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Continuous Dependence of the Solution of a System of Differential Equations with Impulses on the Initial Condition

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Es werden Anfangswertprobleme für Differentialgleichungssysteme mit Impulseffekt betrachtet. Die Impulse sind in den Momenten realisiert, wenn die Integralkurven einige fixierte Hyperflächen im erweiterten Phasenraum schneiden. Es werden hinreichende Bedingungen für die stetige Abhängigkeit der Lösungen dieser Systeme von den Anfangsbedingungen gefunden.

Рассматриваются задачи с начальными значениями для систем дифференциальных уравнений с импульсным воздействием. Импульсы реализованы в те моменты когда интегральная кривая пересекает некоторые наперёд заданные гиперповерхности в расширенном фазовом пространстве. Находятся достаточные условия непрерывной зависимости решений таких систем от начальных данных.

Initial value problems for systems of differential equations with impulses are considered. The impulses are realized in the moments when the integral curves meet some of previously fixed hypersurfaces in the extended phase space. Sufficient conditions for the continuous dependence of the solutions of these systems on the initial conditions are found.

1. Introduction

Systems of differential equations with impulses provide an adequate mathematical description of numerous phenomena and processes studied by physics, chemistry, radiotechnics, etc. By means of such systems phenomena and processes subject to short-time perturbations during their evolution are studied. The duration of the perturbations is negligeable in comparison with the duration of the phenomena and processes considered, therefore they are regarded as "momentary" of the type of "impulses". The first publications on mathematical theory of systems with impulses were by V. D. Milman and A. D. Myshkis [5, 6]. This new and perspective theory was developed further in the works of A. M. Samoilenko [13, 14], T. Pavlidis [11], V. Raghavendra and M. Rao [12], S. G. Pandit [7, 8], etc. Recently the interest in systems of differential equations with impulses has grown considerably due to the numerous applications of these systems to mathematical control theory. The first two monographs dedicated to this subject appeared by A. Halanai and D. Veksler [4] and by S. G. Pandit and S. G. Deo [10]. Systems of differential equations with impulses are a subject of study of the present paper as well.

2. Preliminaries

Consider the hypersurfaces

$$\sigma_i : t = t_i(x) \qquad (t \in \mathbf{R};\, x \in D;\, i \geqq 1), \tag{1}$$

where D is a domain in $\mathbf{R}^n$. Denote by P_t a point with current coordinates $(t, x(t))$

such that the law of its motion is described by means of:

a) the set of hypersurfaces σ_i $(i \geqq 1)$;

b) a set of functions $I_i: D \to \mathbf{R}^n$ $(i \geqq 1)$;

c) a system of ordinary differential equations

$$\frac{dx}{dt} = f(t, x) \qquad (t \neq \tau_i; i \geqq 1) \tag{2}$$

where $f: S \to \mathbf{R}^n$, $S = \{(t, x) \mid t \geqq 0, x \in D\}$. The points

$$\tau_i \ (i \geqq 1): 0 < \tau_1 < \tau_2 < \dots, \tag{3}$$

are the moments in which the mapping point P_t meets the hypersurfaces σ_i (note that P_t meets σ_i only in the moments τ_i);

d) the equalities

$$x(\tau_i + 0) - x(\tau_i - 0) = I_{j_i}(x(\tau_i)) \qquad (i \geqq 1) \tag{4}$$

where j_i is the number of the hypersurface met by the point P_t in the moment τ_i. In general it is possible that $i \neq j_i$ (examples can be given such that for some i the inequality $i < j_i$ holds as well as examples for which $i = j_i$ or $i > j_i$). We assume further that $x(\tau_i) = x(\tau_i - 0)$, i.e. the function x is left continuous in the points τ_i $(i \geqq 1)$.

The set of objects a)—d) is called a *system of differential equations with impulses*. The law of the motion of P_t is called a *solution* of the system of differential equations with impulses and the curve described by this point is called an *integral curve* of the system of differential equations with impulses. In the present paper the initial value problem for such a system is considered with the initial condition

$$x(0) = x_0, \qquad x_0 \in D. \tag{5}$$

We shall give a brief description of the motion of the point P_t along the integral curve of the initial value problem for a system of differential equations with impulses. The initial position of P_t is the point $(0, x_0)$. For $0 \leqq t \leqq \tau_1$ the point P_t moves along the integral curve of the system (2) with initial condition (5). In the moment τ_1 the point P_t meets the hypersurface σ_{j_1} in the point (τ_1, x_1) where $x_1 = x(\tau_1)$ and "jumps instantly" from the position (τ_1, x_1) into the position $(\tau_1, x_1 + I_{j_1}(x_1))$. Further on it moves along the integral curve of (2) with the initial point $(\tau_1, x_1 + I_{j_1}(x_1))$ until the moment τ_2 in which it meets the hypersurface σ_{j_2}, "jumps" again, etc. If after a "jump" the point P_t meets a hypersurface from (1), then a new "jump" in this moment is not realized. The solution of the system of differential equations with impulses is a piecewise continuous function with points of discontinuity of 1st type in which it is left continuous.

Further on we shall use the following notation: $x_i = x(\tau_i)$, $x_i^+ = x_i + I_{j_i}(x_i)$ $(i \geqq 1)$, $\tau_0 = 0$, $j_0 = 0$, $t_0(x) = 0$ for $x \in D$, $x_0^+ = x_0$; by $x(t, \tau^*, x)$ we denote the solution of the system of differential equations with impulses with the initial condition $x(\tau^*, \tau^*, x^*) = x^*$; $x(t) = x(t, 0, x_0)$; $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbf{R}^n$ and $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^n$; $\varrho(A, B)$ is the Euclidean distance between the non-empty sets A and B in $\mathbf{R}^n$. The set of points $(t, x) \in \mathbf{R} \times \mathbf{R}^n$ satisfying the inequality $t - \tau^* \geqq L \|x - x^*\|$ where (τ^*, x^*) is a fixed point in $\mathbf{R} \times \mathbf{R}^n$ and L is a positive constant will be denoted by $K(\tau^*, x^*, L)$.

The qualitative theory of systems of differential equations with impulses in the case when the functions t_i $(i \geqq 1)$ are constant, therefore $t_i(x) = \tau_i$, $x \in D$ (i.e. when the impulses are realized in fixed moments), is comparatively better developed than in the case when the functions t_i are not constant. One of the reasons for this is that in general it is possible for the integral curve of the system with impulses to meet repeatedly one and the same hypersurface from (1). This phenomenon is called "beating". When the phenomenon "beating" is present, the integral curve may be not defined for t sufficiently large. Hence one cannot claim that, e.g., the solution of the system of differential equations with impulses depends continuously on the initial condition in a given finite interval since the solution may be not defined in this interval. In the following example the phenomenon "beating" is observed.

Example 1: Let $n = 1$, $D = (0, +\infty)$, $t_i(x) = \arctan x + i\pi$ ($x \in D$; $i \geqq 1$). Then for every choice of the functions f and I_i ($i \geqq 1$) such that

a) f satisfies conditions for the existence and uniqueness of the solution of (2) for $t \geqq 0$;
b) $I_i(x) > 0$ for $x \in D$,

the integral curve $(t, x(t, 0, x_0))$ "beats" on the hypersurface (in this case on the curve) $t = t_1(x)$ for every choice of the initial point $(0, x_0)$ such that $x_0 > 0$. What is more, in this case the solution of the equation with impulses is not defined for $t \geqq \pi/2$.

3. Main results

3.1 Continuous dependence on the initial condition when the phenomenon "beating" is absent

Denote by (A) the following conditions:

(A1) The function $f = f(t, x)$ is continuous on its first argument on $[0, \infty)$ and is uniformly Lipschitz on its second argument on D with a constant L.

(A2) $\|f(t, x)\| \leqq M$ $((t, x) \in S)$ for a constant $M > 0$.

(A3) The functions $t_i = t_i(x)$ are Lipschitz on D with constants $L_i < 1/M$ ($i \geqq 1$).

(A4) The inequalities

$$0 < t_1(x) < t_2(x) < \dots \qquad (x \in D) \tag{6}$$

hold and, uniformly on D, $t_i(x) \to +\infty$ for $i \to \infty$.

(A5) $t_i\big(x + I_i(x)\big) \leqq t_i(x)$ ($x \in D$; $i \geqq 1$).

(A6) The integral curve $\big(t, x(t)\big)$ of the system of differential equations with impulses $\big(x(t) = x(t, 0, x_0)\big)$ does not leave the set S for $t \in I$ where

$$I = \begin{cases} (0, +\infty) & \text{if the } \tau_i \text{ are a finite number,} \\ \bigcup\limits_{i=1}^{\infty} (\tau_{i-1}, \tau_i] & \text{if the } \tau_i \text{ are infinitely many.} \end{cases}$$

We shall show that the condition (A) is sufficient for the absence of the phenomenon "beating". For this purpose we shall use the following

Theorem 1 [12]: *Let the conditions* (A2), (A3), (A5) *and* (A6) *are satisfied. Assume further that* $(\tau_{i+1}, x_{i+1}) \in K(\tau_i, x_i, 1/M)$ ($i \geqq 1$). *Then the integral curve* $\big(t, x(t)\big)$ *of the system of differential equations with initial condition* (5) *meets each of the hypersurfaces* (1) *not more than once.*

By means of Theorem 1 we get the following

Theorem 2: *Let the condition* (A) *hold. Then the integral curve* $\big(t, x(t)\big)$ *meets each of the hypersurfaces* (1) *not more than once and the solution* $x(t) = x(t, 0, x_0)$ *exists for all* $t \geqq 0$.

Proof: If $\big(t, x(t)\big)$ for $t > 0$ meets no hypersurface from (1), then the proof is trivial. Suppose that, for $t > 0$, $\big(t, x(t)\big)$ meets at least one hypersurface from (1). Conditions (A1) and (A6) imply the existence and uniqueness of the solution $x = x(t)$ for $t \in I$. Since for $\tau_i < t \leqq \tau_{i+1}$ this solution coincides with the solution of the integral equation

$$x(t) = x_i^+ + \int_{\tau_i}^{t} f(\tau, x(\tau))\, d\tau \qquad (i \geqq 1), \tag{7}$$

then by the condition (A2) for $t = \tau_{i+1}$ we obtain

$$\|x_{i+1} - x_i^+\| \leqq M(\tau_{i+1} - \tau_i), \tag{8}$$

i.e., the conditions of Theorem 1 are satisfied. Then, according to Theorem 1, $(t, x(t))$ meets each of the hypersurfaces (1) not more than once.

Finally we shall show that the solution $x = x(t)$ is continuable for $t \geqq 0$. In fact, if the integral curve $(t, x(t))$ meets a finite number of hypersurfaces from (1) for $t > 0$, then the claim follows directly from conditions (A1) and (A6). Suppose that $(t, x(t))$ meets infinitely many hypersurfaces from (1). We shall show that $j_{i+1} > j_i$ $(i \geqq 1)$. Suppose that this is not true, i.e. $j_{i+1} < j_i$ (the equality $j_{i+1} = j_i$ is impossible since $(t, x(t))$ meets each of the hypersurfaces (1) no more than once). Then, according to (6), $t_{j_{i+1}}(x) < t_{j_i}(x)$, $x \in D$. By this we obtain

$$\tau_{i+1} = t_{j_{i+1}}(x_{i+1}) < t_{j_i}(x_{i+1}). \tag{9}$$

Condition (A5) implies the inequality

$$t_{j_i}(x_i^+) \leqq t_{j_i}(x_i) = \tau_i. \tag{10}$$

Let $\tau' \in (\tau_i, \tau_{i+1})$ be an arbitrary point. Then, according to condition (A3),

$$t_{j_i}(x(\tau')) - \tau_i \leqq t_{j_i}(x(\tau')) - t_{j_i}(x_i^+) < \frac{1}{M} \|x(\tau') - x_i^+\|. \tag{11}$$

By (7) and condition (A2) we obtain the estimate $\|x(\tau') - x_i^+\| \leqq M(\tau' - \tau_i)$. Then we have $t_{j_i}(x(\tau')) - \tau_i < \tau' - \tau_i$ which implies

$$t_{j_i}(x(\tau')) < \tau'. \tag{12}$$

Consider the function φ, $\varphi(t) = t_{j_i}(x(t)) - t$ on $[\tau', \tau_{i+1}]$. Since the function $x = x(t)$ on (τ', τ_{i+1}) is a solution of the integral equation (7), it is continuous in this interval, hence φ is also continuous. From (9) and (12) we deduce $\varphi(\tau')\,\varphi(\tau_{i+1}) < 0$. Hence there exists a point $\tau'' \in (\tau', \tau_{i+1})$ satisfying $\varphi(\tau'') = 0$, i.e. $\tau'' = t_{j_i}(x(\tau''))$. From this it follows that $(t, x(t))$ meets σ_{j_i} in the moment τ''. This contradicts the fact that for $\tau_i < t < \tau_{i+1}$ the integral curve $(t, x(t))$ meets no hypersurface from (1). Hence the inequality $j_{i+1} > j_i$ holds, i.e.

$$1 \leqq j_1 < j_2 < \cdots. \tag{13}$$

By this, since j_i are positive integers, we obtain $j_i \to +\infty$ for $i \to \infty$, hence, according to condition (A4),

$$\lim_{i\to\infty} \tau_i = \lim_{i\to\infty} t_{j_i}(x_i) = +\infty. \tag{14}$$

Since the solution $x = x(t)$ is defined in each of the intervals $(\tau_i, \tau_{i+1}]$ $(i \geqq 0)$ we conclude that it is continuable for all $t \geqq 0$ ∎

It is possible for the integral curve of the system of differential equations with impulses, in spite of the condition (A), to meet not all hypersurfaces from (1), as it is seen by the following

Example 2: Let $n = 1$, $D = (0, +\infty)$, $\sigma_i : t = x + i$ $(i \geqq 1)$, $dx/dt = 0$, $\Delta x(t)|_{t=\tau_i} = -x/2$ $(i \geqq 1)$. The condition (A) is satisfied for $M = 1/2$, $L_i = 1$ $(i \geqq 1)$. In spite of this for $x_0 > 2$ the integral curve $(t, x(t))$ does not meet the hypersurface (in this case the curve) σ_2.

Theorem 3: *Let the conditions* (A1) *and* (A6) *hold. Assume further that the functions* $t_i = t_i(x)$ $(i \geqq 1)$ *are continuous in* D *and that the integral curve* $(t, x(t))$ *for* $t \geqq 0$ *meets at least one hypersurface from* (1). *Then*

$$\left(\tau_{i+1} - t_s(x_{i+1})\right)\left(\tau_i - t_s(x_i^+)\right) \geqq 0 \qquad (i, s \geqq 0).$$

Proof: From conditions (A1) and (A6) it follows that the solution $x = x(t)$ exists and is unique in each of the intervals $(\tau_i, \tau_{i+1}]$ $(i \geqq 0)$. Consider the function φ on $[\tau_i, \tau_{i+1}]$,

$$\varphi(t) = t - t_s\left(x^*(t)\right), \qquad x^*(t) = \begin{cases} x_i^+, & t = \tau_i, \\ x(t), & \tau_i < t \leqq \tau_{i+1}. \end{cases}$$

From the continuity of the function x^* it follows that φ is continuous. Suppose that $\left(\tau_{i+1} - t_s(x_{i+1})\right)\left(\tau_i - t_s(x_i^+)\right) < 0$. Then $\varphi(\tau_{i+1})\,\varphi(\tau_i) < 0$, which implies that there exists a point $\tau' \in (\tau_i, \tau_{i+1})$ such that $\varphi(\tau') = 0$, i.e. $\tau' = t_s\left(x(\tau')\right)$ which means that $(t, x(t))$ meets the hypersurface σ_s in the moment τ'. This result contradicts the manner in which the moments τ_i $(i \geqq 1)$ are defined ∎

Consider the following condition

(B) $t_i(x) < t_{i+1}\left(x + I_i(x)\right) \qquad (x \in D;\, i \geqq 1).$

Theorem 4: *Let the conditions* (A) *and* (B) *be satisfied. Then the integral curve* $(t, x(t))$ *meets each of the hypersurfaces* (1) *precisely once.*

Proof: At first we shall show that if $(t, x(t))$ meets σ_{j_i} in the moments τ_i, then $j_i = i$ $(i \geqq 1)$. We shall prove this claim by induction. Suppose that $(t, x(t))$ meets σ_{j_1} in the moment τ_1. Then $\tau_1 = t_{j_1}\left(x(\tau_1)\right)$. Suppose that $j_1 > 1$. Then, according to (6), we get

$$\tau_1 = t_{j_1}\left(x(\tau_1)\right) = t_{j_1}(x_1) > t_1(x_1). \tag{15}$$

By the first inequality of (6) we find $\tau_0 = 0 < t_1(x_0^+)$. This together with (15) contradicts Theorem 3. Suppose that $j_s = s$ for $s = 1, 2, \ldots, i$ and that $(t, x(t))$ meets $\sigma_{j_{i+1}}$ in the moment τ_{i+1}, i.e. $\tau_{i+1} = t_{j_{i+1}}\left(x(\tau_{i+1})\right)$. According to (13) we have $j_{i+1} \geqq i + 1$. Suppose that $j_{i+1} > i + 1$. By condition (B) and the inductive assumption we obtain the inequality

$$t_{i+1}(x_i^+) - \tau_i = t_{i+1}\left(x_i + I_i(x_i)\right) - t_i(x_i) > 0. \tag{16}$$

From the assumption $j_{i+1} > i + 1$, the inequality

$$t_{i+1}(x_{i+1}) - \tau_{i+1} = t_{i+1}(x_{i+1}) - t_{j_{i+1}}(x_{i+1}) < 0 \tag{17}$$

follows. But (16) and (17) contradict Theorem 3.

Now we shall show that the integral curve $(t, x(t))$ meets the hypersurface σ_1. Suppose the contrary. Then for $t > 0$ the integral curve $(t, x(t))$ meets no hypersurface from (1) at all (above we have shown that if the integral curve meets hypersurfaces from (1), then the first one of them is σ_1). Hence,

$$t < t_1\left(x(t)\right) \qquad \text{for } t \geqq 0. \tag{18}$$

In fact, if we suppose that there exists a point $\tau' > 0$ such that $\tau' \geqq t_1\left(x(\tau')\right)$, then for the function φ, $\varphi(t) = t_1\left(x(t)\right) - t$, we find $\varphi(0) > 0$ and $\varphi(\tau') \leqq 0$. From this follows that there exists a point τ'' such that $\varphi(\tau'') = 0$, i.e. $(t, x(t))$ meets σ_1. By (18), using the conditions (A2) and (A3), we obtain $t - t_1(x_0) < t_1\left(x(t)\right) - t_1(x_0) < L_1 M t$, i.e. $t < t_1(x_0)/(1 - L_1 M) = \text{const}$. This contradicts inequality (18). Now suppose that

$(t, x(t))$ meets successively the hypersurfaces $\sigma_1, \sigma_2, \ldots, \sigma_i$ in the moments $\tau_1, \tau_2, \ldots, \tau_i$, respectively, and that for $t > \tau_i$ it meets no hypersurface from (1). Then $t < t_{i+1}(x(t))$ for all $t > \tau_i$. By this we obtain the inequality $t < (1 - L_{i+1}M)^{-1} (t_{i+1}(x_i) - L_{i+1}M\tau_i)$ which leads to a contradiction ∎

Definition 1: Let T be an arbitrary positive constant. We shall say that the solution $x = x(t)$ of a system of differential equations with impulses defined for $0 \leqq t \leqq T$ *depends continuously on the initial condition* $x(0) = x_0$ if for every choice of $\varepsilon > 0$ and $\eta > 0$ there exists $\delta = \delta(\varepsilon, \eta) > 0$ such that each solution $y = y(t)$ of the system considered with initial codition $y(0) = y_0$, for which $\|x_0 - y_0\| < \delta$, satisfies the inequality $\|x(t) - y(t)\| < \varepsilon$ for $0 \leqq t \leqq T$ and $|t - \tau_i| > \eta$ where $\tau_1, \tau_2, \ldots$ are the moments in which the integral curve $(t, x(t))$ meets the hypersurfaces (1).

Consider the following condition

(C) The functions I_i $(i \geqq 1)$ are continuous in D.

Theorem 5: *Let the conditions* (A)—(C) *hold. Then the solution $x = x(t)$ of the system of differential equations with impulses depends continuously on the initial condition $x(0) = x_0$ for $0 \leqq t \leqq T$ where T is an arbitrary positive constant.*

Proof: If for $0 \leqq t \leqq T$ the integral curve $(t, x(t))$ meets no hypersurface from (1), then the proof follows directly from [3: Ch. 5, § 2, Theorem 2.1]. Suppose that, for $0 \leqq t < T$, $(t, x(t))$ meets at least one hypersurface from (1). Then from (14) it follows that there exists a nonnegative integer p such that $0 = \tau_0 < \tau_1 < \ldots < \tau_p < T \leqq \tau_{p+1}$. According to Theorem 4 $(t, x(t))$ meets successively, precisely once, each one of the hypersurfaces (1). Hence, for $0 \leqq t < T$, it meets successively the hypersurfaces $\sigma_1, \sigma_2, \ldots, \sigma_p$. From (7) for $\tau_i < t \leqq \tau_{i+1}$ we deduce that $x = x(t)$ satisfies the integral equation

$$x(t) = x(\tau_i) + I_i(x(\tau_i)) + \int_{\tau_i}^{t} f(\tau, x(\tau))\, d\tau.$$

By this inductively we obtain

$$x(t) = x_0 + \sum_{\tau_i < t} I_i(x_i) + \int_0^t f(\tau, x(\tau))\, d\tau, \qquad 0 \leqq t \leqq T. \tag{19}$$

Let $y = y(t)$ be a solution of the system of differential equations with impulses with initial condition $y(0) = y_0$ and the moments, in which the integral curve $(t, y(t))$ meets the hypersurfaces (1) for $0 < t < T$, be $\eta_1, \eta_2, \ldots, \eta_q$, $q = q(T, y_0)$. Analogously to (19) for the solution y we obtain

$$y(t) = y_0 + \sum_{\eta_i < t} I_i(y_i) + \int_0^t f(\tau, y(\tau))\, d\tau, \qquad 0 \leqq t \leqq T, \tag{20}$$

where $y_i = y(\eta_i)$ $(i \geqq 1)$. Introduce the notations $\theta_i' = \min(\tau_i, \eta_i)$, $\theta_i'' = \max(\tau_i, \eta_i)$, $\Omega_i = (\theta_i', \theta_i'']$ $(i \geqq 1)$, $\Omega = [0, T] \setminus (\Omega_1 \cup \Omega_2 \cup \ldots)$. It is easily seen that for $t \in \Omega$ the number of the addends in the right-hand sides of (19) and (20) is one and the same. Hence

$$\|x(t) - y(t)\| \leqq \|x_0 - y_0\| + \sum_{\tau_i \leqq \eta_i < t} \|I_i(x(\tau_i)) - I_i(y(\eta_i))\|$$

$$+ \int_0^t \|f(\tau, x(\tau)) - f(\tau, y(\tau))\|\, d\tau,$$

$t \in \Omega$, by which we find

$$\|x(t) - y(t)\| \leqq \|x_0 - y_0\| + \sum_{\tau_i \leqq \eta_i < t} \|I_i(x(\tau_i)) - I_i(y(\tau_i))\|$$
$$+ \sum_{\eta_i < \tau_i < t} \|I_i(x(\eta_i)) - I_i(y(\eta_i))\| + \int_0^t L \|x(\tau) - y(\tau)\| \, d\tau$$
$$+ \sum_{\tau_i \leqq \eta_i < t} \|I_i(y(\tau_i)) - I_i(y(\eta_i))\| + \sum_{\eta_i < \tau_i < t} \|I_i(x(\tau_i)) - I_i(x(\eta_i))\|, \tag{21}$$

$t \in \Omega$. Suppose that the inequality $\tau_1 \leqq \eta_1$ holds. We shall show that

$$\|x(\tau_1) - y(\tau_1)\| \leqq \exp(L\tau_1) \|x_0 - y_0\|. \tag{22}$$

In fact, by (21) for $0 \leqq t \leqq \tau_1$ we get the inequality

$$\|x(t) - y(t)\| \leqq \|x_0 - y_0\| + \int_0^t L \|x(\tau) - y(\tau)\| \, d\tau,$$

from which, using the Lemma of Grónwall-Bellman, we deduce $\|x(t) - y(t)\| \leqq \|x_0 - y_0\| \exp(Lt)$. By this for $t = \tau_1$ we obtain (22). Put $h_1 = t_1(y(\tau_1)) - \tau_1$ and $h_2 = \eta_1 - t_1(y(\tau_1))$. From condition (A3) we deduce $h_1 = t_1(y(\tau_1)) - t_1(x(\tau_1)) \leqq L_1 \|y(\tau_1) - x(\tau_1)\|$, by which, according to (22), we obtain $h_1 \leqq L_1 \exp(L\tau_1) \|x_0 - y_0\|$. Since

$$\|y(\tau_1) - y(\eta_1)\| \leqq M(\eta_1 - \tau_1) = M(h_1 + h_2), \tag{23}$$

the estimate

$$h_2 = t_1(y(\eta_1)) - t_1(y(\tau_1)) \leqq L_1 \|y(\eta_1) - y(\tau_1)\| \leqq L_1 M(h_1 + h_2)$$

holds which implies $h_2 \leqq L_1 M h_1/(1 - L_1 M)$. Having in mind these inequalities we find the estimate

$$\eta_1 - \tau_1 = h_1 + h_2 \leqq \frac{L_1 \exp(L\tau_1)}{1 - L_1 M} \|x_0 - y_0\|. \tag{24}$$

By (23) and (24) we obtain the inequality

$$\|y(\tau_1) - y(\eta_1)\| \leqq \frac{L_1 M \exp(L\tau_1)}{1 - L_1 M} \|x_0 - y_0\|. \tag{25}$$

We shall estimate the expression $\|x(\eta_1 + 0) - y(\eta_1 + 0)\|$ provided that $\eta_1 < \tau_2$ (further on we shall show that if $\|x_0 - y_0\|$ is sufficiently small, then $\theta''_{i-1} < \theta_i'$ $(i \geqq 1)$, $\theta_0' = \theta_0'' = 0$, which implies $\eta_1 < \tau_2$):

$$\begin{aligned}\|x(\eta_1 + 0) - y(\eta_1 + 0)\| &\leqq \|x(\eta_1 + 0) - x(\tau_1 + 0)\| + \|x(\tau_1 + 0) - y(\eta_1 + 0)\| \\ &\leqq M(\eta_1 - \tau_1) + \|x(\tau_1) + I_1(x(\tau_1)) - y(\eta_1) - I_1(y(\eta_1))\| \\ &\leqq M(\eta_1 - \tau_1) + \|x(\tau_1) - y(\eta_1)\| + \|I_1(x(\tau_1)) - I_1[(y(\eta_1))\| \\ &\leqq M(\eta_1 - \tau_1) + \|x(\tau_1) - y(\tau_1)\| + \|y(\tau_1) - y(\eta_1)\| \\ &\quad + \|I_1(x(\tau_1)) - I_1(y(\tau_1))\| + \|I_1(y(\tau_1)) - I_1(y(\eta_1))\|.\end{aligned}$$

By the last inequality, using successively (24), (22) and (25), we obtain

$$\|x(\eta_1 + 0) - y(\eta_1 + 0)\| \leqq \frac{(1 + L_1 M) \exp(L\tau_1)}{1 - L_1 M} \|x_0 - y_0\|$$
$$+ \|I_1(x(\tau_1)) - I_1(y(\tau_1))\| + \|I_1(y(\tau_1)) - I_1(y(\eta_1))\|. \tag{26}$$

If $\tau_1 > \eta_1$, we obtain inequalities analogous to (22), (24)—(26) with the only difference that the roles of the moments τ_1, η_1 and of the functions x, y have changed. More precisely, define the functions

$$z_i, \qquad z_i(t) = \begin{cases} x(t) & \text{if } \tau_i > \eta_i \\ y(t) & \text{if } \tau_i \leqq \eta_i \end{cases} \qquad (0 \leqq t \leqq T).$$

Then the following inequalities hold:

$$\theta_1'' - \theta_1' = |\eta_1 - \tau_1| \leqq \frac{L_1 \exp(L\theta_1')}{1 - L_1M} \|x_0 - y_0\|,$$

$$\|z_1(\theta_1'') - z_1(\theta_1')\| \leqq \frac{L_1M \exp(L\theta_1')}{1 - L_1M} \|x_0 - y_0\|,$$

$$\|x(\theta_1') - y(\theta_1')\| \leqq \exp(L\theta_1') \|x_0 - y_0\|,$$

$$\|x(\theta_1'' + 0) - y(\theta_1'' + 0)\| \leqq \frac{(1 + L_1M) \exp(L\theta_1')}{1 - L_1M} \|x_0 - y_0\|$$
$$+ \|I_1(x(\theta_1')) - I_1(y(\theta_1'))\| + \|I_1(z_1(\theta_1'')) - I_1(z_1(\theta_1'))\|. \tag{27}$$

Assume that $\theta''_{i-1} < \theta_i'$, $i = 1, 2, \ldots, p + 1$ (further on we shall show that these inequalities are fulfilled for sufficiently small values of $\|x_0 - y_0\|$). Consider the solutions $x(t, \theta''_{i-1}, x(\theta''_{i-1} + 0))$ and $y(t, \theta''_{i-1}, y(\theta''_{i-1} + 0))$, $(i \geqq 1)$ of the system of differential equations with impulses which coincide for $t > \theta''_{i-1}$ with the solutions $x = x(t)$ and $y = y(t)$, respectively. Analogously to the inequalities (27) we obtain the following ones:

$$\theta_i'' - \theta_i' \leqq \frac{L_i \exp(L(\theta_i' - \theta''_{i-1}))}{1 - L_iM} \|x(\theta''_{i-1} + 0) - y(\theta''_{i-1} + 0)\|,$$

$$\|z(\theta_i'') - z(\theta_i')\| \leqq \frac{L_iM \exp(L(\theta_i' - \theta''_{i-1}))}{1 - L_iM} \|x(\theta''_{i-1} + 0) - y(\theta''_{i-1} + 0)\|,$$

$$\|x(\theta_i') - y(\theta_i')\| \leqq \exp(L(\theta_i' - \theta''_{i-1})) \|x(\theta''_{i-1} + 0) - y(\theta''_{i-1} + 0)\|, \tag{28}$$

$$\|x(\theta_i'' + 0) - y(\theta_i'' + 0)\| \leqq \frac{(1 + L_iM) \exp(L(\theta_i' - \theta''_{i-1}))}{1 - L_iM} \|x(\theta''_{i-1} + 0)$$
$$- y(\theta''_{i-1} + 0)\| + \|I_i(x(\theta_i')) - I_i(y(\theta_i'))\|$$
$$+ \|I_i(z_i(\theta_i'')) - I_i(z_i(\theta_i'))\|.$$

Let $\varepsilon > 0$ be arbitrarily. Then from (28) for $i = p + 1$ and from condition (C) it follows that there exists a positive constant $\delta_p = \delta_p(\varepsilon) > 0$ such that the inequality $\|x(\theta_p'' + 0) - y(\theta_p'' + 0)\| < \delta_p$ implies the inequalities

$$\theta''_{p+1} - \theta'_{p+1} < \varepsilon, \qquad \|z_{p+1}(\theta''_{p+1}) - z_{p+1}(\theta'_{p+1})\| < \varepsilon,$$
$$\|x(\theta'_{p+1}) - y(\theta'_{p+1})\| < \varepsilon, \qquad \|x(\theta''_{p+1} + 0) - y(\theta''_{p+1} + 0)\| < \varepsilon.$$

Analogously, for each $i = p, p - 1, \ldots, 1$ we define successively the positive constants $\delta_{p-1}, \delta_{p-2}, \ldots, \delta_0$ such that the inequality $\|x(\theta''_{i-1} + 0) - y(\theta''_{i-1} + 0)\| < \delta_{i-1}$ implies the inequalities

$$\begin{aligned} &\theta_i'' - \theta_i' < \varepsilon, && \|z_i(\theta_i'') - z_i(\theta_i')\| < \varepsilon, \\ &\|x(\theta_i') - y(\theta_i')\| < \varepsilon, && \|x(\theta_i'' + 0) - y(\theta_i'' + 0)\| < \min(\varepsilon, \delta_i). \end{aligned} \tag{29}$$

From this it follows that if $\|x_0 - y_0\| < \delta_0$, then for every $i = 1, 2, \ldots, p + 1$ the left-hand sides of each of the inequalities (28) are smaller than ε. Using these results we conclude that if $\eta > 0$ is an arbitrary constant, then for sufficiently small $\|x_0 - y_0\|$ the following relations are fulfilled:

a) $\sum_{i=1}^{p+1} |\eta_i - \tau_i| < \eta$.

b) $\theta_i'' \leqq \theta'_{i+1}$, i.e. $\max(\tau_i, \eta_i) \leqq \min(\tau_{i+1}, \eta_{i+1})$, $i = 1, 2, \ldots, p$.

c) If $\tau_p < T < \tau_{p+1}$, then the number of the meetings of the integral curve $(t, y(t))$ with the hypersurfaces (1) for $0 \leqq t \leqq T$ is precisely p. In fact, if we put $\eta = \min(T - \tau_p, \tau_{p+1} - T)$, then from a) it follows directly that $p = q$. Note that for $T = \tau_{p+1}$ both equalities $p = q$ and $p + 1 = q$ are possible.

d) Directly from a) we obtain $\{t \in [0, T] \mid |t - \tau_i| > \eta\} \subset \Omega$.

From inequality (21), using inequalities (29), condition (C) and the above conclusions it follows that for the arbitrarily chosen positive $\varepsilon > 0$ and $\eta > 0$ for sufficiently small $\|x_0 - y_0\|$ the inequality

$$\|x(t) - y(t)\| \leqq \varepsilon + \int_0^t L \|x(\tau) - y(\tau)\| \, d\tau \qquad (0 \leqq t \leqq T, |t - \tau_i| > \eta)$$

holds by which, according to the inequality of Gronwall-Bellman, we obtain

$$\|x(t) - y(t)\| \leqq \varepsilon \exp(LT) \; (0 \leqq t \leqq T, |t - \tau_i| > \eta; i \geqq 1) \; \blacksquare$$

In the following theorem we shall prove the continuous dependence on the initial condition of the solution $x = x(t)$ of the system of differential equations with impulses substituting for condition (B) the following less restrictive condition

(D) $t_s(x_i^+) \neq \tau_i \; (i \geqq 1; s \geqq i)$.

Note that by the conditions of the following theorem it is possible for the integral curve $(t, x(t))$ not to meet each one of the hypersurfaces (1), i.e. the claim of Theorem 4 is no more true.

Theorem 6: *Let conditions* (A), (C) *and* (D) *be fulfilled. Then the solution $x = x(t)$ of the system of differential equations with impulses depends continuously on the initial condition $x(0) = x_0$ for $0 \leqq t \leqq T$, where T is an arbitrary positive constant.*

Proof: First we shall show that if $\|x_0 - y_0\|$ is small enough, then for $0 \leqq t \leqq T$ the integral curves $(t, x(t))$ and $(t, y(t))$ either both do not meet, or both meet the same hypersurfaces from (1). Assume that $(t, x(t))$ meets no hypersurfaces from (1) for $0 \leqq t \leqq T$. Denote

$$A = \{(t, x) \mid 0 \leqq t \leqq T, x = x(t)\}, \quad B = \{(t, x) \mid t = t_1(x), x \in \overline{D}\}$$

where $\overline{D}$ is the closure of D. Since A and B are closed, A is bounded and $A \cap B = \emptyset$, then $\varrho(A, B) = \varepsilon_1 > 0$. Applying [3: Chap. 5, § 2, Theorem 2.1], we conclude that if $\|x_0 - y_0\|$ is small enough, then $\|x(t) - y(t)\| < \varepsilon_1$ holds for $0 \leqq t \leqq T$ from which it follows that $(t, y(t))$ for $0 \leqq t \leqq T$ meets no hypersurface from (1).

Now assume that the integral curve $(t, x(t))$ meets some hypersurfaces from (1) for $0 \leqq t \leqq T$. We shall prove that if

$$t_s(x_i^+) < \tau_i < t_{s+1}(x_i^+), \tag{30}$$

then $s + 1 = j_{i+1}$, i.e. $(t, x(t))$ in the moment τ_{i+1} meets the hypersurface τ_{s+1}. In fact, if we assume that $j_{i+1} > s + 1$, then, according to (6), $t_{s+1}(x_{i+1}) < t_{j_{i+1}}(x_{i+1}) = \tau_{i+1}$.

This inequality and the second one of (30) contradict Theorem 3. If we assume that $j_{i+1} < s$, then the inequality $\tau_{i+1} = t_{j_{i+1}}(x_{i+1}) < t_s(x_{i+1})$ and the first one of (30) contradict Theorem 3. If we assume that $j_{i+1} = s$, then by (30) we obtain

$$\tau_{i+1} - \tau_i < t_{j_{i+1}}(x_{i+1}) - t_s(x_i^+) = t_s(x_{i+1}) - t_s(x_i^+) \leqq \frac{1}{M} \|x_{i+1} - x_i^+\|$$

which contradicts inequality (8).

Having in mind that $t_s(x_i^+) \to +\infty$ for $s \to \infty$ and by condition (D) we conclude that for any $i \geqq 1$ there exists a nonnegative integer s such that the inequalities (30) hold. Hence

$$t_{j_{i+1}-1}(x_i^+) < \tau_i < t_{j_{i+1}}(x_i^+). \tag{31}$$

In order to show that the integral curves $(t, x(t))$ and $(t, y(t))$ meet the same hypersurfaces for $0 \leqq t \leqq T$ it suffices to show that if $\|x_0 - y_0\|$ is small enough, then

$$t_{j_{i+1}-1}(y_i^+) < \eta_i < t_{j_{i+1}}(y_i^+) \qquad (i = 1, 2, \ldots, p), \tag{32}$$

where $y_i^+ = y(\eta_i) + I_{j_i}(y(\eta_i))$ and j_i is the number of the hypersurface met by $(t, x(t))$ in the moment τ_i. The two integral curves meet the hypersurface σ_1 in the moments τ_1 and η_1, respectivelly. Assume that for $i = 2, 3, \ldots, s - 1$ the inequalities (32) hold, i.e. the two integral curves meet successively the hypersurfaces $\sigma_1, \sigma_{j_2}, \ldots, \sigma_{j_s}$. We shall show that for $\|x_0 - y_0\|$ small enough the inequalities (32) hold for $i = s$ as well which implies that $(t, x(t))$ and $(t, y(t))$ meet the hypersurface $\sigma_{j_{s+1}}$ in the moments τ_{s+1} and η_{s+1} ,respectively. In fact, by the inductive assumption analogously to (29) it can be shown that for $\|x_0 - y_0\|$ small enough the inequalities

$$\theta_s'' - \theta_s' = |\tau_s - \eta_s| < \varepsilon, \qquad \|x(\theta_s'' + 0) - y(\theta_s'' + 0)\| < \varepsilon \tag{33}$$

hold. Assume $\tau_s \leqq \eta_s \leqq \theta_s''$ (the case $\tau_s > \eta_s$ is considered analogously). Then

$$\begin{aligned} \|x_s^+ - y_s^+\| &\leqq \|x_s^+ - x(\eta_s)\| + \|x(\eta_s) - y_s^+\| \\ &\leqq M(\eta_s - \tau_s) + \|x(\theta_s'' + 0) - y(\theta_s'' + 0)\| \end{aligned}$$

by which, according to (33), we obtain

$$\|x_s^+ - y_s^+\| \leqq (M + 1)\, \varepsilon. \tag{34}$$

From the first inequality of (33) and inequality (34) it follows that

$$\varrho\big((\tau_s, x_s^+), (\eta_s, y_s^+)\big) \leqq (M + 2)\, \varepsilon. \tag{35}$$

By (31) the point (τ_s, x_s^+) belongs to the open set

$$G = \{(t, x) \mid t_{j_{s+1}}(x) < t < t_{j_{s+1}}(x), x \in D\}.$$

From (35) it follows that for sufficiently small $\varepsilon > 0$ the inclusion $(\eta_s, y_s^+) \in G$ is fulfilled which implies (32) for $i = s$. By induction we obtain that for $\|x_0 - y_0\|$ small enough the inequalities (32) are satisfied for $i = 1, 2, \ldots, p$. If $(t, x(t))$ meets no hypersurface from (1) for $0 \leqq t \leqq T$, since for $\|x_0 - y_0\|$ small enough the integral curve $(t, y(t))$ also meets no hypersurface from (1), by [3: Chap. 5, § 2, Theorem 2.1] we conclude that the solution $x = x(t)$ depends continuously on the initial condition.

Assume that the integral curve $(t, x(t))$ meets some hypersurfaces from (1) for $0 \leqq t \leqq T$. By Theorem 2 it meets each hypersurface no more than once. Then by condition (A4) we conclude that $(t, x(t))$ meets a finite number of hypersurfaces $\sigma_{j_1}, \ldots, \sigma_{j_p}$ ($j_1 = 1$) from (1). For $\|x_0 - y_0\|$ small enough $(t, y(t))$ meets the same hypersurfaces for $0 \leqq t \leqq T$. Further on the proof of the theorem is similar to that of

Theorem 5. In fact, analogously to the inequalities (21), (28) we obtain the inequalities

$$\|x(t) - y(t)\| \leqq \|x_0 - y_0\| + \sum_{\tau_i,\eta_i<t} \|I_{j_i}(x(\tau_i)) - I_{j_i}(y(\tau_i))\|$$
$$+ \sum_{\eta_i<\tau_i<t} \|I_{j_i}(x(\eta_i)) - I_{j_i}(y(\eta_i))\| + \int_0^t L\,\|x(\tau) - y(\tau)\|\,d\tau$$
$$+ \sum_{\tau_i\leqq\eta_i<t} \|I_{j_i}(y(\tau_i)) - I_{j_i}(y(\eta_i))\| + \sum_{\eta_i<\tau_i<t} \|I_{j_i}(x(\tau_i)) - I_{j_i}(x(\eta_i))\|, \quad t \in \Omega,$$

$$\theta_i'' - \theta_i' \leqq \frac{L_{j_i} \exp\left(L(\theta_i' - \theta_{i-1}'')\right)}{1 - L_{j_i}M} \|x(\theta_{i-1}'' + 0) - y(\theta_{i-1}'' + 0)\|,$$

$$\|z(\theta_i'') - z(\theta_i')\| \leqq \frac{L_{j_i}M \exp\left(L(\theta_i' - \theta_{i-1}'')\right)}{1 - L_{j_i}M} \|x(\theta_{i-1}'' + 0) - y(\theta_{i-1}'' + 0)\|,$$

$$\|x(\theta_i') - y(\theta_i')\| \leqq \exp\left(L(\theta_i' - \theta_{i-1}'')\right) \|x(\theta_{i-1}'' + 0) - y(\theta_{i-1}'' + 0)\|,$$

$$\|x(\theta_i'' + 0) - y(\theta_i'' + 0)\| \leqq \frac{\left((1 + L_{j_i}M) \exp L(\theta_i' - \theta_i'')\right)}{1 - L_{j_i}M} \|x(\theta_i'' + 0) - y(\theta_{i-1}'' + 0)\|$$
$$+ \|I_{j_i}(x(\theta_i')) - I_{j_i}(y(\theta_i'))\|$$
$$+ \|I_{j_i}(z_i(\theta_i'')) - I_j(z_i(\theta_i'))\|$$

where the points θ_i' and θ_i'', the function z and the set Ω were introduced in the proof of Theorem 5. From these inequalities repeating the arguments from the proof of Theorem 5, it follows that the solution of the system with impulses depends continuously on the initial condition ∎

3.2 Continuous dependence on the initial condition when the phenomenon "beating" is present

Denote by (**E**) the following group of conditions:

(E1) $m_i \leqq \|I_i(x)\| \leqq M_i$ $(x \in D;\ i \geqq 1)$, where m_i, M_i are positive constants.

(E2) The functions $t_i = t_i(x)$ $(i \geqq 1)$
- (i) are Lipschitz on D with respective constants $L_i < m_i/M(m_i + M_i)$;
- (ii) satisfy the inequalities $t_i\left(x + I_{i+1}(x)\right) < t_{i+1}(x)$, $x \in D$.

(E3) For any $x \in D$ and $i \geqq 1$ there exist neighbourhoods $U_i(x)$, $V_i(x)$ and $W_i(x)$ such that
- (i) $V_i(x)$ is bounded,
- (ii) $U_i(x) \subset V_i(x) \subset W_i(x) \subset D$,
- (iii) $\varrho\left(\mathbf{R}^n \setminus V_i(x), U_i(x)\right) \geqq m_i + M_i$, $\varrho\left(\mathbf{R}^n \setminus W_i(x), V_i(x)\right) \geqq m_i + M_i$,
- (iv) $t_i(z_1) \geqq t_i(z_2)$ for $z_1 \in V_i(x) \setminus U_i(x)$ and $z_2 \in W_i(x) \setminus V_i(x)$.

(E4) For any $x \in D$ and $i \geqq 1$ there exist a unit n-vector $y_i(x)$ and a function d_i: $D \to \mathbf{R}$ such that

$$\varrho_i < d_i(x) \leqq \frac{\langle I_i(z), y_i(x)\rangle}{\|I_i(z)\|} \qquad \left(z \in V_i(x)\right), \qquad \varrho_i = \frac{ML_iM_i}{m_i - ML_im_i}.$$

As it was shown by Example 1, if the integral curve $\left(t, x(t)\right)$ of the system with impulses meets infinitely many times one and the same hypersurface, it is possible for the solution $x = x(t)$ not to be continuable for all $t \geqq 0$. The following theorem contains sufficient conditions by which the integral curve meets a finite number of times each one of the hypersurfaces (1).

Theorem 7 (cf. [14]): *Let the conditions* (A1), (A2), (A4), (A6) *and* (E) *be satisfied. Then the integral curve* $(t, x(t))$ *meets each one of the hypersurfaces* (1) *a finite number of times.*

Theorem 8: *Let the conditions* (A1), (A2), (A4), (A6) *and* (E) *hold. Then the solution* $x = x(t)$ *of the system of differential equations with impulses is continuable for all* $t \geqq 0$.

Proof: If the integral curve $(t, x(t))$ meets a finite number of hypersurfaces from (1), then, having in mind Theorem 7, it follows that the moments τ_i are a finite number which by conditions (A1) and (A6) implies that the solution $x = x(t)$ is defined for all $t \geqq 0$. Assume that $(t, x(t))$ meets infinitely many hypersurfaces from (1). Then by Theorem 7 only a finite number of members of the sequence of positive integers $j_1, j_2, \ldots$ can be equal to $1, 2, \ldots$. Hence $j_i \to +\infty$ for $i \to \infty$ by which, according to condition (A4), we obtain (14) and conclude that $x = x(t)$ is defined for all $t \geqq 0$ ∎

Theorem 9: *Let the conditions* (A1), (A2), (A4), (A6), (C)—(E) *be satisfied. Then the solution* $x = x(t)$ *of the system of differential equations with impulses depends continuously on the initial codition* $x(0) = x_0$ *for* $0 \leqq t \leqq T$, *where* T *is an arbitrary positive constant.*

Proof: If for $0 \leqq t \leqq T$ the integral curve $(t, x(t))$ meets no hypersurface from (1), then the assertion is proved analogously to Theorem 6. Assume that $(t, x(t))$ meets some hypersurfaces from (1) for $0 \leqq t \leqq T$. We shall show that for any $i \geqq 1$ the inequality $j_{i+1} \geqq j_i$ holds. In fact, let us suppose that there exists a positive integer s such that

$$j_{s+1} < j_s. \tag{36}$$

Since for $\tau_s < t \leqq \tau_{s+1}$ the solution $x = x(t)$ coincodes with the solution of the integral equation

$$x(t) = x_s^+ + \int_{\tau_s}^{t} f(\tau, x(\tau))\, d\tau,$$

then by condition (A2) we obtain the estimate

$$\|x_{s+1} - x_s^+\| \leqq M(\tau_{s+1} - \tau_s). \tag{37}$$

By inequalities (37), (6) and condition (E2)/(ii) we find

$$t_{j_{s+1}}(x_s^+) < t_{j_{s+1}+1}(x_s^+) < \cdots < t_{j_s-1}(x_s^+) < t_{j_s}(x_s) = \tau_s. \tag{38}$$

By this and condition (E2)/(i) we obtain

$$\tau_{s+1} - \tau_s < t_{j_{s+1}}(x_{s+1}) - t_{j_{s+1}}(x_s^+) \leqq \frac{1}{M} \|x_{s+1} - x_s^+\|,$$

which contradicts the inequality (37). Hence $j_1 \leqq j_2 \leqq \ldots$. In the proof of Theorem 8 we showed that inequality (14) is satisfied. Having in mind this inequality, inequality (38) and Theorem 7, we conclude that the integral curve $(t, x(t))$ for $0 < t \leqq T$ meets successively a finite number of hypersurfaces σ_i respectively n_i times, where $0 \leqq n_i < +\infty$ $(i = 1, 2, \ldots, p)$, $n_p > 0$ and $p < \infty$. Assume that the integral curve $(t, y(t))$

of the system of differential equations with impulses with initial condition $y(0) = y_0$ for $0 < t \leqq T$ meets successively the hypersurfaces σ_i respectively m_i times, where $0 \leqq m_i < +\infty$ $(i = 1, 2, \ldots, q)$, $m_q > 0$ and $q < \infty$. Let σ_p and σ_q be the last hypersurfaces from (1) met by $(t, x(t))$ and $(t, y(t))$ for $0 \leqq t \leqq T$, respectively. We shall show that for $\|x_0 - y_0\|$ small enough these integral curves meet for $0 < t \leqq T$ the same hypersurfaces an equal number of times. Suppose that this is not true, i.e. $n_i = m_i$ $(i = 1, 2, \ldots, s - 1)$, $n_s < m_s$ and $\tau_l < \eta_l$, where $l = n_1 + \cdots + n_{s-1}$ (the remaining three cases: $n_s < m_s$ and $\tau_l \geqq \eta_l$; $n_s > m_s$ and $\tau_r < \eta_r$; $n_s > m_s$ and $\tau_r \geqq \eta_r$, where $r = m_1 + \cdots + m_s$ are considered analogously). Let $\varepsilon > 0$ be arbitrarily. For $\|x_0 - y_0\|$ small enough, similarly to (35) we obtain the inequality

$$\varrho\big((\tau_l, x_l^+), (\eta_l, y_l^+)\big) \leqq (M + 2)\,\varepsilon. \tag{39}$$

From condition (D) and the fact that $(t, y(t))$ meets at least once the hypersurface σ_s for $t > \eta_l$ it follows that $(\eta_l, y_l^+) \in G$, $G = \{(t, x) \mid t_{s-1}(x) < t < t_s(x), x \in D\}$. Since the set G is open, from (39) it follows that, for ε small enough, $(\tau_l, x_l^+) \in G$ which contradicts the assumption that $(t, x(t))$ for $t > \tau_l$ does not meet the hypersurface σ_s. Further on the proof of the theorem is analogous to that of Theorem 5 ∎

Example 3: Let $n = 1$, $D = (-\infty, +\infty)$ and the hypersurfaces (in this case they are curves) are of the type

$$t = t_i(x) = \begin{cases} 5i - |x|, & x \in [-i, i] \\ 4i, & x \notin [-i, i] \end{cases} \qquad (i \geqq 1).$$

Consider the differential equation with impulses

$$\frac{dx}{dt} = \frac{1}{4}, \qquad x(\tau_i + 0) - x(\tau_i) = \frac{5(-1)^i}{4}.$$

It satisfies the conditions of Theorem 7 for $M = 1/4$, $m_i = M_i = 5/4$, $L_i = 1$, $U_i(x) = (-i - |x|, i + |x|)$, $V_i(x) = (-i - 3 - |x|, i + 3 + |x|)$, $W_i(x) = (-i - 6 - |x|, i + 6 + |x|)$, $d_i(x) = 1$ and if the unit vector $y_i(x)$ (in this case it is one-dimensional) has a coordinate $(-1)^i$. Hence, by Theorem 7, the integral curve $(t, x(t))$ meets each one of the impulse curves $t = t_i(x)$ a finite number of times. E.g., if $x_0 = 0$, i.e. the initial point of $(t, x(t))$ is $(0, 0)$, then $(t, x(t))$ meets the curve σ_1 consecutively twice in the points $(4, 1)$ and $(5, 0)$ and for any $i \geqq 1$ meets the curves σ_{2i} once in the points $(10i, 0)$ and the curves σ_{2i+1} three times each in the points $(10i + 3, 2)$, $(10i + 4, 1)$ and $(10i + 5, 0)$. It is easy to verify that this system satisfies conditions (C) and (D), too. Then from Theorem 9 it follows that the solution of the initial value problem depends continuously on the initial condition.

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