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A Geometric Maximum Principle for Surfaces of Prescribed Mean Curvature in Riemannian Manifolds⁰)

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Sei M eine dreidimensionale Riemannsche Mannigfaltigkeit und f eine Fläche vorgeschriebener mittlerer Krümmung, die in einer Menge $J \cup S \subset M$ mit S als Rand beschränkter mittlerer Krümmung $\mathfrak{H}$ liegt. Unter natürlichen Bedingungen wird bewiesen, daß S völlig in J liegt. Als unmittelbare Konsequenz dieses Resultats ergibt sich eine hinreichende Bedingung für die Existenz von Minimalflächen in einer Menge $J \subset \mathbb{R}^3$, deren Rand S nicht $\mathfrak{H}$-konvex ist.

Пусть M трёхмерное риманoвое многообразие и пусть f поверхность предписанной средней кривизны и лежащая в множестве $J \cup S \subset M$ с краём S ограниченной средней кривизны $\mathfrak{H}$. При естественных условиях доказывается, что S лежит полностью в J. Как непосредственное следствие получается достаточное условие существования минимальных поверхностей в множестве $J \subset \mathbb{R}^3$ край которых не $\mathfrak{H}$-выпукло.

Let M be a three-dimensional Riemannian manifold and let f be some surface of prescribed mean curvature which is restricted to lie in some set $J \cup S \subset M$ with boundary S of bounded mean curvature $\mathfrak{H}$. Assuming natural conditions, we prove that the image of f lies completely in J. An immediate consequence of this result is a sufficient condition for the existence of minimal surfaces in a set $J \subset \mathbb{R}^3$, the boundary S of which is not $\mathfrak{H}$-convex.

0. Introduction

In this paper we shall derive an inclusion theorem for surfaces f of prescribed mean curvature H in a three-dimensional Riemannian manifold M. The decisive quantities which are involved in our result are the absolute values of both, the prescribed mean curvature H and the mean curvature $\mathfrak{H}$ of the boundary S of some including set J, the area of the surface f and the distance from the boundary of f to S. To be more precise, if $f: \Omega \to J \cup S \subset M$ is some conformally parametrized surface which is of prescribed mean curvature H in the interior J; then there exists some constant $c = c(\Lambda, \tau, \varkappa, R)$ depending only on $\Lambda = \max \{|H|_0, |\mathfrak{H}|_0\}$, the injectivity radius τ, an upper bound for the sectional curvature $\varkappa$ and the distance $R = \operatorname{dist}_M \big(f(\partial\Omega), S\big)$ such that $f(\bar{\Omega}) \subset \operatorname{int} J$ provided the area of f is smaller than c.

Thus the main emphasis of the theorem, which also distinguishes this result from the $\mathfrak{H} - \Lambda$ maximum principle by Hildebrandt [11], and Gulliver and Spruck [7], is the fact that the inward mean curvature $\mathfrak{H}$ of the boundary S need not be greater than the absolute value of the prescribed mean curvature H. In particular we allow obstacles S the (inward) mean curvature $\mathfrak{H}$ of which is negative. Exterior domains are therefore typical examples which fit in our framework.

The analytic tool for the proof of our theorem is an estimate by Grüter [5], who used a method from geometric measure theory to prove a pulled back version of the standard monotonicity formula.

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In view of certain existence-regularity results of HILDEBRANDT and KAUL [13] and HILDEBRANDT [12] a direct consequence of the inclusion theorem is a new existence result for minimal surfaces in $\mathbb{R}^3$, which are restricted to lie in J (Proposition 1). Again we are interested in cases where S is not $\mathfrak{H}$-convex (i.e. $\mathfrak{H} \geq 0$ is not satisfied). If $A_{\Gamma,J}$ is the infimum of area of surfaces spanned by the curve Γ in J, then the condition is that $|\mathfrak{H}|_0 < \{-1/4R^2 + \pi/2A_{\Gamma,J}\}^{1/2} - 1/2R$. Two examples illustrate this result. Another application of Theorem 1 appears in Proposition 2, which sharpens a result of BÖHME, HILDEBRANDT and TAUSCH [1: Theorems 12, 13] concerning the existence of extremals for the integral $E(x) = \int x_3 |\nabla x(u, v)|^2 \, du \, dv$. Again a smallness condition on the quantity $A_{\Gamma,J}$ implies existence of an extremal for E.

A further application is treated in [3].

1. Notations and results

We shall adopt here the definition of H-surfaces in Riemannian manifolds given by HILDEBRANDT and KAUL [13], but, in short, repeat the basic concept. Let M be a complete, connected and orientable Riemannian manifold of differentiability class three and $\Omega \subset \mathbb{R}^2$ be an open, connected and bounded set with Lipschitz boundary $\partial\Omega$ and with standard Euclidean metric, put $w = u + iv$, and $u = u_1$, $v = u_2$. The Levi-Civita connection on M will be denoted by D, furthermore $d\colon M \times M \to \mathbb{R}$ stands for the distance function on M and $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, denote the norm and the scalar product on T_pM, respectively. A function $f\colon \Omega \to M$ belongs to the class $H_2^1(\Omega, M)$ if $f \in H_2^1(\Omega, \mathbb{R}^N)$ and $f(\Omega - N) \subset M$, $N \subset \Omega$ denoting some null set (cp. [5: (2.1) Def.]). Here M is thought to be isometrically embedded into some $\mathbb{R}^N$, and $H_2^1(\Omega, \mathbb{R}^N)$ stands for the Sobolev space of $L_2(\Omega, \mathbb{R}^N)$-functions the derivatives of which are again in L_2. The classes $H_2^2(\Omega, M)$ are defined similarly.

In the following let M be three-dimensional and $\varphi\colon U \to \mathbb{R}^3$ denote some chart of an open set $U \subset M$. Then x stands for the representation of f corresponding to that chart. Furthermore, with respect to these coordinates, g_{ik} and Γ_{ij}^k denote the coefficients of the metric and the Christoffel symbols, respectively. Put $g := \det g_{ik}$ and $g^{ik} := (g_{ik})^{-1}$. Consider now a function $\sigma \in C^2(M, \mathbb{R})$ and its level surface $S_c := \{p \in M : \sigma(p) = c\}$, for $c \in \mathbb{R}$, as well as its "interior" $J_c := \{p \in M : \sigma(p) < c\}$. Note that S_c is regular at p, provided $\operatorname{grad}_p \sigma \neq 0$. As usual the gradient vector field $\operatorname{grad}_p \sigma$ for $p \in M$ is given by $\langle \operatorname{grad}_p \sigma, V\rangle = V\sigma$ for any $V \in T_pM$. Also the Hessian tensor Hess σ, the Hessian bilinear form hess σ and the Laplacian Lap σ are defined by

$$\operatorname{Hess}_p \sigma V = D_V \operatorname{grad}_p \sigma, \qquad p \in M,\ V \in T_pM,$$

$$\operatorname{hess}_p \sigma(V, W) = \langle \operatorname{Hess}_p \sigma V, W\rangle, \quad V, W \in T_pM,$$

$$\operatorname{Lap}_p \sigma = \operatorname{trace}(\operatorname{Hess}_p \sigma).$$

The mean curvature $\mathfrak{H}(p)$ of S_c at p with respect to the "interior normal" $-\operatorname{grad}_p \sigma / \|\operatorname{grad}_p \sigma\|$ is defined by

$$\mathfrak{H}(p) = \frac{1}{2\,\|\operatorname{grad}_p \sigma\|} \left\{ \operatorname{Lap} \sigma(p) - \frac{1}{\|\operatorname{grad}_p \sigma\|^2} \operatorname{hess}_p \sigma(\operatorname{grad}_p \sigma, \operatorname{grad}_p \sigma) \right\}.$$

Consider a mapping $f \in H^2_{2,\mathrm{loc}}(\Omega, M) \cap H_2^1(\Omega, M)$ and let $H = H(f)$ be a function of class $L_\infty(\Omega, \mathbb{R})$. Then f is called an H-surface if it satisfies the equation

$$D_{U_\alpha} f_*(U_\alpha) = 2H\big(f(w)\big)\, f_*(U_1) \times f_*(U_2) \tag{1}$$

and $\|f_*(U_1)\| = \|f_*(U_2)\|$, $\langle f_*(U_1), f_*(U_2)\rangle = 0$ a.e. in Ω. Here U_1, U_2 denote the basis fields with respect to u_1, u_2 and $f_*\colon T\Omega \to TM$ is the induced mapping of the tangent bundles. Moreover "$\times$" denotes the cross product on T_pM. Let $w_0 \in \Omega$ and $\Omega_1 \subset \Omega$ be a neighbourhood of w_0 such that $f(\Omega_1)$ is contained in some coordinate neighbourhood $U \subset M$ with some chart $\varphi\colon U \to \mathbb{R}^3$. If $x = \varphi \circ f$ is the representation of f, then (1) implies

$$\Delta x^l + \Gamma^l_{ij} x^i_{u_\alpha} x^j_{u_\alpha} = 2H\bigl(x(w)\bigr)\, g^{lk} \sqrt{g}(x_u \wedge x_v)_k\,^{1)} \qquad \text{a.e. on } \Omega_1 \ (l = 1, 2, 3),$$

$$g_{ij}(x)\, x_u{}^i x_u{}^j = g_{ij}(x)\, x_v{}^i x_v{}^j, \qquad g_{ij}(x)\, x_u{}^i x_v{}^j = 0 \qquad \text{a.e. on } \Omega_1.$$

Here $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ denotes the (Euclidean) Laplacian. Note that H-surfaces are also weak H-surfaces in the sense of [5: cp. (3.5) Def.]. Moreover we use the abbreviations $D(f) = \int_\Omega g_{ij}(x)\, D_\alpha x^i D_\alpha x^j \, du\, dv$ and $\Gamma = f(\partial\Omega)$ to denote the Dirichlet integral and the boundary of f, respectively. Finally, put $R = \operatorname{dist}(\Gamma, S_c) = \inf \{d(\xi, \eta)\colon \xi \in \Gamma, \eta \in S_c\}$; $A(f) =$ area of f, $\Lambda = \max \{|H|_{0,\Omega}, |\mathfrak{H}|_{0,S_c}\}$, where

$$|H|_{0,\Omega} = \operatorname*{ess\,sup}_{\Omega} \bigl|H\bigl(f(w)\bigr)\bigr| \qquad \text{and} \qquad |\mathfrak{H}|_{0,S_c} = \sup_{S_c} |\mathfrak{H}(\xi)|.$$

Let τ be the injectivity radius on $f(\Omega)$ and $\varkappa$ denote an upper bound for the sectional curvature on $f(\Omega)$ (for a precise definition and further properties concerning the injectivity radius and the sectional curvature we refer to GROMOLL, KLINGENBERG and MEYER [4].

Theorem 1: *Let Ω, M, σ, J_c, S_c be defined as above. Assume that* $f \not\equiv$ const *is some surface of class $H_2^1(\Omega, M) \cap C^0(\bar\Omega, M) \cap H^2_{2,\mathrm{loc}}(\Omega, M)$ with the following properties:*

(i) $f(\Omega) \subset J_c \cup S_c$,

(ii) $D_{U_\alpha} f_*(U_\alpha) = 2H(f)\, f_*(U_1) \times f_*(U_2)$ *a.e. on* $\Omega' := \Omega - \Omega^*$, *where* $\Omega^* = f^{-1}(S_c)$,

(iii) $\|f_*(U_1)\| = \|f_*(U_2)\|$, $\langle f_*(U_1), f_*(U_2)\rangle = 0$ *a.e. on* Ω.

Then $f(\bar\Omega) \subset J_c$ *provided that either of the cases* I *or* II *holds:*

(I) $\varkappa \leqq 0$ *and* $A(f) < \dfrac{\pi\varrho^2}{1 + 2\Lambda\varrho + 2^{-1}(2\Lambda\varrho)^2}$, $\varrho := \min\{R, \tau\}$.

(II) $\varkappa > 0$ *and* $A(f) < \dfrac{\pi\varkappa^{-1}}{\dfrac{1}{\sin^2(\varrho\sqrt{\varkappa})} + \dfrac{2\Lambda\varrho}{\sin^2(\varrho\sqrt{\varkappa})} + \dfrac{(2\Lambda)^2}{\varkappa}}$, $\varrho := \min\left\{R, \tau, \dfrac{\pi}{2\sqrt{\varkappa}}\right\}$.

Furthermore f is of class $C^{k,\alpha}(\Omega, M)$ if M belongs to $C^{k+1,\alpha}$ and H is of class $C^{k-1,\alpha}(M, \mathbb{R})$, $k \geqq 2$.

Remarks: 1. Since f is supposed to be continuous on $\bar\Omega$ we have $\tau > 0$ and $\varkappa < \infty$. 2. If, in addition to the other hypotheses M is simply connected then case I holds with $\varrho = R$ provided that $\varkappa \leqq 0$. In fact, this is a consequence of a theorem of Hadamard and Cartan, cf. [4: Section 7.2/Satz]. 3. In view of (iii) we find that $D(f)/2 =$ area of f. 4. The area of f can be estimated by $L^2(\Gamma)$, $L =$ length of $\Gamma = f(\partial\Omega)$, plus suitable error terms, cf. [14].

The following corollaries are simple consequences of the theorem.

Corollary 1: *Suppose that M is a simply connected, complete and orientable Riemannian manifold of class C^3 with non-positive sectional curvature and let $f \in C^0(\bar\Omega, M) \cap H^2_{2,\mathrm{loc}}(\Omega, M) \cap H_2^1(\Omega, M)$ satisfy conditions* (i)—(iii) *of Theorem* 1 *with $H \equiv 0$. Then*

[1]) Here and in the sequel we agree to sum over repeated latin indices $i, j, k \ldots$ from 1 to 3 and over α, β from 1 to 2.

$f(\bar{\Omega}) \subset J_c$ is a minimal surface in M provided that, in addition, $A(f) < \pi R^2/\left(1 + 2\Lambda R + 2^{-1}(2\Lambda R)^2\right)$ *where* $\Lambda = |\mathfrak{H}|_{0,S_c}$. (Note that $M = \mathbb{R}^3$ is possible.)

Corollary 2: *Let the assumption of Theorem 1 hold with* $\varkappa \leqq 0$ *and assume* $\Lambda < \sqrt{\pi/A(f)}$. *Then* $f(\Omega) \subset J_c$ *provided that* $\sqrt{D}\,\dfrac{\Lambda\sqrt{D} + \sqrt{2\pi - \Lambda^2 D}}{2\pi - 2\Lambda^2 D} < \varrho$, $D = 2A(f)$.

Let $\Gamma \subset J \subset \mathbb{R}^3$ denote some closed Jordan arc, then the class $\mathfrak{C}(\Gamma, J)$ is defined by $\mathfrak{C}(\Gamma, J) := \{f \in H_2^1(B, \mathbb{R}^3) : f(\bar{B}) \subset \bar{J}$ a.e., $f|_{\partial B} : \partial B \to \Gamma$ is continuous and weakly monotonic$\}$, where $B = \{(u, v) : u^2 + v^2 < 1\}$. Put $A_{\Gamma,J} = 2^{-1} \inf \{D(f) : f \in \mathfrak{C}(\Gamma, J)\}$, then the existence-regularity results of [12, 13] together with Corollary 1 immediately lead to

Proposition 1: *Let* $\Gamma \subset \operatorname{int} J$ *be a closed Jordan curve with* $\mathfrak{C}(\Gamma, J) \neq \emptyset$ *and suppose* $S = \partial J$ *is of class* C^3, *has bounded principal curvatures and a global parallel surface in J. If* $\Lambda = |\mathfrak{H}|_{0,S}$ *satisfies* $\Lambda < \{-1/4R^2 + \pi/2A_{\Gamma,J}\}^{1/2} - 1/2R$, *then there exists a minimal surface h in J, i.e. (i)—(iii) of Theorem 1 hold with* $H \equiv 0$, $\Omega^* = \emptyset$.

Example 1: Let J be the torus of revolution which is generated by revolving the disk $(\xi_1 - a)^2 + \xi_2^2 < r^2$ about the ξ_2-axis and assume Γ permits $\mathfrak{C}(\Gamma, J) \neq \emptyset$. For $r < a < 2r$ the torus $S = \partial J$ has regions of negative inward mean curvature and thus the $\mathfrak{H} - \Lambda$ maximum principle by HILDEBRANDT [11] and GULLIVER and SPRUCK [6, 7] cannot be applied to solutions of the variational problem $D(f) = \int |\nabla f(u, v)|^2\, du dv \rightsquigarrow$ minimum in $\mathfrak{C}(\Gamma, J)$. On the other hand the maximum absolute value of the mean curvature of S is given by $\Lambda_0 = 2^{-1} \max \{(a + 2r)/r(a + r), |a - 2r|/r(a - r)\}$. Proposition 1 gives the existence of a minimal surface spanned by Γ in J provided $A_{\Gamma,J}$ and R satisfy $\Lambda_0 < \{-1/4R^2 + \pi/2A_{\Gamma,J}\}^{1/2} - 1/2R$. To obtain a numerical example one may assume further that Γ is contained in the torus of revolution that is generated by the disk $(\xi_1 - a)^2 + \xi_2^2 \leqq (0.8r)^2$ and that $r = 2$, $a = 3$. Then $R = 2/5$ leads to the sufficient condition $A_{\Gamma,J} \leqq 0.41$.

Example 2: Let $J = \{\xi \in \mathbb{R}^3 : |\xi| \geqq 1\}$ be the exterior of the unit ball. Then $\mathfrak{H} = -1$, $\Lambda = 1$ and for $R \geqq 1$ Proposition 1 gives the existence of a minimal surface spanned by Γ in J if $A_{\Gamma,J} < \pi/5$. Note that the critical value for $A_{\Gamma,J}$ in this configuration is 3π, since the disk spanned by the circle $\{\xi_3 = 1\} \cap \{|\xi| = 2\}$ has area 3π and touches $|\xi| = 1$ in $(0, 0, 1)$.

Now we are concerned with solutions of the degenerate system

$$\Delta x_1 = -\frac{1}{x_3}(\nabla x_1 \nabla x_3), \quad \Delta x_2 = -\frac{1}{x_3}(\nabla x_2 \nabla x_3), \quad \Delta x_3 = -\frac{1}{x_3}(\nabla x_3 \nabla x_3) + \frac{1}{2x_3}|\nabla x|^2 \tag{2}$$

which turns out to be the system of Euler equation for the integral $E(x) = \int x_3 |\nabla x(u, v)|^2\, du\, dv$, $x = x(u, v)$. Special interest is given to the variational problem $E(\cdot) \rightsquigarrow$ minimum on $\mathfrak{C}(\Gamma, J)$, $J \subset \{\xi_3 \geqq 0\}$, since it describes surfaces of least potential energy under gravitational forces, cf. [1, 2] for various existence results. Proposition 2 improves the corresponding results Theorem 12, 13 in [1].[2])

Proposition 2: *Let* $J = J_\varepsilon = \{\xi \in \mathbb{R}^3 : \xi_3 \geqq \varepsilon\}$, $\varepsilon > 0$, *and let* $h(\Gamma) := \sup \{\xi_3 : \xi \in \Gamma\}$ *denote the height of Γ. Assume that* $f \in \mathfrak{C}(\Gamma, J_\varepsilon)$ *is a solution of* $E(\cdot) \rightsquigarrow$ minimum *on* $\mathfrak{C}(\Gamma, J_\varepsilon)$, *and that either*

$$A(f) = \frac{1}{2}\int_B |\nabla f|^2\, du\, dv < \frac{\pi R^2 \varepsilon^2}{\varepsilon^2 + \varepsilon R + \frac{1}{2} R^2} \quad \text{or} \quad A_{\Gamma,J_\varepsilon} < \frac{\varepsilon}{h(\Gamma)} \frac{\pi R^2 \varepsilon^2}{\varepsilon^2 + \varepsilon R + \frac{1}{2} R^2}.$$

[2]) The constants $2^4 \pi e^{-2} \varepsilon^2$ and $(\varepsilon/h)\,\pi(4\varepsilon/e)^2$ which appear in [1: Theorems 12, 13] have to be replaced by $2^2 \pi e^{-2} \varepsilon^2$ and $(\varepsilon/h)\,\pi(2\varepsilon/e)^2$, because in Lemma 7 of that article $\mathfrak{H}$ denotes 2-times the mean curvatures which is actually used by these authors.

Then $f = f(u, v)$ is contained in the open half space $\{\xi_3 > \varepsilon\}$ and furnishes an analytic solution of (2).

We now turn to the proof of Theorem 1: Let χ denote the characteristic function of $\Omega^* = f^{-1}(S_c)$ and put $\Lambda^*(w) = \chi(w)\,\mathfrak{H}(f(w)) + (1 - \chi(w))\,H(f(w))$. Following an observation of HILDEBRANDT [11], which was also used in [2], we claim that

$$D_{U_\alpha} f_*(U_\alpha) = 2\Lambda^* f_*(U_1) \times f_*(U_2) \qquad \text{a.e. on } \Omega. \tag{3}$$

In fact, (3) is obvious on $\Omega - \Omega^*$, while it is a consequence of the conformality relations on Ω^*. We refer to [2, 11] for explicit calculations. Introduce local coordinates $\varphi\colon U \to \mathbb{R}^3$, where $\Omega_1 \subset \Omega$ fulfils $f(\Omega_1) \subset U \subset M$, and let $x(w) = \varphi \circ f(w)$. Then (3) yields

$$\Delta x^l + \Gamma^l_{ij}\{x_u{}^i x_u{}^j + x_v{}^i x_v{}^j\} = 2\Lambda^*(w)\sqrt{g}\, g^{ln}(x_u \wedge x_v)^n \tag{4}$$

a.e. on Ω_1 and for $l = 1, 2, 3$. By virtue of $|\Lambda^*|_{0,\Omega} \leqq \Lambda < \infty$ and arguments from L_p-theory one immediately infers $f \in H^2_{p,\mathrm{loc}}(\Omega, M) \cap C^{1,\alpha}(\Omega, M)$, for all $p < \infty$ and $\alpha \in (0, 1)$. In view of $f \in C^1(\Omega, M)$ and (4) we see that $|\Delta x| \leqq \text{const}\, |\nabla x|$ a.e. in $\tilde{\Omega}$ for every $\tilde{\Omega} \subset\subset \Omega$. Hence a technique of HARTMAN and WINTNER is applicable, cf. [8 to 10]. In particular one obtains the asymptotic expansions

$$2x_w(w) := x_u - ix_v = (a - ib)\,(w - w_0)^\nu + o(|w - w_0|^\nu) \tag{5}$$

for w close to $w_0 \in \Omega$. Here the vectors $a, b \in \mathbb{R}^3$ fulfil the conformality conditions $\|a\| = \|b\|$, $\langle a, b\rangle = 0$ and $\nu = \nu(w_0)$ stands for a non-negative integer. It is now proven as in [2: cf. Lemma 3.11] that (5) in turn implies the density estimate

$$\limsup_{\varrho \to 0} \frac{1}{\varrho^2} \int_{K_\varrho(w_0)} g_{ij}(x)\, D_\alpha x^i D_\alpha x^j\, du\, dv \geqq 2\pi(\nu + 1) \tag{6}$$

where $K_\varrho(w_0) = \{w \in \Omega\colon d(f(w), f(w_0)) < \varrho\}$. Note that (6) holds for every $w_0 \in \Omega$, and for some $\nu \geqq 0$. We are thus in a position to carry over a result of GRÜTER, compare [5: (3.10) Theorem].[3])

Lemma (cf. [5]): *Let f be as above, then the following assertions hold.*

a) *If $\varkappa \leqq 0$ and if $\inf_{\partial\Omega} d(f(w), f(w_0)) \geqq r$ for some $w_0 \in \Omega$ where $0 < r \leqq \tau$, then*

$$(\nu + 1)\, 2\pi r^2 \leqq D(f)\,\{1 + 2\Lambda r + 2^{-1}(2\Lambda r)^2\}.$$

b) *If $\varkappa > 0$ and if $\inf_{\partial\Omega} d(f(w), f(w_0)) \geqq r$ for some $w_0 \in \Omega$ where $0 < r \leqq \min\{\tau, \pi/2\sqrt{\varkappa}\}$, then*

$$\frac{2\pi(\nu + 1)}{\varkappa} \leqq D(f)\left\{\frac{1}{\sin^2(r\sqrt{\varkappa})} + \frac{2\Lambda r}{\sin^2(r\sqrt{\varkappa})} + \frac{(2\Lambda)^2}{\varkappa}\right\}.$$

Observe that the proof of the theorem in [5: (3.10)] applies to our situation even if w_0 is a branch point, i.e. $\nabla x(w_0) = 0$. In fact in this case w_0 may not belong to the class of "good" points, compare the definition of the set A in [5]. However, in view

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[3] (Note that the left-hand side of (3.11) in [5] has to be replaced by $2\pi/\varkappa$ (instead of $2\pi/\sqrt{\varkappa}$).

of what was said before, especially relation (6), it is clear that, in our case, branch points are even "better" points, since $\nu \geqq 1$ then. This, in turn leads to the estimates of the Lemma, as follows now from a repetition of Grüters argument.

Proceeding with the proof of our theorem, we now assume on the contrary to the assertion that there exists some $w_0 \in \Omega^*$. Since $R = \operatorname{dist}(\Gamma, S_c)$ we obtain $\inf_{\partial\Omega} d(f(w), f(w_0)) \geqq R \geqq \varrho$. Putting $r := \varrho$ and $\nu = 0$ in the previous lemma one immediately derives the desired contradiction. We have thus proved that $f(\bar{\Omega}) \subset J_c$. The remaining assertions will follow from potential theory ∎

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