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Traces of Anisotropic Sobolev Spaces with Mixed L_p-Norms on Hyperplanes

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Wir skizzieren einen neuen kurzen Beweis für einen bekannten Spursatz für anisotrope Sobolev-Räume mit gemischten L_p-Normen, der auf Methoden der Interpolationstheorie beruht.

Мы наметим новое короткое доказательство для одной известной теоремы о следе для анизотропных пространств Соболева со смешанными L_p-нормами базирующее на теории интерполяции.

We outline a new short proof of a known trace theorem for anisotropic Sobolev spaces with mixed L_p-norms based on methods of interpolation theory.

1. Introduction

The aim of this paper is to outline a new proof of direct and inverse embedding theorems for traces of anisotropic Sobolev spaces with mixed L_p-norms on hyperplanes in $\mathbf{R}^n$. Our arguments will be quite similar as in [4: 2.9]. First of all let us recall some definitions and results. Let $(p) = (p_1, \ldots, p_n)$, $(1) \leqq (p) \leqq (\infty)$, i.e. $1 \leqq p_1, \ldots, p_n \leqq \infty$. Then one sets

$$L_{(p)}(\mathbf{R}^n) = L_{p_n}(\mathbf{R}, L_{p_{n-1}}(\mathbf{R}, \ldots, L_{p_1}(\mathbf{R}) \ldots)).$$

Let $\mathbf{A}$ be an arbitrary Banach space and $f\colon \mathbf{R}^+ \to \mathbf{A}$. We define

$$\|f \mid L_q^*(\mathbf{A})\| = \left(\int_0^\infty \|f(t) \mid \mathbf{A}\|^q \frac{dt}{t} \right)^{1/q}.$$

We use standard notations for differences and derivatives: $D_j^m = \partial^m/\partial x_j^m$, $\triangle_{j,t}^1 f(x) = f(x_1, \ldots, x_j + t, \ldots, x_n) - f(x)$, and $\triangle_{j,t}^m = \triangle_{j,t}^{m-1} \triangle_{j,t}^1$.

Definition 1: Let $(p) = (p_1, \ldots, p_n)$, $1 \leqq p_1, \ldots, p_n \leqq \infty$.

(i) If $m \in \mathbf{N}$ and $1 \leqq j \leqq n$, then

$$W_{(p),j}^m(\mathbf{R}^n) = \{f \in L_{(p)}(\mathbf{R}^n) \mid \|f \mid W_{(p),j}^m\| = \|f \mid L_{(p)}\| + \|D_j^m f \mid L_{(p)}\| < \infty\}.$$

(ii) If $(m) = (m_1, \ldots, m_n) \in \mathbf{N}^n$, then

$$W_{(p)}^{(m)}(\mathbf{R}^n) = \{f \in L_{(p)}(\mathbf{R}^n) \mid \|f \mid W_{(p)}^{(m)}\| = \|f \mid L_{(p)}\| + \sum_{j=1}^n \|D_j^{m_j} f \mid L_{(p)}\| < \infty\}.$$

(iii) If $s > 0$, $1 \leqq q < \infty$, $1 \leqq j \leqq n$, $k, l \in \mathbf{N}$, $0 \leqq k < s$ and $l > s - k$, then

$$\begin{aligned} B_{(p),q}^{s,j}(\mathbf{R}^n) = \{f \in L_{(p)}(\mathbf{R}^n) \mid \|f \mid B_{(p),q}^{s,j}\| \\ = \|f \mid L_{(p)}\| + \|t^{-(s-k)} \triangle_{j,t}^l D_j^k f \mid L_q^*(L_{(p)})\| < \infty\}. \end{aligned}$$

(iv) If $(s) = (s_1, \ldots, s_n)$ with $s_1, \ldots, s_n > 0$, and $(k) = (k_1, \ldots, k_n) \in \mathbf{N}^n$ and $(l) = (l_1, \ldots, l_n) \in \mathbf{N}^n$ with $(0) \leqq (k) < (s)$ and $(l) > (s - k)$, i.e. $0 \leqq k_i < s_i$ and $l_i > s_i - k_i$ for all admissible i, then

$$B^{(s)}_{(p),q}(\mathbf{R}^n) = \Big\{f \in L_{(p)}(\mathbf{R}^n) \mid \|f \mid B^{(s)}_{(p),q}\| = \|f \mid L_{(p)}\| + \sum_{j=1}^{n} \|t^{-(s_j-k_j)} \triangle^{l_j}_{j,t} D_j^{k_j} f \mid L_q^*(L_{(p)})\| < \infty\Big\}.$$

Proposition 1 (cf. [1: 5.4/p. 73] for (i)): *For $(p) = (p_1, \ldots, p_n)$ with $1 \leqq p_1, \ldots, p_n < \infty$ the following propositions hold:*

(i) ("*Lebesgue's theorem*") *Let $\{f_k\}_{k=1}^{\infty} \subset L_{(p)}(\mathbf{R}^n)$ be a pointwise convergent sequence, $f_k(x) \to f(x)$ if $k \to \infty$. If there exists a function $g \in L_{(p)}(\mathbf{R}^n)$ with $|f_k(x)| < g(x)$ for all $x \in \mathbf{R}^n$, then $f \in L_{(p)}(\mathbf{R}^n)$ and $f_k \to f$ (convergence in $L_{(p)}$).*

(ii) *$C_0^\infty(\mathbf{R}^n)$ is dense in $L_{(p)}(\mathbf{R}^n)$.*

Remark: To prove (ii) one can use Sobolev's mollification method, which works also in the case of mixed L_p-norms, cf. [1: II §§ 5, 6].

Proposition 2 (cf. [1: Theorem 18.2/p. 294]): *All norms in Definition 1/(iii) and (iv) are equivalent to each other for all admissible k, l and (k), (l), respectively.*

Proposition 3: *We have*

(i) $W^{(m)}_{(p)}(\mathbf{R}^n) = \bigcap_{j=1}^{n} W^{m_j}_{(p),j}(\mathbf{R}^n)$, (ii) $B^{(s)}_{(p),q}(\mathbf{R}^n) = \bigcap_{j=1}^{n} B^{s_j,j}_{(p),q}(\mathbf{R}^n)$.

Proposition 4: $\|f \mid W^{(p)}_{(m)}\|^* = \|f \mid L_{(p)}\| + \sum_{\alpha \in J} \|D^\alpha f \mid L_{(p)}\|$ *is an equivalent norm in* $W^{(m)}_{(p)}(\mathbf{R}^n)$, $J = \{\alpha \mid 0 \leqq \alpha_1/m_1 + \ldots + \alpha_n/m_n \leqq 1\}$.

Proposition 5 (cf. [2] and also [1: 11.5/p. 165]): *Let $A = \{x \in \mathbf{R}^n \mid x_1, \ldots, x_n \neq 0\}$ and let M be a function on $\mathbf{R}^n$, such that $x^\alpha D^\alpha M$ is bounded and continuous on A for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leqq \alpha_1, \ldots, \alpha_n \leqq 1$. Then M is a multiplier for $L_{(p)}(\mathbf{R}^n)$, $(1) < (p) < (\infty)$, i.e. $\|F^{-1}MFf \mid L_{(p)}\| \leqq cC\|f \mid L_{(p)}\|$ for all $f \in L_{(p)}(\mathbf{R}^n)$, where c is a constant independent of f and M and where $C = \sup\{x^\alpha D^\alpha M(x) \mid x \in A, 0 \leqq \alpha_1, \ldots, \alpha_n \leqq 1\}$.*

Proposition 6 (cf. [1: Theorem 14.14/p. 235]): *$C_0^\infty(\mathbf{R}^n)$ is dense in $W^{(m)}_{(p)}(\mathbf{R}^n)$, $(1) < (p) < (\infty)$.*

As usual, $S(\mathbf{R}^n)$ stands for the collection of all complex-valued infinitely differentiable rapidly decreasing functions on $\mathbf{R}^n$. The dual space $S' = S'(\mathbf{R}^n)$ is the collection of all tempered distributions on $\mathbf{R}^n$. We recall that F and F^{-1} stand for the Fourier transform and its inverse on S', respectively. We need the following

Lemma 1: *We have*: (i) *The norms* $\|f \mid W^m_{(p),j}\|^* = \|f \mid L_{(p)}\| + \sum_{k=1}^{m} \|D_j^k \mid L_{(p)}\|$ *and* $\|f \mid H^m_{(p),j}\| = \|F^{-1}(1 + x_j^2)^{m/2} Ff \mid L_{(p)}\|$ *are equivalent in* $W^m_{(p),j}(\mathbf{R}^n)$. (ii) $C_0^\infty(\mathbf{R}^n)$ *is dense in* $W^m_{(p),j}(\mathbf{R}^n)$, $(1) < (p) < (\infty)$.

Proof: (i) We show $\|f \mid W^m_{(p),j}\|^* \leqq c\,\|f \mid H^m_{(p),j}\|$ for all $f \in L_{(p)}(\mathbf{R}^n)$. Proposition 5 yields that $M_{k,j}(x) = x_j^k(1 + x_j^2)^{-m/2}$ is a Fourier multiplier for $L_{(p)}$ if $k \leqq m$. By $D_j^k f = cF^{-1}x_j^k Ff$ we obtain the assertion. Now we show $\|f \mid H^m_{(p),j}\| \leqq c\,\|f \mid W^m_{(p),j}\|$. Using again Proposition 5, we find that for $\sigma \in C_0^\infty(\mathbf{R})$ with $\sigma(t) = 0$ if $0 \leqq t \leqq 1/2$, $\sigma(t) = 1$ if $t \geqq 1$ and $\sigma(-t) = -\sigma(t)$ the function M_σ, $M_\sigma(x) = (1 + x_j^2)^{m/2}/$

$/(1+\sigma^m(x_j)\,x_j^m)$, is also a multiplier. Then it follows that

$$\begin{aligned}\|f \mid H^m_{(p),j}\| &= \|F^{-1}(1+x_j^2)^{m/2}\,Ff \mid L_{(p)}\| \\ &= \|F^{-1}M_\sigma(x)\,FF^{-1}(1+\sigma^m(x_j)\,x_j^m)\,Ff \mid L_{(p)}\| \\ &\leqq c\,\|F^{-1}(1+\sigma^m(x_j)\,x_j^m)\,Ff \mid L_{(p)}\| \\ &\leqq c\left(\|f \mid L_{(p)}\| + \|F^{-1}\sigma^m(x_j)\,x_j^m Ff \mid L_{(p)}\|\right) \\ &\leqq c\left(\|f \mid L_{(p)}\| + \|F_1^{-1}\sigma^m(x_j)\,F_1 D_j^m f \mid L_{(p)}\|\right) \leqq c'\,\|f \mid W^m_{(p),j}\|,\end{aligned}$$

where F_1 refers to the 1-dimensional Fourier transform, and where we used that σ^m is a 1-dimensional multiplier. This completes the proof of part (i).

(ii) It is sufficient to remark that because of Proposition 1 the proof in [4: 2.5.1] is applicable in our situation ∎

Lemma 2 (Interpolation): *The following propositions hold:*

(i) $(L_{(p)}(\mathbf{R}^n),\ W^m_{(p),j}(\mathbf{R}^n))_{\theta,q} = B^{\theta m,j}_{(p),q}(\mathbf{R}^n)$.

(ii) $(L_{(p)}(\mathbf{R}^n),\ W^{(m)}_{(p)}(\mathbf{R}^n))_{\theta,q} = B^{\theta(m)}_{(p),q}(\mathbf{R}^n)$.

Proof: Define a strongly continuous semi-group on $L_{(p)}$ by

$$G_j(t)\colon L_{(p)}(\mathbf{R}^n) \to L_{(p)}(\mathbf{R}^n)\colon f(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_j + t, \ldots, x_n).$$

Λ_j denotes the infinitesimal operator of $\{G_j(t)\}_{t\geqq 0}$. As usual one defines Λ_j^m (m-th power of Λ_j). We use Proposition 1 and Lemma 1 and obtain in the same way as in [4: 2.5.1] that the domain of Λ_j^m equals $W^m_{(p),j}(\mathbf{R}^n)$. Now one can use the interpolation formulas from [4: 1.13]. Therefore (i) is proved. (ii) is a consequence of Proposition 3/(i) and the results in [4: 1.13.2 and 1.12.1] ∎

We refer to the trace method of interpolation (cf. [4: 1.8]). Let $\{\mathbf{A}_0, \mathbf{A}_1\}$ be an interpolation couple, $m \in \mathbf{Z}$, $1 \leqq p_0, p_1 \leqq \infty$ and $\eta_0, \eta_1 \in \mathbf{R}$. One sets
$V_m(p_0, \eta_0, \mathbf{A}_0; p_1, \eta_1, \mathbf{A}_1) = \{u = u(t) \mid u$ is a regular $(\mathbf{A}_0 + \mathbf{A}_1)$-- distribution on $\mathbf{R}^+$

$$\text{with } \|u \mid V_m\| = \|t^{\eta_0}u(t) \mid L^*_{p_0}(\mathbf{A}_0)\| + \|t^{\eta_1}u^{(m)}(t) \mid L^*_{p_1}(\mathbf{A}_1)\| < \infty\}.$$

We need the rather technical concept of quasi-linearizable interpolation couples in order to employ theoretical results of interpolation. But for the sake of brevity we omit details on this subject and refer for an exact definition to [4: 1.8 and 1.12].

Theorem 1 (cf. [4: Theorem 1.8.5/(a)]): *Let $\{\mathbf{A}_0, \mathbf{A}_1\}$ be a quasilinearizable interpolation couple. Let $m \in \mathbf{N}$, $1 \leqq p \leqq \infty$ and $\eta_0, \eta_1 \in \mathbf{R}$. If $J = \{j \in \mathbf{Z} \mid 0 \leqq j \leqq m-1$ and $-\eta_0 < j < m - \eta_1\}$, then*

$$R\colon V_m(p, \eta_0, \mathbf{A}_0; p, \eta_1, \mathbf{A}_1) \to \prod_{j\in J} (\mathbf{A}_0, \mathbf{A}_1)_{\theta_j,p}\colon u \mapsto \{u^{(j)}(0)\}_{j\in J}$$

is a retraction. One has $\theta_j = (\eta_0 + j)/(m + \eta_0 - \eta_1)^{-1}$.

Remark: We recall what is meant by a retraction. If $\mathbf{A}$ and $\mathbf{B}$ are two Banach spaces, then a mapping $R\colon \mathbf{A} \to \mathbf{B}$ is called a *retraction* if R is a linear and bounded mapping from $\mathbf{A}$ onto $\mathbf{B}$ and if there exists a linear and bounded mapping S from $\mathbf{B}$ into $\mathbf{A}$ with $RS = \mathbf{E}_\mathbf{B}$ (identity mapping in $\mathbf{B}$).

2. Direct and inverse embedding theorem

Let $(1) \leqq (p) \leqq (\infty)$, $(m) = (m_1, \dots, m_n) \in \mathbf{N}^n$ and $\mathbf{R}_+{}^n = \{x \in \mathbf{R}^n \mid x_n > 0\}$. Then one defines

$$W_{(p)}^{(m)}(\mathbf{R}_+{}^n) = \{f \in L_1^{\text{loc}}(\mathbf{R}_+{}^n) \mid \exists\, g \in W_{(p)}^{(m)}(\mathbf{R}^n) \text{ with } g(x) = f(x) \text{ if } x \in \mathbf{R}_+{}^n\}.$$

This is a Banach space normed by

$$\|f \mid W_{(p)}^{(m)}(\mathbf{R}_+{}^n)\| = \inf \{\|g \mid W_{(p)}^{(m)}(\mathbf{R}^n)\| \mid f = g \mid \mathbf{R}_+{}^n\}.$$

Proposition 7: *The restriction R: $W_{(p)}^{(m)}(\mathbf{R}^n) \to W_{(p)}^{(m)}(\mathbf{R}_+{}^n): g \mapsto g \mid \mathbf{R}_+{}^n$ is a retraction. If $k \in \mathbf{N}$, then there exists a coretraction S which is independent of (m) with $m_n \leqq k$ and (p).*

Proof: The proof is analogous to that in the case of unmixed L_p-norms. We recall only the definition of the coretraction S: For $k \in \mathbf{N}$ let $0 < \gamma_1 < \dots < \gamma_{k+1} < \infty$. For a smooth function f vanishing for large values of x one sets

$$(Sf)(x) = \begin{cases} \sum_{j=1}^{k+1} a_j f(x_1, \dots, -\gamma_j x_n) & \text{for } x_n < 0, \\ f(x) & \text{for } x_n \geqq 0 \end{cases}$$

where a_j fits the claim that Sf possesses continuous derivatives up to order k in $\mathbf{R}^n$ ∎

Lemma 3: *Let $(m) \in \mathbf{N}^n$, $m \in \mathbf{N}$ and let $1 < p < \infty$ and $(1) \leqq (p) < (\infty)$. Then*

$$V_m\left(p, \frac{1}{p}, W_{(p)}^{(m)}(\mathbf{R}^n); p, \frac{1}{p}, L_{(p)}(\mathbf{R}^n)\right) = W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1}).$$

Proof: Let $u(t)(x) = f(x, t)$ be a smooth function contained in $W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1})$ that vanishes for large values of $|x|$. These functions are dense in $W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1})$. By a straightforward computation one obtains that $\|u \mid V_m\|$ and $\|f \mid W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1})\|$ are equivalent to each other. It remains to show that functions of the described type are also dense in V_m. For this purpose take $f \in V_m$, $f(x, t)$, and $\delta > 0$. Then $f(x, t + \delta) \in V_m$ and $f(x, t + \delta) \to f(x, t)$ in V_m if $\delta \to 0$. We apply Sobolev's mollification method to $f(x, t + \delta)$. Let $0 < h < \delta$ and $\omega^1, \dots, \omega^n, \omega^{n+1} \in C_0^\infty(\mathbf{R}^1)$ with $\operatorname{supp} \omega^j \subseteq (1/2, 1]$ and $\int_0^\infty \omega^j(t)\, dt = 1$, $j = 1, \dots, n + 1$. Let $\omega_h{}^j(t) = h^{-1}\omega^j(t/h)$. One sets $\omega_h(x, t) = \omega_h{}^1(x_1)\, \omega_h{}^2(x_2) \dots \omega_h{}^n(x_n)\, \omega_h{}^{n+1}(t)$, and, for $g \in L_{((p),p)}(\mathbf{R}_+{}^{n+1})$, $(g)_h(x, t) = \int_{\mathbf{R}^n} \int_0^\infty \omega_h(x - z, t - \sigma)\, g(z, \sigma)\, dz\, d\sigma$. Then $(g)_h \to g$ holds in V_m for $h \to 0$. Hence $\big(f(x, t + \delta)\big)_h \to f(x, t + \delta)$ in V_m, $h \to 0$. The function $\big(f(x, t + \delta)\big)_h$ can be extended to $\mathbf{R}^n$ via the construction described in Proposition 7 and hence is contained in $W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1})$. Thus, the lemma is proved ∎

Now we come to the main result of this paper.

Theorem 2: *Let $(m) = (m_1, \dots, m_n) \in \mathbf{N}^n$, $m \in \mathbf{N}$ and $(1) < (p) < (\infty)$. Then*

$$R: W_{((p),p)}^{((m),m)}(\mathbf{R}_+{}^{n+1}) \to \prod_{j=0}^{m-1} B_{(p),p}^{\sigma^j}(\mathbf{R}^n): f(x, t) \mapsto \{D_t{}^j f(x, 0)\}_{j=0}^{m-1}$$

is a retraction, where $(\sigma^j) = (\sigma_1{}^j, \dots, \sigma_n{}^j)$ is given by $\sigma_k{}^j = m_k\big(1 - m^{-1}(p^{-1} + j)\big)$, $k = 1, \dots, n$; $j = 0, \dots, m - 1$.

Proof: Because of Lemma 2, the definition of $W^{(m)}_{(p)}(\mathbf{R}^n)$ and the results from [4: Theorems 1.13.2 and 1.12.1] it follows that $\{W^{(m)}_{(p)}(\mathbf{R}^n), L_{(p)}(\mathbf{R}^n)\}$ is a quasi-linearizable interpolation couple. Therefore the above Lemma and Theorem 1 yield that

$$R: W^{((m),m)}_{((p),p)}(\mathbf{R}_+^{n+1}) \to \prod_{j=0}^{m-1} \left(W^{(m)}_{(p)}(\mathbf{R}^n), L_{(p)}(\mathbf{R}^n)\right)_{\theta_j,p}$$

is a retraction. For θ_j one obtains $\theta_j = m^{-1}(p^{-1} + j)$. Hence by Lemma 2 and the relation $(\mathbf{A}_0, \mathbf{A}_1)_{\theta,p} = (\mathbf{A}_1, \mathbf{A}_0)_{1-\theta,p}$ it follows $(\sigma^j) = (m)(1 - \theta_j)$. This completes the proof ∎

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