On Besov Spaces of Variable Order of Differentiation

H.-G. LEOPOLD

Es werden Funktionenräume mit variabler Glattheit betrachtet, die auf dem n-dimensionalen. euklidischen Raum definiert sind. Die Definition dieser Räume erfolgt mit Hilfe einer geeigneten Klasse von Pseudodifferentialoperatoren.

Рассматриваются пространства функций переменной гладкости, определенных на п-мерном эвклидовом пространстве. Эти пространства определены с помощью подходящего класса псевдодифференциальных операторов.

This paper is concerned with function spaces of variable order of differentiation defined on the n-dimensional Euclidean space. The definition of these spaces is closely connected with an appropriate class of pseudodifferential operators.

The paper deals with Besov spaces of variable order of differentiation defined on the Euclidean *n*-space \mathbb{R}^n . In the classical function spaces of Besov type $B_{p,q}^s(\mathbb{R}^n)$ norms can be defined via a resolution of unity in the Fourier image of the function u , which is connected with the symbol $|\xi|^2$ of the Laplacian $-\Delta$. We consider now in this paper decompositions of $\mathbb{R}_r^n \times \mathbb{R}_i^n$ which are induced by symbols $a(x, \xi)$ of appropriate pseudodifferential operators. This means that we may have different resolutions $\{\varphi_i(x,\xi)\}_{i=0}^{\infty}$ of \mathbf{R}_{ξ} ^{*n*} for different $x \in \mathbf{R}_{x}$ *ⁿ*. So we can get locally in different points x different smoothness demands on the function $u(x)$. The function spaces $B_{p,q}^{s,q}(\mathbf{R}^n)$ defined in this way seem to be useful in the study of degenerate elliptic partial differential equations. In Section 1 we recall some facts about pseudodifferential operators and collect those results which will be needed in the sequel. Section 2 contains the definition of an appropriate subleass $S(m, m'; \delta)$ of hypoelliptic symbols, some examples and the definition of the resolution of unity of $\mathbb{R}_x^n \times \mathbb{R}_i^n$ connected with these symbols. In Section 3 we define the function spaces $B_{p,q}^{s,a}(\mathbf{R}^n)$ of variable order of differentiation and describe properties of these spaces.

1. Basic properties of pseudodifferential operators

Let $p(x, \xi)$ be a polynomially bounded complex-valued function defined on $\mathbb{R}_x^n \times \mathbb{R}_i^n$. The pseudodifferential operator $P(x, D_x)$ with symbol $p(x, \xi)$ is defined by

$$
P(x, D_x) u(x) = \frac{1}{(2\pi)^n} \int e^{ixt} p(x, \xi) (Fu) (\xi) d\xi \quad \text{for } u \in S(\mathbf{R}^n),
$$

where $S(\mathbf{R}^n)$ denotes the Schwartz class and $(Fu)(\xi) = \int e^{-i\psi} u(y) dy$ denotes the Fourier transform of u. A function $p(x, \xi)$ belongs to the class $S_{\varrho, \delta}^{m}$ ($-\infty < m < \infty$; $0 \leq \delta \leq \varrho \leq 1, \delta < 1$) if for any multi-indices α , β there exist a constant $c_{\alpha\beta}$ such that

> $|p_{\left(\beta\right)}^{(\alpha)}(x,\xi)|\leq c_{\alpha\beta}\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}$ for $(x, \xi) \in \mathbb{R}_{x}^{n} \times \mathbb{R}_{\xi}^{n}$,

where $p_{(\beta)}^{(a)}(x,\xi) = \partial_{\xi}^a D_x^{\beta} p(x,\xi)$, $\partial_{\xi}^a = \partial^{|\alpha|} \partial_{\xi_1}^{a_1} \dots \partial_{\xi_n}^{a_n}$, $D_x^{\beta} = (-1)^{|\beta|} \partial_x^{\beta}$ and $\langle \xi \rangle$
= $(1 + |\xi|^2)^{1/2}$. We set $S^{-\infty} = \bigcap_{m} S_{\varrho,\delta}^m$. It is easy to see that $\bigcap_{m} S_{\varrho,\delta}^m$ any ρ and δ . The pseudodifferential operator $P(x, D_x)$ with a symbol $p \in S_{\rho,\delta}^{m'}$ maps $S(\mathbb{R}^n)$ continuously into itself and can be extended to a continuous operator from $S'(\mathbf{R}^n)$ into $S'(\mathbf{R}^n)$, the space of all tempered distributions on \mathbf{R}^n . The mapping between $p(x, \xi)$ and $P(x, D_x)$ is a bijection. For $p \in S_{a,b}^m$ we define the semi-norms $|p|_{(l,k)}^{(m)}$ by

$$
|p|_{(l,k)}^{(m)} = \max_{|\alpha| \leq l, |\beta| \leq k} \sup_{(\mathbf{x},\xi)} \left\{ |p_{(\beta)}^{(\alpha)}(x,\xi)| \left\langle \xi \right\rangle^{-m+\varrho|\alpha|-\delta|\beta|} \right\}.
$$
 (1)

Theorem 1: Assume that $0 \leq \delta < \varrho \leq 1$. Let $P_1(x, D_x) \in S_{\varrho, \delta}^{m_1}$ and $P_2(x, D_x) \in S_{\varrho, \delta}^{m_1}$. Then $P(x, D_x) = P_1(x, D_x) P_2(x, D_x)$ belongs to $S_{p, \delta}^{m_1 + m_2}$. For the symbol $p(x, \xi)$ of $P(x, D_x)$ and for any natural N we have the expansion formula

$$
p(x,\xi) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} p_1^{(\alpha)}(x,\xi) p_{2(\alpha)}(x,\xi) + N \sum_{|\gamma| = N} \int \frac{(1-\vartheta)^{N-1}}{\gamma!} r_{\gamma,\theta}(x,\xi) d\vartheta, \qquad (2)
$$

where (Os *oscillatory integral*)

$$
r_{\gamma,\theta}(x,\xi) = \text{Os-}\frac{1}{(2\pi)^n} \int e^{-i\,\nu\eta} p_1^{(\gamma)}(x,\xi+\vartheta\eta) \, p_{2(\gamma)}(x+y,\xi) \, dy \, d\eta \,. \tag{3}
$$

 $\{r_{\nu,\theta}\}_{|\theta|\leq 1}$ is a bounded set of $S_{\varrho,\theta}^{m_1+m_2-|\nu|(\varrho-\delta)}$. Furthermore, for any pair of integers (l,k) there exist constants c, c' and integers l' , k' independent of ϑ such that.

$$
|p_1^{(a)}p_{2(a)}|^{(m_1+m_2-|a|(p-\delta))}_{(l,k)}\leq c\,|p_1|^{(m_1)}_{(l+|a|,k)}\,|p_2|^{(m_1)}_{(l,k+|a|)}\tag{4},
$$

 \emph{and}

$$
|r_{\gamma,\theta}|_{\{l,k\}}^{(m_1+m_1-|\gamma|(\varrho-\delta))} \leq c' \, |p_1|_{\{l',k\}}^{(m_1)} \, |p_2|_{\{l,k'\}}^{(m_2)}.
$$
 (5)

 \sim The theorem gives an estimate of each term of the sum (2) which is obtained by the composition of two pseudodifferential operators. Especially the estimate of the remainder term will be often useful. The proof is a direct consequence of the definition of semi-norms and of [4; Section 2], see also there for details.

Theorem 2: Let $P(x, D_x) \in S_{1,\delta}^0$ and $\delta < 1$. Then for all p with $1 < p < \infty$ there exist integers l, k and a constant c, all independent of $P(x, D_x)$, such that

$$
||P(x, D_x) u || L_p|| \leq c ||p||_{(l,k)}^{(0)} ||u || L_p|| \quad \text{for all } u \in L_p(\mathbf{R}^n).
$$
 (6)

This was proved first by ILLNER [3] in 1975. Later for example BOURDAUD [2] and NAGASE [7] considered non-regular symbols and got weaker conditions on $p(x, \xi)$. But the result is, sharp with respect to the parameter p. There exist smooth functions $m(\xi) \in S_{1,0}^0$ which are not Fourier multipliers in $L_1(\mathbf{R}^n)$ and $L_\infty(\mathbf{R}^n)$ [14; p. 21]. Consequently for the corresponding pseudodifferential operators (6) is not true in the case $p = 1$ and $p = \infty$.

Corollary 1: Let $P(x, D_x) \in S_{1,\delta}^m$, $\delta < 1$, $1 < p < \infty$ and $-\infty < t, m < \infty$. Then there exist integers l , k and a constant c such that

$$
||P(x, D_x) u || H_p' || \leq c ||p|_{(l,k)}^{(m)} ||u || H_p^{(l+m)} || \qquad \text{for } u \in H_p^{(l+m)}(\mathbb{R}^n). \tag{7}
$$

Again the constants are independent of $P(x, D_x)$ and u. $H_p(\mathbf{R}^n)$ denotes the Besselpotential spaces.

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2. Coverings of $\mathbf{R}_{x} \times \mathbf{R}_{\xi}$ ⁿ induced by symbols of pseudodifferential operators

In the following we consider a subclass of the hypoelliptie symbols of slowly varying strength.

Definition 1: Let $0 \le \delta < 1$ and $0 < m' \le m$. A symbol $a(x, \xi) \in S^{m}_{1,\delta}$ belongs to the class $S(m, m'; \delta)$ if there exists a constant $R_a \geq 0$ such that holds: (i) for any multi-indices $\alpha, \beta, \alpha \in \mathbb{N}$ for any multi-indices $\alpha, \beta, \alpha \in \mathbb{N}$ and $\alpha < m' \leq m$. A symbol $a(x, t)$ for any multi-indices $\alpha, \beta, \alpha \in \mathbb{R}_x$ ⁿ and all $\xi \in \mathbb{R}_t$ ⁿ with $|\xi| \geq \frac{|\alpha(x)|\alpha - \xi|}{\alpha -$ 2. Coverings of $\mathbf{R}_{x} \times \mathbf{R}_{\xi} \cdot \mathbf{n}$ induced by symbols of pseudod

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Definition 1: Let $0 \le \delta < 1$ and $0 < m' \le m$. A symbol $a(x, \xi) \in S_1^m$, belongs

to the class $S(m, m'; \delta)$ if there exists a con

(i) for any multi-indices α , β , all $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$ with $|\xi| \geq R_o$ there'holds

$$
|a_{(\beta)}^{(\alpha)}(x,\xi)| \leq c_{\alpha\beta} |a(x,\xi)| \langle \xi \rangle^{-|\alpha|+|\beta|\delta}; \qquad (8)
$$

(ii) there exist constants $c_m > 0$ and $c_m > 0$ such that for all $x \in \mathbb{R}_z^n$ and all $\xi \in \mathbb{R}_t^n$ with $|\xi| \geq R_o$ there holds

$$
c_m \langle \xi \rangle^{m'} \leq |a(x,\xi)| \leq c_m \langle \xi \rangle^{m}.
$$
 (9)

consider a subclass of the hypoelliptic symbols of slowly varying
 $\det 0 \le \delta < 1$ and $0 < m' \le m$. A symbol $a(x, \xi) \in S_{1,\delta}^m$ belongs
 $\vdots \delta$ if there exists a constant $R_a \ge 0$ such that holds:

radices $\alpha, \beta, \text{all } x \in \mathbb$ The symbols of the class $S(m, m'; \delta)$ will be a substitute for the symbol $|\xi|^2$ of the Laplacian which is used in the definition of the usual Besov spaces. Therefore the restrictions $m' > 0$ and $\varrho = 1$ turn out to be natural in view of the following definitions and Theorem 2. *oms* $m' > 0$ and $\varrho = 1$ turn out to be
d Theorem 2.
ymbols $a(x, \xi)$ and $b(x, \xi)$ belonging
st constants c' , c and *R* with
 $0 < c' \leq |a(x, \xi) b^{-1}(x, \xi)| \leq c < \infty$
 $\in \mathbb{R}$." and all $\xi \in \mathbb{R}$." with $|\xi| > R$. 2. Coverings of $\mathbb{R}_2 \times \mathbb{R}_t^m$ induced by symbols of pseudodifferential operation in the following we consider a subclass of the hypoelliptic symbols of slow strength.

In the following we consider a subclass of th

Two symbols $a(x, \xi)$ and $b(x, \xi)$ belonging to $S(m, m'; \delta)$ are called *equivalent* if there exist constants c' , c and R with

$$
0
$$

for all $x \in \mathbb{R}_z^n$ and all $\xi \in \mathbb{R}_z^n$ with $|\xi| \geq R$.
Let us give now some simple examples:

 $\ddot{}$

1., The trivial example is the symbol $a(x, \xi) = \langle \xi \rangle$ of the Bessel-potential operator for all $x \in \mathbb{R}_z^n$ and all $\xi \in \mathbb{R}_z^n$ with $|\xi| \geq R$.

Let us give now some simple examples:

1. The trivial example is the symbol $a(x, \xi)$
 $(I - \Delta)^{1/2}$. This symbol belongs to $S(1, 1; 0)$.

2. Let $a(x) = s + w(x)$ be $(I - \Delta)^{1/2}$. This symbol belongs to $S(1, 1, 0)$.
2. Let $\sigma(x) = s + \psi(x)$ be a real-valued function, s be a constant and ψ be an ele-

ment of $S(\mathbb{R}^n)$. Let $m' = \inf \sigma(x)$, $m = \sup \sigma(x)$ and $0 < m'$. Then $a(x, \xi) = \langle \xi \rangle^{\sigma(x)}$
belongs to $S(m, m'; \delta)$ for any δ with $0 < \delta < 1$. Such symbols and related function **belongs** to $S(m, m'; \delta)$ for any δ with $0 < \delta < 1$. Such symbols and related function spaces were considered by UNTERBERGER and BOKOBZA [14], VISIK and ESKIN [16, 17] and **BEAUZAMY** [1].

3. Let $\sigma(x) = s + \psi(x)$ be a function as in the previous example and t be an arbitrary- real number. Then $a(x, \xi) = \langle \xi \rangle^{\sigma(x)} (1 + \log \langle \xi \rangle^2)^{t/2}$ belongs to $S(m, m'; \delta)$ with $0 < \delta < 1$, $0 < m' < \inf \sigma(x)$ and $m > \sup \sigma(x)$. Symbols of this type were **considered bYUNTERBERGER** and **BOROBZA [15]** and **UNTERBRGER [13].** 3. Let $\sigma(x) = s + \psi(x)$ be a function as in the previous example and *t* be an arbitrary real number. Then $a(x, \xi) = \langle \xi \rangle^{a(x)} (1 + \log \langle \xi \rangle^{2})^{l/2}$ belongs to $S(m, m'; \delta)$ with $0 < \delta < 1$, $0 < m' < \inf \sigma(x)$ and $m > \sup \sigma(x)$. Symbols of

(niii) α may be zero on a domain $\Omega \subset \mathbb{R}_r^n$. Let $0 < m' \leq m$ and k be a natural number with $(m - m') < 2k$. Then the symbol $a(x, \xi) = \langle \xi \rangle^{m'} + \varrho^{2k}(x) \langle \xi \rangle^m$ belongs to $S(m, m'; \delta)$ where $\delta = (m - m')/2k$. If m' and where $\delta = (m - m')/2k$. If m' and m are even numbers, then $a(x, \xi)$ is the symbol of a degenerate partial differential equation. 3 Let $\sigma(x) = s + \psi(x)$ of a tunction as in the previous example and t be an arbitrary real number. Then $a(x, \xi) = \langle \xi \rangle^{a(x)}(1 + \log \langle \xi \rangle)^{1/2}$ belongs to $S(m, m'; \delta)$ with $0 < \delta < 1$, $0 < m' < \inf \sigma(x)$ and $m > \sup \sigma(x)$. Symbols of this

For each symbol $a(x, \xi) \in S(m, m'; \delta)$ we can define variable coverings of $R_x^n \times R_{\xi}^n$. Variable covering means that in different points $x \in \mathbb{R}_z^n$ we may have different coverings of \mathbb{R}_i^n .

The symbol $a(x, \xi)$ induces a variable covering $\{Q_i^{N,a}\}_{i=0}^{\infty}$ of $\mathbb{R}_x^n \times \mathbb{R}_x^n$ by

For each symbol
$$
a(x, \xi) \in S(m, m'; \delta)
$$
 we can define variable coverings of $R_x^n \times \mathbb{R}_i^n$. Variable covering means that in different points $x \in \mathbb{R}_x^n$ we may have different coverings of \mathbb{R}_i^n . Definition 2: Let N be an integer and $a(x, \xi)$ a symbol belonging to $S(m, m'; \delta)$. The symbol $a(x, \xi)$ induces a variable covering $\{Q_j^{N,a}\}_{j=0}^\infty$ of $\mathbb{R}_x^n \times \mathbb{R}_i^n$ by $Q_j^{N,a} = \{(x, \xi) : |a(x, \xi)| < 2^{J+N+j}\}$ if $j = 0, 1, ..., N$, $\tilde{Q_j}^{N,a} = \{(x, \xi) : 2^{J-N+j} < |a(x, \xi)| < 2^{J+N+j}\}$ if $j = N + 1, N + 2, \ldots$ (11) J is a constant which is fixed in such a way that $|\xi| \leq R_a$ always implies $(x, \xi) \in \Omega_0^{1.a}$.

 (10)

 $\label{eq:2} \frac{1}{2}\int_{0}^{\infty}\frac{dx}{\sqrt{2\pi}}\,dx$

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In the case $a(x, \xi) = \langle \xi \rangle$ we get the usual classical dyadic coverings of \mathbf{R}_{ξ}^{n} , independent of x,

which are the basic for the definition of the spaces $B_{p,q}^{s}(\mathbf{R}^{n})$ and $F_{p,q}^{s}(\mathbf{R$ In the case $a(x, \xi) = \langle \xi \rangle$ we ge
which are the basic for the defin
 $= \langle \xi \rangle^{\sigma(x)}$ or $a(x, \xi) = \langle \xi \rangle^{m'} + \varrho^{2k}$
 $x \in \mathbb{R}_{x}^{n}$ we have a dyadic cover (x) $\langle \xi \rangle^m$ the coverings ${\langle \Omega, N, a \rangle_{i=0}^{\infty}}$ are variable. For each fixed $x \in \mathbf{R}_x$ " we have a dyadic covering of \mathbf{R}_t ", but in general these coverings are different from each other. The $\Omega_i^{N,a}$ are open sets and bounded in ξ . For any number j_0 at most $4N-1$ sets $Q_i^{N,a}$ have a non-empty intersection with $Q_i^{N,a}$. $\mathbf{A} \in \mathbf{R}_{\mathbf{z}}^{\mathbf{w}}$ we have a tystudie exceed other. The $\Omega_j^{N,a}$ are op
 $\Omega_j^{N,a}$ have a non-empty interposition 3: Let $\{\Omega_j\}^{\infty}_{\geq 0}$
 \mathbf{A} function system $\{\varphi_j\}_{j=0}^{\infty}$

(i) $\mathbf{a}_i(\mathbf{x}_i, \mathbf$ $\frac{\partial}{\partial s} B_{p,q}^n(\mathbf{R}^n)$ and $F_{p,q}^i(\mathbf{R}^n)$
 $\log S_{p,q}^i(\mathbf{R}^n)$ and $F_{p,q}^i(\mathbf{R}^n)$
 $\log \frac{1}{2}$ are van
 $\log \frac{1}{2}$ and \log LD

(ξ) we get the usual classical dyadic coverings of \mathbb{R}_t^n , independent of x ,

the definition of the spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$. In the case $a(x, \xi)$
 ξ)^{*m*} + $e^{2k}(x)$ $(\xi)^m$ the co

Definition 3: Let $\{Q_i^{N,a}\}_{i=0}^{\infty}$ be a variable covering induced by the symbol $a(x, \xi)$. belongs to $\Phi^{N,a}$ if for all $j = 0, 1, 2, ...$ holds:

-
- *(ii) supp'991*
- function system $\{q(i) \varphi_i(x, \xi) \in C^{\infty}(\mathbf{R})\}$

(ii) $\text{supp}\varphi_j \subset \Omega_j^N$

(iii) $|\varphi_{j(\beta)}^{(a)}(x, \xi)| \leq c_{\alpha\beta}$ are indepe (iii) $|\varphi_{j(\beta)}^{(\alpha)}(x,\xi)| \leq c_{\alpha\beta}\langle \xi \rangle^{-|\alpha|+|\beta|\delta}$ for any multi-indices α and β , where the constants $c_{\alpha\beta}$ are independent of *j*; A function system $\{\varphi_j\}_{j=0}^{\infty}$ belongs to $\Phi^{N,a}$ if for all $j = 0, 1, 2, ...$ holds:

(i) $\varphi_j(x, \xi) \in C^{\infty}(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$ and $\varphi_j(x, \xi) \geq 0$;

(ii) $\sup_{\{\varphi_j^{\{a\}}\}(x, \xi)\} \leq c_a \chi(\xi)^{-|a|+|\beta|b}$ for any multi

$$
(\text{iv})\sum_{j=0}^{\infty}\varphi_j(x,\xi)=c^{\varphi}>0.
$$

 $c_{\alpha\beta}$ are independent of j;

(iv) $\sum_{j=0}^{\infty} \varphi_j(x,\xi) = c^{\varphi} > 0$.

By assumptions (ii) and (iii) we get $\varphi_j \in S^{-\infty}$. The following estimates for the

mi-norms of φ_j are a simple consequence of (ii), (iii), (9

$$
|\hat{a}_j|_{\beta}(\hat{x}, \xi)| \leq c_{\alpha\beta} \langle \xi \rangle^{-|\alpha| + |\beta| \delta} \text{ for any multi-indices } \alpha \text{ and } \beta, \text{ where the constants}
$$
\n
$$
\varphi_i(x, \xi) = c^{\varphi} > 0.
$$
\n
$$
\varphi_j(x, \xi) = c^{\varphi} > 0.
$$
\nsumptions (ii) and (iii) we get

\n
$$
\varphi_j \in S^{-\infty}.
$$
\nThe following estimates for the ms of

\n
$$
\varphi_j
$$
\nare a simple consequence of (ii), (iii), (9) and (11):

\n
$$
|\varphi_j|_{\alpha, k}^{(\alpha)} \leq c_{l k \alpha} \begin{cases}\n2^{-j \kappa/m} & \text{if } \alpha \geq 0, \\
2^{-j \kappa/m'} & \text{if } \alpha < 0.\n\end{cases}
$$
\n(12)

\nall numbers

\nand with constants

\n
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c_{l k \alpha}
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\nindependent of

\nj. Also by (iii), respectively.

l and the a non-empty intersection with Ω_j^{α} .

Definition 3: Let $\{\Omega_j^{N,a}\}_{j=0}^{\infty}$ be a variable coverfunction system $\{\varphi_j\}_{j=0}^{\infty}$ belongs to $\Phi^{N,a}$ if for all

(i) $\varphi_j(x,\xi) \in C^{\infty}(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$ for all real numbers x and with constants c_{ikx} independent of *j*. Also by (iii), respectively (12), it follows that the semi-norms of the φ_i are uniformly bounded in $S_{1,i}^0$. Together with (ii) and (iv) we get in this way that $\sum_{i=1}^{J} \varphi_i(x, \xi) \to c^{\varphi}$ in $S_{1,\delta}^0$ weakly if The following estima
i), (9) and (11):

endent of *j*. Also by (
j are uniformly bound
 $\sum_{j=0}^{J} \varphi_j(x, \xi) \rightarrow c^{\varphi}$ in S_1^{φ}
imply that for every i
 $\rightarrow \infty$ $J \rightarrow \infty$. The weak convergence in $S^0_{1, \delta}$ and Corollary 1 imply that for every $v \in H_p^s(\mathbb{R}^n)$ *Photons (ii)* and (iii) we get $\varphi_j \in S$ ∞ . The following estimates for the of φ_j are a simple consequence of (ii), (iii), (9) and (11):
 $\begin{cases} \n\frac{1}{2} \leq c_{lkx} \left\{ \frac{2^{-jx/m}}{2^{-jx/m'}} \right. & \text{if } x \leq 0, \\
\frac{2^{-jx/m'}}{2^{-jx$

$$
\sum_{j=0}^{J} \varphi_j(x, D_x) v \to c^{\varphi} v \qquad \text{in } H_p^{\varphi}(\mathbb{R}^n) \qquad \text{if } J \to \infty \tag{13}
$$

holds – see also,[4; Chapter 3, § 7] where this fact was proved for $L_2(\mathbb{R}^n)$. But in view of Corollary 1 there are no difficulties to carry over the proof to the case $1 < p$ $< \infty$ and the Bessel-potential spaces for arbitrary real *s*.

It is easy to describe examples of function systems of the above type. Let $\{Q_i^{N,q}\}_{i=0}^\infty$ be a variable covering induced by $a(x, \xi) \in S(m, m'; \delta)$ and *J* be the fixed number from Definition

2. Furthermore let $\varphi \in C^{\infty}(\mathbb{R}_{+}^{n})$ be a real-valued function with $0 \leq \varphi(t) \leq 1$, $\varphi(t) = 1$ if
 $0 \leq t \leq 2^{J-1}$ an holds — see also \mathbf{A}^1 ; Chapter 3, § 7] where this fact was proved for $L_2(\mathbf{R}^n)$. But in view of Corollary 1 there are no difficulties to carry over the proof to the case $1 < p < \infty$ and the Bessel-potential space $\sum_{j=0} \varphi_j(x, D_x) v \to c^{\varphi} v$ in $H_p^s(\mathbf{R}^n)$
holds — see also,[4; Chapter 3, § 7] where th
view of Corollary 1 there are no difficulties to $<\infty$ and the Bessel-potential spaces for arbit:
It is easy to describe examp escribe examples of function systems of the above type. Let
induced by $a(x, \xi) \in S(m, m'; \delta)$ and *J* be the fixed number i
let $\varphi \in C^{\infty}(\mathbb{R}_{+}^{1})$ be a real-valued function with $0 \leq \varphi(t) \leq$
 $|\sup p \varphi \subset \{t : 0 \leq t \leq 2^{J$ $\sum_{j=0} \varphi_j(x, D_x) v \to c^{\varphi} v$ in $H_p^s(\mathbf{R}^n)$
holds — see also,[4; Chapter 3, § 7] where
view of Corollary 1 there are no difficulties
 $<\infty$ and the Bessel-potential spaces for an
It is easy to describe examples of fun

see also
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[4]
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: Chapter 3, § 7] where this fact was proved for Corollary 1 there are no difficulties to carry over the proof to d the Bessel-potential spaces for arbitrary real s. (say to describe examples of function systems of the above type. Toovering induced by $a(x,\xi) \in S(m, m'; \delta)$ and J be the fixed number more let $\varphi \in C^{\infty}(\mathbb{R}_{+})$ be a real-valued function with $0 \leq \varphi(t)^{2J-1}$ and $\text{supp }\varphi \subset \{t: 0 \leq t \leq 2^{J}\}$. Setting $\varphi_j(x,\xi) = \varphi(2^{-j-N} |a(x,\xi)|) - \varphi(2^{-j+N-1} |a(x,\xi)|)$ if $j = 1, 2$, $\varphi_0(x,\xi) = \sum_{k=1}^{2N-1} \varphi(2^{-k+N-1} |a(x,\xi)|)$, have $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ with $c^{\varphi} = 2N - 1$.

and

3. The spaces $B_{p,q}^{s,a}$ of variable order of differentiation

We are now ready to define Besov spaces of variable order of differentiation. Instead of the classical resolution of \mathbf{R}_{ξ} ⁿ which is connected with the symbols $|\xi|^2$ respectively $\langle \xi \rangle$ we use now function systems $\{ \varphi_i(x, \xi) \}_{i=0}^{\infty} \in \Phi^{N,a}$ connected with the symbol $a(x, \xi)$, which may lead to different resolutions of \mathbf{R}_{ξ} ⁿ for different fixed $x \in \mathbb{R}_r^n$. Throughout this section $a(x, \xi)$ is a fixed element of $S(m, m'; \delta)$.

Definition 4: Let $1 < p < \infty$, $0 < q \le \infty$, $-\infty < s < \infty$ and $\{\varphi_j(x,\xi)\}_{j=0}^{\infty}$ be a system belonging to $\Phi^{N,a}$. Then Definition 4: Let $1 < p < c$
a system belonging to $\Phi^{N.a}$. Then
 $B^{s.a}_{p,q}(\mathbf{R}^n) = \{u: u \in S'(\mathbf{R}^n)\}$

On Besov Spaces 73
\nDefinition 4: Let
$$
1 < p < \infty
$$
, $0 < q \le \infty$, $-\infty < s < \infty$ and $[\varphi_j(x, \xi)]_{j=0}^{\infty}$ be
\nsystem belonging to $\Phi^{N,a}$. Then
\n
$$
B_{p,q}^{s,a}(\mathbf{R}^n) = \{u: u \in S'(\mathbf{R}^n) \text{ and } ||u| B_{p,q}^{s,a}||^{(p_j)} < \infty \},
$$
\n
$$
||u| B_{p,q}^{s,a}||^{(p_j)} = \left(\sum_{j=0}^{\infty} 2^{jaq} ||\varphi_j(x, D_x) u |L_p||^2\right)^{1/q} \text{ if } q < \infty,
$$
\n
$$
||u| B_{p,\infty}^{s,a}||^{(p_j)} = \sup_j 2^{js} ||\varphi_j(x, D_x) u |L_p||.
$$
\nOf course the norms $||u| B_{p,q}^{s,a}||^{(p_j)}$ depend on the chosen system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$. But

Of course the norms $||u|| B^{s,a}_{p,q}||^{(p_j)}$ depend on the chosen system $\{\varphi_i\}_{i=0}^{\infty} \in \Phi^{N,a}$. But this is not the case for the spaces $B_{p,q}^{s,a}(\mathbb{R}^n)$ itself. This will be proved in Theorem 4. But a first we prove the embedding of the spaces $B_{p,q}^{s,a}(\mathbb{R}^n)$ in the scale of the usual Bessel-potential spaces. In this theorem $B_{p,q}^{s,a}(\mathbf{F})$ denotes the function spaces which are defined by (14) and an arbitrary fixed system $\{\varphi_i\}_{i=0}^{\infty}$. Of course the norms $||u||B_{p,q}^{s,q}||^{(\phi_j)}$ depend on the chosen system $\{\varphi_j\}_{p=0}^{\infty} \in \Phi^{N,q}$. But
is is not the case for the spaces $B_{p,q}^{s,q}(\mathbf{R}^n)$ itself. This will be proved in Theorem 4.
it a first we prove the $\begin{aligned}\n\left\|\left\{\mathbf{B}_{\mu}^{(p)}\right\|=\sup_{j} 2^{j} s \cdot \|\varphi_{j}(\mathbf{x}, D_{x}) u \mid L_{p}\|.\right\}.\n\end{aligned}$

forms $\left\|u \mid B_{p,q}^{s,q}\right\|_{\mathcal{F}_{\mu}}\left\{\text{depend on the chosen system } \{\varphi_{j}\}_{j=0}^{\infty} \in \Phi^{N,q}.\n\text{ But } \mathbf{B}_{\mu} \in \mathbb{R} \text{ for the spaces } B_{p,q}^{s,q}(\mathbf{R}^{n}) \text{ itself. This will be proved in Theorem 4. } \n$ this is not the case for the spaces $B_{p,q}^{s,q}(\mathbf{R}^{n})$ itself. This will be proved in T.
 But a first we prove the embedding of the spaces $B_{p,q}^{s,q}(\mathbf{R}^{n})$ in the scale of
 Bessel-potential spaces. In this theor So we have the embedding of the spaces $B_{p,q}^{p,q}(\mathbf{R}^n)$ in the scale of the usual
tential spaces. In this theorem $B_{p,q}^{p,q}(\mathbf{R}^n)(\mathbf{R}^n)$ denotes the function spaces which
ed by (14) and an arbitrary fixed syste

Theorem 3: Let $\{\varphi_i\}_{j=0}^{\infty}$ be a fixed system belonging to $\Phi^{N,a}$ and $1 < p < \infty$, $0 < q$
 $\leq \infty$, $-\infty < s < \infty$.

(i) For $s \geq 0$ *we have*

$$
H_p^o(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(p)}(\mathbf{R}^n) \hookrightarrow H_p^{\times}(\mathbf{R}^n)
$$
 (15)

(ii) For $s \leq 0$ *we have*

$$
H_p^{e}(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(\varphi)}(\mathbf{R}^n) \hookrightarrow H_p^{*}(\mathbf{R}^n)
$$
\n(16)

$$
if x < sm \text{ and } \varrho > sm'.
$$

Proof: *Step* 1. We get by the monotonicity of the l_q -spaces and by a simple calculation the first elementary embedding (ii) $For s \leq 0$ we h
 $H_p^o(\mathbf{R}^n) \hookrightarrow$
 $if \times \leq sm \text{ and } \varrho > s$

Proof: *Step* 1. We

lation the first elements
 $B_{p,\infty}^{s+\epsilon, a} (p) \hookrightarrow$

if $1 \leq p \leq \infty, -\infty$

$$
r \leq \geq 0 \text{ we have}
$$

\n
$$
H_p^{e}(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(p)}(\mathbf{R}^n) \hookrightarrow H_p^{s}(\mathbf{R}^n)
$$
\n
$$
m' \text{ and } \rho > sm.
$$

\nor $s \leq 0$ we have
\n
$$
H_p^{e}(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(p)}(\mathbf{R}^n) \hookrightarrow H_p^{s}(\mathbf{R}^n)
$$
\n
$$
m \text{ and } \rho > sm'.
$$
\n
$$
f: Step 1. We get by the monotonicity of the l_q -spaces and by a simple calcul-
\nthe first elementary embedding
\n
$$
B_{p,\infty}^{s+\epsilon,a} \xrightarrow{(q)} \hookrightarrow B_{p,q_1}^{s,a} \xrightarrow{(q)} \hookrightarrow B_{p,q_2}^{s-\epsilon,a} \xrightarrow{(p)}
$$
\n
$$
0 < \infty, -\infty < s < \infty, 0 < q \leq \infty, 0 < q_1 \leq q_2 \leq \infty \text{ and } \epsilon > 0.
$$
\n
$$
2 \text{ Without loss of generality, we may assume that } \{m\}^{\infty} \text{ is a system with}
$$
$$

if $1 < p < \infty$, $-\infty < s < \infty$, $0 < q \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ and $\varepsilon > 0$.

(ii) For $s \le 0$ we have
 $H_p^o(\mathbf{R}^n) \hookrightarrow B_{p,q}^{s,a(\phi)}(\mathbf{R}^n) \hookrightarrow H_{p}^{\times}(\mathbf{R}^n)$ (16)
 $\begin{align*}\n\text{if } s < sm \text{ and } g > sm'.\n\end{align*}$

Proof: Step 1. We get by the monotonicity of the l_q -spaces and by a simple calculation t $c^p = 1$. We introduce a second system of smooth functions $\{\varphi_j^*\}_{j=0}^{\infty}$, where the $\begin{aligned}\n\delta_i & \text{if } x < sm \text{ and } q > sm'. \\
\text{Proof: Step 1. We get by the monotonicity of the } l_q\text{-spaces and by a simple calculation the first elementary embedding}\n\end{aligned}\n\begin{aligned}\nB_{p,\alpha}^{s+cs} & \text{if } 1 < p < \infty, \ -\infty < s < \infty, \ 0 < q \leq \infty, \ 0 < q_1 \leq q_2 \leq \infty \text{ and } \varepsilon > 0.\n\end{aligned}\n\tag{17}\n\begin{aligned}\n\delta_i & \text{if } 1 < p <$ $if x < sm$
Proof:
lation the
 $If 1 < p <$
 $Step 2$.
 $c^p = 1$. W
are indepe $B_{p,q_1}^{s,a(p)} \hookrightarrow B_{p,q_2}^{s-a(p)} \hookrightarrow B_{p,q}^{s-c,a(p)}$
 $\langle s \rangle \langle \infty, 0 \rangle \langle q \rangle \leq \infty, 0 \langle q_1 \rangle \leq q_2 \leq \infty$ and $\varepsilon > 0$.

ss of generality we may assume that $\{\varphi_j\}_{j=0}^{\infty}$ is a system

is a second system of smooth functions $\{\$ $\begin{array}{l} \text{aclcu-} \ \text{(17)} \ \text{with } \circ \\ \text{we } \varphi_j^* \ \text{rties:} \end{array}$ $\begin{aligned}\n\text{if } &\mathbf{z} < sm \text{ and } \mathbf{e} > sm'. \\
\text{Proof: Step 1. We get} \\
\text{lation the first element} \\
\text{B}_{p,\mathbf{e}}^{s+\epsilon,\mathbf{a}}(\mathbf{e}) &\hookrightarrow \text{B}_{p,q}^{s,a}, \\
\text{if } &1 < p < \infty, -\infty < s \\
\text{Step 2. Without loss } &c^{\mathbf{e}} &= 1. \text{ We introduce a} \\
\text{are independent of } & \mathbf{z} \text{ and} \\
\text{op}_j^*(\xi) &= 1 \text{ on } \mathbf{$

$$
\varphi_j^*(\xi) = 1
$$
 on supp φ_j , supp $\varphi_j^* \subset \Omega_j^{N,a,*}$,

$$
|\varphi_i^{*(\alpha)}(\xi)| \leq c_{\alpha} \langle \xi \rangle^{-|\alpha|}
$$
 for all α and c_{α} and independent of *j*.

Step 2. Without loss of generality we may assume that
$$
\{\varphi_j\}_{j=0}^{\infty}
$$
 is a $c^{\varphi} = 1$. We introduce a second system of smooth functions $\{\varphi_j^*\}_{j=0}^{\infty}$, we are independent of x and therefore we will write $\varphi_j^*(\xi)$, with the following $\varphi_j^*(\xi) = 1$ on supp φ_j , supp $\varphi_j^* \subset \Omega_j^{N,a,*}$, $|\varphi_j^{*(a)}(\xi)| \leq c_a \langle \xi \rangle^{-|a|}$ for all α and c_a and independent of j. where $\Omega_j^{N,a,*} = \{(x,\xi): \langle \xi \rangle < \max(1 + R_a, c_m^{-1/m'} 2^{(j+j+N+1)/m'})\}$ if $j = 0, 1, ..., N$; $\Omega_j^{N,a,*} = \{(x,\xi): c_m^{-1/m} 2^{(j+j-N-1)/m} < \langle \xi \rangle < c_m^{-1/m'} 2^{(j+j+N+1)/m'}\}$ if $j = N + 1, N + 2, \ldots$. The existence of such systems $\{\varphi_j^*\}_{j=0}^{\infty}$ can be shown in analogy to the exact end of the previous section. We put $\varphi_j^*(\xi) = \varphi(2^{-N-j-1}c_m \langle \xi \rangle^m) - \varphi(2^{N-j}c_m \langle \xi \rangle^m)$ for $j = N + 1, N + 2, \ldots$. Now it is easy to verify that the semi-norm be estimated as the semi-norms of φ_j in (12). Also in view of (9) and (

The existence of such systems $\{\varphi_j^*\}_{j=0}^\infty$ can be shown in analogy to the example at the

$$
\varphi_i^*(\xi) = \varphi(2^{-N-j-1}c_{m'}\langle \xi \rangle^{m'}) - \varphi(2^{N-j}c_{m'}\langle \xi \rangle^{m})
$$

The existence of such systems $\{\varphi_j^*\}_{j=0}^{\infty}$ can be shown in analogy to the example at the
end of the previous section. We put
 $\varphi_j^*(\xi) = \varphi(2^{-N-j-1}c_m\langle \xi \rangle^m) - \varphi(2^{N-j}c_m\langle \xi \rangle^m)$
for $j = N + 1$, $N + 2$, Now it for $j = N + 1$, $N + 2$, Now it is easy to verify that the semi-norms of φ_j^* can be estimated as the semi-norms of φ_j in (12). Also in view of (9)'and (11) it holds

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\n
$$
\Omega_j^{N,a} \subset \Omega_j^{N,a,*} \text{ for all } j. \text{ Hence by Theorem 1 we obtain}
$$
\n
$$
\varphi_i^*(D_x) \varphi_j(x, D_x) = \varphi_j(x, D_x) + R_j(x, D_x)
$$
\nand\n
$$
|r_j|_{(l,k)}^{(a)} \leq c_{lk\mu} 2^{-j\gamma}
$$
\nfor arbitrary real numbers μ and Ω when the coefficients μ are independent of Λ

and

$$
|r_j|_{(l,k)}^{(\mu)} \le c_{ik\mu\gamma} 2^{-j\gamma} \tag{19}
$$

for arbitrary real numbers μ and γ , where the constants $c_{ik\mu\nu}$ are independent of *j*. This yields for $J = 1, 2, \ldots$ 74 H. G. LEOPOLD
 $Q_j^{N,a} \subset Q_j^{N,a,*}$ for all j. I
 $\varphi_j^*(D_z) \varphi_j(x, D_z)$

and
 $|r_j|_{(l,k)}^{(u)} \leq c_{lk\mu\gamma} 2^{-j}$

for arbitrary real number

This yields for $J = 1, 2, ...$
 $u = \sum_{j=0}^{J-1} \varphi_j(x, D_z)$

with
 $R^J(x, D_x) = \sum_{j=J}^{\infty} I$

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\n2,
$$
N_a = Q_j N_a
$$
, for all *j*. Hence by Theorem 1 we obtain
\n $\varphi_j^*(D_z) \varphi_j(x, D_z) = \varphi_j(x, D_z) + R_j(x, D_z)$ (18)
\nand
\n $|r_j|_{(l,k)}^{(p)} \leq c_{lk\mu_2} 2^{-j\gamma}$ (19)
\nfor arbitrary real numbers μ and γ , where the constants $c_{lk\mu_2}$ are independent of *j*.
\nThis yields for $J = 1, 2, ...$
\n $u = \sum_{j=0}^{J-1} \varphi_j(x, D_z) u + \sum_{j=J}^{\infty} \varphi_j^*(D_z) \varphi_j(x, D_z) u - R^J(x, D_z) u$ (20)
\nwith
\n $R^J(x, D_z) = \sum_{j=J}^{\infty} R_j(x, D_z)$. (21)
\nBy (19) we get the convergence of the infinite series (21) in $S_{1,s}^0$. The semi-norms of
\n $R^J(x, D_z)$ can be estimated by
\n $|r^J|_{(l,k)}^{(0)} \leq_{l} c_{lk} 2^{-J}$,
\nwhere the constants c_{lk} are independent of *J*.
\n*Step 3.* Let $s \leq 0$ and $z < s$ m fixed. Then by Corollary 1 and (22) we have

•

$$
R^{J}(x, D_{x}) = \sum_{j=J}^{\infty} R_{j}(x, D_{x}).
$$
\n(21)

By (19) we get the convergence of the infinite series. (21) in $S^0_{1,\delta}$. The semi-norms of $R^{J}(x, D_x)$ can be estimated by

$$
|r^J|_{(l,k)}^{(0)} \leq c_{lk} 2^{-J},\tag{22}
$$

Find k are independent of $J = \sum_{j=0}^{n} p_j(x, D_x) u + \sum_{j=0}^{\infty} p_j^*(D_x) p_j(x, D_x) u - R^J$

with
 $R^J(x, D_x) = \sum_{j=0}^{\infty} R_j(x, D_x)$.

By (19) we get the convergence of the infinite series (21)
 $R^J(x, D_x)$ can be estimated by
 $|r'^{[0$ where the constants c_{lk} are independent of J .
Step 3. Let $s \leq 0$ and $\varkappa < sm$ fixed. Then by Corollary 1 and (22) we have *f*
 $u = \sum_{j=0}^{J-1} \varphi_j(x, D_x) u + \sum_{j=J}^{\infty} \varphi_j^*$

with
 $R^J(x, D_x) = \sum_{j=J}^{\infty} R_j(x, D_x)$

By (19) we get the convergence of t
 $R^J(x, D_x)$ can be estimated by
 $|r^J|_{(l,k)}^{(0)} \leq c_{lk} 2^{-J}$,

where the constants c_{lk} are i This implies that the inverse operator of where the constants c_{lk} are independent of *J*.
 Step 3. Let $s \le 0$ and $\varkappa < sm$ fixed. Then by Corollary 1 and (22) we have $||R^J(x, D_x) | L(H_p^{\times}, H_p^{\times})|| \le c_{\kappa p} 2^{-J}$. This implies that the inverse operator of $I + R^J(x$ identity. th
 R^{*I*}(*x*, *D_{<i>z*}) = $\sum_{j=J}^{\infty} R_j(x, D_x)$.
 V (19) we get the convergence of the infinite series (21) in *S*^{*n*}
 S_i^0
 $S_i^1(x, D_x)$ can be estimated by
 $|r^I|_{(l,k)}^{(0)} \leq c_{lk}2^{-J}$,

here the constants c_{lk $I + R^{J}(x, D_x)$ exists and belongs also to $L(H_p^{\bullet \bullet}, H_p^{\bullet \bullet})$ if $J \geq J_0(x, p)$. *I* stands for the get the convergence of the infinite series (21) in $S_{1,3}^0$. The bestimated by
 $\begin{aligned}\n\binom{0}{l,k} &\leq_{l}c_{lk}2^{-J}, \\
\text{constants } c_{lk} \text{ are independent of } J. \\
\text{Let } s \leq 0 \text{ and } z < sm \text{ fixed. Then by Corollary 1 a} \\
L(H_{p^*}, H_{p^*}) \leq c_{kp}2^{-J}. \\
\text{This implies that the inverse of } L(H_{p^*}, H_{p^*}) \text{ if$ where the constants c_{lk} are
 $Step 3$. Let $s \le 0$ and
 $||R^{J}(x, D_{x})||L(H_{p}^{*}, H_{p}^{*})|| \le$
 $I + R^{J}(x, D_{x})$ exists and b

identity.
 $Step 4$. Let $s' = \varkappa/m$, u
 $v_{j} = \begin{cases} \varphi_{j}(x, D_{x}) & u \\ \varphi_{j}^{*}(D_{x}) & \varphi_{j} \end{cases}$

As a cons *c*₀ $L(H_{p^*}, H_{p^*})$ if $J \geq J_0(x, p)$. *I* stands
and
if $j = 0, 1, ..., J_0 - 1$,
if $j = J_0, J_0 + 1, ...$
he system $\{\varphi_j^*\}_{j=0}^{\infty}$ and of (7) in Corolla
 $c^{2j s'} \|\varphi_j(x, D_x) u \mid L_p\|$
ivial estimate gives

P.

$$
u + R^{\circ}(x, D_x) \text{ exists and belongs also to } L(H_p^*, H_p^*) \text{ if } \underline{J} \geq J_0(x, p).
$$

identity.
Step 4. Let $s' = \varkappa/m$, $u \in B_{p,1}^{s',a(\varphi)}(\mathbb{R}^n)$ and

$$
v_j = \begin{cases} \varphi_j(x, D_x) u & \text{if } j = 0, 1, ..., J_0 - 1, \\ \varphi_j^*(D_x) \varphi_j(x, D_x) u & \text{if } j = J_0, J_0 + 1, ... \end{cases}
$$
As a consequence of the properties of the system $\{\varphi_j^*\}_{j=0}^{\infty}$ and of (7) in
see that

$$
||\varphi_j^*(D_x) \varphi_j(x, D_x) u || H_p^*|| \leq c2^{js'} ||\varphi_j(x, D_x) u || L_p||
$$

if $j = J_0, J_0 + 1, ...$ Since $\varkappa < 0$, a trivial estimate gives

$$
||\varphi_j(x, D_x) u || H_p^*|| \leq c' 2^{-js'} ||\varphi_j(x, D_x) u || L_p||
$$

As a consequence of the properties of the system $\{\varphi_j^*\}_{j=0}^\infty$ and of (7) in Corollary 1, we Equence of the properties of the system $\{\varphi_j^*\}_{j=0}^{\infty}$ and of (7) in C
 $\|\varphi_j^*(D_x) \varphi_j(x, D_x) u \| H_p^*\| \leq c2^{js'} \|\varphi_j(x, D_x) u \| L_p \|$

$$
||\varphi_j^*(D_x) \varphi_j(x, D_x) u \mid H_p^*\| \leq c2^{js'} ||\varphi_j(x, D_x) u \mid L_p||
$$

if $j = J_0, J_0 + 1, \ldots$ Since $\varkappa < 0$, a trivial estimate gives

$$
||\varphi_j(x, D_x) u \mid H_p^{\kappa}|| \leq c' 2^{-J_{\sigma} \sigma'} 2^{j \sigma'} ||\varphi_j(x, D_x) u \mid L_p||
$$

if $j = 0, 1, 2, ..., J_0 - 1$. *c* and *c'*-are independent of *j*. We obtain

$$
\sum_{j=0}^{\infty} ||v_j|| H_{p^*}|| \leq \max (c, c' 2^{-J \cdot s'}) ||u|| B_{p,1}^{s',a} ||^{(p_j)}.
$$

 $v_j = \begin{cases} \varphi_j(x, D_x) & \text{if } j = 0, 1, ..., J_0 - 1, \\ \varphi_j^*(D_x) & \varphi_j(x, D_x) & \text{if } j = J_0, J_0 + 1, ... \end{cases}$

As a consequence of the properties of the system $\{\varphi_j^*\}_{j=0}^{\infty}$ and of (7) in Corol

see that
 $||\varphi_j^*(D_x) \varphi_j(x, D_x) u | H_p^*|| \leq c2^{$ If $j = 0, 1, 2, ..., J_0 - 1$. c and c'are independent of *j*. We obtain
 $\sum_{j=0}^{\infty} ||v_j|| H_p^*|| \leq \max(c, c' 2^{-J_0 s'}) ||u|| B_{p,1}^s||^{(p_j)}$.

Together with (20) this implies that $\sum_{j=0}^{\infty} v_j = u + R^J(x, D_x) u$ belongs to $H_p^*({\bf R}^n)$.

$$
||u||H_p^{s}|| \leq c'' ||(I + R^J)^{-1}||L(H_p^{s}, H_p^{s})|| ||u|| |B_{p,1}^{s',a}||^{(p_j)}
$$

Because of the result of the third step the same must be true for u and we get
 $||u||H_p^*|| \le c'' ||(I + R^J)^{-1}||L(H_p^*, H_p^*)|| ||u||B_p^{\frac{s'}{2}}||^{(q_j)}$

if $s \le 0$, $\varkappa < sm$ and $s' = \varkappa/m$. Now the right-hand side of (16) follows in view

Fogether with (20) this implies that $\sum_{j=0}^{\infty} v_j = u + R^J(x, D_x) u$ belongs to $H_p^*(\mathbb{R}^n)$.
Because of the result of the third step the same must be true for u and we get
 $||u \, | H_p^*|| \le c'' ||(I + R^J)^{-1} | L(H_p^*, H_p^*)|| ||u \, | B_p^*||^{$ *Step 5.* In the case $s > 0$ the proof is simpler. Let $0 < x < sm'$ and $s' = x/m'$. If

•

is absolutely convergent in $L_p(\mathbf{R}^n)$. We get on this way $u \in L_p(\mathbf{R}^n)$ and $||u \nmid L_p||$ $\leq c ||u|| B_{p,1}^{s',q} ||^{(p,j)}$. On the other hand it follows by (18), (19) and Corollary 1.

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\nis absolutely convergent in
$$
L_p(\mathbf{R}^n)
$$
. We get on this way $u \in L_p(\mathbf{R}^n)$ and $||u \nmid L_p||$
\n $\leq c ||u|| B_{p,1}^{s'}||^{(q_p)}$. On the other hand it follows by (18), (19) and Corollary 1
\n $\sum_{j=0}^{\infty} ||\varphi_j(x, D_x) u \nmid H_p^*|| \leq c_{sp} \sum_{j=0}^{\infty} |\varphi_j^*|^{(-s)}_{(l,k)}||\varphi_j(x, D_x) u \nmid L_p|| + c_{sp} \sum_{j=0}^{\infty} |r_j|^{(-s)}_{(l,k)}||u \nmid L_p||$
\n $\leq c \sum_{j=0}^{\infty} 2^{js'} ||\varphi_j(x, D_x) u \nmid L_p|| + c' ||u ||L_p|| \leq c''||u ||B_{p,1}^{s'}||^{(s_p)}$.
\nSo we get $u \in H_p^*(\mathbf{R}^n)$ with $||u||H_p^*|| \leq c' ||u|| B_{p,1}^{s'}||^{(s_p)}$ and again the right-hand side
\nof (15) follows in view of (17).
\nStep 6. The proof will be completed by showing that the left-hand sides of (15)
\nand (16) hold. But this is a simple consequence of the semi-norm estimates (12) and
\nCorollary 1 **II**
\nWe will prove now that equivalent symbols define the same spaces and that the

of (15) follows in view of (17).

Step .6. The proof will be completed by showing that the left-hand sides of (15) and (16) hold. But this is a simple consequence of the semi-norm estimates (42) and

We will prove now that equivalent symbols define the same spaces and that the definition of these spaces is independent of the choosen system $\{\varphi_i(x, \xi)\}_{i=0}^{\infty}$. Let. $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols of the class $S(m, m'; \delta)$, that means we have $0 < c_1 \leq |a(x, \xi)| b^{-1}(x, \xi)| \leq c_2 < \infty$ if $x \in \mathbb{R}_z^n$, $\xi \in \mathbb{R}_\xi^n$ and $|\xi| \geq R$. Without loss We will prove now that equivalent symbols define the same spaces and that
definition of these spaces is independent of the choosen system $\{\varphi_i(x, \xi)\}_{j=0}^{\infty}$.
 $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols of the clas denote two function systems belonging to $a(x, \xi)$ and $b(x, \xi)$, respectively. Then there $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols of the class $S(m$
have $0 < c_1 \leq |a(x, \xi) b^{-1}(x, \xi)| \leq c_2 < \infty$ if $x \in \mathbb{R}_x^n, \xi \in \mathbb{R}_t^n$ and of generality we may assume that $R \geq \max(R_a, R_b)$. $\{\varphi_i\}_{i=0}^{\infty} \in$
denote tw exist numbers $j_0(R)$ and $i_0(R)$ such that all (x, ξ) with $|\xi| \leq R$ do not belong to the-We will prove now that definition of these spaces $a(x, \xi)$ and $b(x, \xi)$ be two eq
have $0 < c_1 \leq |a(x, \xi)| b^{-1}(x)$ of generality we may assume
denote two function system
exist numbers $j_0(R)$ and $i_0($ sets $\Omega_j^{N,a}$ and Ω_i $\frac{1}{a_0}$ and $i > i_0$, respectively. Hence we get They we may assume that $R \geq$ may
vo function systems belonging to abers $j_0(R)$ and $i_0(R)$ such that a
i and $\Omega_i^{M,b}$ if $j > j_0$ and $i > i_0$, re
supp φ_j , \cap supp $\psi_i = \emptyset$ if $j \notin \mathcal{J}(i)$, Step 6. The proof will be completed by showing that the left-hand s
and (16) hold. But this is a simple consequence of the semi-norm estimat
Corollary 1 **T**
We will prove now that equivalent symbols define the same spaces have $0 < c_1 \leq |a(x, \xi) b^{-1}(x, \xi)| \leq c_2 < \infty$ if $x \in \mathbb{R}_x^n, \xi \in \mathbb{R}_i^n$ and $|\xi| \geq R$. With
of generality we may assume that $R \geq \max(R_a, R_b)$. $\{\varphi_i\}_{i=0}^{\infty} \in \Phi^{N,a}$ and $\{\psi_i\}_{i=0}^{\infty}$
denote two function systems bel

$$
\operatorname{supp} \varphi_i \cap \operatorname{supp} \psi_i = \emptyset \text{ if } j \notin \mathcal{J}(i),
$$

of generality we may assume that
$$
R \ge \max(R_a, R_b)
$$
. $(\varphi_i)_{i=0}^{\infty} \in \Phi^{N,a}$ and $\{\psi_i\}_{i=0}^{\infty} \in \Phi^{M,b}$
denote two function systems belonging to $a(x, \xi)$ and $b(x, \xi)$, respectively. Then there
exist numbers $j_0(R)$ and $i_0(R)$ such that all (x, ξ) with $|\xi| \le R$ do not belong to the
sets $\Omega_i^{N,a}$ and $\Omega_i^{M,b}$ if $j > j_0$ and $i > i_0$, respectively. Hence we get

$$
supp \varphi_i \cap supp \psi_i = \emptyset
$$
 if $j \notin \mathcal{I}(i)$,
where,

$$
\mathcal{J}(i) = \{j : 0 \le j \le \max(j_0, i + I - J + M + N + H)\} \qquad \text{if } 0 \le i \le i_0,
$$

$$
\mathcal{J}(i) = \{j : \max\{J, i + I - (M + N + H)\} \le i \ne j - I \le i + I + M + N + H\}
$$
(23)

$$
if i > i_0
$$

and H fulfils $2^{-H} \le c_1 \le c_2 \le 2^H$. Therefore we obtain in the case $j \notin \mathcal{J}(i)$ by Theorem 1

$$
\psi_i(x, D_x) \varphi_i(x, D_x) = R_{ij}(x, D_x),
$$

$$
|r_{ij}|_{(l,k)}^{(m_1 + m_1 - l(1 - \delta))} \le c_{Llk} |\psi_i|_{(l,k)}^{(m_1)} |\varphi_j|_{(l,k)}^{(m_2)},
$$

$$
|p_j|_{(l,k)}^{(m_2)}
$$
(25)
The constants c_{Llk} may increase in dependence on L and (l, k) but they are always independent of i and j . In the classical case, that means $\overline{\psi}_i(\xi)$ and $\overline{\varphi}_j(\xi)$ are independent

and *H* fulfils $2^{-H} \leq c_1 \leq c_2 \leq 2^H$. Therefore we obtain in the case $j \notin \mathcal{J}(i)$ by Theorem 1

$$
\psi_i(x, D_x) \varphi_i(x, D_x) = R_{ij}(x, D_x),
$$

where for each natural number L the semi-norms of $R_{ij}(x, D_x)$ can be estimated by \bar{L}

Ifils
$$
2^{-H} \le c_1 \le c_2 \le 2^H
$$
. Therefore we obtain in the case $j \notin \mathcal{J}(i)$ by Theorem 1
\n $\psi_i(x, D_x) \varphi_j(x, D_x) = R_{ij}(x, D_x)$, (24)
\n r each natural number *L* the semi-norms of $R_{ij}(x, D_x)$ can be estimated by
\n $|r_{ij}|_{(l,k)}^{(m_1+m_1-L(1-\delta))} \le c_{Llk} |\psi_i|_{(l',k)}^{(m_1)} |\varphi_j|_{(l,k')}^{(m_2)}$. (25)
\ntants c_{Llk} may increase in dependence on *L* and (l, k) but they are always

The constants c_{Llk} may increase in dependence on L and (l, k) but they are always independent of *i* and *j*. In the classical case, that means $\bar{\psi}_i(\xi)$ and $\bar{\varphi}_i(\xi)$ are independent of x, the terms $R_{ij}(x, D_x)$ do not exist because supp $\bar{\varphi}_j \cap \text{supp } \bar{\varphi}_i = \emptyset$ always yields $\bar{\varphi}_i(D_x) \bar{\varphi}_i(D_x) = 0$. The constants c_{Llk} may if
independent of *i* and *j*. If
dent of *x*, the terms R_i
yields $\bar{\psi}_i(D_x) \bar{\phi}_j(D_x) = 0$.

(i) = { $j: \max (J, i + I - (M + N + H)) \leq i + J \leq i + I + M + N -$
 $if i > i_0$

d *H* fulfils $2^{-H} \leq c_1 \leq c_2 \leq 2^H$. Therefore we obtain in the case $j \notin \mathcal{J}(i)$ by $\mathcal{J}(\hat{J}(i))$
 $\psi_i(x, D_x)\varphi_j(x, D_x) = R_{ij}(x, D_x)$,

here for each natural numbe Theorem 4: Let $a(x, \xi)$ and $b(x, \xi)$ be equivalent symbols and $\{\varphi_i\}_{i=0}^{\infty} \in \Phi^{N,a}$, $\{\psi_i\}_{i=0}^{\infty}$ $\begin{array}{c} \text{dent} \ \text{yield} \ \text{T1} \ \in \boldsymbol{\varPhi^s} \ 0 < 0 \end{array}$ $\begin{aligned} \text{hence } \mathbf{a} \in \mathbb{R}^{n} \text{ is a } (x, \xi) \text{ and } b(x, \xi) \text{ be equivalent symbols and } \{ \varphi_j \}_{j=0}^{\infty} \in \Phi^{N,a}, \ \{ \psi_i \}_{i=0}^{\infty}, \ \text{for } i \leq p \leq \infty, \\ \text{hence } \text{hence } \mathbf{a} \in \mathbb{R}^{n} \text{ is a } (x, \xi) \text{ and } b(x, \xi), \text{ respectively. If } 1 < p < \infty, \end{aligned}$ dent of x, the terms $R_{ij}(x, D_x)$ do not exist becaus
yields $\bar{\psi}_i(D_x) \bar{\phi}_j(D_x) = 0$.
Theorem 4: Let $a(x, \xi)$ and $b(x, \xi)$ be equivalent so
 $\in \Phi^{M,b}$ be two systems belonging to $a(x, \xi)$ and $b(x, \xi)$
 $0 < q \leq \infty$ and $-\infty <$ $0 < q \leq \infty$ and $-\infty < s < \infty$, then $||u|| B^{s,a}_{p,q}||^{(p_j)}$ and $||u|| B^{s,b}_{p,q}||^{(p_i)}$ are equivalent *quasi-norms in* $B_{p,q}^{s,a}(\mathbf{R}^n)$.

Proof: It is easy to see that both $||u||B^{s,a}_{p,q}||^{(p_j)}$ and $||u||B^{s,b}_{p,q}||^{(p_i)}$ are quasi-norms. In order to prove their equivalence we use the preceding considerations. Also we may assume $c^p=1$. $\begin{array}{l} S \rightarrow \infty \quad and \quad -\infty \quad s, \text{ and } \quad s \text{ and }$

Step 1. Let $u \in B_p^{s,a(\varphi)}$. Then by Theorem 3 *u* belongs also to $H_p^{\star}(\mathbf{R}^n)$ if x is suitably chosen. Now we obtain from (13), (12) and Corollary 1 , for $i = 0, 1, 2, \ldots$ and arbitrary fixed $\epsilon > 0$ the estimates

$$
\begin{aligned} ||\psi_i(x, D_x) u || L_p|| \, . \\ &\leq \sum_{j=0}^{\infty} ||\psi_i(x, D_x) \varphi_j(x, D_x) u || L_p|| + 2^{-i(s+t)} ||u || H_p^*||. \end{aligned} \tag{26}
$$

Step 2. Let $s > 0$ and $\varkappa < sm'$ be fixed. From (26) and (24) we get

 $\sum_{i=0}^{\infty} 2^{i s q} ||\psi_i(x, D_x) u || L_p||^q$

$$
\leq \sum_{i=0}^{\infty} 2^{isq} \left(\sum_{j \in \mathcal{J}(i)} \|\psi_i(x, D_x) \varphi_j(x, D_x) u \mid L_p\| + \sum_{j \in \mathcal{J}(i)} \|R_{ij}(x, D_x) u \mid L_p\| \right)^q + c_{\epsilon} \|u \| H_p^*\|^q.
$$

The terms of the first sum will be estimated by Theorem 2 and (12). The estimate of the remainder terms in the second sum will be taken by (25) if we choose there for a fixed $\varepsilon' > 0$, $m_1 = ms + \varepsilon' m$, $m_2 = \varepsilon' m$ and L so large such that $m_1 + m_2 - L(1 - \delta)$ \leq *x* holds. Then the semi-norms of ψ_i and $\dot{\psi}_j$ in (25) can be estimated again by (12) and we obtain

$$
||u||B_{p,q}^{s,b}||^{(v_{0})q} \leq c \sum_{i=0}^{\infty} 2^{isq} \left(\sum_{j \in \mathcal{J}(i)} c' ||\varphi_{i}|^{(0)}_{(l,k)} ||\varphi_{j}(x, D_{x}), u || L_{p}|| \right.
$$

+
$$
\sum_{j=0}^{\infty} c_{L} 2^{-i(s+\epsilon')} 2^{-j\epsilon'} ||u || H_{p}^{*}|| \right)^{q} + c_{\epsilon} ||u || H_{p}^{*}||^{q}
$$

$$
\leq c' \sum_{\ell=0}^{\infty} 2^{jsq} ||\varphi_{j}(x, D_{x}) ||u || L_{p}||^{q} + (c_{\epsilon'} + c_{\epsilon}) ||u || H_{p}^{*}||^{q}
$$

$$
\leq c_{\varphi} ||u || B_{p,q}^{s,a}||^{(v_{j})q}.
$$

The last estimate follows in view of the embedding (15). Also we have used the shape of the set $\mathcal{J}(i)$. Because of our assumptions, the same must be true if we change the role of $\{\psi_i\}_{i=0}^{\infty}$ and $\{\varphi_j\}_{j=0}^{\infty}$ and so we get the converse inequality.

Step 3. Let $s \leq 0$ and $\varkappa < s$ be fixed. The proof of the equivalence will be the same as in the second step, if we choose $m_1 = m's + \varepsilon' m', m_2 = \varepsilon' m'$ and take a corresponding modification in the semi-norm estimates of the ψ_i .

Corollary 2: The definition of the space $B_{p,q}^{s,a}(\mathbf{R}^n)$ is independent of the chosen system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and also of the choice of the constants J and N in the definition of the sets $\{\Omega_j^{N,a}\}_{j=0}^{\infty}$.

Convention: In the sequel we shall not distinguish between equivalent quasinorms. In this sense we shall write $||u||B_{p,q}^{s,a}||$ instead of $||u||B_{p,q}^{s,a}||^{(p_j)}$.

Corollary 3: Symbols $a(x, \xi)$ and $b(x, \xi)$ which are equivalent in the sense of (10) define the same function spaces. For all admissible parameters p , q and s there exist positive constants c' and c such that c' $||u||B_{p,q}^{s,a}|| \le ||u||B_{p,q}^{s,b}|| \le c ||u||B_{p,q}^{s,a}||$ holds. If $b(x,\xi)$ is especially an elliptic pseudodifferential operator of the order m, then the space $B_{p,q}^{s,b}(\mathbf{R}^n)$ coincides with the classical Besov space $B_{p,q}^{sm}(\mathbf{R}^n)$ for $1 < p < \infty$, $0 < q \leq \infty$ and $-\infty < s < \infty$.

Theorem 5: For $-\infty < s < \infty$, $1 < p < \infty$ and $0 < q \le \infty$, $B_{p,q}^{s,a}(\mathbf{R}^n)$ is a \sim quasi-Banach space (Banach space if $1 \leq q \leq \infty$), which is independent of the choice On Besov Spaces 77

of the system $\{\varphi_i\}_{i=0}^{\infty} \in \Phi^{N,a}$, and we have $S(\mathbf{R}^n) \hookrightarrow B^{s,a}_{p,q}(\mathbf{R}^n) \hookrightarrow S'(\mathbf{R}^n)$. Furthermore,
 $if \ -\infty < s < \infty$, $1 < p < \infty$ and $0 < q < \infty$, then $S(\mathbf{R}^n)$ is dense in $B^{s,a}_{p,q}(\mathbf{R$

the system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, and we have $S(\mathbf{R}^n) \hookrightarrow B^{s,a}_{p,q}$
 $-\infty < s < \infty$, $1 < p < \infty$ and $0 < q < \infty$, then $S(\text{Proof: The equivalence of quasi-norms } || \cdot || B^{s,a}_{p,q} ||^{\{r_p\}}$,
 $\sum_{j=0}^{\infty} \in \Phi^{N,a}$, was proved in Theorem 4. Also in view o Proof: The equivalence of quasi-norms $\|\cdot\| B_{p,q}^{s,a} \|^{(\varphi_j)}$, defined by different systems $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, was proved in Theorem 4. Also in view of Theorem 3 we get from the well-known embeddings of the classical Bessel-potential spaces in $S'(\mathbf{R}^n)$ and of $S(\mathbf{R}^n)$ in $H_p^{\bullet}(\mathbf{R}^n)$, respectively, the assertion about the embeddings. Thus we need only to show the completness in order to prove the first part of the theorem.

Step 1. Let $\{u_i\}_{i=1}^{\infty}$ be a fundamental sequence in $B_{p,q}^{s,a}(\mathbb{R}^n)$ which we consider with respect to a fixed quasi-norm $\|\cdot\| B_{p,q}^{s,q}\|^{\{op\}}$. Then the embedding shows that $\{u_i\}_{i=1}^{\infty}$ is also a fundamental sequence in $S'(\mathbf{R}^n)$ with the limit element $u \in S'(\mathbf{R}^n)$. On the other hand, for each fixed $j = 0, 1, 2, ...$, $\{\varphi_j(x, D_x) u_i\}_{i=1}^{\infty}$ is a fundamental sequence in Proot: The equivalence of quasi-norms $||\cdot||B_{p,q}^{p,q}||^{(p_i)}$, defined by different systems $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, was proved in Theorem 4. Also in view of Theorem 3 we get from the well-known embeddings of the classic $L_p(\mathbf{R}^n)$ with the limit element $u^j \in L_p(\mathbf{R}^n)$. Then by $u_i \to u$ in $S'(\mathbf{R}^n)$ we get $\varphi_j(x, D_x)$ u_i $\rightarrow \varphi_i(x, D_x)$ *u* in $L_p(\mathbb{R}^n)$ if $l \rightarrow \infty$. Now it follows by standard arguments that *u* belongs to $B_{p,q}^{s,a}(\mathbf{R}^n)$ and that u_l converges in $B_{p,q}^{s,a}(\mathbf{R}^n)$ to u . Hence $B_{p,q}^{s,a}(\mathbf{R}^n)$ is completely. *Step* **1**. Let $\{u_i\}_{i=1}^{\infty}$ be a fundamental sequence in $B_{p,q}^{s,a}(\mathbf{R}^n)$ which we consider with spect to a fixed quasi-norm $\|\cdot\|B_{p,q}^{s,a}\|^{(sp)}$. Then the embedding shows that $\{u_i\}_{i=1}^{\infty}$ is consider and be the set of the complete since of the set of the properties of the set of the set of the first of the complete since of the first of the calso a fundamental sequence in B_p^s respect to a fixed quasi-norm $\|\cdot\|B_{p,q}^s$ Now the completiess in otter to prove the instead of the theorem.

Let $\{u_i\}_{i=1}^n$ be a fundamental sequence in $B_{p,q}^{s,q}(\mathbf{R}^n)$ which we consider
 α a fixed quasi-norm $\|\cdot\|B_{p,q}^{s,q}(\mathbb{R}^n)$. Then the embedd *s* of the system if $\cdots \infty < s <$

Froof: The $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$, well-known er $S(\mathbb{R}^n)$ in $H_p^*(\cdot)$ only to show t
 $S(\mathbb{R}^n)$ in $H_p^*(\cdot)$ only to show t
 \cdots $Slep$ 1. Let respect to a fi also a fundame h

and $c^{\varphi} = 1$. For any natural *M* we write $\varphi^M(x, D_x) = \sum_{r=1}^{M} \varphi_r(x, D_x)$. Then in analogy *j0*

\n [pletely.\n \n- \n Step 2: We prove now the second part of the theorem. Let
$$
\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}
$$
 be fixed and $c^{\varphi} = 1$. For any natural M , we write $\varphi^M(x, D_x) = \sum_{j=0}^M \varphi_j(x, D_x)$. Then in analogy with (18) and (19) we get by Theorem 1\n
$$
\varphi_j(x, D_x) \varphi^M(x, D_x)
$$
\n
$$
\varphi_j(x, D_x) \varphi^M(x, D_x)
$$
\n
$$
\mathbf{r} = \begin{cases}\n\varphi_j(x, D_x) + R_{M,j}(x, D_x) & \text{if } j \leq M - 2N, \\
R_{M,j}(x, D_x) & \text{if } j \geq M + 2N,\n\end{cases}
$$
\n

\n\n The remainder terms $R_{M,j}(x, D_x)$ always belong to $S^{-\infty}$ and the semi-norms of them can be estimated for each fixed $\varepsilon > 0$ and each real x , independently of j and M , by $|r_{M,j}|_{(l,k)}^{(u)} \leq c2^{-(s+t)j}2^{-M}$. The constant c depends on l, k, s, \varkappa and ε but not on j and M .\n $R_{M,j}(x, D_x) = R_{M,j}(x, D_x) \text{ and } R_{M,j}(x, D_x) = R_{M,j}(x,$

The remainder terms $R_{M,j}(x, D_x)$ always belong to $S^{-\infty}$ and the semi-norms of them can be-estimated for each fixed $\varepsilon > 0$ and each real x, independently of *j* and M, The remainder
can be estimat
by $|r_{M,j}|_{(l,k)}^{(s)} \le$
j and M. ed for each fixed $\varepsilon > 0$ and each real x, independently of *f* and *M*, $c2^{-(s+t)i}2^{-M}$. The constant *c* depends on *l*, *k*, *s*, *x* and *ε* but not on

Step 3. Let $u \in B^{s,a}_{p,q}, q < \infty$ and $x < \min(\mathit{sm}', \mathit{sm})$ fixed. Setting $u_M = \varphi^M(x, D_x) u$, we have in view of the previous step

$$
= \begin{cases} h_{M,j}(x, D_x) & \text{if } j \leq M \\ \varphi_j(x, D_x) & \text{if } M = 2N < j \\ \varphi_j(x, D_x) & \text{if } M = 2N < j \end{cases}.
$$

The remainder terms $R_{M,j}(x, D_x)$ always belong to $S^{-\infty}$ and the semi-
can beestimated for each fixed $\varepsilon > 0$ and each real x, independently
by $|r_{M,j}||_{(l,k)}^{(l,k)} \leq c2^{-(s+\epsilon)j}2^{-M}$. The constant c depends on l, k, s, x and s
j and M.
Step 3. Let $u \in B_{p,q}^{s,a}, q < \infty$ and $x < \min(\varepsilon m', \varepsilon m)$ fixed. Setting $u_M =$
we have in view of the previous step
 $||u - u_M||B_{p,q}^{s,a}||^q = \sum_{j=0}^{\infty} 2^{jsq} ||\varphi_j(x, D_x) u - \varphi_j(x, D_x) \varphi^M(x, D_x) u ||L_p||^q$
 $\leq c_2 2^{-Mq} ||u|| H_p^*||^q + c \sum_{j=M-4N+2}^{\infty} 2^{jsq} ||\varphi_j(x, D_x) u ||L_p||^q$
 $\leq c2^{-Mq} ||u|| B_{p,q}^{s,a}||^q + c \sum_{j=M-4N+2}^{\infty} 2^{jsq} ||\varphi_j(x, D_x) u ||L_p||^q$

The last estimate follows in view of Theorem 3.

(

It is now obvious that $u_M \to u$ in $B_{p,q}^{s,a}$ if $M \to \infty$.

Step 4. Let $x < \min(\text{sm}', s\text{m})$ and $\rho > \max(\text{sm}, s\text{m}')$. Then it is clear that $u \in B_{p,q}^{s,a}$ implies $u \in H_p^{\mathbf{x}}$. The pseudodifferential operator $\varphi^M(x, D_x)$ belongs to $S^{-\infty}$ and therefore, by Corollary 1, u_M becomes an element of H_p^e . $S(\mathbb{R}^n)$ is dense in $H_p^e(\mathbb{R}^n)$. Hence there exists a sequence $u_{M,J} \in S(\mathbb{R}^n)$ such that $u_{M,J} \to u_M$ in H_p^e if $J \to \infty$. Because we had fixed $\varrho > \max(\mathrm{sm}, \mathrm{sm}')$, now Theorem 3 ensures that the sequence $u_{M,J}$

converges in $B_{p,q}^{s,a}$ to *u* if $M, J \to \infty$. This proves the density of $S(\mathbb{R}^n)$ in $B_{p,q}^{s,a}(\mathbb{R}^n)$ if $q < \infty$

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converges in $B_{p,q}^{s,a}$ to u if $M, J \to \infty$. This proves the dense $q < \infty$
 Property 1974 The equivalent quasi-norms in the following theorem

estimate for pseudodifferential operators $B(x, D_x)$ w The equivalent quasi-norms in the following theorem contain also an a priori estimate for pseudodifferential operators $B(x, D_x)$ whose symbols are equivalent to an arbitrary fixed symbol $a(x, \xi)$ of the class $S(m, m'; \delta)$. This shows that the spaces $B_{n,q}^{s,a}(\mathbf{R}^n)$ will be useful in the study of degenerate partial differential equations or other suitable pseudodifferential operators belonging to the class $S(m, m'; \delta)$. We recall that the elements of $S(m, m'; \delta)$ are hypoelliptic. Hence we can always construct for $B(x, D_x) \in S(m, m'; \delta)$ parametrices $Q(x, D_x) \in S_{1, \delta}^{-m'}$ such that bes $B_{p,q}^{s,a}(\mathbf{R}^n)$ will be useful in the study of degenerate partial differential equation
other suitable pseudodifferential operators belonging to the class $S(m, m';$
recall that the elements of $S(m, m'; \delta)$ are hypoel converges in $B_{p,q}^{s,a}$ to u if $M, J \to \infty$. This proves the $q < \infty$ **I**

The equivalent quasi-norms in the following theo

estimate for pseudodifferential operators $B(x, D_x)$ w

to an arbitrary fixed symbol $a(x, \xi)$ of th Fine equivalent quasi-norms in the settimate for pseudodifferential operate

to an arbitrary fixed symbol $a(x, \xi)$ of

spaces $B_{p,q}^{s,a}(\mathbf{R}^n)$ will be useful in the stud

or other suitable pseudodifferential of

We re **are equivalential** operators $B(x, D_x)$ whose symbols
 bo an arbitrary fixed symbol $d(x, \xi)$ of the class $S(m, m'; \delta)$. This

spaces $B_{pq}^{*a}(\mathbf{R}^n)$ will be useful in the study of degenerate partial different

or other

$$
B(x, D_x) Q(x, D_x) = I + R(x, D_x), \qquad Q(x, D_x) B(x, D_x) = I + R'(x, D_x)
$$

and *R*, $R' \in S^{-\infty}$ holds - see [4; Section 2, § 5] or [8; Chapter III, § 3], [9; Chapter IV, § 1].

Theorem 6: Let $a(x, \xi)$ and $b(x, \xi)$ be two equivalent symbols belonging io $S(m, m'; \delta)$, *and* $Q(x, D_x)$ denotes a parametrix for $B(x, D_x)$. If $-\infty < s < \infty$, $1 < p < \infty$ and $0 < q \le \infty$, then

 $||B(x, D_x) u||B_{p,q}^{s-1,q}|| + ||u||B_{p,q}^{s-1,q}||$ and $||Q(x, D_x) u||B_{p,q}^{s+1,q}|| + ||u||B_{p,q}^{s-1,q}||$

Proof: Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and without loss of generality we may assume that *J* in Definition 2 is fixed with respect to R_a and R . The constant R occurs by the definition of equivalence $-$ see (10). Also we choose a second system ${\{\psi_i\}}_{i=0}^{\infty} \in \Phi^{N+1.a}$ where additionally holds $\psi_i(x, \xi) = 1$ on supp φ_i if $j = 1, 2, ...$ By the construction at the end of Section 2 it is easily seen that such a system always exists. *gare equivalent quasi-norms in* B_i^i
 $\text{Proof: Let } \{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and in Definition 2 is fixed with redefinition of equivalence — see

where additionally holds $\psi_i(x, \xi)$

at the end of Section 2 it is easimally *ve* may assume that *J*
stant *R* occurs by the
ystem $\{\psi_i\}_{i=0}^{\infty} \in \Phi^{N+1,a}$
.. By the construction
vays exists.
some estimates which
 $(x, D_x) + R_{2i}(x, D_x)$,
(27)
(x, D_x) are equivalent quasi-norms in $B_{p,q}^{s,a}(\mathbf{R}^n)$.

Proof: Let $(\varphi_j)_{p=0}^{\infty} \in \Phi^{N,a}$ and without loss of gener

in Definition 2 is fixed with respect to R_a and R. T

definition of equivalence – see (10). Also we ch in Definition 2
definition of equ
where additional
at the end of See
for a set of See
 $Step 1$. Let x
will be useful in
 $\varphi_j(x, D_x) B(x, D_y)$
where
 $R_{1j}(x, L)$
and
 $R_{2j}(x, L)$

Step 1. Let $x < \min(0, sm)$ and $\varepsilon > 0$ be fixed. We note some estimates which will be useful in the next step. If $j = N + 1, N + 2, \ldots$, we have

definition of equivalence – see (10). Also we choose a second system
$$
\{\psi_j\}_{j=0}^{\infty} \in \Phi^{N+1,a}
$$
 where additionally holds $\psi_j(x, \xi) = 1$ on supp φ_j if $j = 1, 2, \ldots$. By the construction at the end of Section 2 it is easily seen that such a system always exists. Step 1. Let $x < \min(0, sm)$ and $\varepsilon > 0$ be fixed. We note some estimates which will be useful in the next step. If $j = N + 1, N + 2, \ldots$, we have\n
$$
\varphi_j(x, D_x) B(x, D_x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x) \psi_j(x, D_x) + R_{1j}(x, D_x) + R_{2j}(x, D_x),
$$
\nwhere\n
$$
R_{1j}(x, D_x) = \varphi_j(x, D_x) B(x, D_x) - \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x)
$$
\nand\n
$$
R_{2j}(x, D_x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x) \left(1 - \psi_j(x, D_x)\right).
$$
\nConsequently $R_{1j}(x, D_x)$ denotes the remainder term in Theorem 1 which is obtained by the equation $R_{2j}(x, D_x) = \sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)} \left(1 - \psi_j(x, D_x)\right).$

(27)
\nwhere
\n
$$
R_{1j}(x, D_x) = \varphi_j(x, D_x) B(x, D_x) - \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x)
$$
\nand
\n
$$
R_{2j}(x, D_x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x) \left(1 - \psi_j(x, D_x)\right).
$$
\nConsequently $R_{1j}(x, D_x)$ denotes the remainder term in Theorem 1 which is obtained

$$
R_{2j}(x, D_x) = \left(\sum_{|\alpha|
$$

 $R_{1j}(x, D_x) = \varphi_j(x, D_z) B(x, D_x) - \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x)$

and
 $R_{2j}(x, D_x) = \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x) \left(1 - \psi_j(x, D_x)\right).$

Consequently $R_{1j}(x, D_x)$ denotes the remainder term in Theorem 1 whi where
 $R_{1j}(x, D_x) = \varphi_j(x, D_x) B(x, D_x) - \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x)$

and
 $R_{2j}(x, D_x) = \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)}\right)(x, D_x) \left(1 - \psi_j(x, D_x)\right).$

Consequently $R_{1j}(x, D_x)$ denotes the remainder term in Theore large (in dependence on \varkappa , ε and δ), we get by (5) and (12) $|r_{1j}|_{(l,k)}^{(\varkappa)} \leq c2^{-j(s+\varepsilon)}$. Also Consequently $R_{1j}(x, D_x)$ denotes the remainder term in
by the composition of $\varphi_j(x, D_x)$ and $B(x, D_x)$. Hence
large (in dependence on x, ε and δ), we get by (5) and
the semi-norms of $R_{2j}(x, D_x)$ can be estimated i assumption $\psi_j(x, \xi) = 1$ on supp φ_j and get $|r_{2j}|_{(l,k)}^{(x)} \le c' 2^{-j(\sigma+\epsilon)}$. Finally (8) and the properties of the system $\{\varphi_j(x, \xi)\}_{j=0}^{\infty}$ guarantee that the semi-norms of the first $R_{2j}(x, D_x) = \left(\sum_{|a| \le L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right) (x, D_x) \left(1 - \varphi_j(x, D_x)\right).$
Consequently $R_{1j}(x, D_x)$ denotes the remainder term in Theorem 1 which is obtained
by the composition of $\varphi_j(x, D_x)$ and $B(x, D_x)$. Hence, if we cho pseudodifferential operator on the right-hand side of (27) can be estimated by $R_{2j}(x, D_x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)$ *

<i>c* Consequently $R_{1j}(x, D_x)$ denotes the remain by the composition of $\varphi_j(x, D_x)$ and $B(x)$
 large (in dependence on x, ε and δ), we get the semi-norms of R_{2

$$
|\varphi_j^{(a)}b_{(a)}|_{(l,k)}^{(0)} \leq c_{La} \sup_{(z,\ell)\in\mathcal{Q}_j^{(N,a)}} \{ |b(x,\xi)| \langle \xi \rangle^{-|a|(1-\delta)}\}
$$

and, in view of (10) and (11) , by

$$
\left|\sum_{|\alpha|\n
$$
(28)
$$
$$

In all three cases the constants c, c' and c'' are independent of i .

Step 2. Let $u \in B_{p,q}^{s,a}$ and $x < \min(0, sm)$. Because $\varphi_i \in S^{-\infty}$, it is straightforward to see that we have by Corollary 1 and Theorem 3

$$
\|\varphi_j(x,D_x) B(x,D_x) u \|L_p\| \leq c_N \|u\| B^{s,a}_{p,q}\|^{(\varphi_j)}
$$

if $j = 0, 1, ..., N$. Together with the results of the first step this yields.

$$
\sum_{j=0}^{\infty} 2^{j(s-1)q} ||\varphi_j(x, D_x) \hat{B}(x, D_x) u ||L_p||^q
$$
\n
$$
\leq c_N' ||u || B_{p,q}^{s,a} ||^{(v_j)q}
$$
\n
$$
+ \sum_{j=N+1}^{\infty} 2^{j(s-1)q} (||(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} b_{(\alpha)}) (x, D_x) \psi_j(x, D_x) u ||L_p|| + ||R_{1j}(x, D_x) u + R_{2j}(x, D_x) u ||L_p||)^q
$$
\n
$$
\leq c_N' ||u || B_{p,q}^{s,a} ||^{(v_j)q} + c \sum_{j=N+1}^{\infty} 2^{j s q} ||\psi_j(x, D_x) u ||L_p||^q + c' ||u || H_p^{\times} ||^q
$$
\n
$$
\leq c''||u || B_{p,q}^{s,a} ||^{(v_j)q}
$$

We used again Theorem 2, and Corollary 1 and Theorem 3, respectively. Moreover by Theorem 4 the quasi-norms of u defined by $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ are equivalent. Hence we have proved in this step

$$
||B(x, D_x) u || B^{s-1,a}_{p,q}|| + ||u || B^{s-1,a}_{p,q}|| \leq c ||u || B^{s,a}_{p,q}||.
$$

Step 3. To prove the converse inequality, we use that the symbol of a parametrix $Q(x, D_x)$ can be estimated for any multi-indices α and β , all $x \in \mathbb{R}_x$ ⁿ and all $\xi \in \mathbb{R}_x$ ⁿ with $|\xi| \ge R_{g}$ by

$$
|q^{(a)}_{(\beta)}(x,\,\xi)|\,\leqq\,c_{q\alpha\beta}\,\,|b(x,\,\xi)|^{-1}\,\langle\xi\rangle^{-|a|+|\beta|\delta}\,.
$$

We choose $N^* \ge N$ such that $(x, \xi) \in \Omega_{N^*}^{N,a}$ always implies $|\xi| \ge R_q$. If $j = N^* + 1$, $N^* + 2, \ldots$, we have

$$
\varphi_j(x,D_x)=\left(\sum_{|\alpha|
$$

where.

$$
R_{3j}(x, D_x) = \left(\varphi_j(x, D_x) Q(x, D_x) - \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)}\right)(x, D_x)\right) B(x, D_x)
$$

+
$$
\left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} q_{(\alpha)}\right)(x, D_x) \left(1 - \varphi_j(x, D_x)\right) B(x, D_x)
$$

+
$$
\varphi_j(x, D_x) \left(I - Q(x, D_x) B(x, D_x)\right).
$$

 (29)

Let $x < min (0, sm)$ and $\varepsilon > 0$ be fixed. Then in complete analogy to the estimates Let \varkappa < min (0, sm) and $\varepsilon > 0$ be fixed. Then in complete ana of the first step we can choose *L* in such a way that $|r_{3j}|_{l,k}^{(s)} \leq$ of the first step we can choose L in such a way that $|r_{3i}|_{(l,k)}^{(x)} \leq c2^{-j(s+\epsilon)}$ and

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$$
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$$
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\nLet $x < \min(0, sm)$ and $\varepsilon > 0$ be fixed. Then in complete analogy to the estimates of the first step we can choose L in such a way that $|r_{3j}|_{(i,k)}^{\{n\}} \leq c2^{-j(s+\varepsilon)}$ and $\left|\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} q_{(a)}\right|_{(i,k)}^{\{0\}} \leq c' 2^{-j}$ (31)
\nholds. The constants c and c' are again independent of j . Now straightforward computations as in the second step establish the inequalities $||u| |B_{p,q}^{s,a}||^{(q_j)} \leq c ||B(x, D_x) u |B_{p,q}^{s-1,a}||^{(v_j)} + c_{N^*} ||u | B_{p,q}^{s-1,a}||^{(v_j)}$.

holds. The constants *c* and *c'* are again independent of *j*. Now straightforward com- \int_0^1
 \int_0^1
 \int_0^1
 \int_0^1
 \int_0^1

$$
||u||B_{p,q}^{s,a}||^{ \{ \varphi_{j} \} } \leq c \; ||B(x,D_x)| u||B_{p,q}^{s-1,a}||^{ \{ \varphi_{j} \} } + c_{N^{\bullet}} \; ||u||B_{p,q}^{s-1,a}||^{ \{ \varphi_{j} \} }
$$

and; in view of Theorem 4'

 c' $||u \mid B^{s,a}_{p,q}|| \leq ||B(x, D_x) u \mid B^{s-1,a}_{p,q}|| + ||u \mid B^{s-1,a}_{p,q}||$

with a constant $c' > 0$. Together with (29) this proves the first part of the theorem.

*Step*₄. The other-case may be derived similarly. We change in (27) and (30) the roles of $B(x, D_x)$ and $Q(x, D_x)$. Thus we get

c'
$$
||u||B_{p,q}^{s,a}|| \leq ||B(x, D_x) u||B_{p,q}^{s-1,a}|| + ||u||B_{p,q}^{s-1,a}||
$$

\nconstant $c' > 0$. Together with (29) this proves the first part of the theorem.
\nThe other case may be derived similarly. We change in (27) and (3)
\n $B(x, D_x)$ and $Q(x, D_x)$. Thus we get
\n $\varphi_j(x, D_x) Q(x, D_z) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} q_{(a)}\right)(x, D_x) \psi_j(x, D_x) + R_{4j}(x, D_x)$
\n $\varphi_j(x, D_x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} b_{(a)}\right)(x, D_x) \psi_j(x, D_x) Q(x, D_x) + R_{5j}(x, D_x)$,
\nthe semi-norms of the remainder terms may be estimated as in the for
\nin (28) and (21) we get now the proof of the second part of the theorem.

and

•

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 $\label{eq:1} \begin{array}{l} \mathbf{r}_{\text{in}} = \frac{1}{2} \left(\begin{array}{c} 1 \\ 2 \end{array} \right) \end{array}$

$$
\varphi_j(x,D_x)=\left(\sum_{|s|
$$

where the semi-norms of the remainder terms may be estimated as in the foregoing step. Using (28) and (31) we get now the proof of the second part of the theorem incomplete analogy to the second and third steps I

At the end of this section we will illustrate what variable smoothness or variable order of differentiation means. Let X be an open subset of \mathbb{R}^n . We weaken the condition of equivalence and call two symbols of the class $S(m, m'; \delta)$ *equivalent with respect to X if* end of this section we will
differentiation means. Let
 α of equivalence and call
ect to \overline{X} if
 $0 < c_1 \leq |a(x, \xi)| b^{-1}(x, \xi)|$
all $x \in X$ and $\xi \in \mathbb{R}$." with (a) $\left(x, D_x \right) \psi_i(x, D_x) Q(x, D_x) +$

rder terms may be estimated a

w the proof of the second part

third steps
 E illustrate what variable smood

X be an open subset of \mathbf{R}_i

two symbols of the class $S(n \le c_2 < \infty$
 $|\$ complete analogy to the second and third

At the end of this section we will illust

order of differentiation means. Let X be

condition of equivalence and call two s

with respect to X if
 $0 < c_1 \leq |a(x, \xi)| b^{-1}(x, \xi)| \leq c_2$

$$
0
$$

 $fR.$ For $\varepsilon > 0$ and $u \in S'(\mathbf{R}^n)$ we set $S'(\mathbf{R}^n)$ we ε .

$$
0 < c_1 \leq |a(x,\xi)|b^{-1}(x,\xi)| \leq c_2 < \infty
$$
\n
$$
\text{all } x \in X \text{ and } \xi \in \mathbf{R}_{\xi}^n \text{ with } |\xi| \geq R. \text{ For } \varepsilon > 0 \text{ and } u \in S'.
$$
\n
$$
(\text{supp } u)_\varepsilon = \{x \colon x = y + h \text{ with } y \in \text{supp } u \text{ and } |h| \leq \varepsilon\}.
$$

Theorem 7: Let X be a fixed open subset of \mathbf{R}_x ⁿ and $a(x, \xi)$, $b(x, \xi)$ be two symbols *of* $S(m, m'; \delta)$ which are equivalent with respect to $X \cdot If - \infty < \delta < \infty$, $1 < p < \infty$, (supp u)_{ϵ} = { x : $x = y + h$ with $y \in \text{supp } u$ and $|h| \leq \epsilon$ }.

Theorem 7: Let X be a fixed open subset of \mathbb{R}_x^n and $a(x, \xi)$, $b(x, \xi)$ be the of $S(m, m'; \delta)$ which are equivalent with respect to X . If $-\infty < s < \infty$

$$
c' \ ||u \ ||B^{s,a}_{p,q}|| \leqq ||u \ ||B^{s,b}_{p,q}|| \leqq c \ ||u \ ||B^{s,a}_{p,q}||
$$

 $0 < q \leq \infty$ and $\varepsilon > 0$, then there exists positive constants c' and c such that
 $c' ||u|| B_{p,q}^{s,a}|| \leq ||u|| B_{p,q}^{s,b}|| \leq c ||u|| B_{p,q}^{s,a}||$

holds for all $\hat{u} \in S'(\mathbf{R}^n)$ with $(\text{supp } u)_\varepsilon \subset X$. The constants c' and c are *Of U.*

(supp $u_i = \{x : x = y + n \text{ with } y \in \text{supp } u \text{ and } |n| \ge \varepsilon\}$.

Theorem 7: Let X be a fixed open subset of \mathbb{R}_x^n and $a(x, \xi)$, $b(x, \xi)$ be two symbols $S(m, m'; \delta)$ which are equivalent with respect to X. If $-\infty < s < \infty$, $1 < p < \infty$ $\{\psi_i\}_{i=0}^{\infty} \in \Phi^{M,b}$ belonging to $a(x, \xi)$ -and $b(x, \xi)$, respectively. Let also $\epsilon > 0$ be fixed. Then we can choose a smooth function χ_{ϵ} such that (supp u), $= \{x : x = y + h \text{ with } y \in \text{supp } u\}$

Theorem 7: Let X be a fixed open subset of \mathbb{R}_x^n ,
 $\int S(m, m'; \delta)$ which are equivalent with respect to X.
 $\langle q \leq \infty \text{ and } \varepsilon > 0 \text{, then there exists positive cons}$
 $c' ||u || B_{p,q}^{s,a}|| \leq ||u || B_{p,q}^{s,b}|| \$ **Proof:** In view of Theorem 4 we can fix two arbitrary systems $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{N,a}$ and $\{\varphi_j\}_{j=0}^{\infty} \in \Phi^{M,b}$ belonging to $a(x, \xi)$ -and $b(x, \xi)$, respectively. Let also $\varepsilon > 0$ be fixed.

nen we can choose a s

$$
\Phi^{m,v} \text{ belonging to } a(x, \xi) \text{ and } b(x, \xi), \text{ respectively. Let } \xi
$$

can choose a smooth function χ_{ϵ} such that

$$
\chi_{\epsilon}(x) = \begin{cases} 0 & \text{if } x \in X, \\ 1 & \text{if } x \in X \text{ and dist } (x, \partial X) \ge \varepsilon, \end{cases}
$$

$$
|D^{\gamma}\chi_{\epsilon}(x)| \le c_{\gamma} \varepsilon^{-|\gamma|} \qquad \text{for all multi-indices } \gamma \text{ and } x \in \mathbb{R}_{\mathbf{x}}^{\mathbf{x}}
$$

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holds. By Theorem 1 we have

$$
\varphi_j(x, D_x) \chi_c(x) = \left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} \chi_{\epsilon(a)}\right)(x, D_x) + R_{ji}^L(x, D_x),
$$
\nwith\n
$$
|r_{ji}^L|_{(l,k)}^{(m_1 - L(1 - \delta))} \leq c_{Llk} |\varphi_j|_{(l,k)}^{(m_1)} |\chi_c|_{(l,k)}^{(0)}.
$$
\nNow, in analogy to (23)–(25) and with the same meaning of the we get for all multi-indices α

$$
|r_{j\epsilon}^L|_{(l,k)}^{(m_1-L(1-\delta))}\leq c_{Llk} |{\varphi}_j|_{(l',k)}^{(m_1)}|{\chi}_\epsilon|_{(l,k')}^{(0)}.
$$

Now, in analogy to (23)—(25) and with the same meaning: of the constants *i*_{*0*} and *j*₀' $\langle v, (2, 2) \rangle$.

With $\frac{|r_{\mu}^{L}|_{(i,k)}^{(m_1 - L(1 - \delta))}}{2} \leq c_{Llk} \frac{|q_{j}|_{(i,k)}^{(m_1)}| \chi_{t}|_{(i,k)}^{(0)}}$.

Now, in analogy to (23)—(2 $|r_{j\ell}^{L} \vert_{(l,k)}^{(m_1 - L(1-\delta))} \le c_{Llk} \vert \varphi_j \vert_{(l',k)}^{(m_1)} \vert \chi_{\ell} \vert_{(l,k')}^{(0)}$
analogy to (23) - (25) and with the same me
r all multi-indices α
supp $\langle \varphi_j^{(a)} \chi_{\ell(a)} \rangle$ a supp $\psi_i = \varnothing$ if $j \in \mathcal{J}(i)$,
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$$
\text{supp } (\varphi_j^{(\alpha)} \chi_{\epsilon(\alpha)}) \cap \text{supp } \psi_i = \varnothing \quad \text{if } j \in \mathcal{J}(i),
$$

On Besov Spaces 81\nholds. By Theorem 1 we have\n
$$
\varphi_i(x, D_x) \chi_i(x) = \left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_i^{(\alpha)} \chi_{t(\alpha)} \right) (x, D_x) + R_{ji}^L(x, D_x),
$$
\nwith\n
$$
|r_{ji}^L|_{(l,b)}^{(m_L - L(1-\delta))} \leq c_{Llk} |\varphi_j|_{(l,b)}^{(m_L)} | \chi_i|_{(l,b)}^{(n_L)}.
$$
\nNow, in analogy to (23) - (25) and with the same meaning of the constants i_0 and j_0 we get for all multi-indices α \nsupp $(\varphi_j^{(\alpha)} \chi_{t(\alpha)}) \cap \text{supp } \psi_i = \emptyset$ if $j \notin \mathcal{J}(i)$,\nwhere again\n
$$
\mathcal{J}(i) = \{j : 0 \leq j \leq \max (j_0, i + I - J + M + N + H) \} \text{ if } 0 \leq i \leq i_0,
$$
\n
$$
\mathcal{J}(i) = \{j : \max (J, i + I - (M + N + H)) \leq j + J \leq i + I + M + N + H) \} \text{ if } i > i_0
$$
\nand *H* fulfils $2^{-H} \leq c_1 \leq c_2 \leq 2^{\Pi}$. Then we obtain in the case $j \notin \mathcal{J}(i)$ \n
$$
\psi_i(x, D_x) \varphi_j(x, D_x) \chi_i(x) = \psi_i(x, D_x) \left(\left(\sum_{|\alpha| < L} \frac{1}{\alpha!} \varphi_j^{(\alpha)} \chi_{t(\alpha)} \right) (x, D_x) + R_{ji}^L(x, D_x) \right)
$$
\n
$$
= R_{ij}(\alpha, D_x),
$$
\nwhere for each natural number L^* the semi-norms of $R_{ij}(\alpha, D_x)$ can be estimated by

$$
\mathcal{J}(i) = \{j: \max\left(J, i + I - (M + N + H)\right) \leq j + J \leq i + I + M + N + H)\}
$$
\nif $i > i_0$
\nand H fulfils $2^{-H} \leq c_1 \leq c_2 \leq 2^{\pi}$. Then we obtain in the case $j \notin \mathcal{J}(i)$
\n $\psi_i(x, D_x) \varphi_j(x, D_x) \chi_i(x) = \psi_i(x, D_x) \left(\left(\sum_{|a| < L} \frac{1}{\alpha!} \varphi_j^{(a)} \chi_{\epsilon(a)} \right) (x, D_x) + R_{ji}^L(x, D_x) \right)$ \n
$$
= R_{ij}(x, D_x),
$$
\nwhere for each natural number L^* the semi-norms of $R_{ij\epsilon}(x, D_x)$ can be estimated by\n
$$
|r_{ij\epsilon}|_{(l,k)}^{(m_1+m_1-L^*(1-\delta))} \leq c_{L^*lk} |\psi_i|_{(l',k)}^{(m_1)} |\varphi_j|_{(l',k')}^{(m_2)} |\chi_{\epsilon}|_{(l,k'}^{(0)},)
$$
\nSuppose that (supp u), $\infty \in X$. Then we have always $\chi_i u = u$. Now it is not hard to see that the rest of the proof is a simple modification of the proof of Theorem 4 and

where for each natural number L^* the semi-norms of $R_{ij\epsilon}(x, D_x)$ can be estimated by

$$
|r_{ij\epsilon}|_{(l,k)}^{(m_1+m_2-L^*(1-\delta))}\leq c_{L^{\bullet}/k} \, |\psi_i|_{(l',k)}^{(m_1)} \, |\varphi_j|_{(l'',k')}^{(m_2)} \, |\chi_{\epsilon}|_{(l,k')}^{(0)}.
$$

Suppose that (supp $u)_e \subset X$. Then we have always $\chi_e u = u$. Now it is not hard to see that the rest of the proof is a simple modification of the proof of Theorem 4 and we omit it **^I**

'Remark: Let $a(x, \xi) = \langle \xi \rangle^{m'} + e^{2k} \langle x \rangle \langle \xi \rangle^{m}$ be a symbol of $S(m, m'; \delta)$ as described in the fourth example in Section 2. Furthermore let x_0 be a interior point of the set $\varOmega = \{x\colon \varrho(x) = 0\}$. and $K_{\sigma}(x_0) = \{x\colon |x-x_0| < \sigma\} \subset \Omega$ denotes a neighbourhood of x_0 . Then $a(x,\xi)$ is equivalent to $\langle \xi \rangle^{m'}$ with respect to $K_{\sigma}(x_0)$. Hence an clement *u* of $B_{p,q}^{s,a}$ belongs locally in x_0 to the classical Besov-spaces $B_{p,q}^{sm'}$. On the other hand, if $\rho(x_1) \neq 0$ holds, then the symbol $a(x, \xi)$ is equivalent to $\langle \xi \rangle^{m'}$ with respect to a suitable neighbourhood of x_1 . Thus $u \in B^{s,a}_{p,q}$ belongs locally in x_1 to to $\langle \xi \rangle^{m}$ with respect to a suitable neighbourhood of x_1 . Thus $u \in B_{p,q}^{s,a}$ belongs locally in x_1 to the classical Besov space $B_{p,q}^{s,a}$. Consequently for an element *u* of the space $B_{p,q}^{s,a}$ we may have in different points of \mathbf{R}_x ⁿ locally different smoothness properties. $|r_{ij}|_{(l,k)}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k)}|_{m}^{(n,k$

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