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Error Estimates for the Explicit Finite Difference Method Solving the Stefan Problem

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Die Arbeit beschäftigt sich mit einer klassischen Differenzenmethode zur Lösung der eindimensionalen Zwei-Phasen-Stefan-Aufgabe. Unter Voraussetzungen an die Daten, die eine Regularität der Temperatur u und der freien Grenze s von

$$u \in W_2^{1,1}(\Omega_T) \cap L_\infty(0, T; W_2^1(\Omega)) \quad \text{und} \quad s \in W_4^1(0, T)$$

sichern, werden L_2-Konvergenzordnungen von $O(h)$ für die Näherungen von u und s bewiesen. Zusätzlich wird ein Beispiel mit schwächerer Regularität betrachtet.

В работе исследуется явный метод конечных разностей для одномерной двухфазной задачи Стефана. При предположениях о данных обеспечивающих гладкость температуры u и свободной границы s вида

$$u \in W_2^{1,1}(\Omega_T) \cap L_\infty(0, T; W_2^1(\Omega)) \quad \text{и} \quad s \in W_4^1(0, T)$$

доказывается порядок точности $O(h)$ в норме L_2 для аппроксимации решений u и s. Кроме того, рассматривается пример более слабой регулярности.

The paper deals with a well-known finite difference method for solving the one-dimensional two-phase Stefan problem. Under assumptions on the data which assure a regularity

$$u \in W_2^{1,1}(\Omega_T) \cap L_\infty(0, T; W_2^1(\Omega)) \quad \text{and} \quad s \in W_4^1(0, T)$$

of the temperature u and the moving boundary s L_2-convergence rates $O(h)$ for the approximations of u and s are derived. Additionally, a case of much weaker regularity is considered.

1. Introduction

This paper is concerned with the explicit finite difference method first described in [2] for the general enthalpy problem. Considering the special case of the one-dimensional two-phase Stefan problem, we give error estimates both for temperature and moving boundary.

Sections 2, 3 and 4 contain the problem, the method and regularity results, respectively. The error estimates are derived in Section 5 using the ideas of our previous papers [5–7]. The result on the moving boundary is based on a new proof and allows to deal with the case of nonconsistent initial and boundary temperature. This is the subject of the final Section 6.

2. The problem

Let be $\Omega = \{x: 0 < x < 1\}$, $\Gamma = \{0, 1\}$, $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \Gamma \times (0, T)$. We consider the two-phase Stefan problem

$$\left.\begin{array}{l} (cu)_t - u_{xx} = 0 \quad \big((x, t) \in \Omega_T \text{ and } x \neq s(t)\big), \\ u = g(x, t) \quad \big((x, t) \in \Gamma_T\big), \\ u = 0 \quad \text{and} \quad \lambda s_t = [u_x] \quad \big(x = s(t),\, t \in (0, T)\big), \\ u = u_0(x) \quad (x \in \Omega,\, t = 0), \qquad s = s_0 \quad (t = 0) \end{array}\right\} \tag{S}$$

with the notation $[v(x)] = v(x + 0) - v(x - 0)$ and the following specification on the data and coefficients:

$$c = \begin{cases} c_1 & \text{if } x < s(t) \\ c_2 & \text{if } x > s(t), \end{cases} \qquad g = \begin{cases} g_1(t) & \text{if } x = 0 \\ g_2(t) & \text{if } x = 1, \end{cases}$$

where c_1, c_2, λ, $s_0 \in \Omega$ are positive constants, g and u_0 are functions defined and bounded on Γ_T and Ω, respectively. For definiteness we suppose that $g_1 > 0$ and $g_2 < 0$ $\big(t \in (0, T)\big)$, $u_0 > 0$ $(x < s_0)$ and $u_0 < 0$ $(x > s_0)$.

If the pair (u, s) is a solution of problem (S), then u and

$$H = a(u) + \lambda v, \quad \text{where} \quad v \in \theta(u) \tag{2.1}$$

with

$$a(u) = \begin{cases} c_1 u & \text{if } u \geqq 0 \\ c_2 u & \text{if } u < 0, \end{cases} \qquad \theta(u) = \begin{cases} 1 & \text{if } u > 0 \\ [0, 1] & \text{if } u = 0 \\ 0 & \text{if } u < 0, \end{cases}$$

satisfy the equations

$$\left.\begin{array}{ll} H_t - u_{xx} = 0 & \big((x, t) \in \Omega_T\big), \\ u = g(x, t) & \big((x, t) \in \Gamma_T\big), \\ H = H_0(x) & (x \in \Omega;\, t = 0), \end{array}\right\} \tag{E}$$

where $H_0 = a(u_0) + \lambda v_0$ and $v_0 \in \theta(u_0)$. The functions H and v are called *enthalpy* and *liquid fraction*, respectively.

A *weak solution of problem* (E) is a pair (H, u) of functions $H \in L_2(\Omega_T)$ and $u \in W_2^{1,0}(\Omega_T)$, connected by relation (2.1) and satisfying both the boundary condition $u = g$ and the integral identity

$$\int_0^T \int_\Omega (-H\phi_t + u_x\phi_x)\, dx\, dt = \int_\Omega H_0(x)\, \phi(x, 0)\, dx \tag{2.2}$$

for all test functions $\phi \in W_2^{1,1}(\Omega_T)$ vanishing on Γ_T and for $x \in \Omega$, $t = T$.

More results on problems (S) and (E) and their interrelation can be found in the recent monograph [4].

3. The finite difference method

We define the following sets of grid points:

$$\omega = \{x = ih,\, i = 1, \ldots, N - 1,\, Nh = 1\}, \qquad \gamma = \Gamma, \qquad \overline{\omega} = \omega \cup \gamma,$$

$$\omega_\tau = \{t = n\tau,\, n = 1, \ldots, M,\, M\tau = T\}, \qquad \omega_T = \omega \times \omega_\tau, \qquad \gamma_T = \gamma \times \omega_\tau$$

and denote the approximations of u, H, v and s by y, w, ν and ξ, respectively. Further we make use of the scalar product

$$(y, w) = \sum_{x \in \omega} h y(x)\, w(x) + \frac{1}{2} \sum_{x \in \gamma} h y(x)\, w(x)$$

and the following norms:

$$\|y\| = (y, y)^{1/2}, \qquad \|y\|_1 = (|y|, 1), \qquad \|y\|_\infty = \sup_{x \in \omega} |y(x)|,$$

$$\|y]| = \Big(\sum_{x \in \omega \cup \{1\}} h y^2(x)\Big)^{1/2}, \qquad \|y\|_{\omega_T} = \big(\sum \tau\, \|y(t)\|^2\big)^{1/2}.[1]$$

Algorithm 3.1: *We choose* $\tau = \varkappa h^2$, $0 < \varkappa \leqq c_0/3$, *where* $c_0 = \min\{c_1, c_2\}$ *and set* $y = u_0$ $(x \in \overline{\omega})$, $w = H_0$ $(x \in \omega)$ *and* $\xi = s_0$ *at time* $t = 0$. *We suppose* $\check{y}, \check{w}, \check{\nu}, \check{\xi}$. *(i.e. the values of* y, w, ν, ξ *at time level* $\check{t}$*) to be known. At time* $\check{t} = t + \tau \in \omega_\tau$ *we determine* w, y, ν, ξ *as follows*

$$w_{\bar{t}} - \check{y}_{\bar{x}x} = 0 \qquad (x \in \omega), \tag{3.1}$$

$$y = \begin{cases} (w - \lambda)/c_1 & \text{if } w \geqq \lambda \\ 0 & \text{if } w \in (0, \lambda) \\ w/c_2 & \text{if } w \leqq 0 \end{cases} \quad (x \in \omega), \qquad y = P_t^2 g \quad (x \in \gamma),$$

$$\nu = \begin{cases} \big(w - a(y)\big)/\lambda & \text{if } x \in \omega \\ \theta(y) & \text{if } x \in \gamma, \end{cases} \qquad \xi = (\nu, 1).[2]$$

We define

$$x^+(t) = \max\{x \in \omega : w(x, t) > \lambda\}, \qquad x^-(t) = \min\{x \in \omega : w(x, t) < 0\}.$$

Corollary 3.2: *We have* $x^+(t) + h/2 \leqq \xi(t) \leqq x^-(t) - h/2$ $(t \in \omega_\tau)$.

Lemma 3.3: *We have* $h \leqq x^-(t) - x^+(t) \leqq 2h$ $(t \in \omega_\tau)$, *in particular, at any time* $t \in \omega_\tau$ *one of the following eight relations is valid:*

a) $x^+ = x^- - h = \check{x}^+ + h = \check{x}^- - h$,
e) $x^+ = x^- - 2h = \check{x}^+ - h = \check{x}^- - 2h$,
b) $x^+ = x^- - 2h = \check{x}^+ = \check{x}^- - 2h$,
f) $x^+ = x^- - h = \check{x}^+ = \check{x}^- - h$,
c) $x^+ = x^- - h = \check{x}^+ = \check{x}^- - 2h$,
g) $x^+ = x^- - h = \check{x}^+ + h = \check{x}^-$;
d) $x^+ = x^- - h = \check{x}^+ - h = \check{x}^- - 2h$,
h) $x^+ = x^- - 2h = \check{x}^+ = \check{x}^- - h$.

Proof: The assertion is obvious for $t = 0$. We suppose that it is true at time $\check{t}$ and verify it for $t = \check{t} + \tau \in \omega_\tau$. The maximum principle applied to scheme (3.1) yields $w(x) > \lambda$ $(x < \check{x}^+)$ and $w(x) < 0$ $(x > \check{x}^-)$. If $\check{x}^- - \check{x}^+ = 2h$ then we obtain $w(\check{x}^+) = (1 - 2\varkappa/c_1)\big(\check{w}(\check{x}^+) - \lambda\big) + \varkappa\check{y}(\check{x}^+ - h) + \lambda > \lambda$ and $w(\check{x}^-) = (1 - 2\varkappa/c_2)\,\check{w}(\check{x}^-) + \varkappa\check{y}(\check{x}^- + h) < 0$, which means either case a) or b) or c), corresponding to $w(\check{x}^+ + h) > \lambda$, $w(\check{x}^+ + h) \in [0, \lambda]$ and $w(\check{x}^+ + h) < 0$. Now suppose that the relation $\check{x}^- - \check{x}^+ = h$ holds. If $w(\check{x}^-) < 0$, then we obtain the cases d) or e) or f), depending on $w(\check{x}^+)$: either $w(\check{x}^+) < 0$ or $w(\check{x}^+) \in [0, \lambda]$ or $w(\check{x}^+) > \lambda$. The two remaining cases $w(\check{x}^-) > \lambda$ and $w(\check{x}^-) \in [0, \lambda]$ lead us to g) and h) since $w(\check{x}^-) \geqq 0$ implies $w(\check{x}^+) > \lambda$ as will be shown now. From

$$0 \leqq w(\check{x}^-) = \check{w}(\check{x}^-) + \varkappa\Big(\big(\check{w}(\check{x}^+) - \lambda\big)/c_1 - 2\check{w}(\check{x}^-)/c_2 + \check{y}(\check{x}^- + h)\Big)$$

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[1]) Throughout this paper summation over $t \in \omega_\tau$ is simply denoted by $\sum$ (instead of $\sum_{t \in \omega_\tau}$).
[2]) For the definition of P_t^2 see Section 5.

we conclude $0 > \check{w}(\check{x}^-)' > -(1 - 2\varkappa/c_2)^{-1} \varkappa(\check{w}(\check{x}^+) - \lambda)/c_1$. The combination of the latter inequality and $w(\check{x}^+) > \check{w}(\check{x}^+) + \varkappa(-2(\check{w}(\check{x}^+) - \lambda)/c_1 + \check{w}(\check{x}^-)/c_2)$ indeed yields

$$w(\check{x}^+) > (c_1(c_2 - 2\varkappa))^{-1} ((c_1 - 2\varkappa)(c_2 - 2\varkappa) - \varkappa^2)(\check{w}(\check{x}^+) - \lambda) + \lambda \geqq \lambda$$

since $(c_1 - 2\varkappa)(c_2 - 2\varkappa) - \varkappa^2 \geqq (c_0 - 2\varkappa)^2 - \varkappa^2 = (c_0 - 3\varkappa)(c_0 - \varkappa) \geqq 0$ ∎

Corollary 3.4: *We have* $|\xi(t) - \xi(\check{t})| \leqq h$ $(t \in \omega_\tau)$.

Remark 3.5: The preceding assertion is equivalent to $|\xi(t) - \xi(\check{t})| \leqq C\tau^{1/2}$ $(t \in \omega_\tau)$, which means that the piecewise linear prolongation $\tilde{\xi}(t)$ of ξ onto $(0, T)$ obeys a Hölder condition with exponent 1/2.[3])

4. Regularity

All the results are derived under the following assumptions on the data of problem (S):

$$\left.\begin{array}{l} u_0 \in W_2{}^1(\Omega), \quad u_0(x) > 0 \quad (x < s_0), \quad u_0(x) < 0 \quad (x > s_0) \\ g_i \in W_2{}^1(0, T) \text{ and } 0 < g_0 \leqq (-1)^{i+1} g_i(t) \quad (t \in (0, T)), \quad i = 1, 2 \\ u_0(0) = g_1(0), \quad u_0(1) = g_2(0). \end{array}\right\} \quad (4.1)$$

Lemma 4.1: *There are constants* $0 < d < 1/2$ *and* $C > 0$ *such that*

$$d < x^+(t) < x^-(t) < 1 - d \quad (t \in \omega_\tau) \quad \text{and} \quad \sup_{t \in \omega_\tau} \|y(t)\|_\infty < C.$$

Proof: The linear operator $Lv = \beta(x, t)\, v_{\bar{t}} - \check{v}_{\bar{x}x}$ with

$$\beta = \begin{cases} (a(y) - a(\check{y}))/(y - \check{y}) & \text{if } y \neq \check{y} \\ c_0 & \text{if } y = \check{y} \end{cases}$$

defines a monoton scheme since $\tau/h^2 < c_0/2$. Obviously, $Ly = a(y)_{\bar{t}} - \check{y}_{\bar{x}x} = -\lambda\xi_{\bar{t}} \times \delta(x, \tilde{x})/h$ with either $\tilde{x} = \check{x}^+ + h$ (cases a), b), c), g), h)) or $\tilde{x} = \check{x}^+$ (cases d), e), f))[4]). It is easily realized that there are constants $\underline{C} > 0$, $0 < \underline{d} < 1/2$ defining the straight line $\underline{u}(x) = \underline{C}(\underline{d} - x)$ such that $u_0 \geqq \underline{u}$ and $g_0 - (g_0 + \|g\|_{L_\infty(\Gamma_T)})\, x \geqq \underline{u}$ $(x \in \omega)$ and $x^+(0) \geqq \underline{d}$. Obviously, $L\underline{u} = 0$ $(x \in \omega)$. We suppose that $\check{x}^+ \geqq \underline{d}$ and $\check{y} \geqq \underline{u}$ $(x \in \omega)$ is valid for some $t \in \omega_\tau$. If $\xi_{\bar{t}} \leqq 0$, then $Ly \geqq 0$ $(x \in \omega)$, hence $y \geqq \underline{u}$ $(x \in \omega)$ by the maximum principle. Consequently, $x^+ \geqq \underline{d}$. If $\xi_{\bar{t}} > 0$, then $x^+ \geqq \check{x}^+ \geqq \underline{d}$ and thus $y(x^+) > 0 > \underline{u}(x^+)$. This implies $y \geqq \underline{u}$ $(x \in \omega)$, again by the maximum principle. Analogously we show that $x^- \leqq 1 - \bar{d}$ and $y \leqq \bar{u}(x)$ $(x \in \omega)$ with $\bar{u}(x) = \bar{C}(1 - \bar{d} - x)$ and constants $\bar{C} > 0$, $0 < \bar{d} < 1/2$. The proof is complete with $d = \min(\underline{d}, \bar{d})$ and $C = \max(\underline{C}, \bar{C})$ ∎

Lemma 4.2: *There exists a constant* $C > 0$ *such that*

$$\|\check{y}_{\bar{x}}(t)\|_\infty \leqq C(1 + \|y_{\bar{t}}(t)\|) \qquad (t \in \omega_\tau).$$

Proof: We suppose $x \leqq \tilde{x}$ $(x \in \omega)$ with $\tilde{x}$ defined as in the proof of Lemma 4.1. Obviously, there exists a grid point $x' \in \omega$, $x' \leqq \tilde{x}$ such that $|\check{y}_{\bar{x}}(x')| \leqq |(\check{y}(0) - 0)/x^+|$. If $x > x'$, then we have

$$\check{y}_{\bar{x}}(x) = \check{y}_{\bar{x}}(x') + \sum_{x' \leqq x'' < x} h\check{y}_{\bar{x}x}(x'') = \check{y}_{\bar{x}}(x') + \sum_{x' \leqq x'' < x} h(a(y))_{\bar{t}}\,(x'').$$

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[3]) Throughout this paper C denotes a generic constant not depending on h.

[4]) By $\delta(x, \tilde{x})$ we mean Kronecker's delta.

A similar relation holds if $x < x'$. Thus we obtain

$$|\check{y}_{\bar{x}}(x)| \leqq \|g\|_{L_\infty}/d + \|y_i\| \max\{c_1, c_2\} \qquad (x \leqq \bar{x})$$

since $|\check{y}(0)/x^+| < \|g\|_{L_\infty}/d$ and $|(a(y))_i| \leqq |y_i| \max\{c_1, c_2\}$. If $x > \bar{x}$, then we proceed analogously ∎

Lemma 4.3: *There exists a constant $C > 0$, such that*

$$\|y_i\|_{\omega_T} \leqq C \quad \textit{and} \quad \sup_{t \in \omega_\tau} \|y_{\bar{x}}(t)]| \leqq C.$$

Proof: We multiply the equation

$$(a(y))_i - \check{y}_{\bar{x}x} + \lambda \xi_i \delta(x, \bar{x})/h = 0 \qquad ((x, t') \in \omega_T)$$

by $\tau h y_i(x, t')$ and sum up over $x \in \omega$, $t' \in \omega_\tau$ and $t' \leqq t$ $(0 < t < T)$. Using Lemma 4.2 and the relation

$$y_i(\bar{x})\, \xi_i \geqq 0 \qquad (t \in \omega_\tau), \tag{4.2}$$

which is easily verified in all cases a)—h), we obtain the inequality

$$\sum \tau \|y_i(t')\|^2 + \|y_{\bar{x}}(t)]|^2 \leqq C \left(\|y_{\bar{x}}(0)]|^2 + \sum_{x \in \gamma} \sum \tau\, |y_i(x, t')|^2\right).$$

Finally the estimates

$$\|y_{\bar{x}}(0)]|^2 \leqq \|u_0\|^2_{W_2^1(\Omega)}, \qquad \sum_{x \in \gamma} \sum \tau\, |y_i(x, t')|^2 \leqq \sum_{x \in \gamma} \|g(x)\|^2_{W_2^1(0,T)}$$

yield the result ∎

Lemma 4.4: *There exists a constant $C > 0$, such that*

$$\sum \tau \|\check{y}_{\bar{x}}(t)]|^4_\infty \leqq C.$$

Proof: With the notation used in the proof of Lemma 4.2 we have

$$\check{y}_{\bar{x}}^2(x) = \check{y}_{\bar{x}}^2(x') + \sum_{x' \leqq x'' < x} h \check{y}_{\bar{x}x}(\check{y}_x + \check{y}_{\bar{x}})(x''), \quad \text{if } x' < x \leqq \bar{x}$$

and similar relations in the other cases. Hence,

$$|\check{y}_{\bar{x}}(x)|^2 \leqq (\|g\|_{L_\infty}/d)^2 + 2\, \|y_i\|\, \|\check{y}_{\bar{x}}]|\, \max\{c_1, c_2\} \qquad (x \in \omega).$$

Lemma 4.2 yields the estimate $\|\check{y}_{\bar{x}}]|^2 \leqq C(1 + \|y_i\|)$ in a first step and the final result in a second one ∎

Lemma 4.5: *There exists a constant $C > 0$, such that*

$$\sum \tau\, |\xi_i(t)|^4 \leqq C.$$

Proof: Suppose that $\xi_i \neq 0$. Then

$$0 < \lambda \xi_i^2 = h(\xi_i \check{y}_{\bar{x}x} - \xi_i (a(y))_i)(\bar{x}) \leqq h \xi_i \check{y}_{\bar{x}x}(\bar{x})$$

due to (4.2). Consequently, we have

$$\lambda\, |\xi_i| \leqq h\, |\check{y}_{\bar{x}x}(\bar{x})| = |\check{y}_x(\bar{x}) - \check{y}_{\bar{x}}(\bar{x})| \leqq \|\check{y}_{\bar{x}}]|_\infty.$$

Lemma 4.4 completes the proof ∎

Corollary 4.6: *We have $\sum \tau \|\nu_i(t)\|_1^2 \leqq C$.*

This follows via $\nu_i = \xi_i \delta(x, \bar{x})/h$.

Remark 4.7: Defining suitable prolongations $\tilde{y}$, $\tilde{v}$ and $\tilde{\xi}$ of the grid functions y, v and ξ onto the regions Ω_T and $(0, T)$ respectively and considering their convergence to some functions u, v and s (for details cf. [2, 3]), we find that u, s and $H = a(u) + \lambda v$ form a unique solution of (S) and (E) with the following regularity properties:

$$u \in W_2^{1,1}(\Omega_T) \cap L_\infty\big(0, T; W_2^1(\Omega)\big), \qquad u_x \in L_4\big(0, T; L_\infty(\Omega)\big),$$
$$s \in W_4^1(0, T) \quad \text{and} \quad v_t \in L_2\big(0, T; L_1(\Omega)\big).$$

Moreover, $d < s(t) < 1 - d$ $\big(t \in (0, T)\big)$ holds.

5. Error estimation

We define the functions

$$\phi_2(x'; x) = \frac{1}{h^2}\begin{Bmatrix} h - |x - x'| & \text{if } x' \in (x - h, x + h) \\ 0 & \text{if } x' \notin (x - h, x + h) \end{Bmatrix} \qquad (x \in \Omega),$$

$$\phi_2(t'; t) = \frac{1}{\tau^2}\begin{Bmatrix} \tau - |t - t'| & \text{if } t' \in (t - \tau, t + \tau) \\ 0 & \text{if } t' \notin (t - \tau, t + \tau) \end{Bmatrix} \quad (t \in [\tau, T]),$$

$$\phi_2(t'; t) = \frac{1}{\tau^2}\begin{Bmatrix} \tau & \text{if } t' \leqq t \\ \tau - t' + t & \text{if } t' \in (t, t + \tau) \\ 0 & \text{if } t' \geqq t + \tau \end{Bmatrix} \quad \big(t \in (0, \tau)\big):$$

Then $\phi(x', t') = \phi_2(x'; x)\, \phi_2(t'; t - \tau/2)$ $\big((x, t) \in \omega_T\big)$ is a suitable test function for the identity (2.1) and we obtain the equation

$$(\bar{P}H)_{\bar{t}} - (P\check{u})_{\bar{x}x} = 0 \qquad \big((x, t) \in \omega_T\big) \tag{5.1}$$

with the notations

$$\bar{P}H = \begin{cases} P_t P_x^2 H & \text{if } t > 0 \\ P_x^2 H_0 & \text{if } t = 0, \end{cases} \qquad Pu = \begin{cases} P_t^2 u & \text{if } x \in \omega \\ P_t^2 g & \text{if } x \in \gamma, \end{cases}$$

$$P_x^2 u(x) = \int_{-\infty}^{\infty} \phi_2(x'; x)\, u(x')\, dx', \qquad P_t u(t) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} u(t')\, dt'$$

$$P_t^2 u(t) = \int_0^\infty \phi_2(t'; t + \tau/2)\, u(t')\, dt'.$$

Introducing the errors $z = y - Pu$, $\zeta = \xi - P_t s$ and

$$V(x, t) = \begin{cases} \sum\limits_{t < t' \leqq T} \tau \exp(-t')\, z(x, t') & \text{if } t < T \\ 0 & \text{if } t \geqq T, \end{cases}$$

and defining the operator $Ay = \begin{cases} -y_{\bar{x}x} & \text{if } x \in \omega \\ 0 & \text{if } x \in \gamma, \end{cases}$ we have the following result.

Theorem 5.1: *The errors z and ζ satisfy the estimate*

$$\|z\|_{\omega_T}^2 + \sum \tau (AV, V)(t) + (AV, V)(0) \leqq \varepsilon \sum \tau\, |\zeta(t)|^2 + C\varepsilon^{-1} h^2$$

for sufficiently small $\varepsilon > 0$.

Proof: We subtract the equations (3.1) and (5.1) and obtain

$$\left(a(y) - \tilde{P}\right)_{\bar{t}} + \lambda(\nu - \bar{P}v)_{\bar{t}} + A\check{z} = \eta_{\bar{t}} \qquad (t \in \omega_\tau) \tag{5.2}$$

and $a(y) - \tilde{P} + \lambda(\nu - \bar{P}v) = \chi$ $(t = 0)$ with $\eta = \bar{P}a(u) - \tilde{P}$,

$$\chi = a(u_0) - P_x{}^2 a(u_0) + \lambda(v_0 - P_x{}^2 v_0) \quad \text{and} \quad \tilde{P} = \begin{cases} a(Pu) & \text{if } t \in \omega_\tau \\ P_x{}^2 a(u_0) & \text{if } t = 0. \end{cases}$$

Following the proof of Theorem 4 from [5], we multiply equation (5.2) by $\tau V(\check{t})$ $(t \in \omega_\tau)$ in the sense of the scalar product and sum up over $t \in \omega_\tau$. We get the inequality

$$\|z\|_{\omega_T}^2 + \sum \tau\left(R(t) + (AV, V)(t)\right) + (AV, V)(0) \leqq C(\|\eta\|_{\omega_T}^2 + \|\chi\|_1{}^2)$$

where $R(t)$ is equal to $\exp(t)\,(\nu - \bar{P}v, z)(t)$ up to a positive multiplicative constant and where the right side is estimated using

$$\|\eta\|_{\omega_T} \leqq C(h + \tau)\, \|u\|_{W_2{}^{1,1}(\Omega_T)} \quad \text{and} \quad \|\chi\|_1 \leqq Ch.$$

In the following we verify the inequality

$$R(t) \geqq -C\left(\varepsilon(|\zeta|^2 + \|z\|^2) + \varepsilon^{-1}(\|\psi\|^2 + \|\varphi\|_\infty^2 + h^2)\right) \qquad (\varepsilon > 0) \tag{5.3}$$

and the estimates

$$\sum \tau \|\psi\|_1{}^2 \leqq C\tau^2 \|v_t\|_{L_2(0,T;L_1(\Omega))}^2, \qquad \sum \tau \|\varphi\|_\infty^2 \leqq Ch^2 \|u_x\|_{L_2(0,T;L_\infty(\Omega))}^2 \tag{5.4}$$

which will complete the proof together with $\|\psi\|_{\omega_T}^2 \leqq Ch^{-1} \sum \tau \|\psi\|_1{}^2$ and Remark 4.7.

First we fix $t \in \omega_\tau$ and define

$$\nu_\Delta(x, t) = \theta_\Delta\left(y(x, t)\right) \quad (x \in \bar{\omega}), \qquad v_\Delta(x, t) = \theta_\Delta\left(u(x, t)\right) \quad \left((x, t) \in \Omega_T\right)$$

with

$$\theta_\Delta(u) = \begin{cases} \Delta^{-1}u + \nu(x^+ + h, t) & \text{if } u \in \left(-\nu(x^+ + h, t)\,\Delta, \left(1 - \nu(x^+ + h, t)\right)\Delta\right) \\ \theta(u) & \text{otherwise.} \end{cases}$$

If $\Delta \to 0$, then

$$\nu_\Delta \to \nu \qquad (x \in \bar{\omega}) \tag{5.5}$$

by construction. Further, $v_\Delta(x, t)$ converges pointwise to $v(x, t)$ if $u(x, t) \neq 0$, i.e.

$$v_\Delta \to v \quad \text{a.e. on } \Omega_T. \tag{5.6}$$

Now we consider the term $(\nu_\Delta - \bar{P}v_\Delta, z)(t)$ as a sum $R_1 + R_2 + R_3$ of

$$R_1 = \left(\theta_\Delta(y) - P_x{}^2\theta_\Delta(P_t{}^2u), z - \varphi_\Delta\right), \qquad R_2 = (\psi_\Delta, z - \varphi_\Delta),$$

$$R_3 = (\nu_\Delta - \bar{P}v_\Delta, \varphi_\Delta).$$

with grid functions

$$\varphi_\Delta = (P_t{}^2u)(x_\Delta{}^*) - (P_t{}^2u)(x),$$

$$\psi_\Delta = P_x{}^2\theta_\Delta(P_t{}^2u) - P_tP_x{}^2\theta_\Delta(u)$$
$$= P_x{}^2\left(\theta_\Delta(u(x, t^*)) - P_t\theta_\Delta(u)\right) = P_x{}^2\left(v_\Delta(x, t^*) - P_tv_\Delta(x, t)\right)$$

and numbers

$$x_\Delta{}^* = x_\Delta{}^*(x, t) \in (x - h, x + h) \quad \text{and} \quad t^* = t^*(x, t) \in (t - \tau/2, t + 3/2\tau)$$

whose existence follows from the continuity of u in (x, t) (cf. [8]), the continuity of $P_t^2 u$ in x and the continuity of θ_Δ in u via the generalized mean value theorem of integral calculus. Now we may establish the estimates

$$R_1 = \left(\theta_\Delta(y) - \theta_\Delta(P_t^2 u(x_\Delta^*)), y - P_t^2 u(x_\Delta^*)\right) \geqq 0,$$

$$|R_2| \leqq C(\varepsilon \|z\|^2 + \varepsilon^{-1} \|\psi_\Delta\|^2 + \|\varphi_\Delta\|^2) \qquad (\varepsilon > 0),$$

$$|R_3| \leqq C(\varepsilon \|\nu_\Delta - \bar{P} v_\Delta\|_1^2 + \varepsilon^{-1} \|\varphi_\Delta\|_\infty^2)$$

and obtain

$$(\nu_\Delta - \bar{P} v_\Delta, z)(t) \geqq -C\left(\varepsilon(\|z\|^2 + \|\nu_\Delta - \bar{P} v_\Delta\|_1^2) + \varepsilon^{-1}(\|\psi_\Delta\|^2 + \|\varphi_\Delta\|_\infty^2)\right). \quad (5.7)$$

Further we have

$$|\varphi_\Delta| \leqq P_t^2 |u(x) - u(x_\Delta^*)| \quad \text{and} \quad |\psi_\Delta| \leqq \tau\left(\bar{P} |(v_\Delta)_t| + \bar{P} |(v_\Delta)_t (\hat{t})|\right).$$

Now we consider inequality (5.7) under $\Delta \to 0$. Using the relations (5.5) and (5.6), we find

$$(\nu_\Delta - \bar{P} v_\Delta, z) \to (\nu - \bar{P} v, z),$$

$$\|\nu_\Delta - \bar{P} v_\Delta\|_1 \to \|\nu - \bar{P} v\|_1 \leqq |\zeta| + 2h, \qquad \bar{P} |(v_\Delta)_t| \to \bar{P} |v_t|,$$

$$\psi_\Delta \to \psi \quad \text{where} \quad \|\psi\|_1 \leqq C\tau(P_t \|v_t\|_{L_1(\Omega)} + P_t \|v_t(\hat{t})\|_{L_1(\Omega)})$$

and

$$\varphi_\Delta \to \varphi \quad \text{where} \quad \|\varphi\|_\infty \leqq ChP_t^2 \|u_x\|_{L_\infty(\Omega)}.$$

Thus (5.3) and (5.4) are proved ∎

Theorem 5.2: *The errors z and ζ obey the estimate*

$$\sum \tau\sigma |\zeta(t)|^2 \leqq C\left(\|z\|_{\omega_T}^2 + \sum \tau(AV, V)(t) + (AV, V)(0) + h^2\right)$$

with a suitable weight function $\sigma(t) \geqq 0$ (see Remark 5.3).

Proof: We define grid points $x^{(1)}, x^{(2)} \in \omega$ such that $x^{(2)} - x^{(1)} = h$ and $x^{(1)} \leqq s(t') \leqq x^{(2)}$ $\left(t' \in (t - \tau/2, t + \tau/2)\right)$ and grid functions

$$\underline{r}(t) = \min\{x^+, x^{(1)}\}, \qquad \bar{r}(t) = \max\{x^-, x^{(2)}\},$$

$$\varrho(x, t) = \begin{cases} \dfrac{x}{\underline{r}(t)} & \text{if } 0 \leqq x \leqq \underline{r}(t) \\ 1 & \text{if } \underline{r}(t) < x \leqq \bar{r}(t) \\ \dfrac{1 - x}{1 - \bar{r}(t)} & \text{if } \bar{r}(t) < x \leqq 1, \end{cases}$$

$$W(x, t) = \begin{cases} \sum\limits_{t < t' \leqq T} \tau\sigma\zeta\varrho(x, t') & \text{if } t < T \\ 0 & \text{if } t \geqq T, \end{cases}$$

where the function $\sigma(t) \geqq 0$ is supposed to satisfy the condition

$$\sup_{t \in \omega_\tau} |\sigma(A\varrho, \varrho)(t)| \leqq C. \quad (5.8)$$

Obviously, we have $0 \leqq \varrho \leqq 1$, $W_{\bar{t}} = -\sigma\varrho\zeta$ and

$$(AW, W)(t') \leqq C \sum \tau\sigma^2\zeta^2(A\varrho, \varrho)(t) \qquad (t' \in \omega_\tau \cup \{0\}). \quad (5.9)$$

We multiply equation (5.2) by $\tau W(\check{t})$ in the sense of the scalar product and sum up over $t \in \omega_\tau$. The term generated by $A\tilde{z}$ leads to the estimate

$$|\sum \tau(A\tilde{z}, \check{W})(t)| \leq C\left(\varepsilon \sum \tau\sigma\zeta^2(t) + \varepsilon^{-1}\left((AV, V)(0) + \sum \tau(AV, V)(\check{t})\right)\right). \tag{5.10}$$

This is due to the relations (5.8), (5.9) and the following ones:

$$\sum \tau(A\tilde{z}, \check{W})(t) = \tau(Az, W)(0) - \sum \tau \exp(t)(AV_i, W)(t),$$

$$-\sum \tau \exp(t)(AV_i, W)(t) = -(AV, W)(0) + \sum \tau\left(\exp(t)(A\check{V}, W_i) + (\exp(t))_i (A\check{V}, \check{W})\right),$$

$$\tau |(Az, W)(0)| \leq \varepsilon(AW, W)(0) + \tau^2(Az, z)(0)/(4\varepsilon) \qquad (\varepsilon > 0),$$

$$\tau^2(Az, z)(0) \leq Ch^2,$$

$$|(A\check{V}, W_i)| = |\sigma\zeta(A\check{V}, \varrho)| \leq \varepsilon\sigma^2\zeta^2(A\varrho, \varrho) + (AV, V)/(4\varepsilon),$$

$$|(AV, W)(t)| \leq \varepsilon(AW, W)(t) + (AV, V)(t)/(4\varepsilon).$$

Transformation of the remaining terms implies

$$\sum \tau\left((a(y) - \tilde{P})_i + \lambda(\nu - \bar{P}v)_i - \eta_i, \check{W}\right)$$
$$= -(\chi - \eta, W)(0) + \sum \tau\sigma\zeta\left(a(y) - a(Pu) + \lambda(\nu - \bar{P}v) - \eta, \varrho\right)(t). \tag{5.11}$$

Taking into account that $(\nu - \bar{P}v, \varrho) = (\nu - \bar{P}v, 1) = \xi - P_t s = \zeta$ $(t \in \omega_\tau)$, since $\nu - \bar{P}v = 0$ $(x < \underline{r}$ or $x > \bar{r})$, and that $|a(y) - a(Pu)| \leq C|z|$, relation (5.11) yields the estimate

$$\sum \tau\left((a(y) - \tilde{P})_i + \lambda(\nu - \bar{P}v)_i - \eta_i, \check{W}\right) \leq \sum \tau\sigma\zeta^2(t)$$
$$- \varepsilon\left((AW, W)(0) + \sum \tau\sigma\zeta^2(t)\right) - C\varepsilon^{-1}(\|z\|_{\omega_T}^2 + \|\eta\|_{\omega_T}^2 + \|\chi\|_1^2). \tag{5.12}$$

We obtain the final result by summarizing the estimates (5.10), (5.12) and applying the relations (5.8), (5.9) while choosing $\varepsilon > 0$ sufficiently small ∎

Remark 5.3: We may choose $\sigma(t) = d = \text{const}$. Indeed, noting that $(A\varrho, \varrho) = 1/\underline{r} + 1/(1 - \bar{r})$ and according to Lemma 4.1 and Remark 4.7 the weight function σ satisfies relation (5.8).

Consequently, combining Theorem 5.1 and Theorem 5.2 we obtain the following main result.

Theorem 5.4: *Under the assumptions* (4.1) *the solution* y, ξ *determined by Algorithm* 3.1 *obeys the estimates*

$$\|y - Pu\|_{\omega_T} \leq Ch \quad \text{and} \quad \left(\sum \tau(\xi - P_t s)^2(t)\right)^{1/2} \leq Ch.$$

Remark 5.5: In [6, 7] we obtained error estimates of order $O(h^{1/2})$ for implicit difference methods solving the general enthalpy problem and the two-phase Stefan problem.

6. Nonconsistent initial and boundary data

We consider problem (S) specifying the data as follows: $s_0 = 0$, $H_0 = c_2 g_2 = \text{const} < 0$, $g_1 = \text{const} > 0$. Then the moving boundary $s(t)$ satisfies the relation

$$\lim_{t \to 0} t^{-1/2} s(t) = \text{const} > 0. \tag{6.1}$$

This is due to $\lim_{t\to 0} s(t)/s^*(t) = 1$, where $s^*(t)$ is the moving boundary of the "half-space" problem corresponding to (S), i.e. where Ω is replaced by $\Omega^* = (0, \infty)$, and due to $s^*(t) = c^* t^{1/2}$ with some constant $c^* > 0$ (cf. [1]).

We introduce problems (E_δ) and (S_δ) which are derived from (E) and (S) respectively by replacing H_0 by some function H_0^δ. The corresponding solutions are denoted by u^δ, H^δ and s^δ. We choose, in particular,

$$H_0^\delta(x) = \begin{cases} c_1\big(g_1 - (g_1 - g_2)\, x/\delta\big) + \lambda & \text{if } 0 < x \leqq s_0^\delta \\ c_2\big(g_1 - (g_1 - g_2)\, x/\delta\big) & \text{if } s_0^\delta < x \leqq \delta \\ c_2 g_2 & \text{if } \delta < x \leqq 1 \qquad (0 < \delta < 1) \end{cases}$$

with $s_0^\delta = g_1\delta/(g_1 - g_2)$. Now we compute the finite difference approximation y, ξ of problem (S_δ) according to Algorithm 3.1. Recalling the proof of Lemma 4.1, we find that $\xi(t)$ (and thus $s^\delta(t)$) are bounded away from $x = 1$, but we have $\underline{d} = s_0^\delta = g_1\delta/(g_1 - g_2)$. This requires an analysis how the constants $C > 0$ in the estimates from Section 4 do depend on δ.

Lemma 6.1: *The solution u^δ, H^δ of problem (E_δ) satisfies the estimates*

$$\|u^\delta\|_{W_2^{1,1}(\Omega_T)} \leqq C\delta^{-1/2},$$

$$\|u_x^\delta\|_{L_2(0,T;L_\infty(\Omega))} \leqq C\delta^{-\alpha} \quad (\alpha > 0), \qquad \|v_t^\delta\|_{L_2(0,T;L_1(\Omega))} \leqq C\delta^{-1/2}$$

where $C > 0$ does not depend on δ.

Proof: The first assertion follows from the estimates

$$\|u_t^\delta\|_{L_2(\Omega_T)} + \sup_{(0,T)} \|u_x^\delta\|_{L_2(\Omega)} \leqq C\, \|(u_0^\delta)_x\|_{L_2(\Omega)}, \qquad \|(u_0^\delta)_x\|_{L_2(\Omega)} \leqq C\delta^{-1/2}.$$

In analogy to the proof of Lemma 4.2 we have

$$\|u_x^\delta(t)\|_{L_\infty(\Omega)} \leqq C\big(1/s^\delta(t) + \|u_t^\delta(t)\|_{L_1(\Omega)}\big)$$

and thus

$$\int_0^T \|u_x^\delta(t)\|^2_{L_\infty(\Omega)}\, dt \leqq C \left(\int_0^T dt/\big(s^\delta(t)\big)^2 + \|u_t^\delta\|^2_{L_2(\Omega_T)} \right).$$

Relation (6.1) and Remark 6.4 yield

$$\int_0^T dt/\big(s^\delta(t)\big)^2 \leqq C\delta^{-2\alpha} \int_0^T dt/\big(s(t)\big)^{2-2\alpha} \leqq C\delta^{-2\alpha} \qquad (\alpha > 0)$$

which verifies the second assertion. Finally, the third estimate is a consequence of the first and second one taking into account that both

$$\lambda \int_\Omega v_t^\delta\, dx = -\int_\Omega \big(a(u^\delta)\big)_t\, dx + u_x^\delta(1) - u_x^\delta(0) \quad \text{and} \quad \int_\Omega |v_t^\delta|\, dx = \left| \int_\Omega v_t^\delta\, dx \right|$$

hold, the latter due to $v_t^\delta = 0$ $\big(x \neq s^\delta(t)\big)$ ∎

Corollary 6.2: *The finite difference solution y, ξ and the solution u^δ, s^δ of problem (S_δ) satisfy the estimates*

$$\|y - Pu^\delta\|_{\omega_T} \leqq C\delta^{-1/2-\alpha}h \quad \textit{and} \quad \Big(\sum \tau(\xi - P_t s^\delta)^2\, (t)\Big)^{1/2} \leqq C\delta^{-1-\alpha}h \quad (\alpha > 0).$$

To complete our consideration, we give a result on the approximation of problem (S) by problem (S_δ).

Lemma 6.3: *The solutions of* (S) *and* (S_δ) *satisfy the estimates*

$$\|u^\delta - u\|^2_{L_2(\Omega_T)} \leqq C\delta^3 \quad \text{and} \quad \int_0^T \sigma(t) \left(s^\delta(t) - s(t)\right)^2 dt \leqq C\delta^3$$

with a suitable weight function $\sigma \in L_\infty(0, T)$, $\sigma \geqq 0$ (*see Remark* 6.4).

Proof: First we realize the regularity property $u^\delta, u \in W_2^{1,0}(\Omega_T)$ (e.g. by multiplying equation (3.1) by $\tau h(\check{w} - (c_1 g_1 + \lambda)(1 - x) - c_2 g_2 x)$, summing up over $x \in \omega$ and $t \in \omega_\tau$ and passing to the limit $h \to 0$). Thus,

$$\phi(x, t) = V(x, t) = \int_t^T \exp(-t')(u^\delta - u)(x, t')\, dt'$$

is a suitable test function for the integral identity

$$\int_0^T \int_\Omega \left(-(H^\delta - H)\,\phi_t + (u^\delta - u)_x \phi_x\right) dx\, dt = \int_\Omega (H_0^\delta - H_0)\,\phi(x, 0)\, dx \qquad (6.2)$$

and, moreover, belongs to $L_\infty(0, T; W_2^1(\Omega))$. We obtain

$$\int_0^T \int_\Omega \left((u^\delta - u)^2 + V_x^2\right) dx\, dt + \int_\Omega V_x^2(x, 0)\, dx \leqq \left|\int_\Omega (H_0^\delta - H_0)\, V\, dx\right|.$$

With the definition

$$\chi(x) = \begin{cases} -\int_x^\delta (H_0^\delta - H_0)(x')\, dx' & \text{if } x \leqq \delta \\ 0 & \text{if } x > \delta, \end{cases}$$

i.e. $\chi_x = H_0^\delta - H_0$ we have

$$\left|\int_\Omega (H_0^\delta - H_0)\, V\, dx\right| = \left|\int_\Omega \chi V_x(x, 0)\, dx\right| \leqq \frac{1}{2}\int_\Omega V_x^2(x, 0)\, dx + \frac{1}{2}\int_\Omega \chi^2\, dx$$

and, finally, the first assertion since $\int_\Omega \chi^2\, dx \leqq C\delta^3$ holds. Next we define $\underline{s} = \min\{s, s^\delta\}$, $\bar{s} = \max\{s, s^\delta\}$ $(t \in (0, T))$,

$$\varrho(x, t) = \begin{cases} \dfrac{x}{\underline{s}(t)} & \text{if } 0 \leqq x \leqq \underline{s} \\ 1 & \text{if } \underline{s} < x \leqq \bar{s} \\ \dfrac{1 - x}{1 - \bar{s}(t)} & \text{if } \bar{s} < x \leqq 1, \end{cases} \qquad W(x, t) = \int_t^T \sigma\varrho(s^\delta - s)\, dt'$$

with an arbitrary function $\sigma \in L_\infty(0, T)$, $\sigma \geqq 0$ satisfying the condition

$$\sup_{[0,T]} \sigma(t) \int_\Omega \varrho_x^2(x, t)\, dx \leqq C.$$

Then $\phi = W$ is another test function for the identity (6.2) and belongs to $L_\infty(0, T; W_2^1(\Omega))$. In analogy to Theorem 5.2 we derive the estimate

$$\lambda \int_0^T \sigma(s^\delta - s)^2\, dt \leqq C\left(\int_0^T \int_\Omega \left((u^\delta - u)^2 + V_x^2\right) dx\, dt + \int_\Omega \left(V_x^2(x, 0) + \chi^2\right) dx\right).$$

which proves the second assertion using the first one ∎

Remark 6.4: Comparing the problems (S) and (S_δ), the maximum principle yields $u^\delta \geqq u$ $((x, t) \in \Omega_T)$ since $u_0^\delta \geqq u_0$. This implies $s^\delta \geqq s$ $(t \in (0, T))$, i.e. $\underline{s} = s$. For small t (which is the crucial case) we have $\sigma \int_\Omega \varrho_x^2 \, dx = \sigma(1/s + 1/(1 - \bar{s})) \leqq 2\sigma/s$.

Thanks to the property (6.1) we may choose $\sigma = ct^{1/2}$ with some constant $c > 0$, not depending on t.

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