Zeltschrift für Analysis und hire Anwendungen Bd.8 (2)1989,8. 121— 130

The Spectrum and Quadrature Formulas of Spherical Space Forms

F. PRÜFER

Es wird das Spektrum von kompakten -Riemannschen Mannigfaltigkeiten der konstanten Schnittkrümmung 1, deren Fundamentalgruppe zyklisch ist, berechnet. Daraus ergeben sich Folgerungen für den ersten Eigenwert dreidimensionaler sphärischer Raumformen und für Quadraturformein. **Es wird das Spektrum von k**
Schnittkrümmung 1, deren Fu
Folgerungen für den ersten E.
Quadraturformeln.
Вычисляется спектр компак
кривизны 1 с циклической ф
для первого собственного зн
и для квадратурных формул.
The spec

Вычисляется спектр компактных римановых многообразий постоянной сегментарной кривизны 1 с циклической фундаментальной группой. Отсюда получаются следствия для первого собственного значения трехмерных сферических пространственных форм

The spectrum of compact Riemannian manifolds of constant sectional curvature 1 with cyclic fundamental group is computed. From this corollaries on the first eigenvalue of 3-dimensional spherical space forms and quadrature formulas are obtained.

1. Introduction

Let *M* be a compact connected Riemannian manifold of constant sectional curvature 1. Then M is the orbit space of a finite group acting freely and orthogonally on a sphere-of radius 1. Spherical space forms are completely classified by *J. A. WOLF* [13]. Let Δ be the Laplacian acting on the space of C^{∞} -functions on *M*. It has a purely discrete spectrum consisting of nonnegative eigenvalues with finite multi- plicities. We denote the spectrum of the operator Δ by spec (M) . There are a few explicit results on the spectrum of spherical space forms. For example, the spectra of homogeneous spherical space forms and certain lens spaces were computed, see $[7, 8, 10, 11]$. However, we do not know the spectrum of a lot of other spherical space forms. Using Ikecla's generating function (see [2, 3]), a method to compute explicitly the spectrum of M was given in [11].

The main tool of the present paper is a formula for the dimension of the space of G-automorphic homogeneous harmonic polynomials of degree *j* ($j \in N_0$) proved in [7], where $G \subset O(n + 1)$ is a finite group acting freely on S^{2m+1} . Here, we study this formula for cyclic groups in more detail.

The algebraic computation of the eigenvalues of orbit manifolds is closely related to lattice point problems. In Theorem 1 we give this reformulation for lens spaces. This lattice problem is an efficient method to conrpute the spectrum of spherical space forms. For 3-dimensional lens spaces the corresponding lattice problem was already obtained in [2, 131. Since the computations are purely algebraic we give only short comments.

The purpose of Theorem 2 is to list all the first eigenvalues of 3-dimensional spherical space forms. To one's surprise there are only 6 possibilities for λ_1 . Moreover, the first eigenvalue gives a few informations on the structure of the fundamental group of *M*. In certain cases $\lambda_1(M)$ and the dimension of the corresponding eigenspace

contain the whole information on the geometry of M . In these cases the results are stronger than the results in [2].

In the last section we apply our results on small eigenvalues to derive integration formulas on the unit sphere for homogeneous polynomials. Such formulas are interesting for a few practical problems. In [5] was given a 19-design with 3600 nodes. Using the binary icosahedral group we find a 19-design with 840 nodes.

2. The result^s

Let $G = \langle C \rangle$ be a cyclic subgroup of order q and let *C* be conjugates in $O(2m + 2)$

to the element
 $C' = \begin{pmatrix} R(p_1/q) & 0 \\ 0 & R(p_{m+1}/q) \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$ to the element

$$
C' = \begin{pmatrix} R(p_1|q) & 0 \\ 0 & R(p_{m+1}|q) \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}
$$

where p_1, \ldots, p_{m+1} are integers prime to q. Then S^{2m+1}/G is a spherical space form, where p_1, \ldots, p_{m+1} are integers prime to q. Then S^{2m+1}/G is a spherical space form
which is a so-called lens space, and we have $S^{2m+1}/G = L(q; p_1, \ldots, p_{m+1})$. Further
more, let $J(q; p_1, \ldots, p_{m+1}; x)$ denote the numbe more, let $J(q; p_1, ..., p_{m+1}; x)$ denote the number of lattice points $(a_1, ..., a_{m+1})$ $\in \mathbb{Z}^{m+1}$ such that *a a* so-called lens space, and we have $S^{2m+1}/G = L(q; p_1, ..., p_{m+1})$. Further let $J(q; p_1, ..., p_{m+1}; x)$ denote the number of lattice points $(a_1, ..., a_{m+1})$ and that $|a_1| + \cdots + |a_{m+1}| = x$, $a_1p_1 + \cdots + a_{m+1}p_{m+1} \equiv 0 \pmod{q}$, wh

(i)
$$
|a_1| + \cdots + |a_{m+1}| = x
$$
,

(ii) $a_1p_1 + \cdots + a_{m+1}p_{m+1} \equiv 0 \pmod{q}$, where $q, p_1, ..., p_{m+1}$ and x are given integers.

Theorem 1: Let $L = L(q; p_1, ..., p_{m+1})$ be a lens space of dimension $2m + 1$ *(rn €* N).

(i) For even q the spectrum of L is given by the eigenvalues $\mu_{2i} = 4j(j + m)$, $j \in N_0$

\n- (i)
$$
a_1p_1 + \cdots + a_{m+1}p_{m+1} \equiv 0 \pmod{q}
$$
, where q, p_1, \ldots, p_{m+1} and \times are given integers.
\n- Theorem 1: Let $L = L(q; p_1, \ldots, p_{m+1})$ be a lens space of dimension $2m + 1$ ($m \in \mathbb{N}$).
\n- (i) For even q the spectrum of L is given by the eigenvalues $\mu_{2j} = 4j(j + m)$, $j \in \mathbb{N}_0$ with the multiplicities\n
$$
d(2j, L) = \sum_{e=0}^{j} {m + e - 1 \choose m - 1} J(q; p_1, \ldots, p_{m+1}; 2j - 2e).
$$
\n
\n- (ii) For odd q the spectrum of L is given by the eigenvalues $\mu_{2j} = 4j(j + m)$ $(j \in \mathbb{N}_0)$ with the multiplicities (1) and by the eigenvalues $\mu_{2j+1} = (2j + 1) (2j + 1 + 2m)$.
\n

(ii) For odd q the spectrum of L is given by the eigenvalues $\mu_{2j} = 4j(j + m)$ $(j \in N_0)$ with the multiplicities (1) and by the eigenvalues $\mu_{2i+1} = (2j+1) (2j+1+2m)$, $j \geq j_0$ ($j \in \mathbb{N}$), *with the multiplicities d*(2*j*, *L*) = $\sum_{\ell=0}^{j} \binom{m+\ell}{m}$
d(2*j*, *L*) = $\sum_{\ell=0}^{j} \binom{m+\ell}{m}$
r odd *q* the spectrum
multiplicities (1) $\ell \in \mathbb{N}$, with the multiplicities
 $d(2j + 1, L) = \sum_{\ell=0}^{j}$
s the smallest positive $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $J(q;$
 L is given
 by the e
 *i*ties
 $+e-1$
 $m-1$

eger such t $\begin{array}{l} n+e-1 \ m-1 \end{array}$
 $\begin{array}{l} u m \text{ of } L \text{ is} \end{array}$
 $\begin{array}{l} v \text{ and } \text{ } \text{ } b g \text{ is} \end{array}$
 $\begin{array}{l} \sum\limits_{i=0}^{n} \binom{m+1}{m} \end{array}$

$$
j \geq j_0 \ (j \in \mathbb{N}), \ with \ the \ multiplicities
$$
\n
$$
d(2j+1, L) = \sum_{\varrho=0}^j {m+\varrho-1 \choose m-1} J(q; p_1, \dots, p_{m+1}; 2j+1-2\varrho).
$$
\nHere, j_0 is the smallest positive integer such that there exists a solution $(a_1, \dots, a_{m+1}) \in \mathbb{Z}$ of $|a_1| + \dots + |a_{m+1}| = 2j_0 + 1$, $a_1p_1 + \dots + a_{m+1}p_{m+1} \equiv 0 \pmod{q}$.
\nWe define the following types of groups:

m + 1

We define the following typs of groups:

- (I) $Z_q = \langle C \rangle$, the cyclic group of order $q, q \in \mathbb{N}$.
- (II) $Z_q \times D_{2^l}$, where $(q, 2^l) = 1$ and $D_{2^l} = \langle A, B \rangle$, $A^r = B^{2^l} = I$, $BAB^{-1} = A^{-1}$ and $l, \nu \geq 3$, v odd. *(I)* $Z_q = \langle C \rangle$, the cyclic group of order $q, q \in \mathbb{N}$.
 (II) $Z_q \times D_{2^i}$, where $(q, 2\nu) = 1$ and $D_{2^i} = \langle A, B \rangle$, $A^* = B^{2^i} = I$, $BAB^{-1} = A$, and $l, \nu \geq 3$, ν odd.
 III) $Z_q \times D_{4^i}^*, (q, 2\nu) = 1$, where $D_{4^$
- *(III)* $Z_q \times D_{4\nu}^*$, $(q, 2\nu) = 1$, where $D_{4\nu}^* = \langle A, B \rangle$, $A^{2\nu} = B^4 = I$, $BAB^{-1} = A^{-1}$ and $\nu \geq 2$.
- $XPX^{-1} = Q$, $XQX^{-1} = PQ$, $PQP^{-1} = Q^{-1}$ and $l \in N$.
- $(Z_0 \times 0^*, (q, 6) = 1$, where $0^* = \langle X, P, Q, R \rangle$ as for T'_{24} and $RXR^{-1} = X^{-1}$, $RPR^{-1} = QP, RQR^{-1} = Q^{-1}.$
- (VI) $Z_q \times I^*$, $(q, 30) = 1$, where $I^* = \langle U, W \rangle$, $U^2 = (UW)^3 = W^5$, $U^4 = I$. $Z_q \times T_{3^l 8}, (q, \ XPX^{-1} = Q) \nonumber \ Z_q \times O^*, (q, \ RPR^{-1} = Q) \nonumber \ Z_q \times I^*, (q, \ \text{(It is $I^* \cong 2 \times 2$ matrices)}$ (It is $I^* \cong SL(2, 5)$, where $SL(2, 5)$ denotes the multiplicative group of 2×2 matrices of determinant 1 with entries in the field of 5 elements.)

Theorem 2: *Let .M be a 3-dimensional spherical space form of' curvature* 1 *and* $\text{spec } (M) = \{0, \lambda_1, \lambda_2, \ldots\}, 0 < \lambda_1 \leq \lambda_2 \leq \ldots$

(i) We have $\lambda_1 \in \{3, 8, 24, 48, 80, 168\}$ *and*

 $\lambda_1=3$ iff $M\cong S^3$, $\lambda_1 = 8$ *i*ff $\pi_1(M) \cong Z_q$, $q \geq 2$, 1) We have $\lambda_1 \in \{3, 8, 24, 48, 80, 168\}$ and
 $= 3$ iff $M \cong S^3$,
 $= 8$ iff $\pi_1(M) \cong Z_q$, $q \geq 2$,
 $= 24$ iff $\pi_1(M)$ is isomorphic to a group of typ
 $= 48$ iff $\pi_1(M) \cong Z_q \times T'_{3'8}$, $(q, 6) = 1$, $l \in N$,
 $= 8$

 $\lambda_1 = 24$ *iff* $\pi_1(M)$ *is isomorphic to a group of typ* (II) *or* (III),

 $\lambda_1 = 168$ *iff* $\pi_1(M) = Z_q \times I^*$, $(q, 30) = 1$.

(ii) Further, let $d(\lambda_1, M)$ denote the dimension of the eigenspace corresponding to λ_1 . *Then* $d(\lambda_1, M) \geq 3$ *i*ff *M is homogeneous.*

'Now, let *Hi* be the vector space of homogeneous polynomials of degree *i* in the Now, let H_j be the vector space of homogeneous polynomials of degree j in the variables x_1, x_2, x_3 and x_4 . A quadrature formula of degree j is a set of r points $y_1, \ldots y_r$ on S^3 and constants c_1, \ldots, c_r

on
$$
S^3
$$
 and constants $c_1, ..., c_r$ such that
\n
$$
\frac{1}{\omega_{3s}} \int_{S^*} f \, do = \sum_{j=1}^r c_j f(y_j) \quad \text{for all } f \in H_k \, (0 \leq k \leq j),
$$

where *do* is the invariant measure of S^3 and $\omega_3 = 2\pi^2$. Quadrature formulas for S^3 are **rare.** For a result see [5]. Using the Theorems 1 and 2 we can easily give special quadrature formulas of degree less than 19. Hables x_1, x_2, x_3 and x_i . A quadrature formula of degree *f* is a set of *r* points g_1 , ... g_i
 S^3 and constants $c_1, ..., c_r$ such that
 $\frac{1}{\omega_3} \int_{S^3} f d\omega = \sum_{j=1}^r c_j f(y_j)$ for all $f \in H_k$ ($0 \le k \le j$),

here

morphic if $f \circ T = f$ for each $T \in G$.
Theorem 3: Let $F_i \subset SO(4)$ $(i = 1, ..., 5)$ be finite groups acting freely on the puddrature formulas of degree less than 19.

Let G be a finite subgroup of $O(2m + 2)$. A function $f: S^{2m+1} \to C$ is called G-auto-
 norphic if $f \circ T = f$ for each $T \in G$.

Theorem 3: Let $F_i \subset SO(4)$ $(i = 1, ..., 5)$ be finite g on S^3 and constants $c_1, ..., c_r$ such that
 $\frac{1}{\omega_3} \int f d\omega = \sum_{j=1}^r c_j f(y_j)$ for all $f \in H_k$ $(0 \le k \le j)$,

where $d\omega$ is the invariant measure of S^3 and $\omega_3 = 2\pi^2$. Quadrature formulas for S^3

are rare. For a *Then we have the following quadrature 'formulas:* $\begin{array}{ll} \text{are rare.} \ \text{quadratu} \ \text{Let G} \ |\ \text{morphic}\ \text{if} \ \text{In } \text{e} \ \text{where} \ \text{of} \ \text{Then we} \ \text{if} \ \text{then we} \ \text{if} \$ $\frac{1}{\omega_3} \int f d\omega = \sum_{j=1}^r c_j f(y_j)$ for all $f \in H_k$ $(0 \le k \le j)$
is the invariant measure of S^3 and $\omega_3 = 2\pi^2$. Q
For a result see [5]. Using the Theorems 1 and 2
re formulas of degree less than 19.
be a finite subgro Formulas of degree less than 13.

a finite subgroup of $O(2m + 2)$. A function $f: S^{2m+1} \to C$ is called
 $\circ T = f$ for each $T \in G$.

a 3: Let $F_i \subseteq SO(4)$ $(i = 1, ..., 5)$ be *finite groups' acting freely*

dius 1 and $F_1 \cong D'_{24}$ be a finite subgroup of $O(2m + 2)$. A function $f: S$
 $f \circ T = f$ for each $T \in G$.

em 3: Let $F_i \subseteq SO(4)$ $(i = 1, ..., 5)$ be finite gradius 1, and $F_1 \cong D'_{24}$, $F_2 \cong T^*$, $F_3 \cong O^*$, F_4

thave the following quadrature fo

24TEF,

f \in *H_i* (0 \leq *j* \leq 5) and y_1 is any zero of the first nontrivial F₁-automorphic *eigenfunction of* 53• have the following quadrature formulas:
 $\frac{1}{w_3} \int_{S^4} f d\sigma = \frac{1}{24} \sum_{T \in F_1} f(Ty_1),$
 $f \in H_j$ ($0 \leq j \leq 5$) and y_1 is any zero of

eigenfunction of S^3 .
 $\frac{1}{w_3} \int_{S^4} f d\sigma = \frac{1}{24} \sum_{T \in F_1} f(Tx),$
 $f \in H_j$ ($\frac{1}{\omega_3} \int_{S^4} f \, do = \frac{1}{24} \sum_{T \in F_1} f(Ty_1),$
 $f \in H_j$ ($0 \leq j \leq 5$) and y_1 is any zero of the f

eigenfunction of S^3 .
 $\frac{1}{\omega_3} \int_{S^4} f \, do = \frac{1}{24} \sum_{T \in F_1} f(Tx),$
 $f \in H_j$ ($0 \leq j \leq 5$) and $x \in S^3$,
 $\frac{1$

(i)
$$
\frac{1}{\omega_3} \int_{S^4} f \, d\sigma = \frac{1}{24} \sum_{\tau \in F_1} f \cdot \frac{f \in H}{\tau} \quad (0 \leq j \leq 5) \text{ or } \text{eigenfunction of } S^3.
$$
\n(ii)
$$
\frac{1}{\omega_3} \int_{S^4} f \, d\sigma = \frac{1}{24} \sum_{\tau \in F_1} f \cdot \frac{f \cdot f}{\tau} \quad (0 \leq j \leq 5).
$$

iii)
$$
\frac{1}{\omega_3} \int_{S^1} f \, d\sigma = \frac{1}{48} \sum_{T \in F_1} f(Tx),
$$

 $f \in H_j$, $(0 \leq j \leq 7)$ and $x \in S^3$,

$$
f \in H_j \ (0 \le j \le 5) \ and \ x \in S^3,
$$

\n(iii)
$$
\frac{1}{\omega_3} \int f \, d\sigma = \frac{1}{48} \sum_{T \in F_1} f(Tx),
$$

\n
$$
f \in H_j \ (0 \le j \le 7) \ and \ x \in S^3,
$$

\n(iv)
$$
\frac{1}{\omega_3} \int f \, d\sigma = \frac{1}{120} \sum_{T \in F_1} f(Tx),
$$

\n
$$
f \in H_j \ (0 \le j \le 11) \ and \ x \in S^3,
$$

\n(v)
$$
\frac{1}{\omega_3} \int f \, d\sigma = \frac{1}{940} \sum f(Ty_2),
$$

(v)
$$
\frac{1}{\omega_{3}} \int f d\sigma = \frac{1}{840} \sum_{T \in F_{\bullet}} f(T y_{2}),
$$

 ω_3 $\frac{1}{s}$ $f \in H_j$ $(0 \leq j \leq 5)$ and $x \in S^3$,
 $\frac{1}{\omega_3} \int f d\omega = \frac{1}{48} \sum_{\tau \in F_1} f(Tx),$
 $f \in H_j$ $(0 \leq j \leq 7)$ and $x \in S^3$,
 $\frac{1}{\omega_3} \int f d\omega = \frac{1}{120} \sum_{\tau \in F_1} f(Tx),$
 $f \in H_j$ $(0 \leq j \leq 11)$ and $x \in S^3$,
 $f \in H$, $(0 \leq j \leq 19)$ *and* y_2 *is any zero of the first nontrivial F₅-automorphic eigen function of S:*

3. Spherical space forms and **autoniorphic eigenfunctions**

In this section we will briefly review the facts concerning the G -automorphic eigenfunctions of S^{2m+1} (see [1, 2]). Let S^{2m+1} $(m \in \mathbb{N})$ be the unit sphere centered at the origin in \mathbb{R}^{2m+2} . Furthermore, let *M* be a spherical space form of curvature 1 and dimension $2m + 1$. Then $M = S^{2m+1}/G$, where G is a finite subgroup of $SO(2m + 2)$ and for any $T \in G$ with $T = Id$, 1 is not an eigenvalue of T . Here *Id* denotes the unit matrix in $O(2m + 2)$. The sphere S^{2m+1} is the universal Riemannian covering manifold of M. Let ι be the covering map of S^{2m+1} onto S^{2m+1}/G , ι : $S^{2m+1} \rightarrow S^{2m+1}/G$. **124** F. PRÜFER
 3. Spherical space forms and automorphic eigenfunctions

In this section we will briefly review the facts concerning the *G*-automorphic eigenfunctions of S^{2m+1} (see [1, 2]). Let S^{2m+1} ($m \in N$)

We denote the space of complex valued C^{∞} -functions on *M* by $C^{\infty}(M)$ and denote the Laplacian acting on $C^{\infty}(M)$ and $C^{\infty}(S^{2m+1})$ by Δ and $\tilde{\Delta}$, respectively. The Lapla-
cian has a purely discrete spectrum spec (M). For $k \ge 0$ let H_k be the space of complex valued homogeneous harmonic polynomials of degree *k* on **R**2m+2 . The spectrum of $\tilde{\Delta}$ is well known, see [1]. It contains the eigenvalues $\mu_j = j(j + 2m)$ with multiplicities any $T \in G$ with $T = Id$, 1 is not an eigenting in $O(2m + 2)$. The sphere S^{2m+1} is the of *M*. Let *t* be the covering map of S^{2m+1} content the space of complex valued C^{∞} -function acting on $C^{\infty}(M)$ and C^{\in cian has a purely discrete spectru
plex valued homogeneous harmon
of $\tilde{\Delta}$ is well known, see [1]. It co
plicities
 $d(j) = \binom{j + 2m}{2m} - \binom{j - 2m}{m}$
The eigenfunctions of S^{2m+1}/G
 S^{2m+1} . More precisely, let i: S^{2m and for any $T \in G$ with $T = Id$, 1 is not an eigen
unit matrix in $O(2m + 2)$. The sphere S^{2m+1} is the
manifold of M . Let ι be the covering map of S^{2m+1} or
 N - We denote the space of complex valued C^{∞} -fu

$$
d(j)=\binom{j+2m}{2m}-\binom{j-2+2m}{2m},\qquad j\in\mathbf{N}_0.
$$

The eigenfunctions of S^{2m+1}/G are exactly the G-automorphic eigenfunctions of are exactly the G-automorphic eigenfunctions of $\rightarrow \mathbf{R}^{2m+2}$ be the natural inclusion map. Then i induces the restriction map $i^*: C^{\infty}(\mathbb{R}^{2m+2}) \to C^{\infty}(S^{2m+1})$ and we have the following The eigenfunctions of S^{2m+1}/G are exactly the G-automor S^{2m+1} . More precisely, let i: $S^{2m+1} \rightarrow \mathbb{R}^{2m+2}$ be the natural includes the restriction map $i^*: C^{\infty}(\mathbb{R}^{2m+2}) \rightarrow C^{\infty}(S^{2m+1})$ and we have a Lemma 1

Lemma 1 (see [2]): Let $V(j)$ be the eigenspace with eigenvalue μ_j of $\tilde{\Lambda}$. Then the is the restriction map $i^*: C^{\infty}(\mathbb{R}^{2m+2}) \to C^{\infty}(S^{2m+1})$ and we have the following
Lemma 1 (see [2]): Let $V(j)$ be the eigenspace with eigenvalue μ_j of $\tilde{\Delta}$. Then the
up i^* gives an $O(2m + 2)$ -isomorphism

The following formula is elementary, see [1]: For any $f \in C^{\infty}(S^{2m+1}/G)$, we have $\tilde{\mathcal{A}}(t^*f) = t^*(\Delta f)$. Now it follows

Lemma 2 (see [2]): Let $V(j, G)$ *and* $H(j, G)$ *be the subspaces of* $V(j)$ *and* H_j *consisting of all the G-automorphic elements of* $V(j)$ *and* H_j *, respectively. Then* $(\iota^*)^{-1}$ $V(j, G)$ is the eigenspace with the eigenvalue μ_j of the Laplacian Λ on S^{2m+1}/G and it is *isomorphic to H(j, G). Further, every eigenspace of* Δ *on* S^{2m+1}/G *is obtained in this way.* $V(j, \hat{G})$ is the
isomorphic
way.
Let $d(j, G)$
the calculat
4. The spect
Now we proderived:
 $d(j)$
where q is

Let $d(j, G) = \dim H(j, G)$. Consequently, we need only the values of $d(j, G)$ for the calculation of spec (M) , $M = S^{2m+1}/G$.

4. The spectra of lens spaces

Now we prove our Theorem 1. Recall that in [7] the following formula for $d(j,\,G)$ was derived:

isomorphic to
$$
\tilde{H}(j, G)
$$
. Further, every eigenspace of Δ on S^{2m+1}/G is obtained in this
way.
\nLet $d(j, G) = \dim H(j, G)$. Consequently, we need only the values of $d(j, G)$ for
the calculation of spec (M) , $M = S^{2m+1}/G$.
\n4. The spectra of lens spaces
\nNow we prove our Theorem 1. Recall that in [7] the following formula for $d(j, G)$ was
derived:
\n
$$
d(j, G) = \frac{1}{q} \left\{ 1 + \frac{(-1)^j}{2} \left(1 + (-1)^q \right) \right\} d(j) + \frac{1}{q} \sum_{T \in G} \sum_{k=0}^{[j/2]} \alpha_m(j, k) 2^{j-2k} R(T; j, k) \qquad (j \in \mathbb{N}_0)
$$
\nwhere q is the order of G, \sum' means that we have to sum up only over $T \in G$
\n $\setminus \{Id, -Id\}$, $[j/2]$ is the entire part of $j/2$,
\n
$$
\alpha_m(j, k) = (-1)^k \frac{j+m}{j+m-2k} \left(j+m-k-1 \right)
$$

where q is the order of G, \sum' means that we have to sum up only over $T \in G$ ${e}$ rere q
 $Id, -I$

$$
\alpha_m(j,k) = (-1)^k \frac{j+m}{j+m-2k} \binom{j+m-k-1}{k}.
$$

and

On S
\n
$$
R(T;j,k) = \sum_{\substack{k_1+\cdots+k_{m+1}=j-2k \\ k_1,\ldots,k_{m+1}\geq 0}} \prod_{v=1}^{m+1} \left(\cos \frac{2\pi p_v(T)}{q(T)}\right)^{k_v}.
$$
\n*r*) is the order of $T \in G$ and $\exp \left(\frac{2\pi i p_v(T)}{q(T)}\right)q(T)$

Here $q(T)$ is the order of $T \in G$ and $\exp\left(2\pi i p_r(T)/q(T)\right)$, $\exp\left(-2\pi i p_r(T)/q(T)\right)$ are the eigenvalues of *T* ($1 \leq v \leq m + 1$). Our proof of Theorem 1 is based on an explicit study of this formula for cyclic groups. We have $Z_q = \langle C \rangle$ and *C* is conjugate to *C'* $R(T; j, k) = \sum_{\substack{k_1 + \dots + k_{m+1} = j-2k \ r = 1}} \prod_{\substack{v=1 \ v \leq m}}^{\text{max}} \left\{ \cos \frac{k_1 + \dots + k_{m+1} \geq 0}{k_1 + \dots + k_m \geq 0} \right\}$

Here $q(T)$ is the order of $T \in G$ and $\exp\left\{2\pi \cos \frac{\pi}{2} \right\}$

the eigenvalues of $T (1 \leq v \leq m + 1)$. Our p

s ^{*k*},.....*k*_m.i, ²0

values of T (1 $\leq v \leq m$ +

this formula for cyclic gree

+ 2). Similar to [8] this gi
 $d(j, Z_q) = \sum_{k=0}^{[j/2]} \alpha_m(j, k)$
 $d(i)$ $\prod_{r=1}^{m+1} \left(\cos \frac{2\pi p_r(T)}{q(T)} \right)^{k_r}$

nd $\exp \left(2\pi i p_r(T)/q(T) \right)$, $\exp \left(-1 \right)$. Our proof of Theorem 1 is ba

bups. We have $Z_q = \langle C \rangle$ and C

ives
 $\sum_{\{\varrho_{j-1 k} \leq m+1\}} \sum_{\substack{\epsilon_1, \ldots, \epsilon_{j-1 k} \in \{-1, 1\} \\ \epsilon_1 p_{\varrho_1}$ $\begin{align} \text{ace Forms} \quad & 125 \ \text{2\textit{nip.}}(T)/q(T)) \text{ are} \ \text{used on an explicit} \ \text{is conjugate to } C' \ \text{1.} \quad \text{.} \qquad \text{.} \end{align}$ the order of $T \in G$ and $\exp\left(2\pi i p_r(T)/q(T)\right)$, $\exp\left(-2\pi i p_r(T)\right)$

tes of T ($1 \leq \nu \leq m + 1$). Our proof of Theorem 1 is based on a

formula for cyclic groups. We have $Z_q = \langle C \rangle$ and C is conjug
 $\sum_{k=0}^{[j/2]} Z_q = \sum_{k=0$ q £ *q q-1 2il* JO for *I* 0 (mod q), order of $T \in G$ and $\exp\left(2\pi i p_r(T)/q(T)\right)$, $\exp\left(-2\pi i p_r(T)/q(T)\right)$
 f T ($1 \leq r \leq m + 1$). Our proof of Theorem 1 is based on an explanate for cyclic groups. We have $Z_q = \langle C \rangle$ and *C* is conjugate to

milar to [8] this gi

$$
d(j, Z_q) = \sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_m(j, k) \sum_{1 \leq \varrho_1 \leq \cdots \leq \varrho_{j-1k} \leq m+1} \sum_{\substack{\epsilon_1, \ldots, \epsilon_{j-1k} \in \{-1, 1\} \\ \epsilon_1 p_{\varrho_1} + \cdots + \epsilon_{j-1k} p_{\varrho_j - 1k} = 0 \pmod{q}}} 1.
$$
\nhave used the identities

\n
$$
\prod_{j-2k}^{j-2k} \cos \frac{2\pi p_{\varrho_1} l}{\alpha} = 2^{2k-j} \sum_{\substack{\text{cos } 2\pi l \\ \text{cos } 2\pi}} \left(\epsilon_1 p_{\varrho_1} + \cdots + \epsilon_{j-2k} p_{\varrho_{j-1k}} \right),
$$

Here we have used the identities

and

$$
q = 1 \t q \t t_{1}, \t q = 1, 1 \t q
$$

$$
q = 1 \t 2\pi l \t q \t (0 \text{ for } t \not\equiv 0 \text{ (mod } q),
$$

 $\sum_{t=0}^{\infty} \cos \frac{\pi i}{q} t = \begin{cases} 0 & \text{if } t \equiv 0 \pmod{q}, \\ q & \text{if } t \equiv 0 \pmod{q}. \end{cases}$

The next step is the following

Lemma₃: We have the equality

Here we have used the identities
\n
$$
\lim_{\epsilon_1 p_2 + \cdots + \epsilon_{j-1} p_{\ell} + \cdots + \epsilon_{j-1} p_{\ell} + \cdots + \epsilon_{j-1} p_{\ell} + \cdots + \epsilon_{j-2} p_{\ell} - \epsilon_{j-1}} \text{ (b) (a) }
$$
\nHere we have used the identities
\n
$$
\iint_{r=1}^{r-2k} \cos \frac{2\pi l}{q} = 2^{2k-j} \sum_{\epsilon_1, \ldots, \epsilon_{j-1} \in \{-1, 1\}} \cos \frac{2\pi l}{q} (\epsilon_1 p_{\epsilon_1} + \cdots + \epsilon_{j-2k} p_{\epsilon_{j-1k}});
$$
\nand
\n
$$
\sum_{l=0}^{q-1} \cos \frac{2\pi l}{q} l = \begin{cases} 0 & \text{for } l \equiv 0 \pmod{q}, \\ q & \text{for } l \equiv 0 \pmod{q}. \end{cases}
$$
\n
$$
\text{The next step is the following}
$$
\n
$$
\text{Lemma 3: } We \text{ have the equality}
$$
\n
$$
\sum_{i \geq \epsilon_1 \leq \cdots \leq \epsilon_k \leq m+1} \sum_{\epsilon_1 p_{\epsilon_1} + \cdots + \epsilon_k p_{\epsilon_k} = 0 \pmod{q}} \sum_{\epsilon_1 p_{\epsilon_1} + \cdots + \epsilon_k p_{\epsilon_k} = 0 \pmod{q}} \frac{m+1}{s} \binom{|a_r| + 2s_r}{s_r}.
$$
\n
$$
= \sum_{l=0}^{\lfloor x/2 \rfloor} \sum_{\substack{|a_{l} \mid l + \cdots + |a_{m+l}| = x-2l \\ a_{l} \mid p_{l} + \cdots + a_{m+l} p_{m+l} = 0 \pmod{q}} \frac{m+1}{s_1 \cdots s_{m+l} \geq 0} \binom{m+1}{s} \binom{|a_r| + 2s_r}{s_r}.
$$
\n
$$
\text{Proof: The sum on the left hand side runs over all terms } a_1 p_1 + \cdots + a_{m+l} p_{m+l}
$$
\nwith $a_1 p_1 + \cdots + a_{m+l} p_{m+l} \equiv 0 \pmod{q} \text{ and } |a_1| + \cdots + |a_{m+l}| \leq x.$ Here every term can appear several

. Here every term can appear several times in the sum. Next we have only the possibilities $|a_1| + \cdots + |a_{m+1}| = x - 2l$, $0 \le l \le [x/2]$. This means that the term $a_i p_i$ appears iff p_i *•* $\begin{aligned}\n &= \sum_{l=0}^{\lfloor x/2 \rfloor} \sum_{|a_l|+\cdots +|a_{m+1}|=\mathbf{x}-2l} \sum_{s_1+\cdots+s_{m+1}=l} \prod_{r=1}^{m+1} \binom{|a_r|+2s_r}{s_r}.\n \end{aligned}$ Proof: The sum on the left hand side runs over all terms $a_1p_1 + \cdots + a_{m+1}p_{m+1}$

with $a_1p_1 + \cdots + a_{m+1}p_{m+1}$ $\sum_{i \leq q_1 \leq \cdots \leq q_k \leq m+1} \sum_{\substack{\epsilon_1,\ldots,\epsilon_k \in \{-1,1\} \\ i \equiv 0 \ (a_1 + \cdots + a_{m+1}) = \cdots \leq a_1}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_k \in \{-1,1\} \\ a_1p_1 + \cdots + a_{m+1}p_{m+1} = 0 \pmod{q}}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_k \in \{-1,1\} \\ a_1p_1 + \cdots + a_{m+1}p_{m+1} = 0 \pmod{q}}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_{m+$ appears exactly $|a_i| + 2s_i$ $(0 \le s_i \le [(x - |a_i|)/2])$ times. In this case exactly s_i of the corresponding ε , must be -1 . There are $\binom{|a_i| + 2s_i}{\text{possibilities}}$ for the $(|a_i| + 2s_i)$ possibilities for the choice of these ε ,'s. Further we have $(|a_1| + 2s_1) + \cdots + (|a_{m+1}| + 2s_{m+1}) = x$ and appears exactly $|a_i| + 2s_i$, $(0 \le s_i \le [(x - |a_i|)/2])$ times. In this case exactly s_i
of the corresponding ε , must be -1 . There are $\binom{|a_i| + 2s_i}{s_i}$ possibilities for the
choice of these ε ,'s. Further we have $(|a$ of the corresponding ε , must be -1 . There are $\binom{|a_i|+2s_i}{s_i}$ possibilities for the choice of these ε ,'s. Further we have $(|a_1| + 2s_1) + \cdots + (|a_{m+1}| + 2s_{m+1}) = x$ and for $|a_1| + \cdots + |a_{m+1}| = x - 2l$ we get $s_1 + \cd$ exactly $\sum_{\substack{a_{m+1}p_{m+1} = 0 \pmod{q}}} \sum_{\substack{s_1 + \cdots + s_{m+1} = l \\ s_{m+1}p_{m+1} = 0 \pmod{q}}} \prod_{\substack{s_1, \ldots, s_{m+1} \geq 0}}^{n_1 + \cdots + s_{m+1} = 0} \prod_{\substack{a_{m+1}p_{m+1} = 0 \pmod{q}}}^{n_1 + \cdots + n_{m+1}p_{m+1}} \equiv 0 \pmod{q}$ and $|a_1|$
 $a_1, 0 \leq l \leq [x/2]$. This me $\begin{array}{l} \text{exactly}\ |\ a_i|\,+\,2s_i\ \ \text{(0)}\ \ \text{rresponding}\ \varepsilon,\ \text{must}\ \ \text{these}\ \ \varepsilon.'s.\ \ \text{Further}\ \cdots\,+\,|a_{m+1}|\,=\,\varkappa\,-\ \ \text{term}\ \ \ a_1p_1\,+\,\cdots\,+\ \sum\limits_{s_1,\dots,s_{m+1}\geq 0}\,\frac{m+1}{\prod\limits_{s_1,\dots,s_{m+1}\geq 0}^{n+1}}\binom{|a_i|}{\min\limits_{s_1,\dots,s_{m+1}\geq 0}^{s_1+\cdots,s_{m+1}\geq 0}}\ \end$

$$
\sum_{\substack{s_i+\cdots s_{m+1}=l\\s_1,\ldots,s_{m+1}\geq 0}}\prod_{i=1}^{m+1}\binom{|a_i|+2s_i}{s_i}
$$

times. This completes the proof of the lemma \blacksquare

Now we put

of these
$$
\varepsilon
$$
, 's. Further we have $(|a_1| + 2s_1) + \cdots + |a_{m+1}|$
\n $+ \cdots + |a_{m+1}| = x - 2l$ we get $s_1 + \cdots + s_{m+1} = l$. Now
\nthe term $a_1p_1 + \cdots + a_{m+1}p_{m+1}$ with $|a_1| + \cdots + |a_{m+1}| =$
\n
$$
\sum_{\substack{s_1 + \cdots s_{m+1} = l \\ s_1, \ldots, s_{m+1} \geq 0 \\ s_1, \ldots, s_{m+1} \geq 0}} \prod_{i=1}^{m+1} { |a_i| + 2s_i \choose s_i}
$$
\nThis completes the proof of the lemma
\nwe put
\n $\chi(b_1, \ldots, b_m; x, l) = \sum_{\substack{s_1 + \cdots + s_{m+1} = l \\ s_1, \ldots, s_{m+1} \geq 0}} {b_1 + 2s_1 \choose s_1} \cdots {b_m + 2s_m \choose s_m}$
\n $\times {x - 2l - b_1 - \cdots - b_m + 2s_{m+1} \choose s_{m+1}}$

(4)

126 F. Prüffer
with $b_j \in N_0$ ($1 \leq j \leq m$), $b_1 + \cdots + b_m \leq x - 2l$ and $x, l \in N_0$. A straightforward
calculation shows that χ does not depend on b_1, \ldots, b_m . Therefore we have $\sum_{n=1}^{\infty} m$, $b_1 + \cdots + b_m \le x - 2l$ at
 $x \ne 2l$ are $x \ne 2l$ and b_1, \ldots, b_n
 $x, l) = \chi(0, \ldots, 0; x, l).$

(5) into (3) and permuting the su **d** *x*, *l*
n. The

$$
\chi(b_1, ..., b_m; x, l) = \chi(0, ..., 0; x, l).
$$
\n(5)

Substituting (4) and (5) into (3) and permuting the sums we get

F. PrujFER
\n
$$
N_0 \ (1 \leq j \leq m), \ b_1 + \cdots + b_m \leq x - 2l \text{ and } x, l \in N_0. \ A \text{ straightforward on shows that } \chi \text{ does not depend on } b_1, \ldots, b_m. \text{ Therefore we have}
$$
\n
$$
\chi(b_1, \ldots, b_m; x, l) = \chi(0, \ldots, 0; x, l).
$$
\n(5)\n
$$
\text{diag (4) and (5) into (3) and permuting the sums we get}
$$
\n
$$
d(j, Z_q) = \sum_{\varrho=0}^{\lfloor j/2 \rfloor} A_m(j, \varrho) J(q; p_1, \ldots, p_{m+1}; j - 2\varrho)
$$
\n(6)\n
$$
A_m(j, \varrho) = \sum_{\varrho=0}^{\varrho} \chi_{m-1}(j, k) \qquad \sum_{\varrho=0}^{\lfloor j/2 \rfloor} \chi_{m-1}^{(j)} \left(\frac{2s_1}{j} \right) \left(\frac{2s_m}{j} \right) (j - 2\varrho + 2s_{m+1})
$$

126 F. PrüFER
\nwith
$$
b_j \in N_0
$$
 $(1 \leq j \leq m)$, $b_1 + \cdots + b_m \leq x - 2l$ and $x, l \in N_0$. A straightforward
\ncalculation shows that χ does not depend on $b_1, ..., b_m$. Therefore we have
\n $\chi(b_1, ..., b_m; x, l) = \chi(0, ..., 0; x, l)$.
\nSubstituting (4) and (5) into (3) and permuting the sums we get
\n $d(j, Z_q) = \sum_{e=0}^{\lfloor j/2 \rfloor} A_m(j, \varrho) J(q; p_1, ..., p_{m+1}; j - 2\varrho)$
\nwith
\n
$$
A_m(j, \varrho) = \sum_{k=0}^{\lfloor \varrho \rfloor} \alpha_m(j, k) \sum_{\substack{s_1, ..., s_{m+1} = \varrho - k \\ s_1, ..., s_{m+1} \geq 0}} {\binom{2s_1}{s_1} \cdots \binom{2s_m}{s_m}} {j - 2\varrho + 2s_{m+1} \choose s_{m+1}}.
$$

\nLemma 4: We have $A_m(j, \varrho) = {m + \varrho - 1 \choose \varrho}$ for $j \in N_0, 0 \leq \varrho \leq [j/2], m \in N$.
\nProof: To begin with we find (see [8: p. 169])

Lemma 4: *We have* $A_m(j, \varrho)$

Lemma 4: We have
$$
A_m(j, \rho) = {m + \rho - 1 \choose \rho}
$$
 for $j \in N_0, 0 \le \rho \le [j/2]$. $m \in$
\nProof: To begin with we find (see [8: p. 169])
\n
$$
\sum_{\substack{s_1 + \dots + s_{m+1} = \rho - k \\ s_1, \dots, s_{m+1} \ge 0}} {2s_1 \choose s_1} \cdots {2s_m \choose s_m} {j - 2\rho + 2s_{m+1} \choose s_{m+1}}
$$
\n
$$
= \sum_{\substack{s_1 + \dots + s_m \le \rho - k \\ s_1, \dots, s_m \ge 0}} {2s_m \choose s_1} \cdots {2s_{m-1} \choose s_{m-1}} {j - 2k - 2s_1 - \dots - 2s_{m-1} - s_m \choose \rho - k - s_1 - \dots - s_m}.
$$
\nand the addition formula for binomial coefficients several times now we

Using the addition formula for binomial coefficients several times, now we obtain $A_m(j, \varrho + 1) = A_m(j, \varrho) + A_{m-1}(j + 1, \varrho + 1)$. Therefore our lemma follows by induction with respect to $j \blacksquare$

From Lemma 4 and (6) we obtain the desired formula in Theorem 1. The further statements of Theorem I are easily to prove by means of this formula. We omit here the proofs.

5. The first eigenvalue of **3-dimensional spherical space forms**

The basic theorem of this section is the following. If G is a finite subgroup of $SO(4)$ acting freely on $_1S^3$, then G is isomorphic to one of the groups of typ $(I) - (VI)$, see [12] (also [6, 13]). Moreover, let $S^3/\bar{G_1}$ and S^3/G_2 be spherical space forms. Assume G_1 is isomorphic to G_2 and is not cyclic. Then G_1 is conjugate to G_2 in $O(4)$ such that.
 $S^3/G_1 = S^3/G_2$. The basic the
acting freely of
[12] (also [6, 1
 G_1 is isomorph
 $S^3/G_1 = S^3/G_2$.
We now ma

We now may prove Theorem 2. Next, our Theorem 1 implies Theorem 2 for the groups of typ (I). The further assertions of Theorem 2 we get from the following table.

The calculations are the same in all cases. We decompose the group *0* into conjugate cyclic subgroups and then we determine by means of Theorem 1 its multiplicities. Therefore we consider completely, here only the case $G \cong Z_q \times D'_{2!r}$. In the other cases we give only the corresponding formulas.

Let G, be a group of typ (II). Then $G \cong Z_{q} \times D_{2}$, and we have D_{2} , $\cong \langle A, B \rangle$, *A'* = $B^{2i} = I$, $BAB^{-1} = A^{-1}$ with $l \ge 3$, $q \in N$, $v \ge 3$, *v* odd and $(q, 2v) = 1$. Further, let $Z_q = \langle C \rangle$, $C^q = I$, $CA = AC$, $CB = BC$. A representation $\pi: Z_q \times D_{2^l}$,

 $SO(4)$ acting freely on $S³$ is given by (see [13])

$$
n(A) = \begin{pmatrix} R\left(\frac{1}{v}\right) & 0 & 0 & 0 \ 0 & R\left(-\frac{1}{v}\right) & 0 & 0 \end{pmatrix}, \quad n(B) = \begin{pmatrix} 0 & Id \ R\left(\frac{1}{2^{l-1}}\right) & 0 \end{pmatrix},
$$

$$
n(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 & 0 \ 0 & R\left(-\frac{1}{v}\right) \end{pmatrix}.
$$

$$
n(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 & 0 \ 0 & R\left(\frac{1}{q}\right) & 0 \end{pmatrix}.
$$
Conjugate to $n(Z_q \times D_{2^i\bullet})$ in $O(4)$ and $S^3/G \cong S^3/n(Z_q \times D_{2^i\bullet})$.

Then *G* is conjugate to $\pi(Z_q \times D'_{2'}')$ in $O(4)$ and $S^3/G \cong S^3/\pi(Z_q \times D'_{2'})$. Therefore we can suppose $G = \pi(Z_q \times D'_{2r})$. The following facts are obviously or well known, see [13]. The group G contains the cyclic subgroups $G_{-1} = \langle \pi(B^2AC) \rangle$ of order $2^{l-1}\nu q$ and $G_e = \langle \pi(BA^eC) \rangle$ $(0 \leq e \leq v - 1)$ of order 2^tq . The groups G_e are conjugate subgroups. Further we have $\overline{G_i} \cap G_{i'} = 3$ $(i \neq i')$, where $3 = \langle \pi(B^2C) \rangle$ is the center. of \widetilde{G} . Let $|\mathfrak{Z}|$ be the order of \widetilde{G} . Then $|\mathfrak{Z}| = 2^{t-1}q$. Now (2) yields The group *G* contains the cyclic subgroups $G_{-1} = \langle \pi(B^2 A \check{C}) \rangle$
 $= \langle \pi(BA^qC) \rangle$ $(0 \leq \varrho \leq r - 1)$ of order $2^l q$. The groups G_{ϱ}

s. Further we have $G_i \cap G_{i'} = \varrho(i + i')$, where $\varrho(i) = \langle \pi(B^2 C \check{C}) \rangle$
 $|\varrho(i)| = 2$ $\pi(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 \\ 0 & R\left(\frac{1}{q}\right) \end{pmatrix}$.

Then *G* is conjugate to $\pi(Z_q \times D_{2^*})$ in $O(4)$ and $S^3/G \cong S^3/\pi$

we can suppose $G = \pi(Z_q \times D_{2^*})$. The following facts are obv

see [13]. The group *G* contains the

1 (i)
$$
R\left(\frac{1}{q}\right)
$$

\nThe G is conjugate to $\pi(Z_q \times D_{2^*})$ in $O(4)$ and $S^3/G \cong S^3/\pi(Z_q \times D_{2^*})$. Therefore we can suppose $G = \pi(Z_q \times D_{2^*})$. The following facts are obviously or well known, see [13]. The groups G contains the cyclic subgroups $G_{-1} = \langle \pi(B^2dO) \rangle$ of order $2^{l-1} \gamma q$ and $G_q = \langle \pi(BA^qC) \rangle$ (0 $\leq \varrho \leq r - 1$) of order $2^l q$. The groups G_q are conjugate subgroups. Further we have $G_i \cap G_{r'} = 3$ ($i \neq i'$), where $3 = \langle \pi(B^2C) \rangle$ is the center of G. Let $|3|$ be the order of 3 . Then $|3| = 2^{l-1}q$. Now (2) yields\n
$$
d(2j, G) = \frac{2}{|G|} d(2j) + \frac{1}{|G|} \left\{ \sum_{r \in G_{-1}} \sum_{k=0}^{r} \sum_{r=0}^{j} \alpha_m(2j, k) 4^{j-k} R(T; 2j, k) - \sum_{r \in S} \sum_{k=0}^{r} \alpha_m(2j, k) 4^{j-k} R(T; 2j, k) \right\}
$$
\n
$$
= \frac{1}{2} d(2j, G_{-1}) + \frac{1}{2} \sum_{q=0}^{r-1} d(2j, G_q) - \frac{1}{2} d(2j, 3).
$$
\nSince the groups G_{ϱ} ($0 \leq \varrho \leq r - 1$) are conjugate, we have $d(2j, G_{\varrho}) = d(2j, G_0)$ ($1 \leq \varrho \leq r - 1$). This gives\n
$$
d(2j, Z_q \times D_{2^*}) = \frac{1}{2} d(2j, G
$$

$$
d(2j, Z_q \times D'_{2^{i\bullet}}) = \frac{1}{2} d(2j, G_{-1}) + d(2j, G_0) - \frac{1}{2} d(2j, \mathfrak{Z}).
$$

The groups G_{-1} , G_0 and β are cyclic and we can easily compute the eigenvalues of $\pi(B^2AC)$, $\pi(BC)$ and $\pi(B^2C)$. Then using Theorem 1, we obtain the statements on $Z_q \times D'_{2^{l_y}}$. B F. PRÜFER

Re groups G_{-1} , G_0 and \hat{B} are cyclic and we can easily compute the eigenv
 B^2AC), $\pi(BC)$ and $\pi(B^2C)$. Then using Theorem 1, we obtain the statem
 $\times D_{2^t}$.

For the groups of type III simila

For the groups of type III similar considerations give

$$
d(2j, Z_q \times D_4^*) = \frac{1}{2} d(2j, G_{-1}) + d(2j, G_0) - \frac{1}{2} d(2j, \mathfrak{Z})
$$

where $G_{-1} = \langle \pi(AC) \rangle$, $G_0 = \langle \pi(BC) \rangle$, $\mathfrak{Z} = \langle \pi(B^2C) \rangle$ and

128 F. PatTER
\nThe groups
$$
G_{-1}
$$
, G_0 and β are cyclic and we can easily compute the eigenvalues of $\pi(B^2AC)$, $\pi(BC)$ and $\pi(B^2C)$. Then using Theorem 1, we obtain the statements or $Z_q \times D_{2^n}^*$.
\nFor the groups of type III similar considerations give
\n $d(2j, Z_q \times D_n^*) = \frac{1}{2} d(2j, G_{-1}) + d(2j, G_0) - \frac{1}{2} d(2j, \beta)$
\nwhere $G_{-1} = \langle \pi(AC) \rangle$, $G_0 = \langle \pi(BC) \rangle$, $\beta = \langle \pi(B^2C) \rangle$ and
\n $\pi(A) = \begin{pmatrix} R\left(\frac{1}{2r}\right) & \cdots & 0 \\ 0 & R\left(-\frac{1}{2r}\right) \end{pmatrix}, \qquad \pi(B) = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}, \qquad \pi(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 \\ 0 & R\left(\frac{1}{q}\right) \end{pmatrix}$
\nFor the groups of typ IV we obtain
\n $d(2j, Z_q \times T_{3'0}) = d(2j, G_1) + \frac{1}{2} d(2j, G_2) - \frac{1}{2} d(2j, \beta_3)$
\nwhere $G_i \cong \langle g_i \rangle$ $(i = 1, 2), \beta_1 \cong \langle g_3 \rangle$ with
\n $g_1 = \begin{pmatrix} R\left(\frac{p_1}{2 \cdot 3'q}\right) & 0 \\ 0 & R\left(\frac{p_2}{2 \cdot 3'q}\right) \end{pmatrix}$,

For the groups of typ IV we obtain

$$
\left(\begin{array}{cc} 0 & R\left(-\frac{1}{2\nu}\right)\end{array}\right)
$$

For the groups of typ IV we obtain

$$
d(2j, Z_q \times T'_{3'8}) = d(2j, G_1) + \frac{1}{2} d(2j, G_2) - \frac{1}{2} d(2j, \mathfrak{Z}_1)
$$

where $G_i \cong \langle g_i \rangle$ $(i = 1, 2), \mathfrak{Z}_1 \cong \langle g_3 \rangle$ with

$$
\left(\begin{array}{cc} 0 & R\left(-\frac{1}{2y}\right) \end{array} \right) \qquad \left(\begin{array}{c} -1a & 0 \end{array} \right) \qquad \left(\begin{array}{c} 0 \end{array} \right)
$$
\nFor the groups of typ IV we obtain\n
$$
d(2j, Z_q \times T_{39}) = d(2j, G_1) + \frac{1}{2} d(2j, G_2) - \frac{1}{2} d(2j, \frac{3}{2})
$$
\nwhere $G_i \cong \langle g_i \rangle$ ($i = 1, 2$), $\frac{3}{2i} \cong \langle g_3 \rangle$ with\n
$$
g_1 = \begin{pmatrix} R\left(\frac{p_1}{2 \cdot 3^i q}\right) & 0 \\ 0 & R\left(\frac{p_2}{2 \cdot 3^i q}\right) \end{pmatrix},
$$
\n
$$
g_2 = \begin{pmatrix} R\left(\frac{p_1'}{4 \cdot 3^{i-1} q}\right) & 0 \\ 0 & R\left(\frac{p_2'}{4 \cdot 3^{i-1} q}\right) \end{pmatrix}, \quad g_3 = \begin{pmatrix} R\left(\frac{1}{2 \cdot 3^{i-1} q}\right) & 0 \\ 0 & R\left(\frac{1}{2 \cdot 3^{i-1} q}\right) \end{pmatrix}
$$
\nFor the groups of typ V we have\n
$$
d(2j, Z_q \times O^*) = \frac{1}{2} \{d(2j, G_4) + d(2j, G_5) + d(2j, G_6) - d(2j, \frac{3}{2})\}
$$
\nwhere $G_i \cong \langle g_i \rangle$ ($i = 4, 5, 6$), $\frac{3}{2} = \langle g_3 \rangle$ with\n
$$
g_4 = \begin{pmatrix} R\left(\frac{8+q}{8q}\right) & 0 \\ 0 & R\left(\frac{8-q}{8q}\right) \end{pmatrix}, \quad g_5 = \begin{pmatrix} R\left(\frac{6+q}{6q}\right) & 0 \\ 0 & R\left(\frac{6-q}{6q}\right) \end{pmatrix}
$$
\n
$$
f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

0
$$
R\left(\frac{P^2}{4 \cdot 3^{1-1}q}\right) / \qquad \qquad 0 \qquad R\left(\frac{1}{2 \cdot 3^{1-1}q}\right) /
$$

ne groups of typ V we have

$$
d(2j, Z_q \times O^*) = \frac{1}{2} \{d(2j, G_4) + d(2j, G_5) + d(2j, G_6) - d(2j, 3_2)\}
$$

where $G_i \cong \langle g_i \rangle$ $(i=4,\,5,\,6),\ \mathfrak{Z}_2 = \langle g_8 \rangle$ with

$$
0 \t R\left(\frac{p_2}{4 \cdot 3^{l-1}q}\right) / \t 0 \t R\left(\frac{1}{2 \cdot 3^{l-1}q}\right) /
$$

\n
$$
e \text{ groups of type V we have}
$$

\n
$$
d(2j, Z_q \times O^*) = \frac{1}{2} \{d(2j, G_4) + d(2j, G_5) + d(2j, G_6) - d(2j, 3_2)\}
$$

\n
$$
\approx \langle g_i \rangle \ (i = 4, 5, 6), 3_2 = \langle g_8 \rangle \text{ with}
$$

\n
$$
g_4 = \begin{pmatrix} R\left(\frac{8+q}{8q}\right) & 0 \\ 0 & R\left(\frac{8-q}{8q}\right) \end{pmatrix}, g_5 = \begin{pmatrix} R\left(\frac{6+q}{6q}\right) & 0 \\ 0 & R\left(\frac{6-q}{6q}\right) \end{pmatrix},
$$

\n
$$
g_6 = \begin{pmatrix} R\left(\frac{4+q}{4q}\right) & 0 \\ 0 & R\left(\frac{4-q}{4q}\right) \end{pmatrix}, g_8 = \begin{pmatrix} R\left(\frac{1}{2q}\right) & 0 \\ 0 & R\left(\frac{1}{2q}\right) \end{pmatrix}
$$

\n
$$
e \text{ with for the groups of type VI}
$$

Finally we find for the groups of typ VI

$$
\left(\begin{array}{cc} 0 & R\left(\frac{1}{4q}\right) \end{array}\right) \left(\begin{array}{cc} 0 & R\left(\frac{1}{2q}\right) \end{array}\right)
$$
\nwhere $d(2j, Z_q \times I^*) = \frac{1}{2} \{d(2j, G_7) + d(2j, G_5) + d(2j, G_6) - d(2j, \frac{1}{2})\}$

where $G_7 \cong$ *(97)* with

$$
\approx \langle g_{\gamma} \rangle \text{ with}
$$
\n
$$
g_{\gamma} = \begin{pmatrix} R \left(\frac{10+q}{10q} \right) & 0 \\ 0 & R \left(\frac{10-q}{10q} \right) \end{pmatrix}.
$$
\n
$$
\text{deorem 1 we can complete the proof of}
$$

Using Theorem 1 we can complete the proof of Theorem 2.

6. Special quadrature formulas on $S³$

To prove Theorem 3 we use the following

Lemma 5: Let $G \subset SO(4)$ be a finite subgroup of even order acting freely on S^3 . *If* $x_0 \in S^3$ *is a point with the property* $\tilde{f}(x_0) = 0$ *for each* $\tilde{f} \in V(j, G)$, $1 \leq j \leq l$, then

Theorem 3 we use the following
\n
$$
\begin{aligned}\n\text{na 5:} \text{ Let } G \subseteq SO(4) \text{ be a finite subgroup of even order and} \\
\text{a 3 is a point with the property } \tilde{f}(x_0) = 0 \text{ for each } \tilde{f} \in V(j, G), \\
\frac{1}{\omega_3} \int f \, do &= \frac{1}{|G|} \sum_{T \in G} f(Tx_0) \qquad \text{for any } f \in H_j \ (0 \le j \le l).\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{and } \text{a 3 is a point with the property } \tilde{f}(x_0) = 0 \text{ for each } \tilde{f} \in V(j, G), \\
\text{b 4 is a positive number of integers.}\n\end{aligned}
$$

We omit here the easy proof. Now, the assertions $(i) - (iv)$ of Theorem 3 follow by the table in Section 5 and Lemma 5. Furthermore, using Theorem 1 and our decomposition of $Z_q \times I^*$ we get $d(14, F_5) = d(16, F_5) = d(18, F_5) = 0$ by a straightforward calculation. This completes the proof of Theorem 3.

REFERENCES

- [1] BERGEn, M., GAUDUCHON, P..' and E. MAZET: Le spectre d'une variété Riemannienne. Lect. Notes Math. 194 (1971), $1-251$.
- [2] IKEDA, A.: On the spectrum of a Riemannian manifold of positive constant curvature. Osaka J. Math. 17 (1980), 75-93.
- [3] EKEDA, A.: On the spectrum of a Riemannian manifold of positive constant curvature II. Osaka J. Math. 17 (1980), 691-762.
- [4] 1KEDA, A., and Y., YAMAMOTO: On the spectra of 3-dimensional lens spaces. Osaka J. Math. 16 (1979), 447-469.
- [5] NEUTSCH. W., SCHRÜFER, E., and A. JENNER: Efficient integration on the hypersphere. J. Comp. Phys. 59 (1985), 167-175.
- [6] ORLIK, P.: Seifert manifolds. Lect. Notes Math. 291 (1972), $1-155$.
- [7] PRUFER, F.: On the spectrum and the geometry of a spherical space form II. Ann. Glob. Anal. Geom. 3 (1985), 289-312.
- [8] PRUFER, F.: The spectrum of homogeneous spherical space forms. Math. Nachr. 135 $(1988), 41-51.$
- [9] Риордлн, Д. Ж.: Комбинаторные тождества. Москва: Изд-во Наука 1982.
- [10] SAKAI, T.: On the spectrum of lens spaces. Kodai Math. Scm. Rep. 27 (1975), 249 to 257.
- [11] TSAGAS, G.: The spectrum of the Laplace operator for a spherical space form. Atti Accad. Naz. Lincei, VIII Ser., Rend., Sci. Fis. Math. Nat. 74 (1983), 357-365. [5] NEUTSCH, W., SCHRUFER, E., and A. JENNER: Efficient inte

J. Comp. Phys. 59 (1985), 167-175.

[6] ORLIK, P.: Seifert manifolds. Lect. Notes Math. 291 (1972),

[7] PRÜFER, F.: On the spectrum and the geometry of a sphe

[12] THRELFALL, W., and H. SEIFERT: Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (SchluI3). Math. Ann. 107 (1932), 543-586.

[13] WOLF, J. A.: Spaces of constant curvature. New York: McGraw Hill Book Comp. 1967.

[14] YAMAMOTO, Y.: On the number of lattice points in the square $|x| + |y| \le u$ with a certain congruence condition. Osaka J. Math. 17 (1980), $9-21$.

Manuskriptcingang: 30. 10. 1987; in revidierter Fassung: 05.01. 1988

VERFASSER

Dr. FRIEDBERT PRÜFER Sektion Mathematik der Karl-Marx-Universität Karl-Marx-Platz DDR-7010 Leipzig