Zeitschrift für Analysis
und ihre Anwendungen
Bd. 8 (2) 1989, S. 169–176

On the Behaviour of the Distributional Stieltjes Transformation at the Origin

S. Pilipović

Es wird das Verhalten der Stieltjes-Transformierten solcher Distributionen aus $\mathcal{S}_+'$ im Ursprung 0 untersucht, die ein geeignetes quasiasymptotisches Verhalten in 0^+ besitzen. Diese neuen Ergebnisse werden mit Hilfe bekannter Resultate über das asymptotische Verhalten bei ∞ erhalten. Ferner wird auch ein Satz vom Tauberschen Typ über das Verhalten in 0 bewiesen.

Исследуется поведение трансформации Стильтьеса таких дистрибуций из $\mathcal{S}_+'$ в начале координат 0, которые имеют подходящее квазиасимптотическое поведение в точке 0^+. Эти новые результаты получаются с помощью известных об асимптотическом поведении в точке ∞. Доказывается также один результат Тауберова типа о поведении в точке 0.

The behaviour of the distributional Stieltjes transformation at the origin 0 is investigated for distributions of $\mathcal{S}_+'$ having appropriate quasiasymptotic behaviour at 0^+. These new results follow by known ones for the asymptotic behaviour at ∞. A Tauberian-type result for the behaviour at 0 is also obtained.

1. Notions and known results

The sets of real and natural numbers are denoted by $\mathfrak{R}$ and $\mathfrak{N}$, respectively. $\mathcal{S}(\mathfrak{R}) = \mathcal{S}$ and $\mathcal{S}'(\mathfrak{R}) = \mathcal{S}'$ denote the spaces of rapidly decreasing functions and tempered distributions, respectively. The space $\mathcal{J}'(r)$, $r \in \mathfrak{R} \setminus (-\mathfrak{N})$ is defined in [3] as a subspace of $\mathcal{S}_+' = \{f \in \mathcal{S}'(\mathfrak{R});\ \operatorname{supp} f \subset [0, \infty)\}$ consisting of all f of the form

$$\begin{aligned} &f = D^m F, \text{ for some } m \in \mathfrak{N}_0 = \mathfrak{N} \cup \{0\}, \\ &F \in L^1_{\text{loc}}, \qquad \operatorname{supp} F \subset [0, \infty), \end{aligned} \tag{1}$$

D is the distributional derivative, such that

$$\int_0^\infty |F(t)/(t + x)^{r+m+1}|\, dt < \infty, \quad x > 0. \tag{2}$$

Obviously, if $f \in \mathcal{J}'(r + p)$, then $f^{(p)} \in \mathcal{J}'(r)$, $p \in \mathfrak{N}_0$. We also need the definition of the space $\mathcal{I}'(r)$, $r \in \mathfrak{R} \setminus (-\mathfrak{N})$. This is a subspace of $\mathcal{J}'(r)$ consisting of all $f \in \mathcal{J}'(r)$ for which (1) holds and instead of (2), there holds

$$|F(t)| < C(1 + t)^{r+m-\varepsilon}, \qquad t > 0, \quad \text{for some } C, \varepsilon > 0. \tag{2)*}$$

The distributional Stieltjes transformation of $f = D^m F \in \mathcal{J}'(r)$ is defined by [2]

$$(S_r f)(z) = (r + 1)_m \int_0^\infty \frac{F(t)\, dt}{(z + t)^{r+m+1}}, \qquad z \in \mathfrak{C} \setminus (-\infty, 0], \tag{3}$$

where $(a)_m = a(a+1)\dots(a+m-1)$, $m \in \mathfrak{N}$, $(a)_0 = 1$, $a \in \mathfrak{R}$ and $\mathfrak{C}$ is the set of complex numbers. This is a holomorphic function.

We always denote in this paper by L a function which is *slowly varying at* ∞ (0^+), i.e. which is a continuous positive function defined in $(0, \infty)$ such that $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ $(x \to 0^+)$, $\lambda > 0$. For the properties of such functions we refer to [7]. When in connection with a function L we deal with the point ∞ (0^+) we shall always assume that L is slowly varying at ∞ (0^+).

In our investigations of the distributional Stieltjes transformation the notions of quasiasymptotic behaviour at ∞ and at 0^+ play a fundamental role. These notions are introduced by ZAVIALOV [9]. Note that in [6] we changed slightly the definition of the quasiasymptotic behaviour at 0^+. Recall, $f \in \mathcal{S}_+'$ has the *quasiasymptotic behaviour at* ∞ (0^+) *related to* $k^\alpha L(k)$ $\big((1/k)^\alpha L(1/k)\big)$ *with the limit* $g \in \mathcal{S}_+'$ *if*

$$\lim_{k\to\infty}\left\langle \frac{f(kx)}{k^\alpha L(k)}, \varphi(x)\right\rangle = \langle g(x), \varphi(x)\rangle, \quad \varphi \in \mathcal{S},$$
$$\left(\lim_{k\to\infty}\left\langle \frac{f(x/k)}{(1/k)^\alpha L(1/k)}, \varphi(x)\right\rangle = \langle g(x), \varphi(x)\rangle, \quad \varphi \in \mathcal{S}\right). \tag{4}$$

$\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathcal{S}'$ and $\mathcal{S}$. We include in the definition the case $g = 0$, as well, while in [9] and [6] were assumed $g \neq 0$. It is well known that g in (4) must be of the form $g = Cf_{\alpha+1}$, where

$$f_{\alpha+1}(t) = \begin{cases} H(t)\, t^\alpha/\Gamma(\alpha+1) & \text{if } \alpha > -1, \\ D^n f_{\alpha+n+1}(t) & \text{if } \alpha \leqq -1 \end{cases} \quad (t \in \mathfrak{R})$$

for some $n \in \mathfrak{N}$ with $n + \alpha > -1$ see [8]; H is Heviside's function, Γ is the gamma function. For the properties of the quasiasymptotic behaviour at ∞ we refer to [8] and at 0^+ we refer to [6]. Let us only quote the so-called structural theorem. Let $f \in \mathcal{S}_+'$ have the quasiasymptotic behaviour at ∞ (0^+) related to $k^\alpha L(k)$ $\big((1/k)^\alpha L(1/k)\big)$ with the limit $Cf_{\alpha+1}$; then there is an integer $m_0 \in \mathfrak{N}_0$, $m_0 + \alpha > -1$, such that for every $m \geqq m_0$ there is a locally integrable function F_m with $\operatorname{supp} F_m \subset [0, \infty)$ such that

$$f = F_m^{(m)}, \qquad \lim_{x\to\infty} \frac{F_m(x)}{x^{\alpha+m}} = \frac{C}{\Gamma(\alpha+m+1)} \qquad \left(\lim_{x\to 0^+} \frac{F_m(x)}{x^{\alpha+m}} = \frac{C}{\Gamma(\alpha+m+1)}\right). \tag{5}$$

Note that (5) is proved in [8] and [6] with the assumption $C \neq 0$. One can easily prove that this holds with $C = 0$, as well.

We shall need the following theorem from [5].

Theorem A: *Let* $f \in \mathcal{S}'$ *have the quasiasymptotic behaviour at* ∞ *related to* $k^\alpha L(k)$ *with the limit* $Cf_{\alpha+1}$, *where* $\alpha < r$. *Then*

$$\lim_{k\to\infty} \frac{(ks)^{r-\alpha}}{L(k)} (S_r f)(ks) = \frac{C\Gamma(r-\alpha)}{\Gamma(r+1)}, \tag{6}$$

for any $s \in \Omega_\varepsilon = \{\varrho e^{i\varphi} : \varrho > 0, -\pi + \varepsilon \leqq \varphi \leqq \pi - \varepsilon\}$, *where* $0 < \varepsilon < \pi/2$. *If* $L = 1$, *then*

$$s^{r-\alpha}(S_r f)(s) \to \frac{C\Gamma(r-\alpha)}{\Gamma(r+1)} \quad \textit{uniformly in } \Omega_\varepsilon \textit{ when } |s| \to \infty. \tag{7}$$

This theorem was proved with the assumption $C \neq 0$. But it also holds with $C = 0$, because the main step of its proof is the use of (5) which holds with $C = 0$.

As well, we need from [4] the following theorem.

Theorem B: *The following two statements are true:*

(i) *Let Φ be integrable on $\Re$, supp $\Phi \subset [0, \infty)$ and $C = \int_0^\infty \Phi(t)\,dt$. Then $\lim_{k\to\infty} \Phi(k\cdot)/k^{-1} = C\delta$ in $\mathcal{S}'$ (δ is the delta distribution).*

(ii) *Let $\Phi \in L^1_{\text{loc}}$, supp $\Phi \subset [0, \infty)$, $\lim_{t\to\infty} \Phi(t)/t^{-1}L(t) = C$ and let $L^*(x) = \int_a^x \big(L(t)/t\big)\,dt \to \infty$ as $x \to \infty$ ($a > 0$). Then $\lim_{k\to\infty} \Phi(k\cdot)/k^{-1}L(k) = C\delta$ in $\mathcal{S}'$.*

Note that (i) holds trivially and that (ii) follows from the fact that

$$\int_0^x \Phi(t)\,dt/L^*(x) \to C, \qquad x \to \infty \qquad \text{(L'Hospital's rule)},$$

and so

$$\lim_{k\to\infty} \int_0^{kx} \Phi(t)\,dt/L^*(k) = CH(x) \text{ in } \mathcal{S}' \qquad (x \in \Re).$$

Now, by differentiation we obtain (ii).
Note that we include in Theorem B the case $C = 0$.

Remark: By developing the notion of the quasiasymptotic at 0^+ we proved in [6] the same theorem for the behaviour of $S_r f$ at 0: If $f \in \mathcal{J}'(r)$ has the quasiasymptotic behaviour at 0^+ related to $(1/k)^\alpha L(1/k)$, $\alpha < r$, then (6) holds with $1/k$ instead of k and (7) holds with $|s| \to 0$ instead of $|s| \to \infty$. The aim of this paper is to extend this theorem using Theorem A. We shall also give a Tauberian-type result. It is based on the following theorem from [2: p. 339].

Theorem C: *Let us suppose that for some $m > 0$*

$$\int_0^\infty d\varphi(\lambda)/(\lambda + x)^{m+1} \sim \int_0^\infty d\psi(\lambda)/(\lambda + x)^{m+1}, \qquad x \to \infty,$$

and that the following conditions are satisfied:

1. *φ and ψ are non-decreasing;*
2. *$\psi(x) \to \infty$ as $x \to \infty$;*
3. *for any $C > 1$ there are γ and N, $0 < \gamma < m$, $N > 0$, such that for any $x > y \geq N$, $\psi(x)/\psi(y) < C(x/y)^\gamma$.*

Then, $\varphi(\lambda) \sim \psi(\lambda)$, $\lambda \to \infty$.

2. Abelian-type results

Theorem 1: *Let $F \in \mathcal{J}'(r)$ have the quasiasymptotic behaviour at 0^+ related to $(1/k)^\alpha \times L(1/k)$ with the limit $Cf_{\alpha+1}$. Then:*

(i) *For $r > \alpha$,* $\displaystyle\lim_{x\to 0^+} \frac{(S_r f)(x)}{x^{\alpha-r}L(x)} = \frac{C\Gamma(r-\alpha)}{\Gamma(r+1)}$.

(ii) *For $r < \alpha$, $\lim_{x\to 0^+} (S_r f)(x) = B$, for some $B \in \Re$.*

(iii) *Assume that* $r = \alpha$ *and that* $L_1(t) = L(1/t)$, $t > 0$, *is locally integrable in* (a, ∞), *for some* $a > 0$. *If*

$$\int_a^\infty \frac{L_1(t)}{t}\,dt \begin{cases} < \infty, \text{ then } \lim\limits_{x\to 0^+} (S_r f)(x) = B, \\ = \infty, \text{ then } \lim\limits_{x\to 0^+} \big((S_r f)(x)/\tilde{L}(x)\big) = B \end{cases}$$

for some $B \in \Re$, *where* $\tilde{L}(1/x) = \tilde{L}_1(x) = \int_a^x \big((L_1(t)/t)\big)\,dt$, $x > 0$.

Note that B in part (ii) and in both cases of part (iii) denotes always different constants which depends on C, α and r. This dependence will be clear from the proof. Clearly, (i) is a part of the assertion given in the Remark. We shall give here another proof of this fact.

Proof: Assume that (1) and (2)* hold for f with some $\overline{m} \in \Re_0$. Then for F_m defined by

$$F_m(x) = \int_0^x F_{m-1}(t)\,dt \qquad (x \in \Re;\ m = \overline{m} + 1, \overline{m} + 2, \ldots, F_{\overline{m}} = F)$$

there holds:

$$\begin{aligned} &F_m \text{ is continuous, supp } F_m \subset [0, \infty), \\ &|F_m(t)| \leqq C_m (1 + t)^{r+m-\varepsilon}, \quad t > 0, \quad F_m^{(m)} = f, \\ &(S_r f)(z) = (r + 1)_m \int_0^\infty \frac{F_m(t)}{(z + t)^{r+m+1}}\,dt, \quad z \in \mathfrak{C} \setminus (-\infty, 0] \end{aligned} \tag{8}$$

(for ε see (2)*). Note that the functions F_m are uniquely determined. The structural theorem at 0^+ given in (5) implies that, for $m \geqq \max\{m_0, \overline{m}\}$, $\lim\limits_{x\to 0^+} \big(F_m(x)/x^{\alpha+m}\big) = C/\Gamma(\alpha + m + 1)$. Fix $m > \max\{m_0, \overline{m}\}$ and denote F_m by F again. We have, for $k > 0$ and $z \in \mathfrak{C} \setminus (-\infty, 0]$,

$$\left(\frac{z}{k}\right)^{r+m+1} (S_r f)\left(\frac{z}{k}\right) = (r + 1)_m \left(\frac{z}{k}\right)^{r+m+1} \int_0^\infty \frac{F(t)\,dt}{(t + z/k)^{r+m+1}}$$

$$= (r + 1)_m \left(\frac{z}{k}\right)^{r+m+1} \int_0^\infty \frac{(1/u)^2\,F(1/u)\,du}{(1/u + z/u)^{r+m+1}} = (r + 1)_m \int_0^\infty \frac{u^{r+m-1} F(1/u)\,du}{(u + k/z)^{r+m+1}}.$$

So, we obtain, for $k > 0$ and $z \in \mathfrak{C} \setminus (-\infty, 0]$,

$$\left(\frac{z}{k}\right)^{r+m+1} (S_r f)\left(\frac{z}{k}\right) = (r + 1)_m (S_{r+m}\Phi)\left(\frac{k}{z}\right), \tag{9}$$

where $\Phi(t) = t^{r+m-1} F(1/t)$ for $t > 0$ and $\Phi(t) = 0$ for $t \leqq 0$. Obviously,

$$\lim_{t\to\infty} \big(\Phi(t)/t^{r-\alpha-1} L_1(t)\big) = C/\Gamma(\alpha + m + 1) \tag{10}$$

($L_1(k) = L(1/k)$ is slowly varying at ∞). Because of (8) we have (with suitable C_1)

$$|\Phi(t)| \leqq C_1 t^{-1+\varepsilon} (1 + t)^{r+m-\varepsilon}, \qquad t > 0. \tag{11}$$

((11) shows that Φ is locally integrable on $\Re$.)

(i) Assume $\alpha < r$. Since $r - \alpha - 1 > -1$, (10) implies that Φ has the quasi-asymptotic behaviour at ∞ related to $k^{r-\alpha-1}L_1(k)$ with the limit $C\big(\Gamma(r-\alpha)/\Gamma(\alpha+m+1)\big) f_{r-\alpha}$ [8]. So, Theorem A implies (i).

(ii) Assume $\alpha > r$. Now, from (10) (and (11)) it follows that Φ is integrable on $\mathfrak{R}$. From Theorem B/(i) it follows that Φ has the quasiasymptotic behaviour at ∞ related to k^{-1} with the limit $\tilde{B}\delta$ where $\tilde{B}$ depends on m and Φ (see Theorem B). Theorem implies (ii): We have by (9) $(S_rf)(1/k) = k^{r+m+1}(r+1)_m (S_{r+m}\Phi)(k) \to (r+1)_m \tilde{B}$, $k \to \infty$, i.e. $(S_rf)(t) \to B = (r+1)_m \tilde{B}$, $t \to 0^+$. Note that B does not depend on m.

(iii) Assume $\alpha = r$. We have by (10), $\Phi(t) \sim \tilde{\tilde{B}} t^{-1}L_1(t)$, $t \to \infty$, where $\tilde{\tilde{B}}$ is a suitable constant. If $\int_a^\infty \big((L_1(t)/t)\big)\,dt < \infty$, then from Theorem B/(i) it follows that Φ has the quasiasymptotic behaviour at ∞ related to k^{-1} with the limit $\tilde{B}\delta$ where $\tilde{B}$ depends on m and Φ. Theorem A completes the proof of the first part of (iii). Assume now that $\alpha = r$ and $\int_a^\infty \big((L_1(t)/t)\big)\,dt = \infty$. Then (10), Theorem B/(ii) and Theorem A completes the proof of (iii), because Φ has the quasiasymptotic behaviour at ∞ with the limit $\tilde{B}\delta$ related to $k^{-1}\tilde{L}(k)$ ∎

Let us set for ε, $0 < \varepsilon < \pi/2$,

$$L(0,R) = \{s: |s| < R\}, \qquad \Lambda_\varepsilon = \{\varrho e^{i\varphi}: \varrho > 0, |\varphi| \leqq \pi - \varepsilon\}.$$

Lemma 2: *Let f satisfy the conditions of Theorem 1 with $\alpha \geqq r > -1$. Then the functions*

$$s \to (S_rf)(s), \qquad s \in \Lambda_\varepsilon \cap L(0,R), \qquad \text{for } \alpha > r,$$

$$s \to \frac{1}{\ln s}(S_rf)(s), \qquad s \in \Lambda_\varepsilon \cap L(0,R), \qquad \text{for } \alpha = r,\ L = 1,$$

are bounded ($\ln s = \ln|s| + i\varphi$, $|\varphi| \leqq \pi - \varepsilon$).

Proof: Observe first the case $\alpha > r$. Clearly it is enough to prove that S_rf is bounded in $\Lambda_\varepsilon \cap L(0,R)$. For $z = \varrho e^{i\varphi} \in \Lambda_\varepsilon \cap L(0,R)$ we have $|1/z|/|t + 1/z| = \lambda(t^2 - 2t\lambda\cos\varepsilon + \lambda^2)^{-1/2}$, $\lambda = 1/\varrho > 1/R$. From

$$\begin{aligned} t^2 - 2t\lambda\cos\varepsilon + \lambda^2 &\geqq t^2 + \lambda^2 + (t^2 + \lambda^2)\cos\varepsilon - (t+\lambda)^2\cos\varepsilon \\ &\geqq (t^2+\lambda^2)(1+\cos\varepsilon) - 2(t^2+\lambda^2)\cos\varepsilon \\ &= (t^2+\lambda^2)(1-\cos\varepsilon) > (t+\lambda)^2(1-\cos\varepsilon)/2 \end{aligned}$$

we have ($\lambda = |1/z|$)

$$\left|\frac{1/z}{t+1/z}\right| < \left(\frac{2}{1-\cos\varepsilon}\right)^{1/2} \frac{\lambda}{t+\lambda}, \qquad t > 0, z \in \Lambda_\varepsilon \cap L(0,R). \tag{12}$$

This implies that, for suitable C, $|1/z|/|t+1/z| \leqq C$ $\big(t > 0, z \in \Lambda_\varepsilon \cap L(0,R)\big)$. Since (9) implies

$$(S_rf)(z) = \frac{(r+1)_m}{z^{r+m+1}} \int_0^\infty \frac{\Phi(t)\,dt}{(t+1/z)^{r+m+1}}, \qquad z \in \mathbb{C} \setminus (-\infty, 0],$$

and Φ is integrable, we have

$$|(S_rf)(z)| \leqq C^{r+m+1} \int_0^\infty |\Phi(t)|\,dt < \infty, \qquad z \in \Lambda_\varepsilon \cap L(0,R).$$

Observe now the case $\alpha = r$, $L = 1$. In this case in Theorem 1/(iii), second case, we have $\tilde{L}(x) \sim -\ln x$, $x \to 0^+$. Since the limit $\lim\limits_{x\to\infty} \Phi(x)/x^{-1}$ is finite, with suitable $A, B \in \Re$, and $s \in \Lambda_\varepsilon \cap L(0, R)$, $\lambda = 1/|s|$, (9), (11) and (12) imply

$$(S_r f)(s) \leqq (r+1)_m \left(\int\limits_0^A |\Phi(t)| \left| \frac{1/s}{t + 1/s} \right|^{r+m+1} dt + B \int\limits_A^\infty \frac{1}{t} \left| \frac{1/s}{t+1/s} \right|^{r+m+1} dt \right)$$

$$\leqq (r+1)_m \left(\int\limits_0^A |\Phi(t)|\, dt + B \left(\frac{2}{1 - \cos \varepsilon} \right)^{r+m+1} \int\limits_A^\infty \frac{\lambda^{r+m+1}\, dt}{t(\lambda + t)^{r+m+1}} \right).$$

From the identity

$$\frac{1}{t} \left(\frac{\lambda}{\lambda + t} \right)^{r+m+1} = \frac{1}{t} - \frac{1}{t+\lambda} - \frac{\lambda}{(t+\lambda)^2} - \cdots - \frac{\lambda^{r+m}}{(t+\lambda)^{r+m+1}}$$

we have

$$\int\limits_A^\infty \frac{1}{t} \left(\frac{\lambda}{\lambda + t} \right)^{r+m+1} dt = \left(\ln \frac{t}{t+\lambda} + \frac{\lambda}{t+\lambda} \cdots + \frac{1}{(r+m)} \frac{\lambda^{r+m}}{(t+\lambda)^{r+m}} \right) \Bigg|_A^\infty.$$

We obtained that the integral $\int\limits_A^\infty \ldots$ is bounded independently of λ. This implies that $(S_r f)(s)$, $s \in \Lambda_\varepsilon \cap L(0, R)$, is bounded. Since $1/\ln s$, $s \in \Lambda_\varepsilon \cap L(0, R)$, is bounded, as well, the proof is complete ∎

Assume that the conditions of Lemma 2 hold. We set

$$A(z) = \begin{cases} (S_r f)\left(\dfrac{1}{z}\right) & \text{if } \alpha > r, \\ \dfrac{1}{\ln z} (S_r f)\left(\dfrac{1}{z}\right) & \text{if } \alpha = r,\ L = 1 \end{cases} \qquad \left(z \in \Lambda_\varepsilon + \frac{1}{R} \right). \tag{13}$$

Lemma 2 implies that in both case A is bounded in $\Lambda_\varepsilon + 1/R$. Set $A_1(z) = A(z + 1/R)$, $z \in \Lambda_\varepsilon$.

Lemma 3: *There holds $A_1(z) \to B$ uniformly in Λ_ε when $|z| \to \infty$, where B is from Theorem* 1/(ii) *or* (iii), *second case.*

Proof: We have that A_1 is bounded in Λ_ε and that $A_1(x) \to B$, $x \to \infty$ (Theorem 1/(ii) or (iii)). So Montel's theorem [1: p. 5] implies the assertion ∎

Theorem 4: *Assume that the conditions of Theorem* 1 *hold for f with $\alpha \geqq r > -1$.*
(i) *If $\alpha > r$, then $(S_r f)(z) \to B$, $|z| \to 0$, $z \in \Lambda_\varepsilon$, uniformly.*
(ii) *If $\alpha = r$, $L = 1$, then $(1/\ln z)(S_r f)(z) \to B$, $|z| \to 0$, $z \in \Lambda_\varepsilon$, uniformly.*

Proof: Lemma 3 implies that in both cases $A(z) \to B$, $|z| \to \infty$, $z \in \Lambda_\varepsilon + 1/R$, uniformly. So, this implies the proof of the theorem ∎

3. A Tauberian-type result

Let $r \in \mathfrak{R} \setminus \mathfrak{N}$. Assume that $f = F^{(m)} \in \mathcal{S}_+'$ where F is a non-increasing positive locally integrable function such that $F(x) < Ax^{r+m-\varepsilon}$, $x > 0$, for some $A > 0$. Assume that $s > 1$, $r + m - s > 0$ and that $x^{r+m-s}L_1(x)$, $x > A$, is non-decreasing, where $L(x)$, $x > 0$, is slowly varying at 0^+ and $L_1(x) = L(1/x)$, $x > 0$. With the given assumptions we have

Theorem 5: *Assume that*

$$(S_r f)(x) \sim (r+1)_m \frac{\Gamma(s)}{\Gamma(r+m+1)} \frac{L(x)}{x^{m+r+1-s}}, \qquad x \to 0^+.$$

Then

$$\lim_{k\to\infty} \frac{f(x/k)}{(1/k)^{s-m-1}L(1/k)} = Bf_{s-m} \text{ in } \mathcal{S}',$$

where B is a suitable constant.

Proof: The assumption of the theorem and (9) imply that

$$(S_{r+m}\Phi)(x) \sim \frac{\Gamma(s)}{\Gamma(r+m+1)} \frac{L_1(x)}{x^s}, \quad x \to \infty,$$

where $\Phi(x) = x^{r+m-1}F(1/x)$, $x > 0$, is a non-decreasing function. Let

$$\psi(x) = \begin{cases} x^{r+m-s}L_1(x)/\Gamma(r+m-s+1), & x > A, \\ 0, & x \leqq A. \end{cases}$$

Theorem A implies

$$\int_0^\infty \frac{d\psi(t)}{(x+t)^{r+m}} = (r+m) \int_0^\infty \frac{\psi(t)}{(x+t)^{r+m+1}}\, dt \underset{x\to\infty}{\sim} \frac{(r+m)\,\Gamma(s)}{\Gamma(r+m+1)} \frac{L_1(x)}{x^s}.$$

So,

$$\int_0^\infty (d\Phi/(x+t)^{r+m}) \sim \int_0^\infty \big((d\psi/(x+t)^{r+m})\big) \text{ as } x \to \infty.$$

If we show that for every $C > 1$ there are constants γ and N, $0 < \gamma < r + m - 1$, $N > 0$, such that

$$x > y > N \Rightarrow x^{r+m-s}L_1(x)/y^{r+m-s}L_1(y) = C(x/y)^\gamma, \tag{14}$$

then all the assumptions of Theorem B are satisfied and this theorem implies

$$\Phi(x) \sim \psi(x), \quad x \to \infty. \tag{15}$$

Take $\gamma = r + m - s + \varepsilon$ where $\varepsilon > 0$ such that $\gamma > 0$ and $\varepsilon < s - 1$. With such γ and $x = \lambda y$, $\lambda > 1$, $y > N$, (14) becomes $L_1(\lambda y) \leqq C\lambda^\varepsilon L_1(y)$, and this is true [7: p. 18]; note, N depends on C. So, (15) implies $\varphi(x) \sim x^{r+m-s}L_1(x)/\Gamma(r+m-s+1)$, $x \to \infty$, and thus

$$x^{r+m-1} F\left(\frac{1}{x}\right) \sim \frac{x^{r+m-s}}{\Gamma(r+m-s+1)} L_1(x), \qquad x \to \infty,$$

i.e.

$$F(x) \sim \frac{x^{s-1}}{\Gamma(r+m-s+1)} L(x), \qquad x \to 0^+.$$

Since $f = F^{(m)}$ we have for suitable B the assertion ∎

Acknowledgement. This material is based on work supported by the U. S.—Yugoslav Joint Fund for Scientific and Technological Cooperation, in cooperation with the (IFP) under Grant 838.

REFERENCES

[1] Boas, R. P.: Entire Functions. New York: Acad. Press 1954.
[2] Владимиров, В. С., Дрожжинов, Ю. Н., и Б. И. Завьялов: Многомерные Тауберовы теоремы для обобщенных функций. Москва: Изд-во Наука 1986.
[3] Завьялов, Б. И.: Автомодельная асимптотика электромагнитных форм-факторов и поведение их фурье-образов в окрестности светового конуса. Теор. Мат. Физ. **17** (1973), 178—188.
[4] Костюченко, А. Г., и И. С. Саргсян: Распределение собственных значений. Москва: Изд-во Наука 1979.
[5] Lavoine, J., and D. P. Misra: Abelian theorems for the distributional Stieltjes transformation. Proc. Camb. Phil. Soc. **86** (1979), 287—293.
[6] Pilipović, S., and A. Takači: The quasiasymptotic behaviour of some distributions. Rev. Res. Fac. Sci. Univ. Novi Sad **15** (1985), 37—46.
[7] Pilipović, S., and B. Stanković: Abelian Theorem for the Distributional Stieltjes Transform. Z. Anal. Anw. **6** (1987), 341—349.
[8] Pilipović, S., and B. Stanković: Initial value Abelian theorems for the distributional Stiletjes transform. Studia Math. (to appear).
[9] Seneta, E.: Regularly Varying Functions (Lect. Notes Math. 508). Berlin—Heidelberg—New York: Springer-Verlag 1976.

Manuskripteingang: 02. 06. 1987; in revidierter Fassung 09. 12. 1987

VERFASSER:

Prof. Dr. Stevan Pilipović
Institute of Mathematics, University of Novi Sad
dr. Ilije Djuričića 4
Yu-21000 Novi Sad