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On Besov Spaces of Variable Order of Differentiation

H.-G. Leopold

Es werden Funktionenräume mit variabler Glattheit betrachtet, die auf dem n-dimensionalen euklidischen Raum definiert sind. Die Definition dieser Räume erfolgt mit Hilfe einer geeigneten Klasse von Pseudodifferentialoperatoren.

Рассматриваются пространства функций переменной гладкости, определенных на n-мерном эвклидовом пространстве. Эти пространства определены с помощью подходящего класса псевдодифференциальных операторов.

This paper is concerned with function spaces of variable order of differentiation defined on the n-dimensional Euclidean space. The definition of these spaces is closely connected with an appropriate class of pseudodifferential operators.

The paper deals with Besov spaces of variable order of differentiation defined on the Euclidean n-space $\mathbf{R}^n$. In the classical function spaces of Besov type $B^s_{p,q}(\mathbf{R}^n)$ norms can be defined via a resolution of unity in the Fourier image of the function u, which is connected with the symbol $|\xi|^2$ of the Laplacian $-\Delta$. We consider now in this paper decompositions of $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ which are induced by symbols $a(x, \xi)$ of appropriate pseudodifferential operators. This means that we may have different resolutions $\{\varphi_j(x, \xi)\}_{j=0}^\infty$ of $\mathbf{R}_\xi^n$ for different $x \in \mathbf{R}_x^n$. So we can get locally in different points x different smoothness demands on the function $u(x)$. The function spaces $B^{s,a}_{p,q}(\mathbf{R}^n)$ defined in this way seem to be useful in the study of degenerate elliptic partial differential equations. In Section 1 we recall some facts about pseudodifferential operators and collect those results which will be needed in the sequel. Section 2 contains the definition of an appropriate sublcass $S(m, m'; \delta)$ of hypoelliptic symbols, some examples and the definition of the resolution of unity of $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ connected with these symbols. In Section 3 we define the function spaces $B^{s,a}_{p,q}(\mathbf{R}^n)$ of variable order of differentiation and describe properties of these spaces.

1. Basic properties of pseudodifferential operators

Let $p(x, \xi)$ be a polynomially bounded complex-valued function defined on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$. The pseudodifferential operator $P(x, D_x)$ with symbol $p(x, \xi)$ is defined by

$$P(x, D_x)\, u(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} p(x, \xi)\, (Fu)\, (\xi)\, d\xi \qquad \text{for } u \in S(\mathbf{R}^n),$$

where $S(\mathbf{R}^n)$ denotes the Schwartz class and $(Fu)\,(\xi) = \int e^{-iy\xi} u(y)\, dy$ denotes the Fourier transform of u. A function $p(x, \xi)$ belongs to the class $S^m_{\varrho,\delta}$ $(-\infty < m < \infty;$ $0 \leqq \delta \leqq \varrho \leqq 1, \delta < 1)$ if for any multi-indices α, β there exist a constant $c_{\alpha\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq c_{\alpha\beta} \langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|} \qquad \text{for } (x, \xi) \in \mathbf{R}_x^n \times \mathbf{R}_\xi^n,$$

where $p_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi)$, $\partial_\xi^\alpha = \partial^{|\alpha|}/\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_n}^{\alpha_n}$, $D_x^\beta = (-i)^{|\beta|} \partial_x^\beta$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We set $S^{-\infty} = \bigcap_m S_{\varrho,\delta}^m$. It is easy to see that $\bigcap_m S_{\varrho,\delta}^m = \bigcap_m S_{1,0}^m$ for any ϱ and δ. The pseudodifferential operator $P(x, D_x)$ with a symbol $p \in S_{\varrho,\delta}^m$ maps $S(\mathbf{R}^n)$ continuously into itself and can be extended to a continuous operator from $S'(\mathbf{R}^n)$ into $S'(\mathbf{R}^n)$, the space of all tempered distributions on $\mathbf{R}^n$. The mapping between $p(x, \xi)$ and $P(x, D_x)$ is a bijection. For $p \in S_{\varrho,\delta}^m$ we define the semi-norms $|p|_{(l,k)}^{(m)}$ by

$$|p|_{(l,k)}^{(m)} = \max_{|\alpha| \leq l, |\beta| \leq k} \sup_{(x,\xi)} \{|p_{(\beta)}^{(\alpha)}(x, \xi)| \langle \xi \rangle^{-m+\varrho|\alpha|-\delta|\beta|}\}. \tag{1}$$

Theorem 1: *Assume that $0 \leq \delta < \varrho \leq 1$. Let $P_1(x, D_x) \in S_{\varrho,\delta}^{m_1}$ and $P_2(x, D_x) \in S_{\varrho,\delta}^{m_2}$. Then $P(x, D_x) = P_1(x, D_x) P_2(x, D_x)$ belongs to $S_{\varrho,\delta}^{m_1+m_2}$. For the symbol $p(x, \xi)$ of $P(x, D_x)$ and for any natural N we have the expansion formula*

$$p(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi) + N \sum_{|\gamma|=N} \int_0^1 \frac{(1-\vartheta)^{N-1}}{\gamma!} r_{\gamma,\vartheta}(x, \xi) \, d\vartheta, \tag{2}$$

where (Os *oscillatory integral*)

$$r_{\gamma,\vartheta}(x, \xi) = \text{Os-}\frac{1}{(2\pi)^n} \int e^{-iy\eta} p_1^{(\gamma)}(x, \xi + \vartheta\eta) \, p_{2(\gamma)}(x + y, \xi) \, dy \, d\eta. \tag{3}$$

$\{r_{\gamma,\vartheta}\}_{|\vartheta| \leq 1}$ is a bounded set of $S_{\varrho,\delta}^{m_1+m_2-|\gamma|(\varrho-\delta)}$. Furthermore, for any pair of integers (l, k) there exist constants c, c' and integers l', k' independent of ϑ such that

$$|p_1^{(\alpha)} p_{2(\alpha)}|_{(l,k)}^{(m_1+m_2-|\alpha|(\varrho-\delta))} \leq c \, |p_1|_{(l+|\alpha|,k)}^{(m_1)} \, |p_2|_{(l,k+|\alpha|)}^{(m_2)} \tag{4}$$

and

$$|r_{\gamma,\vartheta}|_{(l,k)}^{(m_1+m_2-|\gamma|(\varrho-\delta))} \leq c' \, |p_1|_{(l',k)}^{(m_1)} \, |p_2|_{(l,k')}^{(m_2)}. \tag{5}$$

The theorem gives an estimate of each term of the sum (2) which is obtained by the composition of two pseudodifferential operators. Especially the estimate of the remainder term will be often useful. The proof is a direct consequence of the definition of semi-norms and of [4; Section 2], see also there for details.

Theorem 2: *Let $P(x, D_x) \in S_{1,\delta}^0$ and $\delta < 1$. Then for all p with $1 < p < \infty$ there exist integers l, k and a constant c, all independent of $P(x, D_x)$, such that*

$$\|P(x, D_x) u \mid L_p\| \leq c \, |p|_{(l,k)}^{(0)} \, \|u \mid L_p\| \quad \textit{for all } u \in L_p(\mathbf{R}^n). \tag{6}$$

This was proved first by ILLNER [3] in 1975. Later for example BOURDAUD [2] and NAGASE [7] considered non-regular symbols and got weaker conditions on $p(x, \xi)$. But the result is sharp with respect to the parameter p. There exist smooth functions $m(\xi) \in S_{1,0}^0$ which are not Fourier multipliers in $L_1(\mathbf{R}^n)$ and $L_\infty(\mathbf{R}^n)$ [14; p. 21]. Consequently for the corresponding pseudodifferential operators (6) is not true in the case $p = 1$ and $p = \infty$.

Corollary 1: *Let $P(x, D_x) \in S_{1,\delta}^m$, $\delta < 1$, $1 < p < \infty$ and $-\infty < t, m < \infty$. Then there exist integers l, k and a constant c such that*

$$\|P(x, D_x) u \mid H_p^t\| \leq c \, |p|_{(l,k)}^{(m)} \, \|u \mid H_p^{t+m}\| \quad \textit{for } u \in H_p^{t+m}(\mathbf{R}^n). \tag{7}$$

Again the constants are independent of $P(x, D_x)$ and u. $H_p^t(\mathbf{R}^n)$ denotes the Bessel-potential spaces.

2. Coverings of $\mathbf{R}_x{}^n \times \mathbf{R}_\xi{}^n$ induced by symbols of pseudodifferential operators

In the following we consider a subclass of the hypoelliptic symbols of slowly varying strength.

Definition 1: Let $0 \leqq \delta < 1$ and $0 < m' \leqq m$. A symbol $a(x, \xi) \in S^m_{1,\delta}$ belongs to the class $S(m, m'; \delta)$ if there exists a constant $R_a \geqq 0$ such that holds:

(i) for any multi-indices α, β, all $x \in \mathbf{R}_x{}^n$ and all $\xi \in \mathbf{R}_\xi{}^n$ with $|\xi| \geqq R_a$ there holds

$$|a^{(\alpha)}_{(\beta)}(x, \xi)| \leqq c_{\alpha\beta}\, |a(x, \xi)|\, \langle\xi\rangle^{-|\alpha|+|\beta|\delta}; \tag{8}$$

(ii) there exist constants $c_{m'} > 0$ and $c_m > 0$ such that for all $x \in \mathbf{R}_x{}^n$ and all $\xi \in \mathbf{R}_\xi{}^n$ with $|\xi| \geqq R_a$ there holds

$$c_{m'}\langle\xi\rangle^{m'} \leqq |a(x, \xi)| \leqq c_m\langle\xi\rangle^m. \tag{9}$$

The symbols of the class $S(m, m'; \delta)$ will be a substitute for the symbol $|\xi|^2$ of the Laplacian which is used in the definition of the usual Besov spaces. Therefore the restrictions $m' > 0$ and $\varrho = 1$ turn out to be natural in view of the following definitions and Theorem 2.

Two symbols $a(x, \xi)$ and $b(x, \xi)$ belonging to $S(m, m'; \delta)$ are called *equivalent* if there exist constants c', c and R with

$$0 < c' \leqq |a(x, \xi)\, b^{-1}(x, \xi)| \leqq c < \infty \tag{10}$$

for all $x \in \mathbf{R}_x{}^n$ and all $\xi \in \mathbf{R}_\xi{}^n$ with $|\xi| \geqq R$.

Let us give now some simple examples:

1. The trivial example is the symbol $a(x, \xi) = \langle\xi\rangle$ of the Bessel-potential operator $(I - \Delta)^{1/2}$. This symbol belongs to $S(1, 1; 0)$.

2. Let $\sigma(x) = s + \psi(x)$ be a real-valued function, s be a constant and ψ be an element of $S(\mathbf{R}^n)$. Let $m' = \inf \sigma(x)$, $m = \sup \sigma(x)$ and $0 < m'$. Then $a(x, \xi) = \langle\xi\rangle^{\sigma(x)}$ belongs to $S(m, m'; \delta)$ for any δ with $0 < \delta < 1$. Such symbols and related function spaces were considered by Unterberger and Bokobza [14], Visik and Eskin [16, 17] and Beauzamy [1].

3. Let $\sigma(x) = s + \psi(x)$ be a function as in the previous example and t be an arbitrary real number. Then $a(x, \xi) = \langle\xi\rangle^{\sigma(x)}(1 + \log\langle\xi\rangle^2)^{t/2}$ belongs to $S(m, m'; \delta)$ with $0 < \delta < 1$, $0 < m' < \inf \sigma(x)$ and $m > \sup \sigma(x)$. Symbols of this type were considered by Unterberger and Bokobza [15] and Unterberger [13].

4. Let $\varrho(x)$ be a real-valued weight function with $\sup |D^\gamma\varrho(x)| \leqq c_\gamma$ for all γ. ϱ may be zero on a domain $\Omega \subset \mathbf{R}_x{}^n$. Let $0 < m' \leqq m$ and k be a natural number with $(m - m') < 2k$. Then the symbol $a(x, \xi) = \langle\xi\rangle^{m'} + \varrho^{2k}(x)\langle\xi\rangle^m$ belongs to $S(m, m'; \delta)$ where $\delta = (m - m')/2k$. If m' and m are even numbers, then $a(x, \xi)$ is the symbol of a degenerate partial differential equation.

For each symbol $a(x, \xi) \in S(m, m'; \delta)$ we can define variable coverings of $R_x{}^n \times \mathbf{R}_\xi{}^n$. Variable covering means that in different points $x \in \mathbf{R}_x{}^n$ we may have different coverings of $\mathbf{R}_\xi{}^n$.

Definition 2: Let N be an integer and $a(x, \xi)$ a symbol belonging to $S(m, m'; \delta)$. The symbol $a(x, \xi)$ induces a variable covering $\{\Omega_j^{N,a}\}_{j=0}^\infty$ of $\mathbf{R}_x{}^n \times \mathbf{R}_\xi{}^n$ by

$$\begin{aligned} \Omega_j^{N,a} &= \{(x, \xi)\colon |a(x, \xi)| < 2^{J+N+j}\} && \text{if } j = 0, 1, \ldots, N,\\ \Omega_j^{N,a} &= \{(x, \xi)\colon 2^{J-N+j} < |a(x, \xi)| < 2^{J+N+j}\} && \text{if } j = N + 1, N + 2, \ldots. \end{aligned} \tag{11}$$

J is a constant which is fixed in such a way that $|\xi| \leqq R_a$ always implies $(x, \xi) \in \Omega_0{}^{1,a}$.

In the case $a(x, \xi) = \langle\xi\rangle$ we get the usual classical dyadic coverings of $\mathbf{R}_\xi^n$, independent of x, which are the basic for the definition of the spaces $B_{p,q}^s(\mathbf{R}^n)$ and $F_{p,q}^s(\mathbf{R}^n)$. In the case $a(x, \xi) = \langle\xi\rangle^{\sigma(x)}$ or $a(x, \xi) = \langle\xi\rangle^{m'} + \varrho^{2k}(x)\langle\xi\rangle^m$ the coverings $\{\Omega_j^{N,a}\}_{j=0}^\infty$ are variable. For each fixed $x \in \mathbf{R}_x^n$ we have a dyadic covering of $\mathbf{R}_\xi^n$, but in general these coverings are different from each other. The $\Omega_j^{N,a}$ are open sets and bounded in ξ. For any number j_0 at most $4N - 1$ sets $\Omega_j^{N,a}$ have a non-empty intersection with $\Omega_{j_0}^{N,a}$.

Definition 3: Let $\{\Omega_j^{N,a}\}_{j=0}^\infty$ be a variable covering induced by the symbol $a(x, \xi)$. A function system $\{\varphi_j\}_{j=0}^\infty$ belongs to $\Phi^{N,a}$ if for all $j = 0, 1, 2, \ldots$ holds:

(i) $\varphi_j(x, \xi) \in C^\infty(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$ and $\varphi_j(x, \xi) \geqq 0$;

(ii) $\operatorname{supp} \varphi_j \subset \Omega_j^{N,a}$;

(iii) $|\varphi_{j(\beta)}^{(\alpha)}(x, \xi)| \leqq c_{\alpha\beta}\langle\xi\rangle^{-|\alpha|+|\beta|\delta}$ for any multi-indices α and β, where the constants $c_{\alpha\beta}$ are independent of j;

(iv) $\sum_{j=0}^{\infty} \varphi_j(x, \xi) = c^\varphi > 0$.

By assumptions (ii) and (iii) we get $\varphi_j \in S^{-\infty}$. The following estimates for the semi-norms of φ_j are a simple consequence of (ii), (iii), (9) and (11):

$$|\varphi_j|_{(l,k)}^{(\varkappa)} \leqq c_{lk\varkappa} \begin{cases} 2^{-j\varkappa/m} & \text{if } \varkappa \geqq 0, \\ 2^{-j\varkappa/m'} & \text{if } \varkappa < 0 \end{cases} \tag{12}$$

for all real numbers $\varkappa$ and with constants $c_{lk\varkappa}$ independent of j. Also by (iii), respectively (12), it follows that the semi-norms of the φ_j are uniformly bounded in $S_{1,\delta}^0$. Together with (ii) and (iv) we get in this way that $\sum_{j=0}^{J} \varphi_j(x, \xi) \to c^\varphi$ in $S_{1,\delta}^0$ weakly if $J \to \infty$. The weak convergence in $S_{1,\delta}^0$ and Corollary 1 imply that for every $v \in H_p^s(\mathbf{R}^n)$

$$\sum_{j=0}^{J} \varphi_j(x, D_x)\, v \to c^\varphi v \quad \text{in } H_p^s(\mathbf{R}^n) \qquad \text{if } J \to \infty \tag{13}$$

holds — see also [4; Chapter 3, § 7] where this fact was proved for $L_2(\mathbf{R}^n)$. But in view of Corollary 1 there are no difficulties to carry over the proof to the case $1 < p < \infty$ and the Bessel-potential spaces for arbitrary real s.

It is easy to describe examples of function systems of the above type. Let $\{\Omega_j^{N,a}\}_{j=0}^\infty$ be a variable covering induced by $a(x, \xi) \in S(m, m'; \delta)$ and J be the fixed number from Definition 2. Furthermore let $\varphi \in C^\infty(\mathbf{R}_+^1)$ be a real-valued function with $0 \leqq \varphi(t) \leqq 1$, $\varphi(t) = 1$ if $0 \leqq t \leqq 2^{J-1}$ and $\operatorname{supp} \varphi \subset \{t: 0 \leqq t \leqq 2^J\}$. Setting

$$\varphi_j(x, \xi) = \varphi(2^{-j-N}\,|a(x, \xi)|) - \varphi(2^{-j+N-1}\,|a(x, \xi)|) \qquad \text{if } j = 1, 2, \ldots$$

and

$$\varphi_0(x, \xi) = \sum_{k=1}^{2N-1} \varphi(2^{-k+N-1}\,|a(x, \xi)|),$$

then we have $\{\varphi_j\}_{j=0}^\infty \in \Phi^{N,a}$ with $c^\varphi = 2N - 1$.

3. The spaces $B_{p,q}^{s,a}$ of variable order of differentiation

We are now ready to define Besov spaces of variable order of differentiation. Instead of the classical resolution of $\mathbf{R}_\xi^n$ which is connected with the symbols $|\xi|^2$ respectively $\langle\xi\rangle$ we use now function systems $\{\varphi_j(x, \xi)\}_{j=0}^\infty \in \Phi^{N,a}$ connected with the symbol $a(x, \xi)$, which may lead to different resolutions of $\mathbf{R}_\xi^n$ for different fixed $x \in \mathbf{R}_x^n$. Throughout this section $a(x, \xi)$ is a fixed element of $S(m, m'; \delta)$.

Definition 4: Let $1 < p < \infty$, $0 < q \leqq \infty$, $-\infty < s < \infty$ and $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ be a system belonging to $\Phi^{N,a}$. Then

$$B_{p,q}^{s,a}(\mathbf{R}^n) = \{u\colon u \in S'(\mathbf{R}^n) \quad \text{and} \quad \|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}} < \infty\},$$

$$\|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(x, D_x)\, u \mid L_p\|^q\right)^{1/q} \quad \text{if } q < \infty, \tag{14}$$

$$\|u \mid B_{p,\infty}^{s,a}\|^{\{\varphi_j\}} = \sup_j 2^{js} \|\varphi_j(x, D_x)\, u \mid L_p\|.$$

Of course the norms $\|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}}$ depend on the chosen system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$. But this is not the case for the spaces $B_{p,q}^{s,a}(\mathbf{R}^n)$ itself. This will be proved in Theorem 4. But a first we prove the embedding of the spaces $B_{p,q}^{s,a}(\mathbf{R}^n)$ in the scale of the usual Bessel-potential spaces. In this theorem $B_{p,q}^{s,a\,(\varphi)}(\mathbf{R}^n)$ denotes the function spaces which are defined by (14) and an arbitrary fixed system $\{\varphi_j\}_{j=0}^{\infty}$.

Theorem 3: *Let $\{\varphi_j\}_{j=0}^{\infty}$ be a fixed system belonging to $\Phi^{N,a}$ and $1 < p < \infty$, $0 < q \leqq \infty$, $-\infty < s < \infty$.*

(i) *For $s \geqq 0$ we have*

$$H_p^{\varrho}(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(\varphi)}(\mathbf{R}^n) \hookrightarrow H_p^{\varkappa}(\mathbf{R}^n) \tag{15}$$

if $\varkappa < sm'$ and $\varrho > sm$.

(ii) *For $s \leqq 0$ we have*

$$H_p^{\varrho}(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(\varphi)}(\mathbf{R}^n) \hookrightarrow H_p^{\varkappa}(\mathbf{R}^n) \tag{16}$$

if $\varkappa < sm$ and $\varrho > sm'$.

Proof: *Step* 1. We get by the monotonicity of the l_q-spaces and by a simple calculation the first elementary embedding

$$B_{p,\infty}^{s+\varepsilon,a\,(\varphi)} \hookrightarrow B_{p,q_1}^{s,a\,(\varphi)} \hookrightarrow B_{p,q_2}^{s,a\,(\varphi)} \hookrightarrow B_{p,q}^{s-\varepsilon,a\,(\varphi)} \tag{17}$$

if $1 < p < \infty$, $-\infty < s < \infty$, $0 < q \leqq \infty$, $0 < q_1 \leqq q_2 \leqq \infty$ and $\varepsilon > 0$.

Step 2. Without loss of generality we may assume that $\{\varphi_j\}_{j=0}^{\infty}$ is a system with $c^{\varphi} = 1$. We introduce a second system of smooth functions $\{\varphi_j^*\}_{j=0}^{\infty}$, where the φ_j^* are independent of x and therefore we will write $\varphi_j^*(\xi)$, with the following properties:

$$\varphi_j^*(\xi) = 1 \text{ on } \operatorname{supp} \varphi_j,\ \operatorname{supp} \varphi_j^* \subset \Omega_j^{N,a,*},$$

$$|\varphi_j^{*(\alpha)}(\xi)| \leqq c_\alpha \langle \xi \rangle^{-|\alpha|} \qquad \text{for all } \alpha \text{ and } c_\alpha \text{ and independent of } j.$$

where

$$\Omega_j^{N,a,*} = \{(x, \xi)\colon \langle \xi \rangle < \max\,(1 + R_a, c_{m'}^{-1/m'}\, 2^{(J+j+N+1)/m'})\} \quad \text{if } j = 0, 1, \ldots, N;$$

$$\Omega_j^{N,a,*} = \{(x, \xi)\colon c_m^{-1/m} 2^{(J+j-N-1)/m} < \langle \xi \rangle < c_{m'}^{-1/m'} 2^{(J+j+N+1)/m'}\} \quad \text{if } j = N+1, N+2, \ldots.$$

The existence of such systems $\{\varphi_j^*\}_{j=0}^{\infty}$ can be shown in analogy to the example at the end of the previous section. We put

$$\varphi_j^*(\xi) = \varphi(2^{-N-j-1} c_{m'} \langle \xi \rangle^{m'}) - \varphi(2^{N-j} c_m \langle \xi \rangle^m)$$

for $j = N+1, N+2, \ldots$. Now it is easy to verify that the semi-norms of φ_j^* can be estimated as the semi-norms of φ_j in (12). Also in view of (9) and (11) it holds

$\Omega_j^{N,a} \subset \Omega_j^{N,a,*}$ for all j. Hence by Theorem 1 we obtain

$$\varphi_j^*(D_x)\,\varphi_j(x, D_x) = \varphi_j(x, D_x) + R_j(x, D_x) \tag{18}$$

and

$$|r_j|_{(l,k)}^{(\mu)} \leq c_{lk\mu\gamma} 2^{-j\gamma} \tag{19}$$

for arbitrary real numbers μ and γ, where the constants $c_{lk\mu\gamma}$ are independent of j. This yields for $J = 1, 2, \ldots$

$$u = \sum_{j=0}^{J-1} \varphi_j(x, D_x)\,u + \sum_{j=J}^{\infty} \varphi_j^*(D_x)\,\varphi_j(x, D_x)\,u - R^J(x, D_x)\,u \tag{20}$$

with

$$R^J(x, D_x) = \sum_{j=J}^{\infty} R_j(x, D_x). \tag{21}$$

By (19) we get the convergence of the infinite series (21) in $S^0_{1,\delta}$. The semi-norms of $R^J(x, D_x)$ can be estimated by

$$|r^J|_{(l,k)}^{(0)} \leq c_{lk} 2^{-J}, \tag{22}$$

where the constants c_{lk} are independent of J.

Step 3. Let $s \leq 0$ and $\varkappa < sm$ fixed. Then by Corollary 1 and (22) we have $\|R^J(x, D_x) \mid L(H_p^\varkappa, H_p^\varkappa)\| \leq c_{\varkappa p} 2^{-J}$. This implies that the inverse operator of $I + R^J(x, D_x)$ exists and belongs also to $L(H_p^\varkappa, H_p^\varkappa)$ if $J \geq J_0(\varkappa, p)$. I stands for the identity.

Step 4. Let $s' = \varkappa/m$, $u \in B^{s',a(\varphi)}_{p,1}(\mathbf{R}^n)$ and

$$v_j = \begin{cases} \varphi_j(x, D_x)\,u & \text{if } j = 0, 1, \ldots, J_0 - 1; \\ \varphi_j^*(D_x)\,\varphi_j(x, D_x)\,u & \text{if } j = J_0, J_0 + 1, \ldots. \end{cases}$$

As a consequence of the properties of the system $\{\varphi_j^*\}_{j=0}^\infty$ and of (7) in Corollary 1, we see that

$$\|\varphi_j^*(D_x)\,\varphi_j(x, D_x)\,u \mid H_p^\varkappa\| \leq c 2^{js'} \|\varphi_j(x, D_x)\,u \mid L_p\|$$

if $j = J_0, J_0 + 1, \ldots$. Since $\varkappa < 0$, a trivial estimate gives

$$\|\varphi_j(x, D_x)\,u \mid H_p^\varkappa\| \leq c' 2^{-J_0 s'} 2^{js'} \|\varphi_j(x, D_x)\,u \mid L_p\|$$

if $j = 0, 1, 2, \ldots, J_0 - 1$. c and c' are independent of j. We obtain

$$\sum_{j=0}^{\infty} \|v_j \mid H_p^\varkappa\| \leq \max(c, c' 2^{-J_0 s'})\, \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}.$$

Together with (20) this implies that $\sum_{j=0}^{\infty} v_j = u + R^J(x, D_x)\,u$ belongs to $H_p^\varkappa(\mathbf{R}^n)$. Because of the result of the third step the same must be true for u and we get

$$\|u \mid H_p^\varkappa\| \leq c'' \|(I + R^J)^{-1} \mid L(H_p^\varkappa, H_p^\varkappa)\|\, \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}$$

if $s \leq 0$, $\varkappa < sm$ and $s' = \varkappa/m$. Now the right-hand side of (16) follows in view of (17).

Step 5. In the case $s > 0$ the proof is simpler. Let $0 < \varkappa < sm'$ and $s' = \varkappa/m'$. If $u \in B^{s',a(\varphi)}_{p,1}(\mathbf{R}^n)$, then it is straightforward to see that $\sum_{j=0}^{\infty} \|\varphi_j(x, D_x)\,u \mid L_p\| \leq c\, \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}$

is absolutely convergent in $L_p(\mathbf{R}^n)$. We get on this way $u \in L_p(\mathbf{R}^n)$ and $\|u \mid L_p\| \leq c \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}$. On the other hand it follows by (18), (19) and Corollary 1

$$\sum_{j=0}^{\infty} \|\varphi_j(x, D_x) u \mid H_p{}^{\varkappa}\| \leq c_{\varkappa p} \sum_{j=0}^{\infty} |\varphi_j{}^*|^{(-\varkappa)}_{(l,k)} \|\varphi_j(x, D_x) u \mid L_p\| + c_{\varkappa p} \sum_{j=0}^{\infty} |r_j|^{(-\varkappa)}_{(l,k)} \|u \mid L_p\|$$

$$\leq c \sum_{j=0}^{\infty} 2^{js'} \|\varphi_j(x, D_x) u \mid L_p\| + c' \|u \mid L_p\| \leq c'' \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}.$$

So we get $u \in H_p{}^{\varkappa}(\mathbf{R}^n)$ with $\|u \mid H_p{}^{\varkappa}\| \leq c'' \|u \mid B^{s',a}_{p,1}\|^{\{\varphi_j\}}$ and again the right-hand side of (15) follows in view of (17).

Step 6. The proof will be completed by showing that the left-hand sides of (15) and (16) hold. But this is a simple consequence of the semi-norm estimates (12) and Corollary 1 ∎

We will prove now that equivalent symbols define the same spaces and that the definition of these spaces is independent of the choosen system $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$. Let $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols of the class $S(m, m'; \delta)$, that means we have $0 < c_1 \leq |a(x, \xi)\, b^{-1}(x, \xi)| \leq c_2 < \infty$ if $x \in \mathbf{R}_x{}^n$, $\xi \in \mathbf{R}_\xi{}^n$ and $|\xi| \geq R$. Without loss of generality we may assume that $R \geq \max(R_a, R_b)$. $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and $\{\psi_i\}_{i=0}^{\infty} \in \Phi^{M,b}$ denote two function systems belonging to $a(x, \xi)$ and $b(x, \xi)$, respectively. Then there exist numbers $j_0(R)$ and $i_0(R)$ such that all (x, ξ) with $|\xi| \leq R$ do not belong to the sets $\Omega_j{}^{N,a}$ and $\Omega_i{}^{M,b}$ if $j > j_0$ and $i > i_0$, respectively. Hence we get

$$\operatorname{supp} \varphi_j \cap \operatorname{supp} \psi_i = \emptyset \text{ if } j \notin \mathcal{J}(i),$$

where

$$\mathcal{J}(i) = \{j : 0 \leq j \leq \max(j_0, i + I - J + M + N + H)\} \qquad \text{if } 0 \leq i \leq i_0,$$

$$\mathcal{J}(i) = \left\{j : \max\left(J, i + I - (M + N + H)\right) \leq i + J \leq i + I + M + N + H\right\} \quad \text{if } i > i_0 \tag{23}$$

and H fulfils $2^{-H} \leq c_1 \leq c_2 \leq 2^H$. Therefore we obtain in the case $j \notin \mathcal{J}(i)$ by Theorem 1

$$\psi_i(x, D_x)\, \varphi_j(x, D_x) = R_{ij}(x, D_x), \tag{24}$$

where for each natural number L the semi-norms of $R_{ij}(x, D_x)$ can be estimated by

$$|r_{ij}|^{(m_1 + m_2 - L(1-\delta))}_{(l,k)} \leq c_{Llk} |\psi_i|^{(m_1)}_{(l',k)} |\varphi_j|^{(m_2)}_{(l,k')}. \tag{25}$$

The constants c_{Llk} may increase in dependence on L and (l, k) but they are always independent of i and j. In the classical case, that means $\bar{\psi}_i(\xi)$ and $\bar{\varphi}_j(\xi)$ are independent of x, the terms $R_{ij}(x, D_x)$ do not exist because $\operatorname{supp} \bar{\varphi}_j \cap \operatorname{supp} \bar{\psi}_i = \emptyset$ always yields $\bar{\psi}_i(D_x)\, \bar{\varphi}_j(D_x) = 0$.

Theorem 4: *Let $a(x, \xi)$ and $b(x, \xi)$ be equivalent symbols and $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, $\{\psi_i\}_{i=0}^{\infty} \in \Phi^{M,b}$ be two systems belonging to $a(x, \xi)$ and $b(x, \xi)$, respectively. If $1 < p < \infty$, $0 < q \leq \infty$ and $-\infty < s < \infty$, then $\|u \mid B^{s,a}_{p,q}\|^{\{\varphi_j\}}$ and $\|u \mid B^{s,b}_{p,q}\|^{\{\psi_i\}}$ are equivalent quasi-norms in $B^{s,a}_{p,q}(\mathbf{R}^n)$.*

Proof: It is easy to see that both $\|u \mid B^{s,a}_{p,q}\|^{\{\varphi_j\}}$ and $\|u \mid B^{s,b}_{p,q}\|^{\{\psi_i\}}$ are quasi-norms. In order to prove their equivalence we use the preceding considerations. Also we may assume $c^{\varphi} = 1$.

Step 1. Let $u \in B^{s,a}_{p,q}(\varphi)$. Then by Theorem 3 u belongs also to $H_p{}^{\varkappa}(\mathbf{R}^n)$ if $\varkappa$ is suitably chosen. Now we obtain from (13), (12) and Corollary 1 for $i = 0, 1, 2, \ldots$ and arbitrary fixed $\varepsilon > 0$ the estimates

$$\|\psi_i(x, D_x)\, u \mid L_p\| \leq \sum_{j=0}^{\infty} \|\psi_i(x, D_x)\, \varphi_j(x, D_x)\, u \mid L_p\| + 2^{-i(s+\varepsilon)} \|u \mid H_p^{\varkappa}\|. \tag{26}$$

Step 2. Let $s > 0$ and $\varkappa < sm'$ be fixed. From (26) and (24) we get

$$\sum_{i=0}^{\infty} 2^{isq} \|\psi_i(x, D_x)\, u \mid L_p\|^q$$
$$\leq \sum_{i=0}^{\infty} 2^{isq} \Big(\sum_{j\in \mathcal{J}(i)} \|\psi_i(x, D_x)\, \varphi_j(x, D_x)\, u \mid L_p\| + \sum_{j\notin \mathcal{J}(i)} \|R_{ij}(x, D_x)\, u \mid L_p\| \Big)^q + c_\varepsilon \|u \mid H_p^{\varkappa}\|^q.$$

The terms of the first sum will be estimated by Theorem 2 and (12). The estimate of the remainder terms in the second sum will be taken by (25) if we choose there for a fixed $\varepsilon' > 0$, $m_1 = ms + \varepsilon' m$, $m_2 = \varepsilon' m$ and L so large such that $m_1 + m_2 - L(1-\delta) \leq \varkappa$ holds. Then the semi-norms of ψ_i and φ_j in (25) can be estimated again by (12) and we obtain

$$\begin{aligned}\|u \mid B_{p,q}^{s,b}\|^{\{\psi_i\}q} &\leq c \sum_{i=0}^{\infty} 2^{isq} \Big(\sum_{j\in \mathcal{J}(i)} c' \, |\psi_i|_{(l,k)}^{(0)} \|\varphi_j(x, D_x)\, u \mid L_p\| \\ &\quad + \sum_{j=0}^{\infty} c_L 2^{-i(s+\varepsilon')} 2^{-j\varepsilon'} \|u \mid H_p^{\varkappa}\| \Big)^q + c_\varepsilon \|u \mid H_p^{\varkappa}\|^q \\ &\leq c'' \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(x, D_x)\, u \mid L_p\|^q + (c_{\varepsilon'} + c_\varepsilon) \|u \mid H_p^{\varkappa}\|^q \\ &\leq c_\varphi \|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}q}.\end{aligned}$$

The last estimate follows in view of the embedding (15). Also we have used the shape of the set $\mathcal{J}(i)$. Because of our assumptions, the same must be true if we change the role of $\{\psi_i\}_{i=0}^{\infty}$ and $\{\varphi_j\}_{j=0}^{\infty}$ and so we get the converse inequality.

Step 3. Let $s \leq 0$ and $\varkappa < sm$ be fixed. The proof of the equivalence will be the same as in the second step, if we choose $m_1 = m's + \varepsilon' m'$, $m_2 = \varepsilon' m'$ and take a corresponding modification in the semi-norm estimates of the ψ_i ∎

Corollary 2: *The definition of the space $B_{p,q}^{s,a}(\mathbf{R}^n)$ is independent of the chosen system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and also of the choice of the constants J and N in the definition of the sets $\{\Omega_j^{N,a}\}_{j=0}^{\infty}$.*

Convention: In the sequel we shall not distinguish between equivalent quasi-norms. In this sense we shall write $\|u \mid B_{p,q}^{s,a}\|$ instead of $\|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}}$.

Corollary 3: *Symbols $a(x, \xi)$ and $b(x, \xi)$ which are equivalent in the sense of* (10) *define the same function spaces. For all admissible parameters p, q and s there exist positive constants c' and c such that $c' \|u \mid B_{p,q}^{s,a}\| \leq \|u \mid B_{p,q}^{s,b}\| \leq c \|u \mid B_{p,q}^{s,a}\|$ holds.*
If $b(x, \xi)$ is especially an elliptic pseudodifferential operator of the order m, then the space $B_{p,q}^{s,b}(\mathbf{R}^n)$ coincides with the classical Besov space $B_{p,q}^{sm}(\mathbf{R}^n)$ for $1 < p < \infty$, $0 < q \leq \infty$ and $-\infty < s < \infty$.

Theorem 5: *For $-\infty < s < \infty$, $1 < p < \infty$ and $0 < q \leq \infty$, $B_{p,q}^{s,a}(\mathbf{R}^n)$ is a quasi-Banach space (Banach space if $1 \leq q \leq \infty$), which is independent of the choice* *of the system* $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, *and we have* $S(\mathbf{R}^n) \hookrightarrow B^{s,a}_{p,q}(\mathbf{R}^n) \hookrightarrow S'(\mathbf{R}^n)$. *Furthermore, if* $-\infty < s < \infty$, $1 < p < \infty$ *and* $0 < q < \infty$, *then* $S(\mathbf{R}^n)$ *is dense in* $B^{s,a}_{p,q}(\mathbf{R}^n)$.

Proof: The equivalence of quasi-norms $\|\cdot \mid B^{s,a}_{p,q}\|^{\{\varphi_j\}}$, defined by different systems $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, was proved in Theorem 4. Also in view of Theorem 3 we get from the well-known embeddings of the classical Bessel-potential spaces in $S'(\mathbf{R}^n)$ and of $S(\mathbf{R}^n)$ in $H_p^{\varkappa}(\mathbf{R}^n)$, respectively, the assertion about the embeddings. Thus we need only to show the completness in order to prove the first part of the theorem.

Step 1. Let $\{u_l\}_{l=1}^{\infty}$ be a fundamental sequence in $B^{s,a}_{p,q}(\mathbf{R}^n)$ which we consider with respect to a fixed quasi-norm $\|\cdot \mid B^{s,a}_{p,q}\|^{\{\varphi_j\}}$. Then the embedding shows that $\{u_l\}_{l=1}^{\infty}$ is also a fundamental sequence in $S'(\mathbf{R}^n)$ with the limit element $u \in S'(\mathbf{R}^n)$. On the other hand, for each fixed $j = 0, 1, 2, \ldots$, $\{\varphi_j(x, D_x)\, u_l\}_{l=1}^{\infty}$ is a fundamental sequence in $L_p(\mathbf{R}^n)$ with the limit element $u^j \in L_p(\mathbf{R}^n)$. Then by $u_l \to u$ in $S'(\mathbf{R}^n)$ we get $\varphi_j(x, D_x)\, u_l \to \varphi_j(x, D_x)\, u$ in $L_p(\mathbf{R}^n)$ if $l \to \infty$. Now it follows by standard arguments that u belongs to $B^{s,a}_{p,q}(\mathbf{R}^n)$ and that u_l converges in $B^{s,a}_{p,q}(\mathbf{R}^n)$ to u. Hence $B^{s,a}_{p,q}(\mathbf{R}^n)$ is completely.

Step 2: We prove now the second part of the theorem. Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ be fixed and $c^{\varphi} = 1$. For any natural M we write $\varphi^M(x, D_x) = \sum_{j=0}^{M} \varphi_j(x, D_x)$. Then in analogy with (18) and (19) we get by Theorem 1

$$\varphi_j(x, D_x)\, \varphi^M(x, D_x) = \begin{cases} \varphi_j(x, D_x) + R_{M,j}(x, D_x) & \text{if } j \leq M - 2N, \\ R_{M,j}(x, D_x) & \text{if } j \geq M + 2N, \\ \varphi_j(x, D_x) \sum\limits_{i=M-4N+2}^{M} \varphi_i(x, D_x) + R_{M,j}(x, D_x) & \text{if } M - 2N < j < M + 2N. \end{cases}$$

The remainder terms $R_{M,j}(x, D_x)$ always belong to $S^{-\infty}$ and the semi-norms of them can be estimated for each fixed $\varepsilon > 0$ and each real $\varkappa$, independently of j and M, by $|r_{M,j}|^{(\varkappa)}_{(l,k)} \leq c 2^{-(s+\varepsilon)j} 2^{-M}$. The constant c depends on $l, k, s, \varkappa$ and ε but not on j and M.

Step 3. Let $u \in B^{s,a}_{p,q}$, $q < \infty$ and $\varkappa < \min\,(sm', sm)$ fixed. Setting $u_M = \varphi^M(x, D_x)\, u$, we have in view of the previous step

$$\begin{aligned} \|u - u_M \mid B^{s,a}_{p,q}\|^q &= \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(x, D_x)\, u - \varphi_j(x, D_x)\, \varphi^M(x, D_x)\, u \mid L_p\|^q \\ &\leq c_{\varkappa} 2^{-Mq} \|u \mid H_p^{\varkappa}\|^q + c \sum_{j=M-4N+2}^{\infty} 2^{jsq} \|\varphi_j(x, D_x)\, u \mid L_p\|^q \\ &\leq c 2^{-Mq} \|u \mid B^{s,a}_{p,q}\|^q + c \sum_{j=M-4N+2}^{\infty} 2^{jsq} \|\varphi_j(x, D_x)\, u \mid L_p\|^q . \end{aligned}$$

The last estimate follows in view of Theorem 3.

It is now obvious that $u_M \to u$ in $B^{s,a}_{p,q}$ if $M \to \infty$.

Step 4. Let $\varkappa < \min\,(sm', sm)$ and $\varrho > \max\,(sm, sm')$. Then it is clear that $u \in B^{s,a}_{p,q}$ implies $u \in H_p^{\varkappa}$. The pseudodifferential operator $\varphi^M(x, D_x)$ belongs to $S^{-\infty}$ and therefore, by Corollary 1, u_M becomes an element of H_p^{ϱ}. $S(\mathbf{R}^n)$ is dense in $H_p^{\varrho}(\mathbf{R}^n)$. Hence there exists a sequence $u_{M,J} \in S(\mathbf{R}^n)$ such that $u_{M,J} \to u_M$ in H_p^{ϱ} if $J \to \infty$. Because we had fixed $\varrho > \max\,(sm, sm')$, now Theorem 3 ensures that the sequence $u_{M,J}$

converges in $B^{s,a}_{p,q}$ to u if $M, J \to \infty$. This proves the density of $S(\mathbf{R}^n)$ in $B^{s,a}_{p,q}(\mathbf{R}^n)$ if $q < \infty$ ∎

The equivalent quasi-norms in the following theorem contain also an a priori estimate for pseudodifferential operators $B(x, D_x)$ whose symbols are equivalent to an arbitrary fixed symbol $a(x, \xi)$ of the class $S(m, m'; \delta)$. This shows that the spaces $B^{s,a}_{p,q}(\mathbf{R}^n)$ will be useful in the study of degenerate partial differential equations or other suitable pseudodifferential operators belonging to the class $S(m, m'; \delta)$. We recall that the elements of $S(m, m'; \delta)$ are hypoelliptic. Hence we can always construct for $B(x, D_x) \in S(m, m'; \delta)$ parametrices $Q(x, D_x) \in S^{-m'}_{1,\delta}$ such that

$$B(x, D_x)\, Q(x, D_x) = I + R(x, D_x), \qquad Q(x, D_x)\, B(x, D_x) = I + R'(x, D_x)$$

and $R, R' \in S^{-\infty}$ holds — see [4; Section 2, § 5] or [8; Chapter III, § 3], [9; Chapter IV, § 1].

Theorem 6: *Let $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols belonging to $S(m, m'; \delta)$, and $Q(x, D_x)$ denotes a parametrix for $B(x, D_x)$. If $-\infty < s < \infty$, $1 < p < \infty$ and $0 < q \leq \infty$, then*

$$\|B(x, D_x)\, u \mid B^{s-1,a}_{p,q}\| + \|u \mid B^{s-1,a}_{p,q}\| \quad \text{and} \quad \|Q(x, D_x)\, u \mid B^{s+1,a}_{p,q}\| + \|u \mid B^{s-1,a}_{p,q}\|$$

are equivalent quasi-norms in $B^{s,a}_{p,q}(\mathbf{R}^n)$.

Proof: Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and without loss of generality we may assume that J in Definition 2 is fixed with respect to R_a and R. The constant R occurs by the definition of equivalence — see (10). Also we choose a second system $\{\psi_j\}_{j=0}^{\infty} \in \Phi^{N+1,a}$ where additionally holds $\psi_j(x, \xi) = 1$ on supp φ_j if $j = 1, 2, \ldots$. By the construction at the end of Section 2 it is easily seen that such a system always exists.

Step 1. Let $\varkappa < \min(0, sm)$ and $\varepsilon > 0$ be fixed. We note some estimates which will be useful in the next step. If $j = N + 1, N + 2, \ldots$, we have

$$\varphi_j(x, D_x)\, B(x, D_x) = \left(\sum_{|\alpha| < L} \frac{1}{\alpha!}\, \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x)\, \psi_j(x, D_x) + R_{1j}(x, D_x) + R_{2j}(x, D_x), \tag{27}$$

where

$$R_{1j}(x, D_x) = \varphi_j(x, D_x)\, B(x, D_x) - \left(\sum_{|\alpha| < L} \frac{1}{\alpha!}\, \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x)$$

and

$$R_{2j}(x, D_x) = \left(\sum_{|\alpha| < L} \frac{1}{\alpha!}\, \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x)\, \big(1 - \psi_j(x, D_x)\big).$$

Consequently $R_{1j}(x, D_x)$ denotes the remainder term in Theorem 1 which is obtained by the composition of $\varphi_j(x, D_x)$ and $B(x, D_x)$. Hence, if we choose L sufficiently large (in dependence on $\varkappa$, ε and δ), we get by (5) and (12) $|r_{1j}|^{(\varkappa)}_{(l,k)} \leq c2^{-j(s+\varepsilon)}$. Also the semi-norms of $R_{2j}(x, D_x)$ can be estimated in this way. We use additionally the assumption $\psi_j(x, \xi) = 1$ on supp φ_j and get $|r_{2j}|^{(\varkappa)}_{(l,k)} \leq c' 2^{-j(s+\varepsilon)}$. Finally (8) and the properties of the system $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ guarantee that the semi-norms of the first pseudodifferential operator on the right-hand side of (27) can be estimated by

$$|\varphi_j^{(\alpha)} b_{(\alpha)}|^{(0)}_{(l,k)} \leq c_{L\alpha} \sup_{(x,\xi) \in \Omega_j^{N,a}} \{|b(x, \xi)|\, \langle \xi \rangle^{-|\alpha|(1-\delta)}\}$$

and, in view of (10) and (11), by

$$\left| \sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)} \right|_{(l,k)}^{(0)} \leqq c'' 2^j. \tag{28}$$

In all three cases the constants c, c' and c'' are independent of j.

Step 2. Let $u \in B_{p,q}^{s,a}$ and $\varkappa < \min(0, sm)$. Because $\varphi_j \in S^{-\infty}$, it is straightforward to see that we have by Corollary 1 and Theorem 3

$$\|\varphi_j(x, D_x) B(x, D_x) u \mid L_p\| \leqq c_N \|u \mid B_{p,q}^{s,a}\|^{\{\psi_j\}}$$

if $j = 0, 1, \ldots, N$. Together with the results of the first step this yields:

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(s-1)q} \|\varphi_j(x, D_x) B(x, D_x) u \mid L_p\|^q \\
&\leqq c_N' \|u \mid B_{p,q}^{s,a}\|^{\{\psi_j\}q} \\
&\quad + \sum_{j=N+1}^{\infty} 2^{j(s-1)q} \left(\left\| \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)} \right) (x, D_x) \psi_j(x, D_x) u \mid L_p \right\| \right. \\
&\quad \left. + \|R_{1j}(x, D_x) u + R_{2j}(x, D_x) u \mid L_p\| \right)^q \\
&\leqq c_N' \|u \mid B_{p,q}^{s,a}\|^{\{\psi_j\}q} + c \sum_{j=N+1}^{\infty} 2^{jsq} \|\psi_j(x, D_x) u \mid L_p\|^q + c' \|u \mid H_p^{\varkappa}\|^q \\
&\leqq c'' \|u \mid B_{p,q}^{s,a}\|^{\{\psi_j\}q}.
\end{aligned}$$

We used again Theorem 2, and Corollary 1 and Theorem 3, respectively. Moreover by Theorem 4 the quasi-norms of u defined by $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ are equivalent. Hence we have proved in this step

$$\|B(x, D_x) u \mid B_{p,q}^{s-1,a}\| + \|u \mid B_{p,q}^{s-1,a}\| \leqq c \|u \mid B_{p,q}^{s,a}\|. \tag{29}$$

Step 3. To prove the converse inequality, we use that the symbol of a parametrix $Q(x, D_x)$ can be estimated for any multi-indices α and β, all $x \in \mathbf{R}_x^n$ and all $\xi \in \mathbf{R}_\xi^n$ with $|\xi| \geqq R_q$ by

$$|q_{(\beta)}^{(\alpha)}(x, \xi)| \leqq c_{q\alpha\beta} |b(x, \xi)|^{-1} \langle \xi \rangle^{-|\alpha|+|\beta|\delta}.$$

We choose $N^* \geqq N$ such that $(x, \xi) \in \Omega_{N^*}^{N,a}$ always implies $|\xi| \geqq R_q$. If $j = N^* + 1, N^* + 2, \ldots$, we have

$$\varphi_j(x, D_x) = \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)} \right) (x, D_x) \psi_j(x, D_x) B(x, D_x) + R_{3j}(x, D_x), \tag{30}$$

where

$$\begin{aligned}
R_{3j}(x, D_x) = &\left(\varphi_j(x, D_x) Q(x, D_x) - \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)} \right) (x, D_x) \right) B(x, D_x) \\
&+ \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)} \right) (x, D_x) \left(1 - \psi_j(x, D_x)\right) B(x, D_x) \\
&+ \varphi_j(x, D_x) \left(I - Q(x, D_x) B(x, D_x)\right).
\end{aligned}$$

Let $\varkappa < \min(0, sm)$ and $\varepsilon > 0$ be fixed. Then in complete analogy to the estimates of the first step we can choose L in such a way that $|r_{3j}|_{(l,k)}^{(\varkappa)} \leqq c2^{-j(s+\varepsilon)}$ and

$$\left| \sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)} \right|_{(l,k)}^{(0)} \leqq c' 2^{-j} \tag{31}$$

holds. The constants c and c' are again independent of j. Now straightforward computations as in the second step establish the inequalities

$$\|u \mid B_{p,q}^{s,a}\|^{\{\varphi_j\}} \leqq c \, \|B(x, D_x)\, u \mid B_{p,q}^{s-1,a}\|^{\{\psi_j\}} + c_{N^*} \|u \mid B_{p,q}^{s-1,a}\|^{\{\psi_j\}}$$

and; in view of Theorem 4

$$c' \, \|u \mid B_{p,q}^{s,a}\| \leqq \|B(x, D_x)\, u \mid B_{p,q}^{s-1,a}\| + \|u \mid B_{p,q}^{s-1,a}\|$$

with a constant $c' > 0$. Together with (29) this proves the first part of the theorem.

Step 4. The other case may be derived similarly. We change in (27) and (30) the roles of $B(x, D_x)$ and $Q(x, D_x)$. Thus we get

$$\varphi_j(x, D_x)\, Q(x, D_x) = \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)} \right) (x, D_x)\, \psi_j(x, D_x) + R_{4j}(x, D_x)$$

and

$$\varphi_j(x, D_x) = \left(\sum_{|\alpha|<L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)} \right) (x, D_x)\, \psi_j(x, D_x)\, Q(x, D_x) + R_{5j}(x, D_x),$$

where the semi-norms of the remainder terms may be estimated as in the foregoing step. Using (28) and (31) we get now the proof of the second part of the theorem in complete analogy to the second and third steps ∎

At the end of this section we will illustrate what variable smoothness or variable order of differentiation means. Let X be an open subset of $\mathbf{R}_x^n$. We weaken the condition of equivalence and call two symbols of the class $S(m, m'; \delta)$ *equivalent with respect to* X *if*

$$0 < c_1 \leqq |a(x, \xi)\, b^{-1}(x, \xi)| \leqq c_2 < \infty$$

holds for all $x \in X$ and $\xi \in \mathbf{R}_\xi^n$ with $|\xi| \geqq R$. For $\varepsilon > 0$ and $u \in S'(\mathbf{R}^n)$ we set

$$(\operatorname{supp} u)_\varepsilon = \{x : x = y + h \text{ with } y \in \operatorname{supp} u \text{ and } |h| \leqq \varepsilon\}.$$

Theorem 7: *Let X be a fixed open subset of $\mathbf{R}_x^n$ and $a(x, \xi)$, $b(x, \xi)$ be two symbols of $S(m, m'; \delta)$ which are equivalent with respect to X. If $-\infty < s < \infty$, $1 < p < \infty$, $0 < q \leqq \infty$ and $\varepsilon > 0$, then there exists positive constants c' and c such that*

$$c' \, \|u \mid B_{p,q}^{s,a}\| \leqq \|u \mid B_{p,q}^{s,b}\| \leqq c \, \|u \mid B_{p,q}^{s,a}\|$$

holds for all $u \in S'(\mathbf{R}^n)$ with $(\operatorname{supp} u)_\varepsilon \subset X$. The constants c' and c are independent of u.

Proof: In view of Theorem 4 we can fix two arbitrary systems $\{\varphi_j\}_{j=0}^\infty \in \Phi^{N,a}$ and $\{\psi_i\}_{i=0}^\infty \in \Phi^{M,b}$ belonging to $a(x, \xi)$ and $b(x, \xi)$, respectively. Let also $\varepsilon > 0$ be fixed. Then we can choose a smooth function χ_ε such that

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \notin X, \\ 1 & \text{if } x \in X \text{ and } \operatorname{dist}(x, \partial X) \geqq \varepsilon, \end{cases}$$

$$|D^\gamma \chi_\varepsilon(x)| \leqq c_\gamma \varepsilon^{-|\gamma|} \qquad \text{for all multi-indices } \gamma \text{ and } x \in \mathbf{R}_x^n$$

holds. By Theorem 1 we have

$$\varphi_j(x, D_x)\,\chi_\varepsilon(x) = \left(\sum_{|\alpha|<L} \frac{1}{\alpha!}\,\varphi_j^{(\alpha)}\chi_{\varepsilon(\alpha)}\right)(x, D_x) + R_{j\varepsilon}^L(x, D_x),$$

with

$$|r_{j\varepsilon}^L|_{(l,k)}^{(m_2-L(1-\delta))} \leqq c_{Llk}\,|\varphi_j|_{(l',k)}^{(m_2)}\,|\chi_\varepsilon|_{(l,k')}^{(0)}.$$

Now, in analogy to (23)—(25) and with the same meaning of the constants i_0 and j_0 we get for all multi-indices α

$$\operatorname{supp}(\varphi_j^{(\alpha)}\chi_{\varepsilon(\alpha)}) \cap \operatorname{supp}\psi_i = \emptyset \qquad \text{if } j \notin \mathcal{J}(i),$$

where again

$$\mathcal{J}(i) = \{j: 0 \leqq j \leqq \max(j_0, i + I - J + M + N + H)\} \qquad \text{if } 0 \leqq i \leqq i_0,$$

$$\mathcal{J}(i) = \{j: \max(J, i + I - (M + N + H)) \leqq j + J \leqq i + I + M + N + H)\} \qquad \text{if } i > i_0$$

and H fulfils $2^{-H} \leqq c_1 \leqq c_2 \leqq 2^H$. Then we obtain in the case $j \notin \mathcal{J}(i)$

$$\psi_i(x, D_x)\,\varphi_j(x, D_x)\,\chi_\varepsilon(x) = \psi_i(x, D_x)\left(\left(\sum_{|\alpha|<L} \frac{1}{\alpha!}\,\varphi_j^{(\alpha)}\chi_{\varepsilon(\alpha)}\right)(x, D_x) + R_{j\varepsilon}^L(x, D_x)\right) = R_{ij\varepsilon}(x, D_x),$$

where for each natural number L^* the semi-norms of $R_{ij\varepsilon}(x, D_x)$ can be estimated by

$$|r_{ij\varepsilon}|_{(l,k)}^{(m_1+m_2-L^*(1-\delta))} \leqq c_{L^*lk}\,|\psi_i|_{(l',k)}^{(m_1)}\,|\varphi_j|_{(l'',k')}^{(m_2)}\,|\chi_\varepsilon|_{(l,k'')}^{(0)}.$$

Suppose that $(\operatorname{supp} u)_\varepsilon \subset X$. Then we have always $\chi_\varepsilon u = u$. Now it is not hard to see that the rest of the proof is a simple modification of the proof of Theorem 4 and we omit it ∎

Remark: Let $a(x, \xi) = \langle\xi\rangle^{m'} + \varrho^{2k}(x)\langle\xi\rangle^m$ be a symbol of $S(m, m'; \delta)$ as described in the fourth example in Section 2. Furthermore let x_0 be a interior point of the set $\Omega = \{x: \varrho(x) = 0\}$ and $K_\sigma(x_0) = \{x: |x - x_0| < \sigma\} \subset \Omega$ denotes a neighbourhood of x_0. Then $a(x, \xi)$ is equivalent to $\langle\xi\rangle^{m'}$ with respect to $K_\sigma(x_0)$. Hence an element u of $B_{p,q}^{s,a}$ belongs locally in x_0 to the classical Besov spaces $B_{p,q}^{sm'}$. On the other hand, if $\varrho(x_1) \neq 0$ holds, then the symbol $a(x, \xi)$ is equivalent to $\langle\xi\rangle^m$ with respect to a suitable neighbourhood of x_1. Thus $u \in B_{p,q}^{s,a}$ belongs locally in x_1 to the classical Besov space $B_{p,q}^{sm}$. Consequently for an element u of the space $B_{p,q}^{s,a}$ we may have in different points of $\mathbf{R}_x^n$ locally different smoothness properties.

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VERFASSER:

Dr. HANS-GERD LEOPOLD
Sektion Mathematik der Friedrich-Schiller-Universität
Universitätshochhaus
DDR-6900 Jena