Zeitschrift für Analysis und ihre Anwendungen
Bd. 8 (1) 1989, S. 83 – 88

 (6)

Comparison Theorems for Nonself-Adjoint Differential Equations of Second **Order**

. E. Müller-Pfeiffer

Bezüglich der Nullstellen der Ableitungen von Lösungen nicht selbstadjungierter Differentialgleichungen zweiter Ordnung werden Vergleichssätze vom Sturm-Picone-Typ bewiesen.

Относительно нулей производных решений не самосопряженных дифференциальных уравнений второго порядка доказываются теоремы сравнения типа Штурма-Пиконе.

Concerning the zeros of the derivatives of solutions for nonself-adjoint second order differential equations Sturm-Picone type comparison theorems are proved.

By the well-known Sturm-Picone theorem solutions u and v of the self-adjoint differential equations

$$
-(P(x) u')' + Q(x) u = 0 \qquad (P, Q \in C, P > 0)^1), \qquad (1)
$$

$$
-(p(x) v')' + q(x) v = 0 \qquad (p, q \in C, p > 0) \qquad (2)
$$

are compared concerning the mutual position of their zeros (cf. [6] or [7]). It is natural to ask if there are similar results for the zeros of the derivatives u' and v' of solutions u and v . An answer for this question is due to LEIGHTON [3, 4].

Theorem 1 (LEIGHTON [3]): Let Q and q be negative on $[a, b]$ and

$$
p(x) \le P(x), \qquad q(x) \le Q(x), \qquad a \le x \le b. \tag{3}
$$

If the derivative u' of a solution u of the equation (1) has consecutive zeros at $x = a$ and $x = b$, then the derivative v' of a nonnull solution v of the equation (2) satisfying $v'(a) = 0$ will have a zero on the interval (a, b) .

In the following this theorem will be extended to the nonself-adjoint equations

$$
-(P(x) u')' + R(x) u' + Q(x) u = 0,
$$

-(p(x) v')' + r(x) v' + q(x) v = 0
(5)

where R, Q, r, $q \in C$, P, $p \in C¹$, and $p(x)$, $P(x) > 0$ on [a, b]. For this end the equations (4) and (5) will be transformed into Riccati differential equations. It is easily seen that the function

$$
u=Pu^{-1}u'
$$

61

¹) All equations and inequalities for functions on intervals are to be understood pointwise, e.g. they are valid for every point of these intervals. In the following, in case of strong inequalities, the functions are, for the sake of clearness, written with their arguments.

is a solution of the Riccati equation **• •** *• •* *****• •*

84 E. MÜLLER-PFEIFFER
\nis a solution of the Riccati equation
\n
$$
y'(x) = -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)
$$
\nif u is a solution to (4). Analogously, equation (5) can be transformed into
\n
$$
z'(x) = -p^{-1}(x) z^2 + p^{-1}(x) r(x) z + q(x)
$$
\nwhere
\n
$$
z = pv^{-1}v'.
$$
\n[3]

$$
z'(x) = -p^{-1}(x) z^2 + p^{-1}(x) r(x) z + q(x) \qquad (8)
$$

where

Lemma *l: Assume*

 $z = pv^{-1}v'$.

E. MülLER-PfelffER
\nion of the Riccati equation
\n
$$
y'(x) = -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)
$$
\nsolution to (4). Analogously, equation (5) can be transformed into
\n
$$
z'(x) = -p^{-1}(x) z^2 + p^{-1}(x) r(x) z + q(x)
$$
\n
$$
z = pv^{-1}v'.
$$
\n(a 1: Assume
\n
$$
p \le P, \quad p^{-1}r \le P^{-1}R, \quad q \le Q \text{ on } [a, b].
$$
\n(10)

p 1 *p Let* $[\alpha, \beta] \subseteq [a, b]$ be a subinterval where the solutions y and z of equations (7) and (8), *respectively, exist and assume that* $y(x) > 0$ *on* $[\alpha, \beta]$ *. Then* $y(\alpha) \ge z(\alpha)$ *implies* $y \ge z$ *and* $y(\beta) \le z(\beta)$ *implies* $y \le z$ *on [* α, β *].* $y'(x) = -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)$
 u is a solution to (4). Analogously, equation (5) c
 $z'(x) = -p^{-1}(x) z^2 + p^{-1}(x) r(x) z + q(x)$

here
 $\downarrow z = pv^{-1}v'$.

Lemma 1: Assume
 $p \le P$, $p^{-1}r \le P^{-1}R$, $q \le Q$ on [a,
 $t [\alpha, \beta] \subset [a, b]$ be where $\lambda z = pv^{-1}v'$.
 Lemma 1: Assume
 $p \le P$, $p^{-1}r \le P^{-1}R$, $q \le Q$
 Let $[\alpha, \beta] \subset [a, b]$ be a subinterval where the screspectively, exist and assume that $y(x) > 0$ on

and $y(\beta) \le z(\beta)$ implies $y \le z$ on $[\alpha, \beta]$.

Pro

$$
-p^{-1}(x) y^2 + p^{-1}(x) r(x) y + q(x) \leq -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x) (11)
$$

for all points of the semistripe $H_+ = \{(x, y) \mid \alpha \leq x \leq \beta, 0 \leq y \leq \alpha\}$ *. Consider the case* $y(\alpha) \geq z(\alpha)$ *implies* $y \leq z$ *.

Proof: By (10) it follows that
* $-p^{-1}(x) y^2 + p^{-1}(x) r(x) y + q(x) \leq -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)$ *(11)

for* 'the case $y(\alpha) \ge z(\alpha)$ and let $z(\alpha) > 0$. Then by a well-known theorem on first order differential equations (cf. [2: p. 91] or [1: p. 27]) it follows that $y \ge z$ in a neigh-. $-p^{-1}(x) y^2 + p^{-1}(x) r(x) y + q(x) \le -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)$ (11)
for all points of the semistripe $H_+ = \{(x, y) \mid \alpha \le x \le \beta, 0 \le y < \infty\}$. Consider
the case $y(\alpha) \ge z(\alpha)$ and let $z(\alpha) > 0$. Then by a well-known theorem on first orde bourhood on the right-hand side of α . Let $\xi \in (\alpha, \beta)$ be a point where $y(\xi) = z(\xi)$.
Then by the same argument we have $y(x) \ge z(x)$ if $x \ge \xi$ and x near to ξ . Hence, the the case $y(\alpha) \ge z(\alpha)$ and let $z(\alpha) > 0$. Then by a well-known theorem on first or differential equations (cf. [2: p. 91] or [1: p. 27]) it follows that $y \ge z$ in a ne bourhood on the right-hand side of α . Let $\xi \in (\alpha, \$ *z(fl)* is likewise a consequence of the named theorem \blacksquare **Proof:** By (10) it follows that
 $-p^{-1}(x) y^2 + p^{-1}(x) r(x) y +$

for all points of the semistripe *B*

the case $y(x) \ge z(x)$ and let $z(x) > 0$

differential equations (cf. [2: p. 91]

bourhood on the right-hand side of

Then by th *•* $-p^{-1}(x) y^2 + p^{-1}(x) r(x) y + q(x) \le -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x) (11)$
for all points of the semistripe $H_+ = \{(x, y) \mid \alpha \le x \le \beta, 0 \le y < \infty\}$. Consider the case $y(\alpha) \ge z(\alpha)$ and let $z(\alpha) > 0$. Then by a well-known theorem on first o *v*(*x*) *y* + *p* (*x*) *y* + *p* (*x*) *r*(*x*) *y* + *q*(*x*) \leq *- P* (*x*) *y* + *P* (*x*) *n*(*x*) *y* + *P* (*x*) *n*(*x*) *y* + *P* (*x*) *n*(*x*) *y* + *Q*(*x*) (11) (*y*(*x*) \geq *z*(*x*) and let $z(x) > 0$. Finally the same argu.

eraph of z cannot cross

is likewise a consequent
 $\begin{array}{r}\n\text{Theorem 2:} \text{ Assume}\\ \text{specificely, such that} \\
u(a) = 0 = u' \\\\ \text{and} \\
v(a) = 0, \\
\text{Then } v' \text{ has a zero on } (a, \text{4) and (5) are identical} \\
\text{Proof: It follows for} \\
u(x) \neq 0, \\
\text{Let us assume that } v'(x) \neq 0 \Leftrightarrow v'(x) \neq 0 \Leftrightarrow v$

Theorem 2: *Assume* (10) *and let u and v be solutions of equations* (4) *and* (5), *re-*
spectively, such that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is likewise a consequence
 $\begin{aligned}\n\text{Theorem 2:} &\quad As \text{sum}\\ \text{spectively, such that} \\
u(a) &= 0 = u'(b) \\
&\qquad \qquad and \\
v(a) &= 0, \qquad v'\\ \text{Then } v' \text{ has a zero on } (a, b)\\
(4) \text{ and } (5) \text{ are identical.}\n\end{aligned}$ *:*

$$
u(a) = 0 = u'(b), \qquad u'(x) = 0 \qquad on [a, b)
$$
 (12)

and

•

$$
v(a) = 0, \qquad v'(a) \neq 0. \tag{13}
$$

Then v' has a zero on (a, b) or v is a constant multiple of u. In the latter case the equations $v'(a) = 0.$

b) or v is a constant multiple of u. In the la

om (12) that
 $0 < u^{-1}(x) u'(x) < \infty$ on (a, b) .
 $v = 0$ on (a, b) . Then, by (13) we have $v(x)$

Proof: It follows from (12) that

$$
u(x) = 0, \qquad 0 < u^{-1}(x) \ u'(x) < \infty \qquad \text{on } (a, b). \tag{14}
$$

Let us assume that $v'(x) = 0$ on (a, b) . Then, by (13) we have $v(x) \neq 0$ on $(a, b]$ and

$$
0 < v^{-1}(x) \, v'(x) < \infty \qquad \text{on } (a, b). \tag{15}
$$

0 $v(a) = 0 = u'(b)$, $u'(x) = 0$ on $[a, b)$ (12)
 $v(a) = 0$, $v'(a) \neq 0$. (13)
 on $[a, b)$ or v is a constant multiple of u . In the latter case the equations

(13) are identical..
 $u(x) \neq 0$, $0 < u^{-1}(x) u'(x) < \infty$ on (a, b) . It follows from (14) and (15) that $y(x)$, $z(x) > 0$ on (a, b) . We prove that $\dot{y} = z$ on *(a, b).* Assuming the contrary let x_0 be a point on (a, b) where $y(x_0) = z(x_0)$. First we discuss the case $z(x_0) > y(x_0)$. Let y_0 be the solution of equation (7) uniquely defined by the initial value $y_0(x_0) = z(x_0)$. Consider y_0 on the left-hand side of x_0 . We prove that y_0 does not exist on the entire interval $(a, x_0]$. The function **Example 12 C c** *d* **c** *d d* Proof: It follows from (12) that
 $u(x) \neq 0$, $0 < u^{-1}(x) u'(x) < \infty$ on (a, b) .

Let us assume that $v'(x) \neq 0$ on (a, b) . Then, by (13) we have $v(x) \neq 0$ on
 $0 < v^{-1}(x) v'(x) < \infty$ on (a, b) .

It follows from (14) and (15) that

$$
d = 1/(y_0 - y), \quad d(x_0) = 1/\delta > 0, \tag{16}
$$

$$
w' + [P^{-1}(x) R(x) - 2P^{-1}(x) y(x)] w - P^{-1}(x) = 0
$$

in a neighbourhood of x_0 (cf. [2: p. 42]). Hence, we have

Comparison Theorems for Differential Equations 85
\nthe
\nthe
\nthe
\nthe
\n
$$
d(x) = \exp\left(\int_{x_0}^{x} \frac{2y - R}{P} dt\right) \left(\frac{1}{\delta} + \int_{x_0}^{x} \frac{1}{P} \exp\left(\int_{x_0}^{t} \frac{R - 2y}{P} dx\right) dt\right)
$$
\n
$$
= \exp\left(\int_{x_0}^{x} \left(2\frac{u'}{u} - \frac{R}{P}\right) dt\right) \left(\frac{1}{\delta} + \int_{x_0}^{x} \frac{1}{P} \exp\left(\int_{x_0}^{t} \left(\frac{R}{P} - 2\frac{u'}{u}\right) dt\right) dt\right)
$$
\n
$$
= \frac{\varphi(x_0) u^2(x)}{u^2(x) \varphi(x)} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2P}\right)
$$
\n
$$
x) = \exp\left(\int_{x_0}^{x} P^{-1}R dt\right), \ c \in (a, b). \text{ In view of } u(x) = [u'(a) + o(1)] (x - a)
$$
\n
$$
u(t) = \int_{x_0}^{x} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2P}\right)
$$
\n
$$
u(t) = \left[\frac{u'(a) + o(1)}{a}\right] (x - a)
$$
\n
$$
u(t) = \int_{x_0}^{x} \frac{1}{\delta} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2P}\right)
$$
\n
$$
u(t) = \int_{x_0}^{x} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2P}\right)
$$
\n
$$
u(t) = \int_{x_0}^{x} \left(\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^{x} \frac{\varphi dt}{u^2P}\right)
$$
\n
$$
u(t)
$$

where $\varphi(x) = \exp \left(\int_a^x P^{-1}R \ dt \right)$, $c \in (a, b)$. In view of $u(x) = [u'(a) + o(1)] (x - a)$ where $\varphi(x) = \exp\left(\int_{c}^{x} P^{-1}R \, dt\right)$, $c \in (a, b)$. In view of $u(x) = [u'(a) + o(1)] (x - a)$
mear to a it follows that $\int_{c}^{x} u^{-2}P^{-1}\varphi \, dt \to -\infty$ when $x \downarrow a$. Hence, there exists a point

 $x_{\delta} \in (\alpha, x_0)$ such that $d(x) \to 0$ when $x \downarrow x_{\delta}$. Consequently, we have

$$
y_0(x) \to +\infty \text{ when } x \downarrow x_\delta. \tag{18}
$$

 $x_b \in (a, x_0)$ such that $d(x) \to 0$ when $x \downarrow x_b$. Consequently, we have
 $y_0(x) \to +\infty$ when $x \downarrow x_b$. (18)

It follows from $d(x) > 0$ on $(x_b, x_0]$ that $y_0(x) > y(x)$ on $(x_b, x_0]$. Hence, $y_0(x)$ is posi-

tive on $(x_b, x_0]$. C $y_0(x) \rightarrow +\infty$ when $x \downarrow x_\delta$. (18)

It follows from $d(x) > 0$ on $(x_\delta, x_0]$ that $y_0(x) \rightarrow y(x)$ on $(x_\delta, x_0]$. Hence, $y_0(x)$ is posi-

tive on $(x_\delta, x_0]$. Consider now the solution $z, z(x_0) = y_0(x_0)$, of equation (8). By Lem it follows that $z \ge y_0$ on $[\alpha, x_0]$, where $\alpha \in (x_\delta, x_0)$ is any point. Hence, in view of (18), z cannot be bounded on $(x_\delta, x_0]$. This, however, contradicts the fact that z exists on the entire interval (a, b) . Hence, $z(x_0) > y(x_0)$ is impossible.

Assume now that $z(x_0) < y(x_0)$. Let y_0 be again the solution of the Riccati equation Assume now that $z(x_0) < y(x_0)$. Let y_0 be again the solution of the Riccati equation (7) defined by $y_0(x_0) = z(x_0)$. We show that y_0 has a zero on the interval (x_0, b) . For this end we use formula (17). Consider the (1) defined by $y_0(x_0) = z(x_0)$. We show that y_0 has a zero on the interval (x_0, b) . For this end we use formula (17). Consider the behaviour of $d(x)$ when x is increasing. The factor $\varphi(x_0) u^2(x) |u^2(x_0) \varphi(x)$ is pos factor $\varphi(x_0)$ $u^2(x)/u^2(x_0)$ $\varphi(x)$ is positive and tends to $\varphi(x_0)$ $u^2(b)/u^2(x_0)$ $\varphi(b)$ when $x \to b$. For the second factor we have near to *a* it follows that $\int u^{-2}P^{-1}\varphi dt \to -\infty$ when $x \downarrow a$. Hence, there exists $x_b \in (a, x_0)$ such that $d(x) \to 0$ when $x \downarrow x_b$. Consequently, we have
 $y_0(x) \to +\infty$ when $x \downarrow x_b$.

It follows from $d(x) > 0$ on $(x, x_0]$ from $d(x) > 0$ on $(x_b, x_0]$,
 y_a, x_0]. Consider now the solution

that $z \geq y_0$ on $[x, x_0]$,

finot be bounded on (x_b)

the entire interval (a, b) .
 $y_0(x_0) = z(x_0)$. We slue to the summary of the summary of the second *x*₁ (*x*₁, *x*₀). Consider now the solid conducts on $(x_6, x_0]$. Consider now the solid the solid the solid the solid the solid the solid on $(x_6, x_6]$, 18), z cannot be bounded on (x_7, x_8) . Assume now that $z(x_0$

$$
\frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^x \frac{\varphi \, dt}{u^2 P} \to \frac{1}{\delta} + \frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^b \frac{\varphi \, dt}{u^2 P}
$$

when $x \to b$. If this limit is non-negative, then there exists a point $\xi \in (x_0, b]$ such that $d(x) \to 0$ when $x \uparrow \xi$, and it follows that $y_0(x) \to -\infty$ when $x \uparrow \xi$. Note that this is also true in the case $\xi = b$ because of $y(x) \to 0$ when $x \uparrow b$. In the case of negative when $x \to b$. If this limit is non-negative, then there exists a point $\xi \in (x_0, b]$ such that $d(x) \to 0$ when $x \uparrow \xi$, and it follows that $y_0(x) \to -\infty$ when $x \uparrow \xi$. Note that this is also true in the case $\xi = b$ becaus limit above we obtain $d(x) \to \sigma < 0$ when $x \uparrow b$ and, consequently, $y_0(x) \to \sigma^{-1} < 0$
when $x \uparrow b$. Since $y_0(x_0) > 0$, in each case the function y_0 has a zero $\xi_0 \in (x_0, b)$. Lemma 1 applied to the functions y_0 and *z* yields the estimate $z \leq y_0$ on $[x_0, \xi_0)$. This estimate, however, contradicts the fact that z is a positive function on (a, b) . Hence, $z(x_0) < y(x_0)$ is also impossible and the assertion $y = z$ on (a, b) is proved. In this case y is a solution of equation (7) as well as of equation (8). This leads to $-P^{-1}y^2$ $+ P^{-1}Ry + Q = -p^{-1}y^2 + p^{-1}ry + q$ on (a, b). Finally, it follows from (10) and $y > 0$ that $P = p$, $R = r$, and $Q = q$. *u* and *v* are solutions of equation (4) with $u(a) = v(a) = 0$. Thus, we obtain $v = cu$ on (a, b) when $x \rightarrow b$. If this limit is non-negative, then there exists a point $\xi \in (x_0, b]$ such that $d(x) \rightarrow 0$ when $x \uparrow \xi$, and it follows that $y_0(x) \rightarrow -\infty$ when $x \uparrow \xi$. Note that this is also true in the case $\xi = b$ becaus

In the special case $R = r = 0$ Theorem 2 was proved by LEIGHTON [3: Corollary].

$$
p \leq P, \qquad p^{-1}r \geq P^{-1}R, \qquad q \leq Q \text{ on } [a, b]. \tag{19}
$$

E. MÜLLER-PFEIFFER

Lemma 2: *Assume*
 $p \le P$, $p^{-1}r \ge P^{-1}R$, $q \le Q$ on [a, b].
 $t [\alpha, \beta] \subset [a, b]$ be a subinterval where the solutions y and z of equations (**E.** MÜLLER-PFEIFFER
 p $\leq P$, $p^{-1}r \geq P^{-1}R$, $q \leq Q$ on [a, b]. (19)
 $\subset [a, b]$ be a subinterval where the solutions y and z of equations (7) and (8),
 \cup , exist and assume that $y(x) < 0$ on [α , β]. Then $y(\alpha$ Lemma 2: Assume
 $p \le P$, $p^{-1}r \ge P^{-1}R$, $q \le Q$ on [a, b]. (19)

Let $[\alpha, \beta] \subset [a, b]$ be a subinterval where the solutions y and z of equations (7) and (8),

respectively, exist and assume that $y(x) < 0$ on $[\alpha, \beta]$. Then *remma 2: Assume*
p $\leq P$, $p^{-1}r \geq P^{-1}R$, $q \leq Q$ on [a, b]. (19)
Let $[\alpha, \beta] \subset [a, b]$ *be a subinterval where the solutions y and z of equations* (7) and (8),
respectively, exist and assume that y(x) $<$ 0 *on* [$p \leq P$, $p^{-1}r \geq P^{-1}R$,
 Let $[\alpha, \beta] \subset [a, b]$ be a subinterval where
 respectively, exist and assume that $y(x)$

and $y(\beta) \leq z(\beta)$ implies $y \leq z$ on $[\alpha, \beta]$.

Proof: Compare the proof of Lemm *ulters* $q \leq Q$ on [a, b]. (19)
 ulters the solutions y and z of equations (7) and (8),
 ut y(x) ≤ 0 on [α , β]. Then $y(\alpha) \geq z(\alpha)$ implies $y \geq z$
 u[α , β].
 f Lemma 1 **i**
 ud let u *and y be so I.emma 2: As*
 $p \leq P$,
 Let $[\alpha, \beta] \subset [a, b]$
 respectively, exist
 and $y(\beta) \leq z(\beta)$
 Proof: Compa
 Theorem 3:
 respectively, such
 $u'(a) =$
 and
 $v(a) \neq (a)$
 Then v has a zero

(4) *and* (5) *are id*

Proof: Compare the proof of Lemma 1 **^I**

Theorem 3: *Assume* (19) and let.u and *v* be solutions of equations (4) and (5), *respectiv'ely, such that* Example the proof of
 e m 3: Assume (19) a
 dy , such that
 $u'(a) = 0 = u(b)$, **• 11001: Compute the proof of Lemma 1 •** *•*
• f i <i>expectively, such that $u'(a) = 0 = u(b)$, $u'(x) \neq 0$ on $(a, b]$ (20)
and $v(a) \neq 0$, $v^{-1}(a) v'(a) \leq 0$. (21)
Phen v has a zero on (a, b) or *v is a co*

$$
u'(a) = 0 = u(b), \qquad u'(x) = 0 \text{ on } (a, b] \tag{20}
$$

•

$$
v(a) \neq 0, \qquad v^{-1}(a) \; v'(a) \leq 0. \tag{21}
$$

Then v has a zero on (a, b) or v is a constant multiple of u. In the latter case the equations (4) and (5) *are identical.*

Proof: It follows from (20) that

$$
u(x) \neq 0, \qquad -\infty < u^{-1}(x) \ u'(x) < 0 \ \text{on} \ (a, b). \tag{22}
$$

u(*x*) = $\frac{1}{2}$ (*x)* in plies $y \leq z$ on $[x, p]$. Then $y(x) \equiv c(x)$, implies $y \equiv z$
 $\leq z(\beta)$ implies $y \leq z$ on $[x, p]$.

Compare the proof of Lemma 1 **a**
 u(*x*) = 0, *u'(x)* + 0 on *(a, b)*
 u'(a) = 0 = *u(b),* Let us assume that $v(x) = 0$ on (a, b) . Then the functions $y = P u^{-1} u'$ and $z = p v^{-1} v'$ exist on [a, b) and (20) and (21) give $z(a) \leq y(a) = 0$. By (22) it follows that $y(x) < 0$ on *(a, b).* Since *u'(a)* = 0 = *u(b)*, *u'(x)* \neq 0 *on* (*a*,
 and
 v(a) \neq 0, *v*⁻¹(*a)v'*(*a)* \leq 0.
 Then v has a zero on (*a, b) or v is a constant mult*

(4) *and* (5) *are identical.*

Proof: It follows from (20)

$$
-p^{-1}(x)y^{2}+p^{-1}(x) r(x) y + q(x) \leq -P^{-1}(x) y^{2}+P^{-1}(x) R(x) y + Q(x)
$$

for all points (x, y) of the semistripe $H_{-} = \{(x, y) | a \le x \le b, -\infty < y \le 0\}$, by the above mentioned theorem on first order- differential equations it follows that $z \leq y$ on $[a, b)$. We prove that $z=y$ on $[a, b)$. Let $x_0 \in (a, b)$ be a point where $z(x_0)$ $\langle y(x_0) \rangle$. Consider the solution y_0 to (7) defined by $y_0(x_0) = z(x_0)$ and apply formula (17). Since δ^{-1} is negative and
 $\frac{u^2(x_0)}{\varphi(x_0)} \int_{x_0}^x \frac{\varphi dt}{u^2 P} \to +\infty$ when $x \uparrow b$ Proof: It follows from (20) that
 $u(x) \neq 0$, $-\infty < u^{-1}(x) u'(x) < 0$ on (a, b) .

Let us assume that $v(x) \neq 0$ on (a, b) . Then the functions $y = Pu^{-1}u'$ and

exist on $[a, b)$ and (20) and (21) give $z(a) \leq y(a) = 0$. By (22) it f Since
 $-p^{-1}(x)^{2}y^{2} + p^{-1}$

Since
 $-p^{-1}(x)^{2}y^{2} + p^{-1}$
 \vdots *r*₀ *r*₀ *r*₀ *r*₁
 $\left[a, b\right)$. We prove

Consider the solution
 $\left(e^{5})^{1}$ is negative
 $\frac{u^{2}(x_{0})}{\varphi(x_{0})} \int_{x_{0}}^{x} \frac{\varphi dt}{u^{2}P}$ \rightarrow *dtion* $-\infty < u^{-1}(x) u'(x) < 0$ on (a, b) .
 dt $v(x) \neq 0$ on (a, b) . Then the functions *y*
 dd (20) and (21) give $z(a) \leq y(a) = 0$. By (2
 d')^{*y*}² + $p^{-1}(x) r(x) y + q(x) \leq -P^{-1}(x) y^2$
 d', *y*) of the semistripe $H_{-} = \$ $p(x) \neq 0$, $-\infty < u^{-1}(x)$
 $p(x) \neq 0$ on (a, b)
 $p(x) \neq 0$ on (a, b)

Since
 $-p^{-1}(x)^{2} + p^{-1}(x) r(x) y$

ints (x, y) of the semistrical
 $p(x, b)$. We prove that $z =$
 $p(x_0) \int_0^x \frac{y}{u^2 P} dx$
 $p(x_0) \int_0^x \frac{y}{u^2 P} dx$
 $p(x_0) \$

$$
\frac{u^2(x_0)}{\varphi(x_0)}\int\limits_{x_0}^x\frac{\varphi dt}{u^2P}\to+\infty\qquad\text{when }x\uparrow b
$$

there exists a point $x_{\delta} \in (x_0, b)$ such that $d(x) \to 0$ when $x \uparrow x_{\delta}$. This leads to $y_0(x) \to -\infty$ when $x \uparrow x_{\delta}$. By Lemma 2 it follows that $z \leq y_0$ on $[x_0, \beta]$, where the point β $\rightarrow -\infty$ when $x \uparrow x_b$. By Lemma 2 it follows that $z \leq y_0$ on $[x_0, \beta]$, where the point β can be chosen arbitrarily on (x_0, x_b) . Hence, z cannot be bounded on $[x_0, x_b)$. This, however, contradicts the fact that *z* is continuous on (a, b) . This proves that $z = y$ on $[a, b)$ and, consequently, the differential equations (4) and (5) are identical \blacksquare **p** $\mathcal{F}(x_0, y_0)$
 p as a point $x_b \in (x_0, b)$ such that $d(x) \to 0$ when $x \uparrow x_b$. This leads to $y_0(x)$

when $x \uparrow x_b$. By Lemma 2 it follows that $z \leq y_0$ on $[x_0, \beta]$, where the point β

nosen arbitrarily on $(x_$ *p*(x_0) *y* u^2P .

there exists a point $x_s \in (x_0, b)$ such that $d(x) \to 0$ when $x \uparrow x_s$. This leads to $y_s \to -\infty$ when $x \uparrow x_s$. By Lemma 2 it follows that $z \leq y_0$ on $[x_0, y_0]$, where the poir can be chosen arbitr ses a point $x_{\delta} \in (x_0, v)$ such
then $x \uparrow x_{\delta}$. By Lemma 2 is
nosen arbitrarily on (x_0, x)
contradicts the fact that
ind, consequently, the diff
a 3: Assume
 $p \le P, q \le Q, (P^{-1}R - p)$
 $\subset [a, b]$ be a subinterval l
 $ly, exist.$ it follows that $z \leq y_0$ on $[x_0, \beta]$, where the point β
 a). Hence, *z* cannot be bounded on $[x_0, x_3)$. This,
 z is continuous on (a, b) . This proves that $z = y$

erential equations (4) and (5) are identical \blacks

Lemma 3: *Assume*

$$
p \leq P, q \leq Q, (P^{-1}R - p^{-1}r)^2 \leq 4(p^{-1} - P^{-1})(Q - q) \text{ on } [a, b]. \tag{23}
$$

Let $[\alpha, \beta] \subset [a, b]$ be a subinterval where the solutions y and z of equations (7) and (8), Lemma 3: Assume
 p $\leq P$, $q \leq Q$, $(P^{-1}R - p^{-1}r)^2 \leq 4(p^{-1} - P^{-1})$ $(Q - q)$ on [a, b]. (23)
 Let $[\alpha, \beta] \subset [a, b]$ be a subinterval where the solutions y and z of equations (7) and (8),
 respectively, exist. Then $y(\alpha) \ge$ $[\alpha, \beta]$.

$$
-p^{-1}(x) y^2 + p^{-1}(x) r(x) y + q(x) \leq -P^{-1}(x) y^2 + P^{-1}(x) R(x) y + Q(x)
$$
\n(24)

for all points (x, y) of the stripe $S = \{(x, y) \mid a \le x \le b, -\infty < y < \infty\}$. Then, as in the proof of Lemma 1, the assertion follows from (24) \blacksquare

Theorem 4: Assume (23) and let u be a solution of equation (4) with $u'(a) = 0$ $= u'(b)$ and $u'(x) \neq 0$ on (a, b) . Further assume that u has a zero c on (a, b) . Then the *derivative v' of a solution v of equation* (5) *has a zero on (a, b) or* **Published Comparison Theorems for Divority of the stripe** $S = \{(x, y) | a \le x \le b, -\text{of Lemma 1, the assertion follows from (24) } \blacksquare$
 Pu 4: Assume (23) and let u be a solution of each $u'(x) \neq 0$ on (a, b) . Further assume that u has v' of a solution v Compar

for all points (x, y) of the stripe $S = \{(\text{the proof of Lemma 1, the assertion for } \text{Theorem 4:} As sume (23) and let
= u'(b) and u'(x) = 0 on (a, b). *Further*
derivative v' of a solution v of equation
 $Pu^{-1}u' = pv^{-1}v'$ on [a, c)
Proof: Since $u'(x) = 0$ on (a, b) , t]
from

$$
\lim_{xtc} y(x) = \lim_{xtc} P(x) \frac{u'(x)}{u(x)} =
$$$ Comparison Theorems for Differential Equation
 ue S = {(x, y) | $a \le x \le b$, $-\infty < y < \infty$ }.

sertion follows from (24) \blacksquare
 and let u be a solution of equation (4) *wit*
 u. Further assume that u has a zero c on (a, urn *x*. Assume (25) and iet u be a solution of equation (x) and u'(x) = nd u'(x) = 0 on (a, b). Further assume that u has a zero c on (a, b). Then v' of a solution v of equation (5) has a zero on (a, b) or
 $Pu^{-1}u' = pv^{-1}v$

$$
Pu^{-1}u' = pv^{-1}v' \qquad on [a, c) \cup (c, b]. \qquad (25)
$$

Proof: Since $u'(x) = 0$ on (a, b) , there is only one zero of u on (a, b) . It follows

$$
x' \circ f \text{ a solution } v \text{ of equation (5) has a zero on } (a, b) \text{ or}
$$
\n
$$
Pu^{-1}u' = pv^{-1}v' \qquad \text{on } [a, c) \cup (c, b].
$$
\n
$$
\therefore \text{ Since } u'(x) = 0 \text{ on } (a, b), \text{ there is only one zero of } u \text{ on } (a, b). \text{ It follows}
$$
\n
$$
\lim_{x \uparrow c} y(x) = \lim_{x \uparrow c} P(x) \frac{u'(x)}{u(x)} = -\frac{1}{\infty}, \qquad \lim_{x \downarrow c} y(x) = \lim_{x \downarrow c} P(x) \frac{u'(x)}{u(x)} = +\infty
$$
\n
$$
\text{is negative on } (a, c) \text{ and positive on } (c, b). \text{ Assume that } v'(x) = 0 \text{ on } (a, b) \text{ and } v' = 0 \text{ on } (a, b) \text{ and}
$$

that $y(x)$ is negative on (a, c) and positive on (c, b) . Assume that $v'(x) \neq 0$ on (a, b) . We then show that $y = z$ on $[a, c) \cup (c, b]$. Assume the contrary and let $x_1 \in (a, c)$ $\sigma(c, b)$ be a point where $y(x_1) \neq z(x_1)$. First we discuss the case $x_1 \in (a, c), y(x_1)$ $u(c, b)$ be a point where $y(x_1) = z(x_1)$. First we discuss the case $x_1 \in (a, c)$, $y(x_1) < z(x_1) < 0$ and consider the function z on the left of x_1 . Let y_1 be the solution of equation (7) defined by $y_1(x_1) = z(x_1)$. The is given by a point where $y(x_1) \neq z(x_1)$. First we

c 0 and consider the function z on the le

(7) defined by $y_1(x_1) = z(x_1)$. The function

y
 $d(x) = \frac{\varphi(x_1) u^2(x)}{u^2(x_1) \varphi(x)} \left(\frac{1}{\delta} + \frac{u^2(x_1)}{\varphi(x_1)} \int_{x_1}^x \frac{dt}{Pu^2} \right)$
 x

$$
(x_1) < 0
$$
 and consider the function z on the left of x_1 . Let y_1 be the solution of
ation (7) defined by $y_1(x_1) = z(x_1)$. The function $d = 1/(y_1 - y)$, $d(x_1) = 1/\delta > 0$,
ven by

$$
d(x) = \frac{\varphi(x_1) u^2(x)}{u^2(x_1) \varphi(x)} \left(\frac{1}{\delta} + \frac{u^2(x_1)}{\varphi(x_1)} \int_{x_1}^z \frac{dt}{Pu^2} \right).
$$
(26)

where $\varphi(x) = \exp\left(\int_{\gamma}^{x} P^{-1}R \, dt\right)$, $a < \gamma < c$ (compare the proof of Theorem 2). The first factor $\varphi(x_1) \cdot u^2(x)/u^2(x_1) \cdot \varphi(x)$ is positive on [*a*, *c*). If the second factor $\frac{1}{\delta} + \frac{u^2(x_1)}{\varphi(x_1)} \int_{\gamma}^{a} \frac{\varphi \$ first factor $\varphi(x_1) u^2(x)/u^2(x_1) \varphi(x)$ is positive on [a, c). If the second factor

$$
\frac{1}{\delta}+\frac{u^2(x_1)}{\varphi(x_1)}\int\limits_{x_1}^a\frac{\varphi\ dt}{P u^2}\leqq 0\,,
$$

'then there exists a point $\xi \in [a, x_1)$ such that $d(x) \to 0$ when $x \downarrow \xi$ and, consequently, $y_1(x) \rightarrow +\infty$ when $x \downarrow \xi$. Hence, by Lemma 3, it follows that the graph of *z* crosses the x-axis on (a, x_1) . This, however, is impossible because we have supposed that $v'(x) = 0$ on (a, b) . In the case $z(x_1) > 0$ the function *z* will be described on the right of x_1 as follows. Because v' is bounded on [a, b] and $v'(x) \neq 0$ on [x₁, b), it is easily seen that $z = pv^{-1}v'$ is bounded from above and $z(x) > 0$ on $[x_1, b)$. Since $y(x) \rightarrow +\infty$ when $x \downarrow c$, there exists a point $x_2 \in (c, b)$ such that $z(x_2) < y(x_2)$. Then, by Lemma 3, or x_1 as ronows. Because *v* is bounded on $[u, v]$ and $v(x) = 0$ on $[x_1, v_1]$, it is easily
seen that $z = pv^{-1}v'$ is bounded from above and $z(x) > 0$ on $[x_1, b]$. Since $y(x) \to +\infty$
when $x \downarrow c$, there exists a point $x_2 \$ we have $z \leq y$ on $[x_2, b)$. Consider the solution y_2 of equation (7) defined by $y_2(x_2)$
= $z(x_2)$ and use the function (26) where x_1 has to be replaced by x_2 . Then, as above one can see that *z* must have a zero on (x_2, b) . This contradicts the hypothesis $v'(x) \neq 0$ on (a, b) . Assume now $z(x_1) < y(x_1)$. The function *z* will be described on the right of x_1 as follows. By using Lemma 3 it is easily seen that there exists a point $\zeta \in (x_1, c)$ such that $z(x) \to -\infty$ when $x \uparrow \zeta$. This implies $z(x) \to +\infty$ when $x \downarrow \zeta$. Hence, in a neighbourhood on the right of ζ we have $z(x) > 0$. This case was already handled above (the case $z(x_1) > 0$) and leads to a contradiction. We state that $y = z$ on [a, c). Analogously, one can prove that $y = z$ on (c, b)

Corollary: Assume $Q(a)$, $Q(b) < 0$ and (23). Let u be a solution to (4) with $u'(a)$ $= 0 = u'(b)$ and $u'(x) \neq 0$ on (a, b) . Then the derivative v' of any solution v to (5) has a *zero on* (a, b) *or* $Pu^{-1}u' = pv^{-1}v'.$

88 E. MÜLLER-PFEIFFER Proof: Without loss of generality we can suppose that $u(a) > 0$ ($u(a) = 0$ would imply $u \equiv 0$). Then it follows from $P(a) u''(a) = Q(a) u(a)$ that $u''(a) < 0$. By $u'(a) = 0$, $u'(x) \neq 0$ on (a, b) , and from $u''(a) < 0$ it easily follows that $u'(x) < 0$ on (a, b). Thus, in view of $u'(b) = 0$ we obtain $(u'(b + h) - u'(b))/h \rightarrow u''(b) \ge 0$ when $h \uparrow 0$. By $Q(b) < 0$ and $u''(b) \ge 0$ it then follows from $P(b)$ $u''(b) = Q(b) u(b)$ that $u(b) \leq 0$. In view of $u'(b) = 0$ the boundary value $u(b) = 0$ would imply that $u \equiv 0$. Hence, we have $u(b) < 0$. Consequently, u has a zero on (a, b) , and Theorem 4 can be $h \uparrow 0$. By $Q(b)$
 $u(b) \leq 0$. In view
Hence, we have Hence, we have $u(b) = 0$ the boundary value $u(b) = 0$ would imply that $u \equiv 0$.

Hence, we have $u(b) < 0$. Consequently, u has a zero on (a, b) , and Theorem 4 can be
 $\{x \in \mathbb{R}^n : \int_a^b f(x) dx \}$ The corollary of Theorem 4 ge $u'(a) = 0$, $u'(x) \neq 0$ on
 (a, b) . Thus, in view of
 $h \uparrow 0$. By $Q(b) < 0$ and
 $u(b) \leq 0$. In view of $u'($

Hence, we have $u(b) <$
 \downarrow
 \downarrow
 \downarrow perce, we have $u(b) <$
 \downarrow
 \downarrow
 \downarrow perce \downarrow
 \downarrow
 \downarrow

The corollary of Theorem 4 generalizes Theorem 1 of **LEIGHTON [3].**

can be strengthened as follows. The derivative *v'* has a zero on (a, *b)* or *v* is a constant multiple of u .

Finally, we consider the case that the function u does not vanish in (a, b) .

Theorem 5: *Let the hypothesis* (10).be *fulfilled and let u and v be solutions of equations (4) and (5), respectively, such that u(x), u'(x)* ± 0 *on [a, b), u'(b)* $= 0$, *and (5), respectively, such that u(x), u'(x)* ± 0 *on [a, b), u'(b)* $= 0$, *and* $0 < p(a) v^{-1}(a) v'(a) \leq P(a) u^{-1}(a) u'(a)$. Then v' ha *multiple of u. In the latter case the differential equations (4) and (5) are identical.* Theorem 5: Let the hy

tions (4) and (5), respect
 $0 < p(a) v^{-1}(a) v'(a) \leq P(c)$

multiple of u. In the latter

The proof can be omit

In [5] by the help of the

comparison theorem is ext

(5) considered on possibly

REFERENCES

The proof can be omitted \blacksquare

In [5], by the help of the transformations (6) and (9) the well-known Sturm-Picone comparison theorem is extended to the nonself-adjoint differential equations (4) and (5) considered on possibly non-compact intervals. $(2 \pi/a) v^{-1}(a) v'(a) \leq P(a) u^{-1}(a)$
 litple of u. In the latter case the

The proof can be omitted

In [5], by the help of the transformation

mericon theorem is extended to

considered on possibly non-considered on possibl The proof can be omin

n [5] by the help of the

mparison theorem is ex

considered on possibly

CFERENCES

HARTMAN, P.: Ordinary

Wiley & Sons 1964.

KAMKE, E.: Differential

& Portig 1950.

LEIGHTON, W.: Some Ele

MÜLLER

S
S
S

- **[1] HARTMAN,** P.: Ordinary Differential Equations. New York—London—Sidney: John INDENCES

HARTMAN, P.: Ordinary Differential Equations.

Wiley & Sons 1964.

KAMKE, E.: Differentialgleichungen reeller Funktie

& Portig 1950.

LEIGHTON, W.: Some Elementary Sturm Theory. Ap

LEIGHTON, W.: More Elementary
- **[2] KAMKE,** E.: Differentialgleichungen reeller Funkinen Leipzig: Akad. Vcrlasges. Geest *F* nerential Equations. New York
hungen reeller Funktionen. Leipzig
htary Sturm Theory. J. Diff. Fou.
	- [3] **LEIGHTON,** V.: Some Elementary Sturm Theory. J. Diff. Equ. 4 (1968), 187-193.
	- **[4] LEIGH-roN,** W.: More Elementary Sturm Theory. Applicable Anal. 3 (1973), 187-203.
	- **[5] M1JLLEH-PFEIFFER,** E.: **Sturmian theory for nonseif-adjoint differential equations of second order. Proc.** Roy. Soc. **Edinburgh** 105A (1987), 337-343. TON, W.: Some Elementary Sturm Theory. (TON, W.: More Elementary Sturm Theory. A

	R. P. P. Elementary Sturm Theory. A

	R. P. P. Elementary Sturm and theory for no

	order. Proc. Roy. Soc. Edinburgh 105A (19

	W. T.: Sturmian
	- **[6] REID,** W. T.: Sturmian Theory for Ordinary Differential Equations (Applied Mathematical Sciences: Vol. 31). New York — **Heidelberg** — Berlin: Springer-Verlag 1980.
	- **[7] SWANSON,** C. A.: Comparison and Oscillation Theory of Linear Differential Equations.

 \cdot

VERFASSER:

Prof. Dr. Eaicii **MULLER-PFELFER** Sektion Mathematik/Physik der Pädagogischen Hochschule "Dr. Th. Neubauer" W. T.: Sturmian Theory for Ordines: Vol. 31). New York—Heidelber

Ser. Vol. 31). New York—Heidelber

ON, C. A.: Comparison and Oscill

Str. – London: Acad. Press 1968.

Manuskripteingang: 09. 10. 1987

VERFASSER:

Prof. Dr DDR-5010 Erfurt