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Comparison Theorems for Nonself-Adjoint Differential Equations of Second Order

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Bezüglich der Nullstellen der Ableitungen von Lösungen nicht selbstadjungierter Differentialgleichungen zweiter Ordnung werden Vergleichssätze vom Sturm-Picone-Typ bewiesen.

Относительно нулей производных решений не самосопряженных дифференциальных уравнений второго порядка доказываются теоремы сравнения типа Штурма-Пиконе.

Concerning the zeros of the derivatives of solutions for nonself-adjoint second order differential equations Sturm-Picone type comparison theorems are proved.

By the well-known Sturm-Picone theorem solutions u and v of the self-adjoint differential equations

$$-\big(P(x)\,u'\big)' + Q(x)\,u = 0 \qquad (P, Q \in C, P > 0)^{1)}, \qquad (x \in [a, b]) \tag{1}$$

$$-\big(p(x)\,v'\big)' + q(x)\,v = 0 \qquad (p, q \in C, p > 0) \tag{2}$$

are compared concerning the mutual position of their zeros (cf. [6] or [7]). It is natural to ask if there are similar results for the zeros of the derivatives u' and v' of solutions u and v. An answer for this question is due to Leighton [3, 4].

Theorem 1 (Leighton [3]): *Let Q and q be negative on $[a, b]$ and*

$$p(x) \leqq P(x), \qquad q(x) \leqq Q(x), \qquad a \leqq x \leqq b. \tag{3}$$

If the derivative u' of a solution u of the equation (1) *has consecutive zeros at $x = a$* and *$x = b$, then the derivative v' of a nonnull solution v of the equation* (2) *satisfying $v'(a) = 0$ will have a zero on the interval $(a, b]$.*

In the following this theorem will be extended to the nonself-adjoint equations

$$-\big(P(x)\,u'\big)' + R(x)\,u' + Q(x)\,u = 0, \qquad (x \in [a, b]), \tag{4}$$

$$-\big(p(x)\,v'\big)' + r(x)\,v' + q(x)\,v = 0 \tag{5}$$

where $R, Q, r, q \in C$, $P, p \in C^1$, and $p(x), P(x) > 0$ on $[a, b]$. For this end the equations (4) and (5) will be transformed into Riccati differential equations. It is easily seen that the function

$$y = Pu^{-1}u' \tag{6}$$

1) All equations and inequalities for functions on intervals are to be understood pointwise, e.g. they are valid for every point of these intervals. In the following, in case of strong inequalities, the functions are, for the sake of clearness, written with their arguments.

is a solution of the Riccati equation

$$y'(x) = -P^{-1}(x)\, y^2 + P^{-1}(x)\, R(x)\, y + Q(x) \tag{7}$$

if u is a solution to (4). Analogously, equation (5) can be transformed into

$$z'(x) = -p^{-1}(x)\, z^2 + p^{-1}(x)\, r(x)\, z + q(x) \tag{8}$$

where

$$z = pv^{-1}v'. \tag{9}$$

Lemma 1: *Assume*

$$p \leqq P, \qquad p^{-1}r \leqq P^{-1}R, \qquad q \leqq Q \text{ on } [a, b]. \tag{10}$$

Let $[\alpha, \beta] \subset [a, b]$ *be a subinterval where the solutions* y *and* z *of equations* (7) *and* (8), *respectively, exist and assume that* $y(x) > 0$ *on* $[\alpha, \beta]$. *Then* $y(\alpha) \geqq z(\alpha)$ *implies* $y \geqq z$ *and* $y(\beta) \leqq z(\beta)$ *implies* $y \leqq z$ *on* $[\alpha, \beta]$.

Proof: By (10) it follows that

$$-p^{-1}(x)\, y^2 + p^{-1}(x)\, r(x)\, y + q(x) \leqq -P^{-1}(x)\, y^2 + P^{-1}(x)\, R(x)\, y + Q(x) \tag{11}$$

for all points of the semistripe $H_+ = \{(x, y) \mid \alpha \leqq x \leqq \beta, 0 \leqq y < \infty\}$. Consider the case $y(\alpha) \geqq z(\alpha)$ and let $z(\alpha) > 0$. Then by a well-known theorem on first order differential equations (cf. [2: p. 91] or [1: p. 27]) it follows that $y \geqq z$ in a neighbourhood on the right-hand side of α. Let $\xi \in (\alpha, \beta)$ be a point where $y(\xi) = z(\xi)$. Then by the same argument we have $y(x) \geqq z(x)$ if $x \geqq \xi$ and x near to ξ. Hence, the graph of z cannot cross the graph of y on $[\alpha, \beta]$. The assertion in the case $y(\beta) \leqq z(\beta)$ is likewise a consequence of the named theorem ∎

Theorem 2: *Assume* (10) *and let* u *and* v *be solutions of equations* (4) *and* (5), *respectively, such that*

$$u(a) = 0 = u'(b), \qquad u'(x) \neq 0 \qquad \text{on } [a, b) \tag{12}$$

and

$$v(a) = 0, \qquad v'(a) \neq 0. \tag{13}$$

Then v' *has a zero on* (a, b) *or* v *is a constant multiple of* u. *In the latter case the equations* (4) *and* (5) *are identical.*

Proof: It follows from (12) that

$$u(x) \neq 0, \qquad 0 < u^{-1}(x)\, u'(x) < \infty \qquad \text{on } (a, b). \tag{14}$$

Let us assume that $v'(x) \neq 0$ on (a, b). Then, by (13) we have $v(x) \neq 0$ on $(a, b]$ and

$$0 < v^{-1}(x)\, v'(x) < \infty \qquad \text{on } (a, b). \tag{15}$$

It follows from (14) and (15) that $y(x), z(x) > 0$ on (a, b). We prove that $y = z$ on (a, b). Assuming the contrary let x_0 be a point on (a, b) where $y(x_0) \neq z(x_0)$. First we discuss the case $z(x_0) > y(x_0)$. Let y_0 be the solution of equation (7) uniquely defined by the initial value $y_0(x_0) = z(x_0)$. Consider y_0 on the left-hand side of x_0. We prove that y_0 does not exist on the entire interval $(a, x_0]$. The function

$$d = 1/(y_0 - y), \qquad d(x_0) = 1/\delta > 0, \tag{16}$$

is a solution of the differential equation

$$w' + [P^{-1}(x)\, R(x) - 2P^{-1}(x)\, y(x)]\, w - P^{-1}(x) = 0$$

in a neighbourhood of x_0 (cf. [2: p. 42]). Hence, we have

$$d(x) = \exp\left(\int_{x_0}^{x} \frac{2y-R}{P}\,dt\right)\left(\frac{1}{\delta} + \int_{x_0}^{x} \frac{1}{P} \exp\left(\int_{x_0}^{t} \frac{R-2y}{P}\,d\tau\right) dt\right)$$

$$= \exp\left(\int_{x_0}^{x} \left(2\frac{u'}{u} - \frac{R}{P}\right) dt\right)\left(\frac{1}{\delta} + \int_{x_0}^{x} \frac{1}{P} \exp\left(\int_{x_0}^{t} \left(\frac{R}{P} - 2\frac{u'}{u}\right) d\tau\right) dt\right)$$

$$= \frac{\varphi(x_0)\, u^2(x)}{u^2(x_0)\, \varphi(x)} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi\, dt}{u^2 P}\right) \tag{17}$$

where $\varphi(x) = \exp\left(\int_c^x P^{-1}R\,dt\right)$, $c \in (a, b)$. In view of $u(x) = [u'(a) + o(1)]\,(x - a)$ near to a it follows that $\int_{x_0}^{x} u^{-2}P^{-1}\varphi\,dt \to -\infty$ when $x \downarrow a$. Hence, there exists a point $x_\delta \in (a, x_0)$ such that $d(x) \to 0$ when $x \downarrow x_\delta$. Consequently, we have

$$y_0(x) \to +\infty \text{ when } x \downarrow x_\delta. \tag{18}$$

It follows from $d(x) > 0$ on $(x_\delta, x_0]$ that $y_0(x) > y(x)$ on $(x_\delta, x_0]$. Hence, $y_0(x)$ is positive on $(x_\delta, x_0]$. Consider now the solution z, $z(x_0) = y_0(x_0)$, of equation (8). By Lemma 1 it follows that $z \geqq y_0$ on $[\alpha, x_0]$, where $\alpha \in (x_\delta, x_0)$ is any point. Hence, in view of (18), z cannot be bounded on $(x_\delta, x_0]$. This, however, contradicts the fact that z exists on the entire interval (a, b). Hence, $z(x_0) > y(x_0)$ is impossible.

Assume now that $z(x_0) < y(x_0)$. Let y_0 be again the solution of the Riccati equation (7) defined by $y_0(x_0) = z(x_0)$. We show that y_0 has a zero on the interval (x_0, b). For this end we use formula (17). Consider the behaviour of $d(x)$ when x is increasing. The factor $\varphi(x_0)\, u^2(x)/u^2(x_0)\, \varphi(x)$ is positive and tends to $\varphi(x_0)\, u^2(b)/u^2(x_0)\, \varphi(b)$ when $x \to b$. For the second factor we have

$$\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi\, dt}{u^2 P} \to \frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{b} \frac{\varphi\, dt}{u^2 P}$$

when $x \to b$. If this limit is non-negative, then there exists a point $\xi \in (x_0, b]$ such that $d(x) \to 0$ when $x \uparrow \xi$, and it follows that $y_0(x) \to -\infty$ when $x \uparrow \xi$. Note that this is also true in the case $\xi = b$ because of $y(x) \to 0$ when $x \uparrow b$. In the case of negative limit above we obtain $d(x) \to \sigma < 0$ when $x \uparrow b$ and, consequently, $y_0(x) \to \sigma^{-1} < 0$ when $x \uparrow b$. Since $y_0(x_0) > 0$, in each case the function y_0 has a zero $\xi_0 \in (x_0, b)$. Lemma 1 applied to the functions y_0 and z yields the estimate $z \leqq y_0$ on $[x_0, \xi_0)$. This estimate, however, contradicts the fact that z is a positive function on (a, b). Hence, $z(x_0) < y(x_0)$ is also impossible and the assertion $y = z$ on (a, b) is proved. In this case y is a solution of equation (7) as well as of equation (8). This leads to $-P^{-1}y^2 + P^{-1}Ry + Q = -p^{-1}y^2 + p^{-1}ry + q$ on (a, b). Finally, it follows from (10) and $y > 0$ that $P = p$, $R = r$, and $Q = q$. u and v are solutions of equation (4) with $u(a) = v(a) = 0$. Thus, we obtain $v = cu$ on (a, b) ∎.

In the special case $R = r = 0$ Theorem 2 was proved by Leighton [3: Corollary].

Lemma 2: *Assume*

$$p \leqq P, \quad p^{-1}r \geqq P^{-1}R, \quad q \leqq Q \text{ on } [a, b]. \tag{19}$$

Let $[\alpha, \beta] \subset [a, b]$ *be a subinterval where the solutions* y *and* z *of equations* (7) *and* (8), *respectively, exist and assume that* $y(x) < 0$ *on* $[\alpha, \beta]$. *Then* $y(\alpha) \geqq z(\alpha)$ *implies* $y \geqq z$ *and* $y(\beta) \leqq z(\beta)$ *implies* $y \leqq z$ *on* $[\alpha, \beta]$.

Proof: Compare the proof of Lemma 1 ∎

Theorem 3: *Assume* (19) *and let* u *and* v *be solutions of equations* (4) *and* (5), *respectively, such that*

$$u'(a) = 0 = u(b), \quad u'(x) \neq 0 \text{ on } (a, b] \tag{20}$$

and

$$v(a) \neq 0, \quad v^{-1}(a)\, v'(a) \leqq 0. \tag{21}$$

Then v *has a zero on* (a, b) *or* v *is a constant multiple of* u. *In the latter case the equations* (4) *and* (5) *are identical.*

Proof: It follows from (20) that

$$u(x) \neq 0, \quad -\infty < u^{-1}(x)\, u'(x) < 0 \text{ on } (a, b). \tag{22}$$

Let us assume that $v(x) \neq 0$ on (a, b). Then the functions $y = Pu^{-1}u'$ and $z = pv^{-1}v'$ exist on $[a, b)$ and (20) and (21) give $z(a) \leqq y(a) = 0$. By (22) it follows that $y(x) < 0$ on (a, b). Since

$$-p^{-1}(x) y^2 + p^{-1}(x)\, r(x)\, y + q(x) \leqq -P^{-1}(x)\, y^2 + P^{-1}(x)\, R(x)\, y + Q(x)$$

for all points (x, y) of the semistripe $H_- = \{(x, y) \mid a \leqq x \leqq b, -\infty < y \leqq 0\}$, by the above mentioned theorem on first order differential equations it follows that $z \leqq y$ on $[a, b)$. We prove that $z = y$ on $[a, b)$. Let $x_0 \in (a, b)$ be a point where $z(x_0) < y(x_0)$. Consider the solution y_0 to (7) defined by $y_0(x_0) = z(x_0)$ and apply formula (17). Since δ^{-1} is negative and

$$\frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2 P} \to +\infty \qquad \text{when } x \uparrow b$$

there exists a point $x_\delta \in (x_0, b)$ such that $d(x) \to 0$ when $x \uparrow x_\delta$. This leads to $y_0(x) \to -\infty$ when $x \uparrow x_\delta$. By Lemma 2 it follows that $z \leqq y_0$ on $[x_0, \beta]$, where the point β can be chosen arbitrarily on (x_0, x_δ). Hence, z cannot be bounded on $[x_0, x_\delta)$. This, however, contradicts the fact that z is continuous on (a, b). This proves that $z = y$ on $[a, b)$ and, consequently, the differential equations (4) and (5) are identical ∎

Lemma 3: *Assume*

$$p \leqq P, q \leqq Q, (P^{-1}R - p^{-1}r)^2 \leqq 4(p^{-1} - P^{-1})\,(Q - q) \text{ on } [a, b]. \tag{23}$$

Let $[\alpha, \beta] \subset [a, b]$ *be a subinterval where the solutions* y *and* z *of equations* (7) *and* (8), *respectively, exist. Then* $y(\alpha) \geqq z(\alpha)$ *implies* $y \geqq z$ *and* $y(\beta) \leqq z(\beta)$ *implies* $y \leqq z$ *on* $[\alpha, \beta]$.

Proof: By (23) it follows that

$$-p^{-1}(x)\, y^2 + p^{-1}(x)\, r(x)\, y + q(x) \leqq -P^{-1}(x)\, y^2 + P^{-1}(x)\, R(x)\, y + Q(x) \tag{24}$$

for all points (x, y) of the stripe $S = \{(x, y) \mid a \leqq x \leqq b, -\infty < y < \infty\}$. Then, as in the proof of Lemma 1, the assertion follows from (24) ▮

Theorem 4: *Assume* (23) *and let* u *be a solution of equation* (4) *with* $u'(a) = 0 = u'(b)$ *and* $u'(x) \neq 0$ *on* (a, b). *Further assume that* u *has a zero* c *on* (a, b). *Then the derivative* v' *of a solution* v *of equation* (5) *has a zero on* (a, b) *or*

$$Pu^{-1}u' = pv^{-1}v' \qquad \text{on } [a, c) \cup (c, b]. \tag{25}$$

Proof: Since $u'(x) \neq 0$ on (a, b), there is only one zero of u on (a, b). It follows from

$$\lim_{x \uparrow c} y(x) = \lim_{x \uparrow c} P(x) \frac{u'(x)}{u(x)} = -\infty, \quad \lim_{x \downarrow c} y(x) = \lim_{x \downarrow c} P(x) \frac{u'(x)}{u(x)} = +\infty$$

that $y(x)$ is negative on (a, c) and positive on (c, b). Assume that $v'(x) \neq 0$ on (a, b). We then show that $y = z$ on $[a, c) \cup (c, b]$. Assume the contrary and let $x_1 \in (a, c) \cup (c, b)$ be a point where $y(x_1) \neq z(x_1)$. First we discuss the case $x_1 \in (a, c)$, $y(x_1) < z(x_1) < 0$ and consider the function z on the left of x_1. Let y_1 be the solution of equation (7) defined by $y_1(x_1) = z(x_1)$. The function $d = 1/(y_1 - y)$, $d(x_1) = 1/\delta > 0$, is given by

$$d(x) = \frac{\varphi(x_1)\, u^2(x)}{u^2(x_1)\, \varphi(x)} \left(\frac{1}{\delta} + \frac{u^2(x_1)}{\varphi(x_1)} \int_{x_1}^{x} \frac{dt}{Pu^2} \right), \tag{26}$$

where $\varphi(x) = \exp\left(\int_{\gamma}^{x} P^{-1}R\, dt \right)$, $a < \gamma < c$ (compare the proof of Theorem 2). The first factor $\varphi(x_1) u^2(x)/u^2(x_1)\, \varphi(x)$ is positive on $[a, c)$. If the second factor

$$\frac{1}{\delta} + \frac{u^2(x_1)}{\varphi(x_1)} \int_{x_1}^{a} \frac{\varphi\, dt}{Pu^2} \leqq 0,$$

then there exists a point $\xi \in [a, x_1)$ such that $d(x) \to 0$ when $x \downarrow \xi$ and, consequently, $y_1(x) \to +\infty$ when $x \downarrow \xi$. Hence, by Lemma 3, it follows that the graph of z crosses the x-axis on (a, x_1). This, however, is impossible because we have supposed that $v'(x) \neq 0$ on (a, b). In the case $z(x_1) > 0$ the function z will be described on the right of x_1 as follows. Because v' is bounded on $[a, b]$ and $v'(x) \neq 0$ on $[x_1, b)$, it is easily seen that $z = pv^{-1}v'$ is bounded from above and $z(x) > 0$ on $[x_1, b)$. Since $y(x) \to +\infty$ when $x \downarrow c$, there exists a point $x_2 \in (c, b)$ such that $z(x_2) < y(x_2)$. Then, by Lemma 3, we have $z \leqq y$ on $[x_2, b)$. Consider the solution y_2 of equation (7) defined by $y_2(x_2) = z(x_2)$ and use the function (26) where x_1 has to be replaced by x_2. Then, as above one can see that z must have a zero on (x_2, b). This contradicts the hypothesis $v'(x) \neq 0$ on (a, b). Assume now $z(x_1) < y(x_1)$. The function z will be described on the right of x_1 as follows. By using Lemma 3 it is easily seen that there exists a point $\zeta \in (x_1, c)$ such that $z(x) \to -\infty$ when $x \uparrow \zeta$. This implies $z(x) \to +\infty$ when $x \downarrow \zeta$. Hence, in a neighbourhood on the right of ζ we have $z(x) > 0$. This case was already handled above (the case $z(x_1) > 0$) and leads to a contradiction. We state that $y = z$ on $[a, c)$. Analogously, one can prove that $y = z$ on $(c, b]$ ▮

Corollary: *Assume* $Q(a), Q(b) < 0$ *and* (23). *Let* u *be a solution to* (4) *with* $u'(a) = 0 = u'(b)$ *and* $u'(x) \neq 0$ *on* (a, b). *Then the derivative* v' *of any solution* v *to* (5) *has a zero on* (a, b) *or* $Pu^{-1}u' = pv^{-1}v'$.

Proof: Without loss of generality we can suppose that $u(a) > 0$ ($u(a) = 0$ would imply $u \equiv 0$). Then it follows from $P(a)\,u''(a) = Q(a)\,u(a)$ that $u''(a) < 0$. By $u'(a) = 0$, $u'(x) \neq 0$ on (a, b), and from $u''(a) < 0$ it easily follows that $u'(x) < 0$ on (a, b). Thus, in view of $u'(b) = 0$ we obtain $\big(u'(b + h) - u'(b)\big)/h \to u''(b) \geqq 0$ when $h \uparrow 0$. By $Q(b) < 0$ and $u''(b) \geqq 0$ it then follows from $P(b)\,u''(b) = Q(b)\,u(b)$ that $u(b) \leqq 0$. In view of $u'(b) = 0$ the boundary value $u(b) = 0$ would imply that $u \equiv 0$. Hence, we have $u(b) < 0$. Consequently, u has a zero on (a, b), and Theorem 4 can be applied ∎

The corollary of Theorem 4 generalizes Theorem 1 of LEIGHTON [3].

Remark: Condition (23) is satisfied if $p \leqq P$, $q \leqq Q$, and $p^{-1}r = P^{-1}R$ on $[a, b]$. Theorem 4 and its corollary are then valid. Concerning the solution v the assertion can be strengthened as follows. The derivative v' has a zero on (a, b) or v is a constant multiple of u.

Finally, we consider the case that the function u does not vanish in (a, b).

Theorem 5: *Let the hypothesis* (10) *be fulfilled and let u and v be solutions of equations* (4) *and* (5), *respectively, such that $u(x)$, $u'(x) \neq 0$ on $[a, b)$, $u'(b) = 0$, and $0 < p(a)\,v^{-1}(a)\,v'(a) \leqq P(a)\,u^{-1}(a)\,u'(a)$. Then v' has a zero on (a, b) or v is a constant multiple of u. In the latter case the differential equations* (4) *and* (5) *are identical.*

The proof can be omitted ∎

In [5] by the help of the transformations (6) and (9) the well-known Sturm-Picone comparison theorem is extended to the nonself-adjoint differential equations (4) and (5) considered on possibly non-compact intervals.

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