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The Spectrum and Quadrature Formulas of Spherical Space Forms

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Es wird das Spektrum von kompakten Riemannschen Mannigfaltigkeiten der konstanten Schnittkrümmung 1, deren Fundamentalgruppe zyklisch ist, berechnet. Daraus ergeben sich Folgerungen für den ersten Eigenwert dreidimensionaler sphärischer Raumformen und für Quadraturformeln.

Вычисляется спектр компактных римановых многообразий постоянной сегментарной кривизны 1 с циклической фундаментальной группой. Отсюда получаются следствия для первого собственного значения трехмерных сферических пространственных форм и для квадратурных формул.

The spectrum of compact Riemannian manifolds of constant sectional curvature 1 with cyclic fundamental group is computed. From this corollaries on the first eigenvalue of 3-dimensional spherical space forms and quadrature formulas are obtained.

1. Introduction

Let M be a compact connected Riemannian manifold of constant sectional curvature 1. Then M is the orbit space of a finite group acting freely and orthogonally on a sphere of radius 1. Spherical space forms are completely classified by J. A. Wolf [13]. Let Δ be the Laplacian acting on the space of C^∞-functions on M. It has a purely discrete spectrum consisting of nonnegative eigenvalues with finite multiplicities. We denote the spectrum of the operator Δ by spec (M). There are a few explicit results on the spectrum of spherical space forms. For example, the spectra of homogeneous spherical space forms and certain lens spaces were computed, see [7, 8, 10, 11]. However, we do not know the spectrum of a lot of other spherical space forms. Using Ikeda's generating function (see [2, 3]), a method to compute explicitly the spectrum of M was given in [11].

The main tool of the present paper is a formula for the dimension of the space of G-automorphic homogeneous harmonic polynomials of degree j ($j \in \mathbb{N}_0$) proved in [7], where $G \subset O(n+1)$ is a finite group acting freely on S^{2m+1}. Here, we study this formula for cyclic groups in more detail.

The algebraic computation of the eigenvalues of orbit manifolds is closely related to lattice point problems. In Theorem 1 we give this reformulation for lens spaces. This lattice problem is an efficient method to compute the spectrum of spherical space forms. For 3-dimensional lens spaces the corresponding lattice problem was already obtained in [2, 13]. Since the computations are purely algebraic we give only short comments.

The purpose of Theorem 2 is to list all the first eigenvalues of 3-dimensional spherical space forms. To one's surprise there are only 6 possibilities for λ_1. Moreover, the first eigenvalue gives a few informations on the structure of the fundamental group of M. In certain cases $\lambda_1(M)$ and the dimension of the corresponding eigenspace contain the whole information on the geometry of M. In these cases the results are stronger than the results in [2].

In the last section we apply our results on small eigenvalues to derive integration formulas on the unit sphere for homogeneous polynomials. Such formulas are interesting for a few practical problems. In [5] was given a 19-design with 3600 nodes. Using the binary icosahedral group we find a 19-design with 840 nodes.

2. The results

Let $G = \langle C \rangle$ be a cyclic subgroup of order q and let C be conjugates in $O(2m+2)$ to the element

$$C' = \begin{pmatrix} R(p_1/q) & \cdot & 0 \\ 0 & \cdot & R(p_{m+1}/q) \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$$

where $p_1, \ldots, p_{m+1}$ are integers prime to q. Then S^{2m+1}/G is a spherical space form, which is a so-called lens space, and we have $S^{2m+1}/G \doteq L(q; p_1, \ldots, p_{m+1})$. Furthermore, let $J(q; p_1, \ldots, p_{m+1}; \varkappa)$ denote the number of lattice points $(a_1, \ldots, a_{m+1}) \in \mathbf{Z}^{m+1}$ such that

(i) $|a_1| + \cdots + |a_{m+1}| = \varkappa$,

(ii) $a_1 p_1 + \cdots + a_{m+1} p_{m+1} \equiv 0 \pmod q$, where $q, p_1, \ldots, p_{m+1}$ and $\varkappa$ are given integers.

Theorem 1: *Let $L = L(q; p_1, \ldots, p_{m+1})$ be a lens space of dimension $2m+1$ ($m \in \mathbf{N}$).*

(i) *For even q the spectrum of L is given by the eigenvalues $\mu_{2j} = 4j(j+m)$, $j \in \mathbf{N}_0$ with the multiplicities*

$$d(2j, L) = \sum_{\varrho=0}^{j} \binom{m+\varrho-1}{m-1} J(q; p_1, \ldots, p_{m+1}; 2j - 2\varrho). \tag{1}$$

(ii) *For odd q the spectrum of L is given by the eigenvalues $\mu_{2j} = 4j(j+m)$ ($j \in \mathbf{N}_0$) with the multiplicities (1) and by the eigenvalues $\mu_{2j+1} = (2j+1)(2j+1+2m)$, $j \geqq j_0$ ($j \in \mathbf{N}$), with the multiplicities*

$$d(2j+1, L) = \sum_{\varrho=0}^{j} \binom{m+\varrho-1}{m-1} J(q; p_1, \ldots, p_{m+1}; 2j+1-2\varrho).$$

Here, j_0 is the smallest positive integer such that there exists a solution $(a_1, \ldots, a_{m+1}) \in \mathbf{Z}^{m+1}$ of $|a_1| + \cdots + |a_{m+1}| = 2j_0 + 1$, $a_1 p_1 + \cdots + a_{m+1} p_{m+1} \equiv 0 \pmod q$.

We define the following typs of groups:

(I) $Z_q = \langle C \rangle$, the cyclic group of order q, $q \in \mathbf{N}$.

(II) $Z_q \times D'_{2^l\nu}$, where $(q, 2\nu) = 1$ and $D'_{2^l\nu} = \langle A, B \rangle$, $A^\nu = B^{2^l} = I$, $BAB^{-1} = A^{-1}$ and $l, \nu \geqq 3$, ν odd.

(III) $Z_q \times D^*_{4\nu}$, $(q, 2\nu) = 1$, where $D^*_{4\nu} = \langle A, B \rangle$, $A^{2\nu} = B^4 = I$, $BAB^{-1} = A^{-1}$ and $\nu \geqq 2$.

(IV) $Z_q \times T'_{3^l 8}$, $(q, 6) = 1$, where $T'_{3^l 8} = \langle X, P, Q \rangle$, $X^{3^l} = P^4 = I$, $P^2 = Q^2$, $XPX^{-1} = Q$, $XQX^{-1} = PQ$, $PQP^{-1} = Q^{-1}$ and $l \in \mathbf{N}$.

(V) $Z_q \times O^*$, $(q, 6) = 1$, where $O^* = \langle X, P, Q, R \rangle$ as for T'_{24} and $RXR^{-1} = X^{-1}$, $RPR^{-1} = QP$, $RQR^{-1} = Q^{-1}$.

(VI) $Z_q \times I^*$, $(q, 30) = 1$, where $I^* = \langle U, W \rangle$, $U^2 = (UW)^3 = W^5$, $U^4 = I$. (It is $I^* \cong SL(2, 5)$, where $SL(2, 5)$ denotes the multiplicative group of 2×2 matrices of determinant 1 with entries in the field of 5 elements.)

Theorem 2: *Let M be a 3-dimensional spherical space form of curvature* 1 *and* spec $(M) = \{0, \lambda_1, \lambda_2, \ldots\}$, $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$.

(i) *We have* $\lambda_1 \in \{3, 8, 24, 48, 80, 168\}$ *and*

$\lambda_1 = 3$ *iff* $M \cong S^3$,
$\lambda_1 = 8$ *iff* $\pi_1(M) \cong Z_q$, $q \geqq 2$,
$\lambda_1 = 24$ *iff* $\pi_1(M)$ *is isomorphic to a group of typ* (II) *or* (III),
$\lambda_1 = 48$ *iff* $\pi_1(M) \cong Z_q \times T'_{3^l 8}$, $(q, 6) = 1$, $l \in \mathbf{N}$,
$\lambda_1 = 80$ *iff* $\pi_1(M) = Z_q \times O^*$, $(q, 6) = 1$,
$\lambda_1 = 168$ *iff* $\pi_1(M) = Z_q \times I^*$, $(q, 30) = 1$.

(ii) *Further, let $d(\lambda_1, M)$ denote the dimension of the eigenspace corresponding to λ_1. Then $d(\lambda_1, M) \geqq 3$ iff M is homogeneous.*

Now, let H_j be the vector space of homogeneous polynomials of degree j in the variables x_1, x_2, x_3 and x_4. A quadrature formula of degree j is a set of r points $y_1, \ldots y_r$ on S^3 and constants $c_1, \ldots, c_r$ such that

$$\frac{1}{\omega_3} \int_{S^3} f\, do = \sum_{j=1}^{r} c_j f(y_j) \qquad \text{for all } f \in H_k \ (0 \leqq k \leqq j),$$

where do is the invariant measure of S^3 and $\omega_3 = 2\pi^2$. Quadrature formulas for S^3 are rare. For a result see [5]. Using the Theorems 1 and 2 we can easily give special quadrature formulas of degree less than 19.

Let G be a finite subgroup of $O(2m + 2)$. A function $f: S^{2m+1} \to C$ is called *G-automorphic* if $f \circ T = f$ for each $T \in G$.

Theorem 3: *Let $F_i \subset SO(4)$ $(i = 1, \ldots, 5)$ be finite groups acting freely on the sphere of radius 1 and $F_1 \cong D'_{24}$, $F_2 \cong T^*$, $F_3 \cong O^*$, $F_4 \cong I^*$ and $F_5 \cong Z_7 \times I^*$. Then we have the following quadrature formulas:*

(i) $$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{24} \sum_{T \in F_1} f(Ty_1),$$

$f \in H_j$ $(0 \leqq j \leqq 5)$ *and y_1 is any zero of the first nontrivial F_1-automorphic eigenfunction of S^3.*

(ii) $$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{24} \sum_{T \in F_2} f(Tx),$$

$f \in H_j$ $(0 \leqq j \leqq 5)$ *and* $x \in S^3$,

(iii) $$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{48} \sum_{T \in F_3} f(Tx),$$

$f \in H_j$ $(0 \leqq j \leqq 7)$ *and* $x \in S^3$,

(iv) $$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{120} \sum_{T \in F_4} f(Tx),$$

$f \in H_j$ $(0 \leqq j \leqq 11)$ *and* $x \in S^3$,

(v) $$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{840} \sum_{T \in F_5} f(Ty_2),$$

$f \in H_j$ $(0 \leqq j \leqq 19)$ *and y_2 is any zero of the first nontrivial F_5-automorphic eigenfunction of S^3:*

3. Spherical space forms and automorphic eigenfunctions

In this section we will briefly review the facts concerning the G-automorphic eigenfunctions of S^{2m+1} (see [1, 2]). Let S^{2m+1} ($m \in \mathbf{N}$) be the unit sphere centered at the origin in $\mathbf{R}^{2m+2}$. Furthermore, let M be a spherical space form of curvature 1 and dimension $2m + 1$. Then $M = S^{2m+1}/G$, where G is a finite subgroup of $SO(2m + 2)$ and for any $T \in G$ with $T \neq Id$, 1 is not an eigenvalue of T. Here Id denotes the unit matrix in $O(2m + 2)$. The sphere S^{2m+1} is the universal Riemannian covering manifold of M. Let ι be the covering map of S^{2m+1} onto S^{2m+1}/G, $\iota: S^{2m+1} \to S^{2m+1}/G$.

We denote the space of complex valued C^∞-functions on M by $C^\infty(M)$ and denote the Laplacian acting on $C^\infty(M)$ and $C^\infty(S^{2m+1})$ by Δ and $\tilde{\Delta}$, respectively. The Laplacian has a purely discrete spectrum spec (M). For $k \geqq 0$ let H_k be the space of complex valued homogeneous harmonic polynomials of degree k on $\mathbf{R}^{2m+2}$. The spectrum of $\tilde{\Delta}$ is well known, see [1]. It contains the eigenvalues $\mu_j = j(j + 2m)$ with multiplicities

$$d(j) = \binom{j+2m}{2m} - \binom{j-2+2m}{2m}, \qquad j \in \mathbf{N}_0.$$

The eigenfunctions of S^{2m+1}/G are exactly the G-automorphic eigenfunctions of S^{2m+1}. More precisely, let $\mathrm{i}: S^{2m+1} \hookrightarrow \mathbf{R}^{2m+2}$ be the natural inclusion map. Then i induces the restriction map $\mathrm{i}^*: C^\infty(\mathbf{R}^{2m+2}) \to C^\infty(S^{2m+1})$ and we have the following

Lemma 1 (see [2]): *Let $V(j)$ be the eigenspace with eigenvalue μ_j of $\tilde{\Delta}$. Then the map* i^* *gives an $O(2m + 2)$-isomorphism* $\mathrm{i}^*: H_j \cong V(j)$, $j \in \mathbf{N}_0$.

Furthermore, the map ι induces the injective map $\iota^*: C^\infty(S^{2m+1}/G) \to C^\infty(S^{2m+1})$. The following formula is elementary, see [1]: For any $f \in C^\infty(S^{2m+1}/G)$, we have $\tilde{\Delta}(\iota^* f) = \iota^*(\Delta f)$. Now it follows

Lemma 2 (see [2]): *Let $V(j, G)$ and $H(j, G)$ be the subspaces of $V(j)$ and H_j consisting of all the G-automorphic elements of $V(j)$ and H_j, respectively. Then $(\iota^*)^{-1}$ $V(j, G)$ is the eigenspace with the eigenvalue μ_j of the Laplacian Δ on S^{2m+1}/G and it is isomorphic to $H(j, G)$. Further, every eigenspace of Δ on S^{2m+1}/G is obtained in this way.*

Let $d(j, G) = \dim H(j, G)$. Consequently, we need only the values of $d(j, G)$ for the calculation of spec (M), $M = S^{2m+1}/G$.

4. The spectra of lens spaces

Now we prove our Theorem 1. Recall that in [7] the following formula for $d(j, G)$ was derived:

$$d(j, G) = \frac{1}{q}\left\{1 + \frac{(-1)^j}{2}\left(1 + (-1)^q\right)\right\} d(j) + \frac{1}{q} \sum_{T \in G}{}' \sum_{k=0}^{[j/2]} \alpha_m(j, k)\, 2^{j-2k} R(T; j, k) \qquad (j \in \mathbf{N}_0) \tag{2}$$

where q is the order of G, $\sum'$ means that we have to sum up only over $T \in G \setminus \{Id, -Id\}$, $[j/2]$ is the intire part of $j/2$,

$$\alpha_m(j, k) = (-1)^k \frac{j + m}{j + m - 2k} \binom{j + m - k - 1}{k}.$$

and

$$R(T; j, k) = \sum_{\substack{k_1+\cdots+k_{m+1}=j-2k \\ k_1,\ldots,k_{m+1}\geqq 0}} \prod_{\nu=1}^{m+1} \left(\cos \frac{2\pi p_\nu(T)}{q(T)}\right)^{k_\nu}.$$

Here $q(T)$ is the order of $T \in G$ and $\exp\left(2\pi i p_\nu(T)/q(T)\right)$, $\exp\left(-2\pi i p_\nu(T)/q(T)\right)$ are the eigenvalues of T ($1 \leqq \nu \leqq m+1$). Our proof of Theorem 1 is based on an explicit study of this formula for cyclic groups. We have $Z_q = \langle C\rangle$ and C is conjugate to C' in $O(2m+2)$. Similar to [8] this gives

$$d(j, Z_q) = \sum_{k=0}^{[j/2]} \alpha_m(j, k) \sum_{1\leqq \varrho_1 \leqq \cdots \leqq \varrho_{j-2k} \leqq m+1} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_{j-2k}\in\{-1,1\} \\ \varepsilon_1 p_{\varrho_1}+\cdots+\varepsilon_{j-2k}p_{\varrho_{j-2k}}\equiv 0 (\mathrm{mod}\, q)}} 1. \tag{3}$$

Here we have used the identities

$$\prod_{\nu=1}^{j-2k} \cos \frac{2\pi p_{\varrho_\nu} l}{q} = 2^{2k-j} \sum_{\varepsilon_1,\ldots,\varepsilon_{j-2k}\in\{-1,1\}} \cos \frac{2\pi l}{q} (\varepsilon_1 p_{\varrho_1} + \cdots + \varepsilon_{j-2k} p_{\varrho_{j-2k}}),$$

and

$$\sum_{l=0}^{q-1} \cos \frac{2\pi l}{q} t = \begin{cases} 0 & \text{for } t \not\equiv 0 \ (\mathrm{mod}\, q), \\ q & \text{for } t \equiv 0 \ (\mathrm{mod}\, q). \end{cases}$$

The next step is the following

Lemma 3: *We have the equality*

$$\sum_{1\leqq \varrho_1 \leqq \cdots \leqq \varrho_\varkappa \leqq m+1} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_\varkappa \in\{-1,1\} \\ \varepsilon_1 p_{\varrho_1}+\cdots+\varepsilon_\varkappa p_{\varrho_\varkappa}\equiv 0(\mathrm{mod}\, q)}} 1$$

$$= \sum_{l=0}^{[\varkappa/2]} \sum_{\substack{|a_1|+\cdots+|a_{m+1}|=\varkappa-2l \\ a_1p_1+\cdots+a_{m+1}p_{m+1}\equiv 0(\mathrm{mod}\, q)}} \sum_{\substack{s_1+\cdots+s_{m+1}=l \\ s_1,\ldots,s_{m+1}\geqq 0}} \prod_{\nu=1}^{m+1} \binom{|a_\nu| + 2s_\nu}{s_\nu}. \tag{4}$$

Proof: The sum on the left hand side runs over all terms $a_1p_1 + \cdots + a_{m+1}p_{m+1}$ with $a_1p_1 + \cdots + a_{m+1}p_{m+1} \equiv 0 \pmod q$ and $|a_1| + \cdots + |a_{m+1}| \leqq \varkappa$. Here every term can appear several times in the sum. Next we have only the possibilities $|a_1| + \cdots + |a_{m+1}| = \varkappa - 2l$, $0 \leqq l \leqq [\varkappa/2]$. This means that the term a_ip_i appears iff p_i appears exactly $|a_i| + 2s_i$ $\left(0 \leqq s_i \leqq [(\varkappa - |a_i|)/2]\right)$ times. In this case exactly s_i of the corresponding ε_ν must be -1. There are $\binom{|a_i| + 2s_i}{s_i}$ possibilities for the choice of these ε_ν's. Further we have $(|a_1| + 2s_1) + \cdots + (|a_{m+1}| + 2s_{m+1}) = \varkappa$ and for $|a_1| + \cdots + |a_{m+1}| = \varkappa - 2l$ we get $s_1 + \cdots + s_{m+1} = l$. Now we conclude easily that the term $a_1p_1 + \cdots + a_{m+1}p_{m+1}$ with $|a_1| + \cdots + |a_{m+1}| = \varkappa - 2l$ appears exactly

$$\sum_{\substack{s_1+\cdots+s_{m+1}=l \\ s_1,\ldots,s_{m+1}\geqq 0}} \prod_{i=1}^{m+1} \binom{|a_i| + 2s_i}{s_i}$$

times. This completes the proof of the lemma ∎

Now we put

$$\chi(b_1, \ldots, b_m; \varkappa, l) = \sum_{\substack{s_1+\cdots+s_{m+1}=l \\ s_1,\ldots,s_{m+1}\geqq 0}} \binom{b_1 + 2s_1}{s_1} \cdots \binom{b_m + 2s_m}{s_m} \times \binom{\varkappa - 2l - b_1 - \cdots - b_m + 2s_{m+1}}{s_{m+1}}$$

with $b_j \in \mathbf{N}_0$ $(1 \leqq j \leqq m)$, $b_1 + \cdots + b_m \leqq \varkappa - 2l$ and $\varkappa, l \in \mathbf{N}_0$. A straightforward calculation shows that χ does not depend on $b_1, \ldots, b_m$. Therefore we have

$$\chi(b_1, \ldots, b_m; \varkappa, l) = \chi(0, \ldots, 0; \varkappa, l). \tag{5}$$

Substituting (4) and (5) into (3) and permuting the sums we get

$$d(j, Z_q) = \sum_{\varrho=0}^{[j/2]} A_m(j, \varrho)\, J(q; p_1, \ldots, p_{m+1}; j - 2\varrho) \tag{6}$$

with

$$A_m(j, \varrho) = \sum_{k=0}^{\varrho} \alpha_m(j, k) \sum_{\substack{s_1+\cdots+s_{m+1}=\varrho-k\\ s_1,\ldots,s_{m+1}\geqq 0}} \binom{2s_1}{s_1} \cdots \binom{2s_m}{s_m} \binom{j - 2\varrho + 2s_{m+1}}{s_{m+1}}.$$

Lemma 4: *We have* $A_m(j, \varrho) = \binom{m + \varrho - 1}{\varrho}$ *for* $j \in \mathbf{N}_0$, $0 \leqq \varrho \leqq [j/2]$, $m \in \mathbf{N}$.

Proof: To begin with we find (see [8: p. 169])

$$\sum_{\substack{s_1+\cdots+s_{m+1}=\varrho-k\\ s_1,\ldots,s_{m+1}\geqq 0}} \binom{2s_1}{s_1} \cdots \binom{2s_m}{s_m} \binom{j - 2\varrho + 2s_{m+1}}{s_{m+1}}$$

$$= \sum_{\substack{s_1+\cdots+s_m\leqq\varrho-k\\ s_1,\ldots,s_m\geqq 0}} 2^{s_m} \binom{2s_1}{s_1} \cdots \binom{2s_{m-1}}{s_{m-1}} \binom{j - 2k - 2s_1 - \cdots - 2s_{m-1} - s_m}{\varrho - k - s_1 - \cdots - s_m}.$$

Using the addition formula for binomial coefficients several times, now we obtain $A_m(j, \varrho + 1) = A_m(j, \varrho) + A_{m-1}(j + 1, \varrho + 1)$. Therefore our lemma follows by induction with respect to j ∎

From Lemma 4 and (6) we obtain the desired formula in Theorem 1. The further statements of Theorem 1 are easily to prove by means of this formula. We omit here the proofs.

5. The first eigenvalue of 3-dimensional spherical space forms

The basic theorem of this section is the following. If G is a finite subgroup of $SO(4)$ acting freely on S^3, then G is isomorphic to one of the groups of typ (I)–(VI), see [12] (also [6, 13]). Moreover, let S^3/G_1 and S^3/G_2 be spherical space forms. Assume G_1 is isomorphic to G_2 and is not cyclic. Then G_1 is conjugate to G_2 in $O(4)$ such that $S^3/G_1 = S^3/G_2$.

We now may prove Theorem 2. Next, our Theorem 1 implies Theorem 2 for the groups of typ (I). The further assertions of Theorem 2 we get from the following table.

The calculations are the same in all cases. We decompose the group G into conjugate cyclic subgroups and then we determine by means of Theorem 1 its multiplicities. Therefore we consider completely here only the case $G \cong Z_q \times D'_{2^l\nu}$. In the other cases we give only the corresponding formulas.

Let G be a group of typ (II). Then $G \cong Z_q \times D'_{2^l\nu}$ and we have $D'_{2^l\nu} = \langle A, B\rangle$, $A^\nu = B^{2^l} = I$, $BAB^{-1} = A^{-1}$ with $l \geqq 3$, $q \in \mathbf{N}$, $\nu \geqq 3$, ν odd and $(q, 2\nu) = 1$. Further, let $Z_q = \langle C\rangle$, $C^q = I$, $CA = AC$, $CB = BC$. A representation π: $Z_q \times D'_{2^l\nu}$

G	$d(0, G)$	$d(2, G)$	$d(4, G)$	$d(6, G)$	$d(8, G)$	$d(10, G)$	$d(12, G)$
$Z_q \times D'_{2^l\nu}$	1	0	1				
$Z_q \times D^*_{4\nu}$	1	0	$2(q > 1, \nu = 2)$ $10(q = 1, \nu = 2)$ $1(q > 1, \nu > 2)$ $5(q = 1, \nu > 2)$				
$Z_q \times T'_{3^l8}$	1	0	0	$1(q > 1$ or $l > 1)$ $7\ (q = l = 1)$			
$Z_q \times O^*$	1	0	0	0	$1\ (q > 1)$ $9\ (q = 1)$		
$Z_q \times I^*$	1	0	0	0	0	0	$1\ (q > 1)$ $13\ (q = 1)$

$\to SO(4)$ acting freely on S^3 is given by (see [13])

$$\pi(A) = \begin{pmatrix} R\left(\frac{1}{\nu}\right) & 0 \\ 0 & R\left(-\frac{1}{\nu}\right) \end{pmatrix}, \qquad \pi(B) = \begin{pmatrix} 0 & Id \\ R\left(\frac{1}{2^{l-1}}\right) & 0 \end{pmatrix},$$

$$\pi(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 \\ 0 & R\left(\frac{1}{q}\right) \end{pmatrix}.$$

Then G is conjugate to $\pi(Z_q \times D'_{2^l\nu})$ in $O(4)$ and $S^3/G \cong S^3/\pi(Z_q \times D'_{2^l\nu})$. Therefore we can suppose $G = \pi(Z_q \times D'_{2^l\nu})$. The following facts are obviously or well known, see [13]. The group G contains the cyclic subgroups $G_{-1} = \langle \pi(B^2AC) \rangle$ of order $2^{l-1}\nu q$ and $G_\varrho = \langle \pi(BA^\varrho C) \rangle$ $(0 \leq \varrho \leq \nu - 1)$ of order $2^l q$. The groups G_ϱ are conjugate subgroups. Further we have $G_i \cap G_{i'} = \mathfrak{Z}$ $(i \neq i')$, where $\mathfrak{Z} = \langle \pi(B^2C) \rangle$ is the center of G. Let $|\mathfrak{Z}|$ be the order of $\mathfrak{Z}$. Then $|\mathfrak{Z}| = 2^{l-1}q$. Now (2) yields

$$\begin{aligned} d(2j, G) &= \frac{2}{|G|}\, d(2j) + \frac{1}{|G|} \Big\{ \sum_{T \in G_{-1}}{}' \sum_{k=0}^{j} \alpha_m(2j, k)\, 4^{j-k} R(T; 2j, k) \\ &\quad + \sum_{\varrho=0}^{\nu-1} \sum_{T \in G_\varrho}{}' \sum_{k=0}^{j} \alpha_m(2j, k)\, 4^{j-k} R(T; 2j, k) \\ &\quad - \nu \sum_{T \in \mathfrak{Z}}{}' \sum_{k=0}^{j} \alpha_m(2j, k)\, 4^{j-k} R(T; 2j, k) \Big\} \\ &= \frac{1}{2}\, d(2j, G_{-1}) + \frac{1}{\nu} \sum_{\varrho=0}^{\nu-1} d(2j, G_\varrho) - \frac{1}{2}\, d(2j, \mathfrak{Z}). \end{aligned}$$

Since the groups G_ϱ $(0 \leq \varrho \leq \nu - 1)$ are conjugate, we have $d(2j, G_\varrho) = d(2j, G_0)$ $(1 \leq \varrho \leq \nu - 1)$. This gives

$$d(2j, Z_q \times D'_{2^l\nu}) = \frac{1}{2}\, d(2j, G_{-1}) + d(2j, G_0) - \frac{1}{2}\, d(2j, \mathfrak{Z}).$$

The groups G_{-1}, G_0 and $\mathfrak{Z}$ are cyclic and we can easily compute the eigenvalues of $\pi(B^2AC)$, $\pi(BC)$ and $\pi(B^2C)$. Then using Theorem 1, we obtain the statements on $Z_q \times D'_{2^l\nu}$.

For the groups of type III similar considerations give

$$d(2j, Z_q \times D^*_{4\nu}) = \frac{1}{2} d(2j, G_{-1}) + d(2j, G_0) - \frac{1}{2} d(2j, \mathfrak{Z})$$

where $G_{-1} = \langle \pi(AC) \rangle$, $G_0 = \langle \pi(BC) \rangle$, $\mathfrak{Z} = \langle \pi(B^2C) \rangle$ and

$$\pi(A) = \begin{pmatrix} R\left(\frac{1}{2\nu}\right) & 0 \\ 0 & R\left(-\frac{1}{2\nu}\right) \end{pmatrix}, \quad \pi(B) = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}, \quad \pi(C) = \begin{pmatrix} R\left(\frac{1}{q}\right) & 0 \\ 0 & R\left(\frac{1}{q}\right) \end{pmatrix}.$$

For the groups of typ IV we obtain

$$d(2j, Z_q \times T'_{3^l 8}) = d(2j, G_1) + \frac{1}{2} d(2j, G_2) - \frac{1}{2} d(2j, \mathfrak{Z}_1)$$

where $G_i \cong \langle g_i \rangle$ $(i = 1, 2)$, $\mathfrak{Z}_1 \cong \langle g_3 \rangle$ with

$$g_1 = \begin{pmatrix} R\left(\frac{p_1}{2 \cdot 3^l q}\right) & 0 \\ 0 & R\left(\frac{p_2}{2 \cdot 3^l q}\right) \end{pmatrix},$$

$$g_2 = \begin{pmatrix} R\left(\frac{p_1'}{4 \cdot 3^{l-1} q}\right) & 0 \\ 0 & R\left(\frac{p_2'}{4 \cdot 3^{l-1} q}\right) \end{pmatrix}, \quad g_3 = \begin{pmatrix} R\left(\frac{1}{2 \cdot 3^{l-1} q}\right) & 0 \\ 0 & R\left(\frac{1}{2 \cdot 3^{l-1} q}\right) \end{pmatrix}.$$

For the groups of typ V we have

$$d(2j, Z_q \times O^*) = \frac{1}{2} \{d(2j, G_4) + d(2j, G_5) + d(2j, G_6) - d(2j, \mathfrak{Z}_2)\}$$

where $G_i \cong \langle g_i \rangle$ $(i = 4, 5, 6)$, $\mathfrak{Z}_2 = \langle g_8 \rangle$ with

$$g_4 = \begin{pmatrix} R\left(\frac{8+q}{8q}\right) & 0 \\ 0 & R\left(\frac{8-q}{8q}\right) \end{pmatrix}, \quad g_5 = \begin{pmatrix} R\left(\frac{6+q}{6q}\right) & 0 \\ 0 & R\left(\frac{6-q}{6q}\right) \end{pmatrix},$$

$$g_6 = \begin{pmatrix} R\left(\frac{4+q}{4q}\right) & 0 \\ 0 & R\left(\frac{4-q}{4q}\right) \end{pmatrix}, \quad g_8 = \begin{pmatrix} R\left(\frac{1}{2q}\right) & 0 \\ 0 & R\left(\frac{1}{2q}\right) \end{pmatrix}.$$

Finally we find for the groups of typ VI

$$d(2j, Z_q \times I^*) = \frac{1}{2} \{d(2j, G_7) + d(2j, G_5) + d(2j, G_6) - d(2j, \mathfrak{Z}_2)\}$$

where $G_7 \cong \langle g_7 \rangle$ with

$$g_7 = \begin{pmatrix} R\left(\frac{10+q}{10q}\right) & 0 \\ 0 & R\left(\frac{10-q}{10q}\right) \end{pmatrix}.$$

Using Theorem 1 we can complete the proof of Theorem 2.

6. Special quadrature formulas on S^3

To prove Theorem 3 we use the following

Lemma 5: *Let $G \subset SO(4)$ be a finite subgroup of even order acting freely on S^3. If $x_0 \in S^3$ is a point with the property $\bar{f}(x_0) = 0$ for each $\bar{f} \in V(j, G)$, $1 \leqq j \leqq l$, then*

$$\frac{1}{\omega_3} \int_{S^3} f\, do = \frac{1}{|G|} \sum_{T \in G} f(Tx_0) \qquad \text{for any } f \in H_j \ (0 \leqq j \leqq l).$$

We omit here the easy proof. Now, the assertions (i)—(iv) of Theorem 3 follow by the table in Section 5 and Lemma 5. Furthermore, using Theorem 1 and our decomposition of $Z_q \times I^*$ we get $d(14, F_5) = d(16, F_5) = d(18, F_5) = 0$ by a straightforward calculation. This completes the proof of Theorem 3.

REFERENCES

[1] Berger, M., Gauduchon, P., and E. Mazet: Le spectre d'une variété Riemannienne. Lect. Notes Math. **194** (1971), 1—251.
[2] Ikeda, A.: On the spectrum of a Riemannian manifold of positive constant curvature. Osaka J. Math. **17** (1980), 75—93.
[3] Ikeda, A.: On the spectrum of a Riemannian manifold of positive constant curvature II. Osaka J. Math. **17** (1980), 691—762.
[4] Ikeda, A., and Y. Yamamoto: On the spectra of 3-dimensional lens spaces. Osaka J. Math. **16** (1979), 447—469.
[5] Neutsch, W., Schrüfer, E., and A. Jenner: Efficient integration on the hypersphere. J. Comp. Phys. **59** (1985), 167—175.
[6] Orlik, P.: Seifert manifolds. Lect. Notes Math. **291** (1972), 1—155.
[7] Prüfer, F.: On the spectrum and the geometry of a spherical space form II. Ann. Glob. Anal. Geom. **3** (1985), 289—312.
[8] Prüfer, F.: The spectrum of homogeneous spherical space forms. Math. Nachr. **135** (1988), 41—51.
[9] Риордан, Д. Ж.: Комбинаторные тождества. Москва: Изд-во Наука 1982.
[10] Sakai, T.: On the spectrum of lens spaces. Kodai Math. Sem. Rep. **27** (1975), 249 to 257.
[11] Tsagas, G.: The spectrum of the Laplace operator for a spherical space form. Atti Accad. Naz. Lincei, VIII Ser., Rend., Sci. Fis. Math. Nat. **74** (1983), 357—365.

[12] THRELFALL, W., and H. SEIFERT: Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß). Math. Ann. **107** (1932), 543–586.
[13] WOLF, J. A.: Spaces of constant curvature. New York: McGraw Hill Book Comp. 1967.
[14] YAMAMOTO, Y.: On the number of lattice points in the square $|x| + |y| \leqq u$ with a certain congruence condition. Osaka J. Math. **17** (1980), 9–21.

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