det(zA_{l+1} + B_{l+1}) \equiv 0$ and $\det(z\hat{A}_l + \hat{B}_l)$

$\equiv 0$ implies $\det(z\hat{A}_{l+1} + \hat{B}_{l+1}) \equiv 0$; then obviously A_k or $\hat{A}_k$ cannot be non-singular, if $\det(zA + B) \equiv 0$. The equations $\det(zA_{l+1} + B_{l+1}) = \det(zA_l + zB_lQ_l + B_lP_l) = \det\left(z^2A_lQ_l + z(A_lP_l + B_lQ_l) + B_lP_l\right) = \det(zA_l + B_l)\det(zQ_l + P_l)$ and $\det(zA_{l+1} + B_{l+1}) = \det(zA_l + zB_lQ_l + B_l) = \det(zA_l + B_l)\det(zQ_l + I)$ prove the conclusions stated above.

For the proof of the first assertion we use the decomposition (1.1) and define

$$U = \operatorname{diag}(I_r, J), \quad V = \operatorname{diag}(W, I_{m-r}), \quad U_0 = \hat{U}_0 = U, \quad V_0 = \hat{V}_0 = V,$$

$$U_i = U_{i-1} + V_{i-1}S_{i-1}, \quad V_i = V_{i-1}R_{i-1}, \quad S_{i-1} = I - R_{i-1} = FQ_{i-1}F^{-1},$$

$$\hat{U}_i = \hat{U}_{i-1} + \hat{V}_{i-1}\hat{S}_{i-1}, \quad \hat{V}_i = \hat{V}_{i-1}, \qquad \hat{S}_{i-1} = I - \hat{R}_{i-1} = F\hat{Q}_{i-1}F^{-1},$$

$i = 1, \ldots, k$. Then obviously S_{i-1} and $\hat{S}_{i-1}$ are projectors onto $\ker(U_{i-1})$ and $\ker(\hat{U}_{i-1})$, respectively, and we obtain $A_i = EU_iF$, $\hat{A}_i = E\hat{U}_iF$, $B_i = EV_iF$ and $\hat{B}_i = E\hat{V}_iF$. Therefore, we only have to prove that the U_l, $\hat{U}_l$ with $l < k$ are singular, whereas U_k and $\hat{U}_k$ are non-singular. The calculation of U_i and $\hat{U}_i$ yields in each case

$$U_i = \begin{pmatrix} I_s & 0 \\ G_i & J + H_i \end{pmatrix}, \qquad \hat{U}_i = \begin{pmatrix} I_s & 0 \\ \hat{G}_i & J + \hat{H}_i \end{pmatrix},$$

$$S_i = \begin{pmatrix} 0 & 0 \\ M_i & Q_i^* \end{pmatrix}, \qquad \hat{S}_i = \begin{pmatrix} 0 & 0 \\ \hat{M}_i & \hat{Q}_i^* \end{pmatrix},$$

where

$$G_i = \sum_{l=0}^{i-1} P_0^*P_1^* \ldots P_{l-1}^*M_l, \qquad \hat{G}_i = \sum_{l=0}^{i-1} \hat{M}_l,$$

$$H_i = \sum_{l=0}^{i-1} P_0^*P_1^* \ldots P_{l-1}^*Q_l^*, \qquad H_i = \sum_{l=0}^{i-1} \hat{Q}_l^*,$$

and the $Q_i^* = I - P_i^*$, $\hat{Q}_i^*$ are projectors onto $\ker(J + H_i)$ and $\ker(J + \hat{H}_i)$, respectively. M_i and $\hat{M}_i$ are arbitrary $(m - s) \times s$-matrices with $Q_i^*M_i = M_i$ and $\hat{Q}_i^*\hat{M}_i = \hat{M}_i$, i.e. $(J + H_i)\,M_i = (J + \hat{H}_i)\,\hat{M}_i = 0$. Theorem 2 then immediately delivers the assertion ∎

Now as a direct corollary we obtain the following theorem, which generalizes a result proved in [7] in a more complicated way.

Theorem 4: *If (A, B) is a regular pencil with the index k, then (A_l, B_l) and $(\hat{A}_l, \hat{B}_l)$ are regular with the index $k - 1$. That means*

$$\operatorname{ind}(A_l, B_l) = \operatorname{ind}(A_{l-1}, B_{l-1}) - 1, \quad \operatorname{ind}(\hat{A}_l, \hat{B}_l) = \operatorname{ind}(\hat{A}_{l-1}, \hat{B}_{l-1}) - 1$$

$$(l = 1, \ldots, k).$$

Proof: The sequences beginning with (A_l, B_l) or $(\hat{A}_l, \hat{B}_l)$ lead to non-singular matrices A_k and $\hat{A}_k$, respectively, in exactly $k - l$ steps ∎

In the next chapter we need a further corollary of Theorem 3.

Theorem 5: *If (A, B) is a regular pencil, then the relations*

$$\ker(A_l) \cap \ker(B_l) = \ker(\hat{A}_l) \cap \ker(\hat{B}_l) = \{0\},$$

$$\operatorname{span}\left(\operatorname{im}(A_l) \cup \operatorname{im}(B_l)\right) = \operatorname{span}\left(\operatorname{im}(\hat{A}_l) \cup \operatorname{im}(\hat{B}_l)\right) = \mathbb{R}^m$$

are valid for $l = 0, 1, 2, \ldots$.

Proof: If we assume $x \neq 0$, $x \in \ker(A_l) \cap \ker(B_l)$ or $x \in \ker(\hat{A}_l) \cap \ker(\hat{B}_l)$, we obtain $A_{l+1}x = A_lx + B_lQ_lx = A_lx + B_lx = 0$, $B_{l+1}x = B_lP_lx = 0$ (using $Q_lx = x$) or $\hat{A}_{l+1}x = \hat{A}_lx + \hat{B}_l\hat{Q}_lx = \hat{A}_lx + \hat{B}_lx = 0$, $\hat{B}_{l+1}x = \hat{B}_lx = 0$, respectively. Hence, no A_k or $\hat{A}_k$ is non-singular, and (A, B) cannot be a regular pencil. Further, $x = A_lu + B_lv = A_{l-1}u + B_{l-1}(Q_{l-1}u + P_{l-1}v)$ or $x = \hat{A}_lu + \hat{B}_lv = \hat{A}_{l-1}u + \hat{B}_{l-1}(\hat{Q}_{l-1}u + v)$ yields $\operatorname{span}(\operatorname{im}(A_{l-1}) \cup \operatorname{im}(B_{l-1})) \supseteq \operatorname{span}(\operatorname{im}(A_l) \cup \operatorname{im}(B_l))$ and $\operatorname{span}(\operatorname{im}(\hat{A}_{l-1}) \cup \operatorname{im}(\hat{B}_{l-1})) \supseteq \operatorname{span}(\operatorname{im}(\hat{A}_l) \cup \operatorname{im}(\hat{B}_l))$. Since for regular pencils $\operatorname{im}(A_k) = \operatorname{im}(\hat{A}_k) = \mathbb{R}^m$ for some k, the theorem is proved ∎

Remark: Neither the two conditions for the (A_l, B_l) nor the two conditions for the $(\hat{A}_l, \hat{B}_l)$ are sufficient for the regularity of the pencil, as the following examples demonstrate.

1. $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

lead to $A_{2i} = A$, $B_{2i} = B$, $A_{2i+1} = \tilde{A}$, $B_{2i+1} = \tilde{B}$ $(i = 0, 1, 2, \ldots)$ using

$$Q_{2i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_{2i+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

lead to

$$\hat{A}_l = \begin{pmatrix} l+1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{using} \quad \hat{Q}_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -l-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (l = 0, 1, 2, \ldots).$$

Supposed the pencil (A, B) is regular, we are allowed (by Theorem 5 and [15: Theorem 2.3]) to choose the projectors Q_i within the chain (1.2) in such a way that

$$Q_jQ_i = 0 \quad (i = 0, \ldots, j-1) \text{ for all } j. \tag{1.3}$$

This implies, in particular, all products of projectors $P_{i_1} \ldots P_{i_n}$, $i_1 < \ldots < i_n$, to become projectors again. Namely, we have

$$(P_{i_1} \ldots P_{i_n})^2 = P_{i_1} \ldots P_{i_{n-1}}(I - Q_{i_n}) P_{i_1} \ldots P_{i_{n-1}}P_{i_n}$$
$$= (P_{i_1} \ldots P_{i_{n-1}})^2 P_{i_n} - P_{i_1} \ldots P_{i_{n-1}}Q_{i_n}P_{i_n} = (P_{i_1} \ldots P_{i_{n-1}})^2 P_{i_n},$$

i.e. the assertion is shown inductively. Introducing the subspaces $S_i = \{z \in \mathbb{R}^m : B_iz \in \operatorname{im}(A_i)\}$, $i \geq 0$, we are able to characterize the dimensions of the nullspaces $\ker(A_{i+1})$ (cf. [4: Lemma 2.1]) as follows:

$$\dim(\ker(A_{i+1})) = \dim(\ker(A_i) \cap S_i). \tag{1.4}$$

Finally, the special choice of the projectors described by (1.3) leads to

$$M_{k-1} := \ker(A_0) + \cdots + \ker(A_{k-1}) = \ker(A_0) \oplus \cdots \oplus \ker(A_{k-1}),$$

$$\dim(M_{k-1}) = \dim(\ker(A_0)) + \sum_{i=0}^{k-2} \dim(\ker(A_i) \cap S_i).$$

§ 2. Analyzing linear differential-algebraic equations

Consider the singular linear ordinary differential equation

$$Ax' + Bx = q, \tag{2.1}$$

where $A, B \in C(\mathcal{J}, L(\mathbb{R}^m))$, $q \in C(\mathcal{J}, \mathbb{R}^m) =: C$, $\mathcal{J}$ is a compact interval, and $A(t)$ is singular on $\mathcal{J}$ but has constant rank r. Those equations are called *normal linear differential-algebraic equations*. Suppose the nullspace $N(t) = \ker(A(t))$ to depend smoothly on t, that is, there is a projector function $Q \in C^1(\mathcal{J}, L(\mathbb{R}^m))$ so that $Q(t)^2 = Q(t)$, $\operatorname{im}(Q(t)) = N(t)$ hold (or equivalently, $N(t)$ is spanned by a base $n_1, \ldots, n_{m-r} \in C^1(\mathcal{J}, \mathbb{R}^m) =: C^1$). Introduce also $P \in C^1(\mathcal{J}, L(\mathbb{R}^m))$, $P(t) = I - Q(t)$.

Notice that the smoothness of $N(\cdot)$ is equivalent to the existence of linear independent functions $n_1, \ldots, n_{m-r} \in C^1$ spanning $N(\cdot)$, i.e. $N(t) = \operatorname{span}\{n_1(t), \ldots, n_{m-r}(t)\}$ for all $t \in \mathcal{J}$ [1]. Namely, for fixed $t_0 \in \mathcal{J}$ we choose $n_1^0, \ldots, n_{m-1}^0 \in \mathbb{R}^m$ to be a base of $N(t_0)$. Then, determine $n_j \in C^1$ to be the solution of the initial value problem $n' = Q'n$, $n(t_0) = n_j$, $j = 1, \ldots, m-r$. Further, we have $Pn_j' = PQ'n_j$, therefore $(Pn_j)' = P'n_j + PQ'n_j = P'n_j - P'Qn_j = P'Pn_j$. Because of $P(t_0)\,n_j(t_0) = 0$, the function Pn_j vanish identically, i.e. $n_j(t) \in N(t)$ for all $t \in \mathcal{J}$, $j = 1, \ldots, m-r$. Since the $n_1, \ldots, n_{m-r}$ are linearly independent, they span $N(\cdot)$. On the other hand, if $N(\cdot)$ is known to be spanned by given $n_1, \ldots, n_{m-r} \in C^1$, the matrix function $Q = F(F^TF)^{-1}F^T$, $F(t) = [n_1(t), \ldots, n_{m-r}(t)] \in L(\mathbb{R}^{m-r}, \mathbb{R}^m)$, has all properties required above.

Since $A = AP$, $Px' = (Px)' - P'x$ for $x \in C^1$, we may reformulate (2.1) to

$$A(Px)' + (B - AP')\,x = q \tag{2.2}$$

what shows that all functions x belonging to $C_N^1 = \{y \in C : Py \in C^1\}$ and satisfying (2.2) should be accepted to be a solution of (2.1) but not only $x \in C^1$. Define the linear map

$$\mathfrak{A}: C_N^1 \to C, \qquad \mathfrak{A}x = A(Px)' + (B - AP')\,x.$$

Clearly, C_N^1 equipped with the norm $\|\cdot\| = \|\cdot\|_\infty + \|(P\cdot)'\|_\infty$ becomes a Banach space, and $\mathfrak{A}$ is bounded. Both, the set C_N^1 and its topology, and also the map $\mathfrak{A}$ are independent of the choice of the projector functions Q, P (cf. [8]).

Now, denote by $\mathcal{N}$ the set of all ordered matrix pairs $\{A, B\}$ having the properties described above. In the following, we are interested only in differential-algebraic equations (2.1) the coefficients of which form a pair belonging to $\mathcal{N}$.

It should be mentioned that constant coefficient equations (2.1) are well-understood via the Kronecker canonical normal form [3], and also by means of the matrix chains constructed in § 1 (cf. [7, 15]). In particular, if (A, B) forms a singular matrix pencil, then the related nullspace $\ker(\mathfrak{A})$ has an infinite dimension (cf. [8]). Supposed the pencil (A, B) is regular, the normal form (1.1) may be used to obtain $\dim(\ker(\mathfrak{A})) = \operatorname{grad}(\det(zA + B))$ as well as to characterize $\operatorname{im}(\mathfrak{A})$. The index of the pencil (A, B) is said to be the *index* of (2.1) in this constant coefficient case. If $\operatorname{ind}(A, B) =: k > 1$, then $\operatorname{im}(\mathfrak{A})$ consists of functions $q \in C$ certain components of which are up to $k-1$ times continuously differentiable. However, even in this constant coefficient case, the formulation of appropriate initial conditions requires informations about the canonical normal form (1.1) included the transformations E, F (what cannot be realized practically) or the computation of some projectors defined in § 1 (cf. [8, 15]).

But what about general equations (2.1) with $\{A, B\} \in \mathcal{N}$? The first problems we should deal with are the questions whether the solutions of the homogeneous equation $Ax' + Bx = 0$ form a finite-dimensional subspace of C_N^1, and how to formulate initial conditions to fix a unique solution.

Supposed $\{A, B\} \in \mathcal{N}$, the map $\mathfrak{A}$ and also equation (2.1) are called *tractable* if $\dim(\ker(\mathfrak{A})) < \infty$. The global index concept of Gear and Petzold [5] generalizes (1.1) for $\{A, B\} \in \mathcal{N}$ (resp. (2.1)) to

$$\left.\begin{aligned} &E(t)\,A(t)\,F(t) = \operatorname{diag}(I_s, J), \\ &E(t)\,B(t)\,F(t) + E(t)\,A(t)\,F'(t) = \operatorname{diag}\left(W(t), I_{m-s}\right). \end{aligned}\right\} \tag{2.3}$$

The *global index* (if it exists) is defined to be the nilpotency index (Riesz index) of J. Clearly, equations (2.1) having a global index are tractable. However, there is no chance to realize (2.3) practically, e.g. for the statement of initial conditions. The matrix chain calculus proposed in [7, 15] for constant coefficient equations and in [12, 13] for index 2 and index 3 equations with $\{A, B\} \in \mathcal{N}$ seems to be a better tool to handle (2.1) practically, e.g. to formulate initial conditions, to investigate the behaviour of numerical methods, to check the index etc. In the following we try to characterize a new class of tractable differential-algebraic equations in terms of their coefficients by means of the matrix chain approach.

Trivially, instead of (2.2) we may write also

$$\left(A + (B - AP')\,Q\right)\{P(Px)' + Qx\} + (B - AP')\,Px = q. \tag{2.4}$$

Now, if $A_1 := A + (B - AP')\,Q$ is non-singular for all $t \in \mathcal{J}$, multiplying by PA_1^{-1} and QA_1^{-1}, respectively, splits up equation (2.4) into the system

$$\left.\begin{aligned} &(Px)' - P'Px + PA_1^{-1}(B - AP')\,Px = PA_1^{-1}q, \\ &Qx + QA_1^{-1}(B - AP')\,Px = QA_1^{-1}q. \end{aligned}\right\}$$

This system is decoupled into a regular explicit ordinary differential equation for the component Px (the state variable) and an explicit assessment determining Qx. We obtain solvability for all continuous right-hand sides, that is $\operatorname{im}(\mathfrak{A}) = C$. The initial condition $P(t_0)\left(x(t_0) - a\right) = 0$ fixes a unique solution of (2.1) (cf. [8]).

Equation (2.1) is called *transferable* (into state variable form) if $A_1(t)$ remains non-singular for all $t \in \mathcal{J}$. By Theorem 3, the matrix $A_1(t)$ is non-singular if and only if the so-called *modified local pencil* $\left(A(t), B(t) - A(t)\,P'(t)\right)$ is regular and has index 1 — or, equivalently, if and only if the so-called *local pencil* $\left(A(t), B(t)\right)$ is regular with index 1 (cf. [8: Theorem A. 13]). This is why transferable differential-algebraic equations are called uniformly index 1 equations (e.g. [4, 5]). Linear and also nonlinear transferable differential-algebraic equations are well-understood (e.g. [8]). They form the simplest class of tractable equations, and they are amenable to numerical methods in a similar way as regular ordinary differential equations are (e.g. [5, 8]). Let us point out that $A_1(t)$ is exactly the first matrix defined within the chain (1.2) when starting with $A_0(t) := A(t)$, $B_0(t) := B(t) - A(t)\,P'(t)$, $Q_0(t) := Q(t)$.

Now, we are going to use the whole chain for each $t \in \mathcal{J}$. Additionally, we are interested in the subspaces

$$N_j(t) \equiv \ker\left(A_j(t)\right),$$
$$S_j(t) \equiv \left\{z \in \mathbb{R}^m : B_j(t)\,z \in \operatorname{im}\left(A_j(t)\right)\right\}$$

which we call *canonical subspaces* of (2.1). By the use of our matrix chain, equation (2.1) can be rewritten in the form (cf. [15])

$$A_k\{P_{k-1}\dots P_1P(Px)' + P_{k-1}\dots P_1Qx + P_{k-1}\dots P_2Q_1x + \cdots$$
$$+ P_{k-1}Q_{k-2}x + Q_{k-1}x\} + B_0PP_1\dots P_{k-1}x = q. \tag{2.5}$$

Supposed the modified local pencil $(A_0(t), B_0(t))$ becomes regular for all $t \in \mathcal{J}$, and ind $(A_0(t), B_0(t)) \leqq k$, $k > 1$, we are able to choose the projector functions in our chain so that the identity

$$Q_j(t)\,Q_i(t) = 0, \quad 0 \leqq i < j \leqq k,\ t \in \mathcal{J} \tag{2.6}$$

is satisfied (cf. § 1). The matrix $A_k(t)$ is everywhere non-singular then. It should be mentioned that (cf. (1.4)) $\dim (N_{i+1}(t)) = \dim (N_i(t) \cap S_i(t))$. Using the identity

$$I = P_0P_1\dots P_{k-1} + Q_0P_1\dots P_{k-1} + \cdots + Q_{k-3}P_{k-2}P_{k-1} + Q_{k-2}P_{k-1} + Q_{k-1}$$

as well as certain properties following from (2.6), we split up equation (2.5) into the system

$$\left.\begin{aligned}
P_0\dots P_{k-1}(Px)' + P_0\dots P_{k-1}A_k^{-1}B_0P_0\dots P_{k-1}x &= P_0\dots P_{k-1}A_k^{-1}q\\
-(Q_0Q_1 + Q_0P_1Q_2 + \dots + Q_0P_1\dots P_{k-2}Q_{k-1})(Px)'&\\
+ Q_0x + \tilde{Q}_0P_0\dots P_{k-1}x &= Q_0P_1\dots P_{k-1}A_k^{-1}q\\
&\ \ \vdots\\
-(Q_{k-3}Q_{k-2} + Q_{k-3}P_{k-2}Q_{k-1})(Px)'&\\
+ Q_{k-3}x + \tilde{Q}_{k-3}P_0\dots P_{k-1}x &= Q_{k-3}P_{k-2}P_{k-1}A_k^{-1}q\\
-(Q_{k-2}Q_{k-1})(Px)' + Q_{k-2}x + \tilde{Q}_{k-2}P_0\dots P_{k-1}x &= Q_{k-2a}P_{k-1}A_k^{-1}q\\
Q_{k-1}x + \tilde{Q}_{k-1}P_0\dots P_{k-1}x &= Q_{k-1}A_k^{-1}q
\end{aligned}\right\}. \tag{2.7}$$

Thereby,

$$\tilde{Q}_{k-j} := Q_{k-j}P_{k-j+1}\dots P_{k-1}A_k^{-1}B_0P_0\dots P_{k-j-1}, \qquad j = 1, \dots, k,$$

may be shown to be also projector functions, i.e. $\tilde{Q}_{k-j}(t)$ projects $\mathbb{R}^m$ onto $N_{k-j}(t)$. In particular, $\tilde{Q}_{k-1}(t)$ projects onto $N_{k-1}(t)$ along $S_{k-1}(t)$. Further, it should be notices that $i < j$ implies

$$\tilde{Q}_j\tilde{Q}_i = Q_jP_{j+1}\dots P_{k-1}A_k^{-1}B_0P_0\dots P_{j-1}Q_iP_{i+1}\dots P_{k-1}A_k^{-1}B_0P_0\dots P_{i-1} = 0,$$

thus the projector functions $\tilde{Q}_j$ satisfy the condition (2.6) also. Taking $\tilde{Q}_j = Q_j$ in advance would lead to $\tilde{Q}_jP_0\dots P_{k-1} = 0$, $j = 0, \dots, k-1$, hence to an easier form of (2.7). This is why we call $\tilde{Q}_j$ canonical projector functions. Note that $\tilde{Q}_0$ is not necessarily continuously differentiable.

Denote $\Pi_k = P_0\dots P_{k-1}$, $M_k = \operatorname{im} (\Pi_k(t_0))$, where $t_0 \in \mathcal{J}$ is fixed. Further, introduce the linear bounded maps

$$\mathfrak{B}\colon C_N^1 \to M_k, \qquad \mathfrak{B}x = \Pi_k(t_0)\,x(t_0), \qquad \mathfrak{L} = (\mathfrak{A}, \mathfrak{B})\colon C_N^1 \to C \times M_k.$$

The initial value problem for (2.1) with the initial condition

$$\Pi_k(t_0)\,(x(t_0) - a) = 0 \tag{2.8}$$

is represented now by the equation $\mathfrak{L}x = (q, \Pi_k(t_0)\,a)$.

Theorem 6: *Assume $\{A, B\} \in \mathcal{N}$. Let $(A(t), B(t) - A(t)\,P'(t))$ be regular and have the index $k \geqq 1$ for all $t \in \mathcal{J}$. If $k > 1$ let (2.6) be satisfied, $Q_j \in C^1(\mathcal{J}, L(\mathbb{R}^m))$, $j = 1$,*

..., $k-1$, *and, moreover,*

$$\left.\begin{aligned}&(P_0P_1\dots P_{k-1})'\,P_0(I-P_1\dots P_{k-1})=0,\\&\big(Q_{j-1}(I-P_j\dots P_{k-1})\big)'\,P_0(I-P_1\dots P_{k-1})=0,\qquad j=1,\dots,k-1\end{aligned}\right\}\tag{2.9}$$

Then $\mathfrak{L}$ is injective, thus $\mathfrak{A}$ is tractable, $\dim\big(\ker(\mathfrak{A})\big)\le \operatorname{rank}\big(\Pi_k(t_0)\big)=:s$. *Provided certain further smoothness related to the projector functions and coefficients is given, it holds that* $\dim\big(\ker(\mathfrak{A})\big)=s$, *and* $\mathfrak{L}^{-1}\big(0,\Pi_k(t_0)\,a\big)$ *depends continuously on* a.

Proof: Condition (2.9) leads to

$$P_0\dots P_{k-1}(Px)'=(\Pi_kPx)'-\Pi_k'Px=(\Pi_kx)'-\Pi_k'\Pi_kx,$$

$$(Q_{j-1}Q_j+\dots+Q_{j-1}P_j\dots P_{k-2}Q_{k-1})\,(Px)'=Q_{j-1}(I-P_j\dots P_{k-1})\,(Px)'$$

$$=Q_{j-1}\big[\big(Q_{j-1}(I-P_j\dots P_{k-1})\,Px\big)'-\big(Q_{j-1}(I-P_j\dots P_{k-1})\big)'\,Px\big]$$

$$=Q_{j-1}(Q_{j-1}Q_jx+\dots+Q_{j-1}P_j\dots P_{k-2}Q_{k-1}x)'-Q_{j-1}\big(Q_{j-1}(I-P_j\dots P_{k-1})\big)'\,\Pi_kx$$

for $j=1,\dots,k-1$. Inserting this expressions in (2.7), and taking into account that $Q_iP_0\dots P_i=0$, $i=0,\dots,k-1$, thus $P_{j-1}\big(Q_{j-1}(I-P_j\dots P_{k-1})\big)'\,\Pi_k=0$, $j=1,\dots,k-1$, we obtain

$$\left.\begin{aligned}&(\Pi_kx)'-\Pi_k'\Pi_kx+\Pi_kA_k^{-1}B_0\Pi_kx=\Pi_kA_k^{-1}q,\\&-Q_0(Q_0Q_1x+\dots+Q_0P_1\dots P_{k-2}Q_{k-1}x)'\\&\quad+\big\{\big(Q_0(I-P_1\dots P_{k-1})\big)'+\tilde{Q}_0\big\}\,\Pi_kx+Q_0x=Q_0P_1\dots P_{k-1}A_k^{-1}q,\\&\qquad\vdots\\&-Q_{k-3}(Q_{k-3}Q_{k-2}x+Q_{k-3}P_{k-2}Q_{k-1}x)'\\&+\big\{\big(Q_{k-3}(I-P_{k-2}P_{k-1})\big)'+\tilde{Q}_{k-3}\big\}\,\Pi_kx+Q_{k-3}x=Q_{k-3}P_{k-2}P_{k-1}A_k^{-1}q,\\&-Q_{k-2}(Q_{k-2}Q_{k-1}x)'\\&\quad+\{(Q_{k-3}Q_{k-1})'+\tilde{Q}_{k-2}\}\,\Pi_kx+Q_{k-2}x=Q_{k-2}P_{k-1}A_k^{-1}q.,\\&\qquad Q_{k-1}x+\tilde{Q}_{k-1}\Pi_kx=Q_{k-1}A_k^{-1}q.\end{aligned}\right\}\tag{2.10}$$

Now, the first assertion becomes evident, since the first equation in (2.10) is decoupled from the other ones, and the initial condition (2.8) fixes exactly one of its solutions. $q=0$, $\Pi_k(t_0)\,a=0$ implies $\Pi_kx=0$. Now, the last equation of (2.10) leads to $Q_{k-1}x=0$, the next to the last one gives $Q_{k-2}x=0$ and so on. Hence $x=\Pi_kx+P_0\dots P_{k-2}Q_{k-2}x+\dots+P_0Q_1x+Q_0x=0$. In a similar way we construct a solution for each nontrivial $\Pi_k(t_0)\,a\in M_k$. But now Π_kx becomes also nontrivial, and we have to assume the existence of all derivatives we need for (2.10) successively ∎

Notice that the assumption of Theorem 6 related to the uniform index k on $\mathcal{J}$ cannot be weakened to $\operatorname{ind}\big(A(t),B(t)-A(t)\,P'(t)\big)\le k$.

Example 1 (cf. [8: § 1.3, Example 1]): Let $A(t)=\begin{pmatrix}1&-t\\1&-t\end{pmatrix}$, $B=2I$, $\mathcal{J}=[0,2]$. Compute $Q(t)=\begin{pmatrix}0&t\\0&1\end{pmatrix}$, $A_1(t)=\begin{pmatrix}1&1+t\\1&3-t\end{pmatrix}$. The modified local pencil is regular for all $t\in\mathcal{J}$. It has index 1 for $t\neq1$, but index 2 at $t=1$. In [8] it is shown that the solution of the homogeneous initial value problem bifurcates at the point of index change.

In Example 1, the change of the index is accompanied by a discontinuity of Q_1. By the next example we point out that even if there is no index change of the modified local pencil, the demanded smoothness of the projector functions is also essential to Theorem 6.

Example 2: For $A(t) = \begin{pmatrix} 1-t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B(t) = I, \mathcal{J} = [0, 2]$, we have

$$Q(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t-1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_1(t) = \begin{pmatrix} 3-t & 1 & 0 & 0 \\ t-1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\dim\left(\ker\left(A_1(t)\right)\right) = \begin{cases} 1 \text{ for } t \neq 1, \\ 2 \text{ for } t = 1. \end{cases}$$

$Q_1(t)$ is not continuous at $t = 1$. Choose

$$Q_1(1) = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Q_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } t \neq 1,$$

hence $Q_1 Q = 0$ is true. Further, compute

$$A_2(1) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2(t) = \begin{pmatrix} 3-t & 1 & 0 & 0 \\ t-1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } t \neq 1.$$

$A_2(t)$ is non-singular, but $A_1(t)$ singular for all $t \in \mathcal{J}$, thus the modified local pencil is everywhere on $\mathcal{J}$ regular and has the (constant) index 2. Note that we have also $(PP_1)'(t)\,P(t)\,Q_1(t) = 0$, $(QQ_1)'(t) = Q'(t)\,Q_1(t) = 0$ for $t \neq 1$. The detailed homogeneous differential algebraic system (2.2) under consideration is

$$((1-t)\,x_1 + x_2)' + 2x_1 = 0, \qquad x_2 = 0, \qquad x_4' + x_3 = 0, \qquad x_4 = 0.$$

The discontinuity of Q_1 at $t = 1$ indicates a singular point of the inherent ordinary differential equation related to $\Pi_2 x$. For arbitrary $\gamma \in \mathbb{R}$, the function x defined by

$$x(t) = \begin{cases} (0, 0, 0, 0)^{\mathrm{T}} & \text{if } t \in [0, 1], \\ (\gamma(t-1), 0, 0, 0)^{\mathrm{T}} & \text{if } t \in [1, 2] \end{cases}$$

belongs to C_N^1 and solves our homogeneous initial value problem as well. Therefore, $\mathfrak{L}$ is not injective.

Next we turn to the question whether (2.1) is solvable at all in the case of $k > 1$. Again, system (2.10) suggests how to proceed. Additionally to the assumptions of Theorem 6 we suppose that

$$\left.\begin{aligned} &\tilde{Q}_j \Pi_k \in C^1(\mathcal{J}, \mathrm{L}(\mathbb{R}^m)), && j = 1, \ldots, k-1, \\ &Q_{j-1}(I - P_j \ldots P_{k-1}) \in C^2(\mathcal{J}, \mathrm{L}(\mathbb{R}^m)), && j = 2, \ldots, k-1, \end{aligned}\right\} \tag{2.11}$$

if $k > 1$. Take $x \in C_N^1$ to consider $q := \mathfrak{A}x$. We have

$$y := \Pi_k x = \Pi_k P x \in C^1, \qquad Q_j x = Q_j P x \in C^1 \text{ for } j = 1, \ldots, k-1.$$

Denote

$$\left.\begin{aligned}
\tilde{p}_{k-1} &= Q_{k-1}A_k^{-1}q,\\
p_{k-1} &= \tilde{p}_{k-1} - \tilde{Q}_{k-1}\Pi_k y,\\
\tilde{p}_{k-j} &= Q_{k-j}P_{k-j+1}\dots P_{k-1}A_k^{-1}q\\
&\quad + Q_{k-j}(Q_{k-j}p_{k-j+1} + \dots + Q_{k-j}P_{k-j+1}\dots P_{k-2}p_{k-1})',\\
p_{k-j} &= \tilde{p}_{k-j} - \tilde{Q}_{k-j}\Pi_k y - \big(Q_{k-j}(I - P_{k-j+1}\dots P_{k-1})\big)'\,\Pi_k y,\\
j &= 2,\dots,k-1.
\end{aligned}\right\} \tag{2.12}$$

Now, successively using (2.10), (2.11) as well as $p_{k-j} = Q_{k-j}x \in C^1$, we derive that $\tilde{p}_{k-j} \in C^1$, $j = 1, \dots, k-1$. Therefore, we obtain

$$\operatorname{im}(\mathfrak{A}) \subseteq \mathcal{R}_k := \{q \in C: \tilde{p}_j \in C^1, j = 1, \dots, k-1\}, \tag{2.13}$$

that means, some components of q have to be continuously differentiable up to $k-1$ times. Notice that y is the solution of the initial value problem

$$\left.\begin{aligned}
y' - \Pi_k' y + \Pi_k A_k^{-1} B_0 y &= \Pi_k A_k^{-1} q,\\
y(t_0) &= \Pi_k(t_0)\,x(t_0).
\end{aligned}\right\} \tag{2.14}$$

Defining $\mathcal{R}_k$ by (2.13) we understand y to be the solution of (2.14) where $\Pi_k(t_0)\,x(t_0)$ is replaced by $\Pi_k(t_0)\,a$, $a \in \mathbb{R}^m$. This makes clear that $\mathcal{R}_k$ possibly concerns further smoothness conditions related to y also.

To prove the solvability of (2.1) for each given $q \in \mathcal{R}_k$, first of all we define $y \in C^1$ as described above. It may be checked easily that $y = \Pi_k y$ holds. Next we determine $p_{k-1}, \dots, p_1$ according to (2.12), but now $q \in \mathcal{R}_k$ implies $p_{k-j} \in C^1$, $j = 1, \dots, k-1$. Moreover, we use the inclusion

$$p_0 := Q_0P_1 \dots P_{k-1}A_k^{-1}q - \{\tilde{Q}_0 + \big(Q_0(I - P_1 \dots P_{k-1})\big)'\}\,\Pi_k y + Q_0(Q_0p_1 + \dots + Q_0P_1 \dots P_{k-2}p_{k-1})'' \in C.$$

Since $p_j = Q_j p_j$, $j = 0, \dots, k-1$, the function $x = p_0 + y + P_0p_1 + \dots + P_0 \dots P_{k-2}p_{k-1}$ belongs to C_N^1. Finally, x can be proved to satisfy (2.1) immediately. Consequently, $\operatorname{im}(\mathfrak{A}) = \mathcal{R}_k$.

Theorem 7: *Let the assumptions of Theorem* 6 (*inclusive that of the second part*) *be valid,* $k > 1$. *Moreover, let* (2.11) *be given. Then the inclusions*

$$C^{k-1} \subseteq \operatorname{im}(\mathfrak{A}) = \mathcal{R}_k \subseteq \{q \in C: Q_{k-1}A_k^{-1}q \in C^1\} \subset C$$

are true. The maps $\mathfrak{A}: C_N^1 \to C$, $\mathfrak{L}: C_N^1 \to C \times M_k$ *are not Fredholm.* $\mathfrak{L}$ *is injective but has no bounded inverse.*

Proof: The assertion is a direct consequence of our considerations above since $\mathcal{R}_k$ is a proper nonclosed subset within C ∎

Corollary 8: $\mathfrak{L}: C_N^1 \to C \times M_k$ *becomes a homeomorphism if and only* $k = 1$.

Theorem 7 suggests to call the differential-algebraic equation under consideration also *index k-tractable.* Note that index-k-tractability, $k \leq 3$, is defined in [12, 13, 16] without using the assumption (2.9) by means of certainly modified matrix chains. This definition does not include the smoothness of Q_1 and Q_2, respectively. But we know from Example 2 that we should include it. For a conjecture to define index-k-tractability also for $k > 3$ unless (2.9) is given, and without using a matrix pencil we refer to [16].

It should be mentioned that Theorems 6, 7 cover the linear prototypes of interesting applications (cf. [10, 15]). On the other hand, the modified local pencil used in the present paper remains regular only for a restricted class of equations.

Example 3: Let

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{pmatrix}.$$

The related equation (2.1) has the global index 3, and it is index-3-tractable in the sense of [13]. However, the modified local pencil is singular.

By Corollary 8, the only initial value problems in differential-algebraic equations,w hich are well-posed, are those where (2.1) is transferable. All higher index (i.e. $k > 1$) equations lead to ill-posed problems in this framework, that is $\mathfrak{L}^{-1}$ is unbounded, and small perturbations of the right-hand sides in (2.1) does not imply necessarily small errors in the solution. Surely, we could look for an appropriate stronger norm on im $(\mathfrak{A})$ to obtain a bounded inverse of $\mathfrak{L}$ in this new setting. But the description of im $(\mathfrak{A})$ is technically rather complicated. When assuming $\tilde{Q}_j = Q_j$, $Q_j' = 0$, $j = 1, \ldots, k-1$, the expressions for im $(\mathfrak{A})$ becomes more transparent, in particular we have then

$$\mathcal{R}_2 = \{q \in C: Q_1A_2^{-1}q \in C^1\},$$

$$\mathcal{R}_3 = \{q \in C: Q_2A_3^{-1}q \in C^1, (Q_1Q_2A_3^{-1}q)' + Q_1P_2A_3^{-1}q \in C^1\},$$

$$\mathcal{R}_4 = \{q \in C: Q_4A_3^{-1}q =: p_3 \in C^1, (Q_2p_3)' + Q_2P_3A_4^{-1}q =: p_2 \in C^1,$$

$$(Q_1p_2 + Q_1P_2p_3)' + Q_1P_2P_3A_4^{-1}q \in C^1\}.$$

But even in this case $\mathcal{R}_k$ is quite complicated for $k > 3$. Besides, up to now, there are no proposals to design numerical methods going well with the modified setting where $\mathfrak{L}^{-1}$ becomes bounded. On the other hand, first proposals to regularize the ill-posed problems as they are given in our original setting are made (e.g. [9, 10]).

Further, the C_N^1, $C \times M_k$-setting and also the C, $C \times M_k$-setting are convenient to study the behaviour of integration methods applied to differential algebraic equations (cf. [14]). Note that both $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ are unbounded when the max-norm $\|\cdot\|_\infty$ is used in all spaces. The reflection of the unboundedness of $\mathfrak{L}^{-1}$ by the integration methods is their instability (cf. [8, 14]). For index 2 equations having a constant projector function P, the instability is weak and only related to certain components. In the consequence, e.g. backward differentiation methods may be managed to work well for those equations. But, unfortunately, it seems to remain the only class where this is possible.

Since, up to now, no numerical methods are really practicable for higher index problems with $k \geqq 3$ (also the effort for the use of special methods designed to solve ill-posed problems is only in its beginning) one should try to apply reduction steps (cf. § 3) to decrease the index k to 2 or 1.

We close this section formulating some simple inequalities which result from (2.10) immediately. Namely, we have

$$\|x\|_\infty \leqq K_1(|b| + \|q\|_\infty + \|q'\|_\infty + \cdots + \|q^{(k-1)}\|_\infty)$$

for $q \in C^{k-1}$, $b = \Pi_k(t_0)\, a$, $x = \mathfrak{L}^{-1}(q, b)$. If $k = 2$, we obtain more precisely

$$\|x\|_\infty \leqq K_2\big(|b| + \|q\|_\infty + \|(QQ_1A_2^{-1}q)'\|_\infty\big),$$

$$\|x\| \leqq K_3\big(|b| + \|q\|_\infty + \|(Q_1A_2^{-1}q)'\|_\infty\big)$$

for all $q \in$ im $(\mathfrak{A})$.

§ 3. Reduction methods

Together with the pair (A, B) now we consider (A^T, B^T), where A^T and B^T denote the transposed matrices. Clearly, (A^T, B^T) is a regular pencil with the index k iff (A, B) is so. Starting with (A^T, B^T) we form the sequences (1.2) by

$$A_{l+1}^T := A_l^T + B_l^T Q_l, \quad B_{l+1}^T := B_l^T P_l, \quad \hat{A}_{l+1}^T := \hat{A}_l^T + \hat{B}_l^T \hat{Q}_l, \quad \hat{B}_{l+1}^T := \hat{B}_l^T, \tag{3.1}$$

where $Q_l = I - P_l$ and $\hat{Q}_l = I - \hat{P}_l$ now are projectors onto ker (A_l^T) and ker$(\hat{A}_l^T)$, respectively. Of course for the sequences $\{(A_l^T, B_l^T)\}_{l=0}^k$ and $\{(\hat{A}_l^T, \hat{B}_l^T)\}_{l=0}^k$ produced by (3.1) the Theorems 3 and 4 are valid. $Q_l = I - P_l$ and $\hat{Q}_l = I - \hat{P}_l$ are projectors onto ker (A_l^T) and ker $(\hat{A}_l^T)$, respectively, iff $R_l = I - S_l := Q_l^T$ and $\hat{R}_l = I - \hat{S}_l := \hat{Q}_l^T$ are projectors along im (A_l) and im $(\hat{A}_l)$, respectively. By transposition we obtain from (3.1) the sequences

$$\begin{aligned} &\hat{A}_0 = A_0 = A, \quad \hat{B}_0 = B_0 = B, \\ &A_{l+1} = A_l + R_l B_l, \quad B_{l+1} = S_l B_l, \\ &\hat{A}_{l+1} = \hat{A}_l + \hat{R}_l \hat{B}_l, \quad \hat{B}_{l+1} = \hat{B}_l, \qquad l = 0, \ldots, k-1, \end{aligned} \tag{3.2}$$

where $R_l = I - S_l$ and $\hat{R}_l = I - \hat{S}_l$ are arbitrary projectors along im (A_l) and im $(\hat{A}_l)$, respectively. Applying the Theorems 3 and 4 to (3.1) we obtain the following statement.

Theorem 9: *If (A, B) is a regular pencil with the index k and the sequences $\{(A_l, B_l)\}_{l=0}^k$, $\{(\hat{A}_l, \hat{B}_l)\}_{l=0}^k$ are produced by (3.2), then the matrices A_l, $\hat{A}_l$ with $l < k$ are singular, whereas A_k and $\hat{A}_k$ are non-singular. The (A_l, B_l) and $(\hat{A}_l, \hat{B}_l)$ are regular pencils with the index $k - l$, that means for $l = 1, \ldots, k$*

$$\text{ind}\,(A_l, B_l) = \text{ind}\,(A_{l-1}, B_{l-1}) - 1, \qquad \text{ind}\,(\hat{A}_l, \hat{B}_l) = \text{ind}\,(\hat{A}_{l-1}, \hat{B}_{l-1}) - 1.$$

Conversely, if A_k or $\hat{A}_k$ are non-singular, then (A, B) is a regular pencil.

This theorem suggests a method for the index reduction in differential-algebraic equations with constant coefficients $Ax'(t) + Bx(t) = q(t)$ considered on the interval $\mathcal{J}$. If we define $\hat{A}_0 = A$, $\hat{B}_0 = B$, $q_0 = q$ and omit the argument t, we obtain from $\hat{A}_0 x' + \hat{B}_0 x = q_0$, by multiplication with $\hat{R}_0$ and differentiation, $\hat{R}_0 \hat{B}_0 x = \hat{R}_0 q_0$ and $(\hat{R}_0 \hat{B}_0)\, x' = (\hat{R}_0 q_0)'$. Adding this equation to the original one, we receive $(\hat{A}_0 + \hat{R}_0 \hat{B}_0) \times x' + \hat{B}_0 x = q_0 + (\hat{R}_0 q_0)'$. The definition $q_1 = q_0 + (\hat{R}_0 q_0)'$ yields $\hat{A}_1 x' + \hat{B}_1 x = q_1$ with the constraint $\hat{R}_0(\hat{B}_0 x - q_0) = 0$. For the new differential-algebraic equation we have got an index reduction by virtue of ind $(\hat{A}_1, \hat{B}_1) =$ ind $(\hat{A}_0, \hat{B}_0) - 1$. Continuing this process we obtain after k steps at last the explicit differential equation $x' = \hat{A}_k^{-1}(q_k - \hat{B}_k x)$ with the constraints $\hat{R}_l(\hat{B}_l x - q_l) = 0$, $l = 0, 1, \ldots, k-1$. Each step of the procedure reduces the index of the pair $(\hat{A}_l, \hat{B}_l)$ by 1; but we have to assume, that all derivatives $(\hat{R}_i q_i)'$ exist. This is a strong restriction for the feasibility of the method. As we will see later, the constraints are fulfilled by $x(t)$ for all $t \in \mathcal{J}$ automatically, if for the initial vector $x(t_0)$ the equations $\hat{R}_l\big(\hat{B}_l x(t_0) - q_l(t_0)\big) = 0$ $(l = 0, 1, \ldots, k-1)$ hold (cf. Lemma 10).

The reduction procedure suggested here permits the following generalization to linear time-variable differential-algebraic equations

$$A(t)\, x'(t) + B(t)\, x(t) = q(t), \tag{3.3}$$

where rank $(A(t)) \equiv$ const, A, B and q are continuous on $\mathcal{J}$. Differentiation of $RBx = Rq$ yields $RBx' + (RB)' x = (Rq)'$, and adding the equations we obtain

$$A_1x' + B_1x = q_1 \tag{3.4}$$

with $A_1 := A + RB$, $B_1 := B + (RB)'$, $q_1 := q + (Rq)'$, provided that all occuring derivatives exist.

Lemma 10: *Let $R(\cdot)$ be an arbitrary differentiable projektor function and $R(\cdot)\, y(\cdot)$ be differentiable. Then $y(t) = 0$ iff $y(t) + R(t)\, y(t)' = 0$ and $R(t_0)\, y(t_0) = 0$.*

Proof: Due to $R(Ry)' + R'(Ry) = (R^2y)' = (Ry)'$ we have $0 = R(y + [Ry]') = Ry - R'Ry + (Ry)' = [Ry]' + (I - R')\, Ry$. This is a homogeneous linear differential equation for Ry, and the initial condition $R(t_0)\, y(t_0) = 0$ yields $(Ry)\,(t) \equiv 0$. Consequently, $0 \equiv -(Ry)' = y$ ∎

Applying our lemma to $y(t) = A(t)\, x'(t) + B(t)\, x(t) - q(t)$, we secure that (3.3) is on C^1 equivalent to (3.4) constrained by the condition $R(t_0)\, \{B(t_0)\, x(t_0) - q(t_0)\} = 0$. If rank $(A_1(t)) \equiv$ const and R_1B_1, R_1q_1 are differentiable, we can repeat the explained procedure for (3.4) and obtain in the same way $A_2x' + B_2x = q_2$ with the additional restrictions $R_1(t_0)\, \{B_1(t_0)\, x(t_0) - q_1(t_0)\} = 0$ for $x(t_0)$. We call (3.3) *reducible*, if the process can be continued until a matrix A_k appears which is continuous and non-singular on $\mathcal{J}$. (3.3) is called *k-reducible* if k is the smallest integer for which in k steps a non-singular A_k is attainable.

Theorem 11: *If (3.3) is k-reducible, then there is a sequence*

$$A_0(t) := A(t), \quad B_0(t) := B(t), \quad q_0(t) := q(t),$$

$$A_{l+1}(t) := A_l(t) + R_l(t)\, B_l(t), \quad B_{l+1}(t) := B_l(t) + [R_l(t)\, B_l(t)]',$$

$$q_{l+1}(t) := q_l(t) + [R_l(t)\, q_l(t)]', \qquad l = 0, 1, \ldots, k-1$$

leading to the ordinary differential equation

$$x'(t) = [A_k(t)]^{-1}\, \{q_k(t) - B_k(t)\, x(t)\},$$

which is under the restrictions $R_l(t_0)\, \{B_l(t_0)\, x(t_0) - q_l(t_0)\} = 0$, $l = 0, 1, \ldots, k-1$, *equivalent to* (3.3).

Assuming $(A(t), B(t))$ to be a regular pencil for each $t \in \mathcal{J}$, we get an index depending on t: $k(t) = \operatorname{ind}\,(A(t), B(t))$. If $k(t) \equiv k$ it seems to be reasonable to define k as the global index of (3.3). But simple examples show that the pencil $(A(t), B(t))$ only in the case $k(t) \equiv 1$ characterizes the solution behaviour (cf. [8: § 1.3]). Therefore, Gear and Petzold [4] called (3.3) to have the global index k, if a continuous non-singular matrix function $E(\cdot)$ and a continuously differentiable non-singular matrix function $F(\cdot)$ exist, so that scaling of (3.3) by $E(t)$ and the transformation $x(t) = F(t) \times y(t)$ lead to the differential-algebraic equation

$$\hat{A}_0 y'(t) + \hat{B}_0(t)\, y(t) = q_0(t), \tag{3.10}$$

where $\hat{A}_0 = \operatorname{diag}\,(I_s, J) = E(t)\, A(t)\, F(t)$, $\hat{B}_0(t) = \operatorname{diag}\,(W(t), I_{m-s}) = E(t)\, A(t)\, F'(t) + E(t)\, B(t)\, F(t)$, $q_0(t) := E(t)\, q(t)$ and J is a constant nilpotent matrix with $J^k = 0$, $J^{k-1} \neq 0$. For all constant projectors $\hat{R}_0$ along im $(\hat{A}_0)$ we obtain $\hat{R}_0$ diag $(W(t), I_{m-s})$

$= \hat{R}_0 \operatorname{diag}(0, I_{m-s})$; therefore, the matrix $\hat{A}_1 = \hat{A}_0 + \hat{R}_0\hat{B}_0(t)$ is constant again. This procedure can be continued; then the global index k means that the sequences $\hat{A}_{l+1} := \hat{A}_l + \hat{R}_l\hat{B}_l(t)$, $\hat{B}_{l+1}(t) := \hat{B}_l(t)$ with arbitrary constant projectors $\hat{R}_l$ along im $(\hat{A}_l)$ are ending with non-singular matrices $\hat{A}_k$.

Theorem 12: *If* (3.3) *has the global index k in the sense of Gear and Petzold, then the transformed equation* (3.5) *is k-reducible. Moreover,* (3.3) *itself is k-reducible in the case of a differentiable $E(\cdot)$.*

The second statement of the theorem is proved in [7]. Since the class of problems covered by the definition of Gear and Petzold is rather restricted, the k-reducibility seems to be a convenient generalization of the property expressed by the global index k. In [7] a generalization of the k-reducibility to quasi-linear problems is suggested, too. CHISTYAKOV [18] and GEAR and PETZOLD [5] considered reduction methods, too. There each step consists in a certain transformation separating the differential part of the system from the algebraical one, followed by the differentiation of the latter part.

REFERENCES

[1] ABRAMOV, A. A.: Personal communication.

[2] ARNOLD, V. I.: Gewöhnliche Differentialgleichungen. Berlin—Heidelberg—New York: Springer-Verlag/Berlin: Dt. Verlag Wiss. 1979.

[3] GANTMACHER, F. R.: Matrizentheorie. Berlin: Dt. Verlag Wiss./Berlin—Heidelberg—New York: Springer-Verlag 1986.

[4] GEAR, C. W., and L. R. PETZOLD: Differential/algebraic systems and matrix pencils. Lect. Notes Math. **973** (1983), 75—89.

[5] GEAR, C. W., and L. R. PETZOLD: ODE methods for the solution of differential algebraic systems. SIAM J. Numer. Anal. **21** (1984), 716—728.

[6] GRIEPENTROG, E.: Transformation concepts for differential-algebraic equations. Banach Cent. Publ. (to appear).

[7] GRIEPENTROG, E.: Transformations of differential-algebraic equations. Proc. Fourth Int. Sem. "Numerical Treatment of Differential Equations", Halle (GDR) May 25—29, 1987.

[8] GRIEPENTROG, E., and R. MÄRZ: Differential-algebraic equations and their numerical treatment (Teubner-Texte Math.: Vol. 88). Leipzig: B. G. Teubner Verlagsges. 1986.

[9] HANKE, M.: On the regularization of index-2 differential-algebraic equations. Preprint. Berlin: Sektion Math. der Humboldt-Univ., Preprint 137 (1987), 1—22.

[10] HANKE, M., MÄRZ, R., and A. NEUBAUER: On the regularization of a class of nontransferable differential-algebraic equations. J. Diff. Equ. (to appear).

[11] HUNDSDORFER, W.-H.: Stability results for θ-methods applied to a class of stiff differential-algebraic equations. Amsterdam: Centre Math. Comp. Sci., Report NM-R8708 (1987), 1—10.

[12] MÄRZ, R.: Index-2 differential-algebraic equations. Preprint. Berlin: Sektion Math. der Humboldt-Univ., Preprint 135 (1987), 1—23.

[13] MÄRZ, R.: Some new results concerning index-3 differential-algebraic equations. J. Math. Anal. Appl. (to appear).

[14] MÄRZ, R.: Higher-index differential-algebraic equations: Analysis and numerical treatment. Preprint. Berlin: Sektion Math. der Humboldt-Univ., Preprint 159 (1987), 1—30; Banach Cent. Publ. (to appear).

[15] MÄRZ, R.: A matrix chain for analyzing differential-algebraic equations. Preprint. Berlin: Sektion Math. der Humboldt-Univ., Preprint 162 (1987), 1—30.

[16] MÄRZ, R.: Recent results on higher-index differential-algebraic equations. Proc. Fourth Int. Sem. "Numerical Treatment of Differential Equations" held in Halle (GDR) May 25—29, 1987.

[17] Rheinboldt, W. C.: Differential-algebraic systems as differential equations on manifolds. Pittsburgh: Dep. Math. Stat. Univ. of Pittsburgh; Techn. Rep. ICMA-83-55 (1983), 1—15.
[18] Чистяков, В. Ф.: К методам решения сингулярных линейных систем обыкновенных дифференциальных уравнений. В книге: Вырожденные системы обыкновенных дифференциальных уравнений (под ред-ей Ю. Е. Бояринцева). Новосибирск: Изд-во Наука 1982, стр. 37—65.

Manuskripteingang: 20. 01. 1988

VERFASSER:

Prof. Dr. E. Griepentrog
Sektion Mathematik
der Ernst-Moritz-Arndt-Universität
Ludwig-Jahn-Str. 15a
DDR-2200 Greifswald

Prof. Dr. Roswitha März
Sektion Mathematik
der Humboldt-Universität
Postfach 1297
DDR-1086 Berlin

Buchbesprechung

J. Musielak (ed.): **Function Spaces.** Proc. Int. Conf. Poznań, August 25—30, 1986 (Teubner-Texte zur Mathematik: Bd. 103). Leipzig: B. G. Teubner Verlagsges. 1988, 196 S.

These proceedings contain 27 lectures presented at the International Conference on "Function Spaces" held on August 25—30, 1986 in Poznań, Poland. The total number of participants was 82, from the following countries: Belgium, Bulgaria, China, Czechoslovakia, FRG, France, GDR, Holland, Hungary, Poland, Rumania, Spain, Sweden, USA and USSR. The proceedings are divided in 4 parts.

Part I, "Orlicz Spaces", contains 9 contributions dealing with geometric properties in Orlicz spaces, minimal Orlicz function spaces, galb conditions, measure of non-compactness and some probability and control system aspects of Orlicz spaces.

Part II, "Other Function Spaces", contains 5 notes on modular function spaces, Riesz spaces, spaces of differentiable functions and Riemann integrable functions.

Part III, "Approximation and Interpolation in Function Spaces", consists of 6 papers devoted to various problems in approximation theory.

Finally, Part IV, "Other Topics in Function Spaces and Banach Spaces", contains 7 contributions on positive contractions in Banach spaces, p-Banach spaces, F- and D_∞-spaces and multivalued maximal accretive mappings.

It is evident from this list that many aspects of the theory of function spaces got attention during this conference; apart from the 27 contributions above, there were 30 lectures which are not included in this volume.

The contents are as follows:

Part I. **Orlicz Spaces**

J. Appell	Measures of non-compactness in ideal spaces
Chen Shutao	Convexity and smoothness of Orlicz spaces. Geometry of Orlicz space I
I. Fazekas	On Banach spaces of type Φ