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On the Number of Stable Local Minima of Some Functionals

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Unter Verwendung der Theorien lokal monotoner Operatoren und eigentlicher Fredholm-Operatoren werden Aussagen über die Anzahl der stabilen lokalen Minima einer Klasse von Funktionalen getroffen und entsprechende Beispiele betrachtet.

Применением теории локально монотонных операторов и собственных операторов Фредгольма получаются результаты о числе устойчивых локальных минимумов некоторого класса функционалов и рассматриваются соответствующие примеры.

Using the theories of locally monotone operators and of proper Fredholm operators, results on the number of stable local minima of some functionals are obtained and related examples are considered.

1. Introduction

Let X be a real Hilbert space, X' its dual space, $\langle f, u \rangle$ the value of $f \in X'$ at $u \in X$. Furthermore, let a functional $\Phi \in C^2(X, \mathbf{R})$ be given. For $g \in X'$ we consider the functional Φ_g on X,

$$\Phi_g(u) = \Phi(u) - \langle g, u \rangle, \qquad u \in X, \tag{1.1}$$

and call a local minimum u of Φ_g *stable* if the second variation of Φ_g at u is positive, i.e. $\delta^2\Phi_g(u; h) > 0$ for all $h \in X$, $h \neq 0$. It is our aim to get propositions on the number of stable local minima of the functionals Φ_g and to study the dependence of this number on $g \in X'$.

Because we are seeking stable local minima, it is not necessary to study the generalized Euler equation for Φ_g,

$$\Phi'(u) = g, \tag{1.2}$$

on the whole domain of definition of the operator Φ'. We investigate the restriction of this operator to the so-called *stability region* ST of the functional Φ, which contains all points $u \in X$ for which the second variation of Φ is positive. The operator Φ' is said to be *locally strictly monotone* (cf. definition in Section 2). Under some assumptions on Φ (cf. Section 3) the determination of the stable local minima of Φ_g is equivalent to the solution of the equation

$$\Phi'(u) = g, \qquad u \in ST. \tag{1.3}$$

For the investigation of the solution set of this equation we assume that Φ' is a proper Fredholm operator of index zero (cf. definitions in Section 2). In the literature one can find propositions on the structure of the solution set of (1.2) for such operators (cf. [1, 4, 5, 8]). These propositions were applied to the investigation of some semilinear boundary value problems in [1, 7]. We will prove a proposition of this type for locally strictly monotone operators.

In Section 2 we present propositions on locally strictly monotone proper Fredholm operators of index zero, which we apply to the investigation of the functionals (1.1) in Section 3. In Section 4 we consider some examples. In Example 1, we are able to completely describe the global structure of the region of stability.

2. Locally strictly monotone operators

In this section we consider an operator $L \in C^1(X, X')$.

Definition: The operator $L \in C^1(X, X')$ is called *locally strictly monotone at the point* $u \in X$ if $\langle L'(u) h, h\rangle > 0$ for all $h \in X$, $h \neq 0$. The *monotonicity region* M_L of L is the set of all points of X in which it is locally strictly monotone. L is said to be *locally strictly monotone* if $M_L \neq \emptyset$. L is called a *(nonlinear) Fredholm operator of index zero* if its Fréchet derivative $L'(u)$ is a linear Fredholm operator of index zero for each $u \in X$. The operator L is called *proper* if $L^{-1}(K)$, for any compact set $K \subset X'$, is compact. The point $u \in X$ is called *regular for* L if $L'(u)$ is a linear homeomorphism of X onto X'. (Cf. e.g. [4].)

Proposition: *Let L be a locally strictly monotone Fredholm operator of index zero. Then every point of its monotonicity region M_L is a regular point for L.*

Proof: From the Fredholm property of L and from Banach's theorem it follows that u is a regular point for L if and only if $\dim\left(\operatorname{Ker} L'(u)\right) = 0$. Let $u \in X$ be no regular point for L. Then there exists an $h \in X$, $h \neq 0$, so that $\langle L'(u) h, k\rangle = 0$ for all $k \in X$. For $k = h$ it follows $\langle L'(u) h, h\rangle = 0$, i.e. $u \notin M$ ∎

Remark: Let L be the operator of the Proposition and let $u \in M_L$. According to the Inverse Function Theorem there exist neighbourhoods U of u and V of $L(u)$ such that L is a homeomorphism of U onto V.

We denote by $\tilde{M} \subset M_L$ a subset for the locally strictly monotone operator L. Let $\tilde{L}$ be the restriction of L to $\tilde{M}$. Then $\tilde{L}\colon \tilde{M} \to L(\tilde{M})$ is a continuous surjective mapping. For $g \in L(\tilde{M})$ we denote by $c(g)$ the cardinal number of the set $\tilde{L}^{-1}(g)$. Using these notations we have

Theorem 1: *Let L be a locally strictly monotone proper Fredholm operator of index zero and let $\tilde{M} \subset M_L$. Then*

(i) *$c(g)$ is finite for each $g \in L(\tilde{M}) \setminus L(\partial\tilde{M})$,*
(ii) *$c(\cdot)$ is constant on every connected component of $L(\tilde{M}) \setminus L(\partial\tilde{M})$.*

Proof: (i) We show that for $g \in L(\tilde{M})$ with $c(g)$ infinite there holds $g \in L(\partial\tilde{M})$. Let $g \in L(\tilde{M})$. If $c(g)$ is infinite, then there exists a sequence $\{v_l\} \subset \tilde{M}$ with $L(v_l) = g$ $(l = 1, 2, \ldots)$. Because $\{g\}$ is compact and L is proper, there exist a subsequence $\{v_{l'}\}$ of $\{v_l\}$ and $v \in \tilde{M} \cup \partial\tilde{M}$ with $v_{l'} \to v$ in X for $l' \to \infty$. L is continuous, consequently $L(v) = g$. If $v \in \tilde{M}$, then there exists a neighbourhood U of v with $U \cap L^{-1}(g) = \{v\}$, by our Remark. But this is a contradiction to $v_{l'} \to v$ for $l' \to \infty$. Consequently, $v \in \partial\tilde{M}$, and $g \in L(\partial\tilde{M})$.

(ii) Let $g \in L(\tilde{M}) \setminus L(\partial\tilde{M})$. Because of $L(\tilde{M}) \setminus L(\partial\tilde{M}) = L(\operatorname{int}\tilde{M}) \setminus L(\partial\tilde{M})$, there holds $g \in L(\operatorname{int}\tilde{M})$. According to (i), we have $\tilde{L}^{-1}(g) = \{u_1 \ldots, u_k\}$. We choose for each $u_i \in \tilde{L}^{-1}(g)$ a neighbourhood $U_i \subset \tilde{M}$ in such a way that L is a homeomorphism of U_i onto $L(U_i)$. According to the Remark, this is always possible, because $g \in L(\operatorname{int}\tilde{M})$ yields $u_i \in \operatorname{int}\tilde{M}$. We can suppose $U_i \cap U_j = \emptyset$ for $i \neq j$; if necessary, we can take smaller neighbourhoods U_i so that this holds. Now, let $W = L(U_1) \cap \ldots \cap L(U_k)$. We show that there is a neighbourhood V of g in W so that $c(\cdot)$ is constant on V. If there were no such V, then there would exist a sequence $\{v_l\} \subset \tilde{M}$ with $L(v_l) \to g$ in X' for $l \to \infty$ and $v_l \notin \tilde{L}^{-1}(W)$, $l \in \mathbb{N}$. Because L is a proper operator and $\{L(v_l)\} \cup \{g\}$ is a compact set, the sequence $\{v_l\}$ contains a subsequence $\{v_{l'}\}$ with $v_{l'} \to v$ in X for $l' \to \infty$. L is continuous, consequently $L(v) = g$. According to $\{v_{l'}\} \subset \tilde{M}$, v is in the closure of $\tilde{M}$. By $L(v) = g$ and $g \notin L(\partial\tilde{M})$ we have $v \in \operatorname{int} \tilde{M}$. Consequently, $v \in \tilde{L}^{-1}(g)$, which is a contradiction to the fact that from $v_{l'} \to v$ for $l' \to \infty$, $v_{l'} \notin \tilde{L}^{-1}(W)$, and $\tilde{L}^{-1}(W)$ being an open set it follows that $v \notin \tilde{L}^{-1}(W)$.

Thus, we have shown that $c(\cdot)$ is constant on a neighbourhood V of an arbitrary point $g \in L(\tilde{M}) \setminus L(\partial\tilde{M})$, which yields that $c(\cdot)$ is constant on every connected component of $L(\tilde{M}) \setminus L(\partial\tilde{M})$ ▮

In the following theorems we make additional assumptions on the monotonicity region of the operator L.

Theorem 2: *Let L be a locally strictly monotone proper Fredholm operator of index zero with monotonicity region $M_L = X$. Then L is a diffeomorphism of X onto X', in particular, $c(g) = 1$ for every $g \in X'$.*

Proof: $M_L = X$ yields that every point of X is regular for L, by the Proposition. Because $L \in C^1(X, X')$ is a proper operator, the theorem follows from the theorem of Banach and Mazur [4: p. 221; 6] ▮

Theorem 3: *Let L be a locally strictly monotone proper Fredholm operator of index zero and let $\tilde{M} \subset M_L$ be a convex subset. Then $\tilde{L}: \tilde{M} \to L(\tilde{M})$ is a homeomorphism, in particular, $c(g) = 1$ for every $g \in L(\tilde{M})$.*

Proof: $\tilde{L}$ is surjective by construction. We show that $\tilde{L}$ is injective. Let $u_1, u_2 \in \tilde{M}$, $u_1 \neq u_2$, with $\tilde{L}(u_1) = \tilde{L}(u_2)$. Then $L(u_1) = L(u_2)$. On the other hand, according to Taylor's theorem $L(u_2) - L(u_1) = \int_0^1 L'(u_1 + sh)\, h\, ds$, where $h = u_2 - u_1$, $h \neq 0$. This implies

$$\langle L(u_2) - L(u_1), h\rangle = \int_0^1 \langle L'(u_1 + sh)\, h, h\rangle\, ds. \tag{2.1}$$

Because of the convexity of $\tilde{M}$, we have $u_1 + sh \in \tilde{M}$ for all $s \in [0, 1]$. Consequently, $\langle L'(u_1 + sh)\, h, h\rangle > 0$ for all $s \in [0, 1]$ so that, by (2.1), it follows $\langle L(u_2) - L(u_1), h\rangle > 0$, which contradicts $L(u_1) = L(u_2)$.

$\tilde{L}$ is continuous because of the continuity of L. The continuity of $\tilde{L}^{-1}$ follows from the fact that L is a local homeomorphism on M_L ▮

3. Application to the calculus of variations

We want to apply the theory of Section 2 to investigate the functionals (1.1). To this end, let the Hilbert space X be compactly imbedded in a real Hilbert space Y. We denote the norms in X, Y by $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. Let $\Phi \in C^2(X, R)$. The operator $\Phi': X \to X'$ is called the *Lagrange operator* of the functional Φ; $\Phi \in C^2(X, \mathbb{R})$ yields $\Phi' \in C^1(X, X')$. We assume that Φ satisfies the following additional assumptions:

(A) For each $u \in X$ and for each $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that

$$|\langle \Phi''(u + k)\, h, h\rangle - \langle \Phi''(u)\, h, h\rangle| \leqq \varepsilon\, \|h\|_X^2 \quad \text{for all } h, k \in X,\ \|k\|_X < \eta(\varepsilon).$$

(B) Gårding inequality: For each $u \in X$ there exist constants $\sigma > 0$, $\varrho > 0$ such that

$$\langle \Phi''(u)\,h, h\rangle \geqq \sigma \|h\|_X^2 - \varrho \|h\|_Y^2 \quad \text{for all } h \in X.$$

(C) Φ' is a proper Fredholm operator of index zero.

We define a functional $\gamma: X \to \mathbf{R}$ by

$$\gamma(u) = \min_{h \in B} \langle \Phi''(u)\,h, h\rangle, \qquad B = \{h \in X \mid \|h\|_Y = 1\}. \tag{3.1}$$

According to (A), (B), $\gamma(u)$ is well-defined for every $u \in X$, cf. [10: p. 200–203]. The set

$$ST = \{u \in X \mid \gamma(u) > 0\}$$

is called the *stability region* of the functional Φ (cf. [2]). If $ST \neq \emptyset$, then Φ' is a locally strictly monotone proper Fredholm operator of index zero with the monotonicity region ST.

An element $u^\circ \in X$ is a stable local minimum of the functional Φ_g if and only if it is a solution of the equation (1.3).

Indeed, at first let u^0 be a stable local minimum of Φ_g. Then u^0 is a solution of the Euler equation $\Phi_g'(u^0) = 0$, therefore, $\Phi'(u^0) = g$. Furthermore, we have

$$\delta^2\Phi_g(u^0; h) = \langle \Phi''(u^0)\,h, h\rangle \qquad \text{for all } h \in X. \tag{3.2}$$

This implies $u^0 \in ST$. Therefore, u^0 is a solution of (1.3).

Conversely, let u^0 be a solution of (1.3). Then the functional Φ_g has a local minimum at u^0. This follows from the facts that, because of $\Phi'(u^0) = g$, u^0 satisfies the Euler equation $\Phi_g'(u^0) = 0$ and that the sufficient Jacobi criterion is fulfilled, because of $\gamma(u^0) > 0$ (cf. [10: p. 200–203]). From $u^0 \in ST$ it follows that u^0 is a stable local minimum of Φ_g, by (3.2) ∎

4. Examples

Let G be a bounded domain in $\mathbf{R}^n$ with sufficient regular boundary. We denote by $L^p(G)$ the space of p-integrable functions with the usual norm $\|\cdot\|_{L^p}$; by $\|\cdot\|_0$ we denote the norm in $L^2(G)$. Furthermore, let $H^1(G)$ be the Sobolev space with the norm $\|\cdot\|_1$, $\|u\|_1^2 = \|u\|_0^2 + (\|\partial u/\partial x^1\|_0^2 + \dots + \|\partial u/\partial x^n\|_0^2)$, and $H_0^1(G)$ the closure of the space $C_0^\infty(G)$ in $H^1(G)$. The dual space of $H_0^1(G)$ we denote by $H^{-1}(G)$, its norm by $\|\cdot\|_{-1}$. Using the notation of the preceding sections, we set $X = H_0^1(G)$, $X' = H^{-1}(G)$, $Y = L^2(G)$. Looking at an element $g \in H^{-1}(G)$ as a regular distribution we can write (cf. [8: p. 262]) $\langle g, u\rangle = \int_G gu\,dx$, $u \in H_0^1(G)$. We set

$$\lambda_1 = \min_{h \in B} \int_G |\nabla h|^2\,dx, \qquad B = \{h \in H_0^1(G) \mid \|h\|_0 = 1\}.$$

The minimum λ_1 exists (cf. [10: p. 200–203]), furthermore,

$$\int_G |\nabla h|^2\,dx \geqq \lambda_1 \int_G h^2\,dx \qquad \text{for all } h \in H_0^1(G).$$

Example 1: Let $1 \leqq n \leqq 6$, let $F \in C^3(\mathbf{R}, \mathbf{R})$ be a function such that, with some constants $K > 0$ and $L > 0$,

$$F'(t) \geqq 0, \qquad |F''(t)| \leqq K, \qquad |F'''(t)| \leqq L \qquad \text{for all } t \in \mathbf{R}, \tag{4.1}$$

and let the functional Φ be defined by

$$\Phi(u) = \int_G \left[\frac{1}{2} |\nabla u|^2 + F(u)\right] dx \qquad \left(u \in H_0^1(G)\right).$$

Then the functionals (1.1) are of the form

$$\Phi_g(u) = \int_G \left[\frac{1}{2} |\nabla u|^2 + F(u) - gu\right] dx \qquad \left(g \in H^{-1}(G)\right).$$

Our goal is to obtain a global description of the region of stability.

Because of the assumptions $n \leqq 6$ and (4.1), there holds $\Phi \in C^2\left(H_0^1(G), \mathbf{R}\right)$. The Fréchet derivatives of Φ are given by

$$\left.\begin{aligned} \langle\Phi'(u), h\rangle &= \int_G [\nabla u \, \nabla h + F'(u) \, h] \, dx \\ \langle\Phi''(u) \, h, k\rangle &= \int_G [\nabla h \, \nabla k + F''(u) \, hk] \, dx \end{aligned}\right\} \qquad \left(h, k \in H_0^1(G)\right).$$

The functional Φ satisfies (A) (this can be proved using the Sobolev imbedding theorems and the continuity of the Nemyckii operator; one can find a detailed proof in [3]). The validity of the Gårding inequality (B) follows immediately if one takes into account the condition $|F''(t)| \leqq K$. In [3] one can find the proof that the Lagrange operator Φ' satisfies the assumption (C). Here we give only the ideas of the proof. First one uses that

(i) Φ' is weakly coercive, i.e. $\|\Phi'(u)\|_{-1} \to \infty$ as $\|u\|_1 \to \infty$;

(ii) $\Phi' = D + P$, where

$$D: H_0^1(G) \to H^{-1}(G), \qquad \langle Du, h\rangle = \int_G \nabla u \, \nabla h \, dx \qquad \left(h \in H_0^1(G)\right),$$

is a homeomorphism of $H_0^1(G)$ onto $H^{-1}(G)$ and, therefore, a proper operator, and

$$P: H_0^1(G) \to H^{-1}(G), \qquad \langle P(u), h\rangle = \int_G F'(u) \, h \, dx \qquad \left(h \in H_0^1(G)\right),$$

is a compact operator. Then, according to [4: p. 103], Φ' is proper. The assertion that Φ' is a Fredholm operator of index zero follows from the fact that $\Phi''(u): H_0^1(G) \to H^{-1}(G)$ for each $u \in H_0^1(G)$ is a compact perturbation of a linear homeomorphism of $H_0^1(G)$ onto $H^{-1}(G)$.

Under additional assumptions on the nonlinearity F we can get more information on the stability region ST of the functional Φ. We consider two special cases for F.

a) Let F satisfy (4.1) and $F''(t) > -\lambda_1$ for all $t \in \mathbf{R}$. Then

$$\langle\Phi''(u) \, h, h\rangle \equiv \int_G [|\nabla h|^2 + F''(u) \, h^2] \, dx > \int_G [|\nabla h|^2 - \lambda_1 h^2] \, dx \geqq 0$$

for all $h \in H_0^1(G)$, $h \neq 0$. According to (3.2), from this follows $\gamma(u) > 0$, therefore, $ST = H_0^1(G)$. Because of Theorem 2 and the investigations of Section 3 the functional Φ_g has for each $g \in H^{-1}(G)$ one and only one stable local minimum in $H_0^1(G)$.

b) We assume $1 \leqq n \leqq 4$. Let $F \in C^4(\mathbf{R}, \mathbf{R})$ satisfy (4.1) and

$$\lim_{t\to-\infty} F''(t) = l < -\lambda_1, \qquad \lim_{t\to+\infty} F''(t) = 0,$$

$$F'''(t) > 0, \qquad |F^{(4)}(t)| \leqq N = \text{const.} \qquad \text{for all } t \in \mathbf{R}.$$

A subset A of a Banach space Z is called a C^1-*manifold of codimension* 1 if for every point $u^0 \in A$ there exist a neighbourhood U of u^0 in Z and a functional $\Gamma \in C^1(U, \mathbf{R})$ such that $\Gamma'(u^0) \neq 0$ and $A \cap U = \{u \in U \mid \Gamma(u) = 0\}$. Now under our assumptions the following holds (the proofs are carried out in [3], using methods of AMBROSETTI and PRODI [1]):

(i) ∂ST *is a closed connected* C^1-*manifold of codimension* 1, $H_0{}^1(G) \setminus \partial ST$ *has exactly two connected components and* ST *is one of them.*

(ii) $\Phi'(\partial ST)$ *is a closed connected* C^1-*manifold of codimension* 1, $H^{-1}(G) \setminus \Phi'(\partial ST)$ *has exactly two connected components and* $\Phi'(ST)$ *is one of them;* $\Phi'(ST) \cap \Phi'(\partial ST) = \emptyset$.

(iii) $c(g) = 1$ *for each* $g \in \Phi'(ST)$.

Example 2: Let $F \in C^3(\mathbf{R}, \mathbf{R})$ be a function with $F'(t)\, t \geqq 0$ for all $t \in \mathbf{R}$. For $u \in L^1(G)$ set $\overline{u} = |G|^{-1} \int_G u\, dx$ with $|G| = \int_G dx$. Then

$$|\overline{u}| \leqq |G|^{-1/2}\, \|u\|_0 \qquad \text{for each } u \in H_0{}^1(G). \tag{4.2}$$

This follows immediately using the Schwarz inequality:

$$|\overline{u}| = |G|^{-1} \left|\int_G u\, dx\right| \leqq |G|^{-1}\, |G|^{1/2}\, \|u\|_0 = |G|^{-1/2}\, \|u\|_0.$$

Now let us set

$$\Phi(u) = \int_G \left[\frac{1}{2}\, |\nabla u|^2 + F(\overline{u})\right] dx \qquad \big(u \in H_0{}^1(G)\big).$$

Then the functionals (1.1) are of the form

$$\Phi_g(u) = \int_G \left[\frac{1}{2}\, |\nabla u|^2 + F(\overline{u}) - gu\right] dx \qquad \big(g \in H^{-1}(G)\big).$$

Under our assumptions it holds that $\Phi \in C^2\big(H_0{}^1(G), \mathbf{R}\big)$. The Fréchet derivatives of Φ are given by

$$\left.\begin{aligned} \langle \Phi'(u), h\rangle &= \int_G [\nabla u\, \nabla h + F'(\overline{u})\, \overline{h}]\, dx \\ \langle \Phi''(u)\, h, k\rangle &= \int_G [\nabla h\, \nabla k + F''(\overline{u})\, \overline{h}\, \overline{k}]\, dx \end{aligned}\right\} \quad \big(h, k \in H_0{}^1(G)\big).$$

The functional Φ satisfies (A) (this can be proved using the Sobolev imbedding theorems and the continuity of the Nemyckii operator, if one takes into account (4.2); for a detailed proof cf. [3]). The Gårding inequality (B) follows immediately with the help of (4.2). The proof that Φ' satisfies the assumption (C) is analogous to Example 1.

We will now investigate the stability region ST of the functional Φ. We define a function $\Gamma: \mathbf{R} \to \mathbf{R}$ by

$$\Gamma(s) = \min_{h\in B} \int_G [|\nabla h|^2 + F''(s)\, \overline{h}^2]\, dx, \qquad B = \{h \in H_0{}^1(G) \mid \|h\|_0 = 1\}.$$

Lemma: (i) $\Gamma(s) > 0$ *if* $F''(s) > -\lambda_1$. (ii) $\Gamma(s) \leqq 0$ *if* $F''(s) \leqq -\lambda_1$.

Proof: (i) Let $h \in B$. If $F''(s) > 0$, then

$$\int_G [|\nabla h|^2 + F''(s)\,\bar{h}^2]\,dx \geqq \int_G |\nabla h|^2\,dx > 0$$

if $0 \geqq F''(s) > -\lambda_1$, then by (4.2)

$$\int_G [|\nabla h|^2 + F''(s)\,\bar{h}^2]\,dx \geqq \int_G |\nabla h|^2\,dx + |G|\,F''(s)\,|G|^{-1}\,\|h\|_0^2$$

$$\geqq \lambda_1 + F''(s) > 0.$$

Because the minimum $\Gamma(s)$ is attained, the assertion is proved.

(ii) Let $F''(s) < -\lambda_1$, $h \in B$ with $\lambda_1 = \int_G |\nabla h|^2\,dx$. Then

$$\Gamma(s) = \int_G [|\nabla h|^2 + F''(s)\,\bar{h}^2]\,dx \leqq \int_G [|\nabla h|^2 - \lambda_1 \bar{h}^2]\,dx$$

$$\leqq \int_G |\nabla h|^2\,dx - \lambda_1\,\|h\|_0^2 = 0. \blacksquare$$

This lemma yields the characterization

$$u \in ST \text{ if and only if } F''(\bar{u}) > -\lambda_1. \tag{4.3}$$

We define sets $S \subset \mathbf{R}$ and $E_s \subset H_0^1(G)$ for each $s \in \mathbf{R}$ by

$$S = \{s \in \mathbf{R} \mid F''(s) > -\lambda_1\} \text{ and } E_s = \{u \in H_0^1(G) \mid \bar{u} = s\}.$$

According to (4.3), $ST = \cup\,\{E_s \mid s \in S\}$. Because of the continuity of F'', S is the union of countably many open intervals. Let $\tilde{S}$ be a connected component of S, i.e. a maximal open interval which is contained in S. We define

$$\widetilde{ST} = \bigcup_{s \in \tilde{S}} E_s. \tag{4.4}$$

This set possesses the following properties:

(i) $\widetilde{ST}$ *is convex.*
To prove this, let $u_1, u_2 \in \widetilde{ST}$. Then $\bar{u}_1, \bar{u}_2 \in \tilde{S}$. For each $\sigma \in [0, 1]$ there holds $\sigma\bar{u}_1 + (1 - \sigma)\,\bar{u}_2 \in \tilde{S}$. Therefore, we have $\sigma u_1 + (1 - \sigma)\,u_2 \in \widetilde{ST}$.

(ii) $\widetilde{ST}$ *is a connected component of* ST, *i.e. there exists no connected set* $A \subset ST$ *with* $\widetilde{ST} \subset A$, $\widetilde{ST} \neq A$.

If such a set A existed, then we could choose $u \in A$ with $u \notin \widetilde{ST}$. Because of the connectedness of A follows $\bar{u} \in \tilde{S}$, which contradicts $u \notin \widetilde{ST}$, by (4.4).

We now apply the theory of Sections 2 and 3. Because every connected component $\widetilde{ST}$ of the stability region ST is convex, by Theorem 2.5 the restriction of Φ' to $\widetilde{ST}$ is a homeomorphism of $\widetilde{ST}$ onto $\Phi'(\widetilde{ST})$. In other words, for arbitrary $g \in H^{-1}(G)$ the functional Φ_g has at most one local minimum in $\widetilde{ST}$. Let $c \in \mathbf{N} \cup \{\infty\}$ be the number of connected components of the stability region ST. Then the functional Φ_g for arbitrary $g \in H^{-1}(G)$ has at most c local minima in ST. According to (4.4), we can easily determine c by determining the number $\hat{c}$ of the connected components of S, because we have $c = \hat{c}$.

REFERENCES

[1] AMBROSETTI, A., and G. PRODI: On the inversion of some differentiable mappings with singularities between Banach spaces. Ann. Mat. Pura Appl. (4) **93** (1972), 231–246.

[2] BECKERT, H.: Variationsrechnung und Stabilitätstheorie. Math. Nachr. **75** (1976), 47–59.

[3] BENKERT, F.: Über den globalen Lösungszusammenhang einiger Scharen von Variationsproblemen. Dissertation. Leipzig: Karl-Marx-Universität 1985.

[4] BERGER, M. S.: Nonlinearity and functional analysis. New York–San Francisco–London: Academic Press 1977.

[5] GEBA, K.: The Leray Schauder degree and framed bordism. In: La théorie des points fixes et ses applications á l'analyse. Séminaire de Mathématiques Supérieures 1973. Montreal: Presses de l'Université 1975.

[6] PLASTOCK, R. A.: Nonlinear Fredholm maps of index zero and their singularities. Proc. Amer. Math. Soc. **68** (1978), 317–322.

[7] QUITTNER, P.: Singular sets and number of solutions of nonlinear boundary value problems. Comment. Math. Univ. Carolin. **24** (1983), 371–385.

[8] SAUT, J. C., and R. TEMAN: Generic properties of nonlinear boundary value problems. Comm. Part. Diff. Equ. **4** (1979), 293–319.

[9] WLOKA, J.: Partielle Differentialgleichungen. Sobolevräume und Randwertaufgaben. Leipzig: BSB B. G. Teubner Verlagsgesellschaft 1982.

[10] ZEIDLER, E.: Nonlinear functional analysis and its applications, Vol. III. New York–Berlin–Heidelberg–Tokyo: Springer-Verlag 1985.

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