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Quadrature and Collocation Methods for Singular Integral Equations on Curves with Corners

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Dedicated to Prof. Dr. S. G. Mikhlin on the occasion of his 80th birthday

In dieser Arbeit werden Näherungsverfahren für singuläre Integralgleichungen auf Kurven mit Ecken betrachtet. Es werden notwendige und hinreichende Stabilitätskriterien für die stückweise konstante ε-Kollokation und für bestimmte Quadraturformelverfahren hergeleitet.

В статье рассматриваются методы приближенного решения сингулярных интегральных уравнений на кривых с углами. Для метода ε-коллокации с кусочно постоянными координатными функциями и для определенных квадратурных методов доказываются необходимые и достаточные условия устойчивости.

This paper is concerned with approximation methods for singular integral equations on curves with corners. Necessary and sufficient conditions for the stability of the piecewise constant ε-collocation and of certain quadrature methods are given.

0. Introduction

0.1. Many boundary value problems of elasticity, aerodynamics, hydrodynamics, fluid mechanics, electromagnetics, acoustics, and other engineering applications can be reduced to a singular integral equation of the form

$$A_\Gamma u(t) := c(t)\, u(t) + \frac{d(t)}{\pi i} \int_\Gamma \frac{u(\tau)}{\tau - t}\, d\tau + \int_\Gamma k(t, \tau)\, u(\tau)\, d\tau = f(t) \qquad (t \in \Gamma), \tag{0.1}$$

where Γ is a closed and piecewise smooth curve in the complex plane, c, d and k are given continuous functions, u is the unknown solution and the first integral is to be interpreted as a Cauchy principal value (see, e.g., [3, 14, 16, 17]). For the numerical solution of this equation spline approximation methods are extensively employed. In fact, collocation and quadrature methods are the most widely used numerical procedures for solving the boundary integral equations of the form (0.1) arising from exterior or interior boundary value problems of applications. (See, e.g., [1, 3, 4].)

If Γ is a closed smooth curve, a fairly complete stability theory and error analysis of collocation methods for (0.1) using smooth splines has been established (see the surveys given in [8, 28, 15: Chap. 17, 26]). A general approach to the stability and error analysis of quadrature methods for (0.1) using equidistant quadrature knots has been developed in [19, 23].

In this paper we present a stability analysis of quadrature and spline collocation methods for (0.1) in the case when Γ is a closed curve with a finite number of corners. For this case, Costabel and Stephan (unpublished) proved the strong ellipticity of the operator A_Γ to be sufficient for the L^2-stability of the piecewise linear collocation. We establish conditions for the stability of the collocation method with piecewise constant trial functions on uniform partitions. Repeating the argumentation in the corresponding proof it can be shown that the strong ellipticity is not necessary for the piecewise linear collocation to be stable.

0.2. The discretization of (0.1) via spline collocation is very simple. We take a finite set of collocation points $\{\tau_k^{(n)}, k = 0, \ldots, n-1\} \subset \Gamma$ and choose a space of spline functions X_n ($\dim X_n = n$) on Γ. For the exact solution $u = A_\Gamma^{-1}f$, we determine an approximation $u_n \in X_n$ by solving the system

$$(A_\Gamma u_n)(\tau_k^{(n)}) = f(\tau_k^{(n)}), \qquad k = 0, \ldots, n-1. \tag{0.2}$$

If X_n is defined on a suitable graded mesh and the degree of the functions in X_n is sufficiently large, then a high order of convergence is to be expected. However, for the sake of simplicity, we restrict our considerations to uniform partitions and piecewise constant splines. Using the arguments of this paper it is not hard to treat special nonuniform meshes (see, e.g., [22]) and spline functions of higher degree, too.

In order to solve the system of equations (0.2) one has to compute $(Au_n)(\tau_k^{(n)})$. If this can not be done analytically, then one has to make use of quadrature rules. In this case we recommend the immediate discretization of equation (0.1) via quadrature rules. Thereby, the singularity subtraction technique is needed to obtain convergent approximation methods. If suitable graded meshes and quadrature rules with high accuracy are used, then a high order of convergence can be achieved (compare the quadrature methods for the unit circle in [19, 23]). For the sake of simplicity, in this paper we use the rectangle rule. However, by the same way a modified rectangle rule can be considered. In fact, a suitable modification of the quadrature weights in a finite number of knots (in the neighbourhood of the corner points) leads to high accuracy of the quadrature rule. It is also possible, but more complicated to investigate composite Newton-Cotes rules, e.g. the composite Simpson rule.

In order to show the nature of our quadrature methods we discretise the equation

$$A_{\mathbf{R}}u(t) = f(t), \qquad t \in \mathbf{R}, \tag{0.3}$$

$$A_{\mathbf{R}}u(t) := a(t)\, u(t) + \frac{1}{\pi i} \int_{\mathbf{R}} \frac{u(\tau)}{\tau - t}\, d\tau + \int_{\mathbf{R}} k(t, \tau)\, u(\tau)\, d\tau. \tag{0.4}$$

Though, for numerical computation, the resulting quadrature methods are not of interest, they are very simple to deduce and give a good motivation for the methods in consideration. Fix $n \in \mathbf{N}$, $-1 < \varepsilon < 1$ and, for $k \in \mathbf{Z}$, set $t_k^{(n)} = k/n$, $\tau_k^{(n)} = (k + \varepsilon)/n$. Using the rule

$$\int_{\mathbf{R}} g(t)\, dt \sim \sum_{j \in \mathbf{Z}} g(t_j^{(n)})\, \frac{1}{n} \tag{0.5}$$

we obtain

$$\begin{aligned} \frac{1}{\pi i} \int_{\mathbf{R}} \frac{u(\tau)}{\tau - \tau_k^{(n)}}\, d\tau &= \frac{1}{\pi i} \int_{\mathbf{R}} \frac{u(\tau) - u(\tau_k^{(n)})}{\tau - \tau_k^{(n)}}\, d\tau \sim \frac{1}{\pi i} \sum_{j \in \mathbf{Z}} \frac{u(t_j^{(n)}) - u(\tau_k^{(n)})}{t_j^{(n)} - \tau_k^{(n)}}\, \frac{1}{n} \\ &\sim \frac{1}{\pi i} \sum_{j \in \mathbf{Z}} \frac{u(t_j^{(n)})}{t_j^{(n)} - \tau_k^{(n)}}\, \frac{1}{n} - u(\tau_k^{(n)})\, \frac{1}{\pi i} \sum_{j \in \mathbf{Z}} \frac{1}{j - k - \varepsilon}. \end{aligned} \tag{0.6}$$

For $\varepsilon \neq 0$, the well-known formula $\cot(\pi x) = \frac{1}{\pi}\left\{\frac{1}{x} + \sum_{j=1}^{\infty}\left[\frac{1}{x-j} + \frac{1}{x+j}\right]\right\}$ yields

$$\frac{1}{\pi i}\int_{\mathbf{R}} \frac{u(\tau)}{\tau - \tau_k^{(n)}}\,d\tau \sim \frac{1}{\pi i}\sum_{j\in\mathbf{Z}} \frac{u(t_j^{(n)})}{t_j^{(n)} - \tau_k^{(n)}}\,\frac{1}{n} - u(\tau_k^{(n)})\,\mathrm{i}\cot(\pi\varepsilon). \tag{0.7}$$

Replacing the integrals in (0.4) by (0.5), (0.7) and substituting $u(\tau_k^{(n)})$ by $u(t_k^{(n)})$ we arrive at

$$\begin{aligned}&[a(\tau_k^{(n)}) - \mathrm{i}\cot(\pi\varepsilon)]\,u_n(t_k^{(n)}) + \frac{1}{\pi i}\sum_{j\in\mathbf{Z}}\frac{u_n(t_j^{(n)})}{t_j^{(n)} - \tau_k^{(n)}}\,\frac{1}{n}\\ &+ \sum_{j\in\mathbf{Z}} k(\tau_k^{(n)}, t_j^{(n)})\,u_n(t_j^{(n)})\,\frac{1}{n} = f(\tau_k^{(n)}), \quad k\in\mathbf{Z}.\end{aligned} \tag{0.8}$$

For $\varepsilon = 1/2$ or $\varepsilon = -1/2$, $\cot(\pi\varepsilon)$ vanishes and the system (0.8) is called the *method of discrete whirls* (see [3]).

If $\varepsilon = 0$, then (0.6) and $\sum_{j=1}^{\infty}\{1/j + 1/(-j)\} = 0$ yields

$$\frac{1}{\pi i}\int_{\mathbf{R}} \frac{u(\tau)}{\tau - t_k^{(n)}}\,d\tau \sim \frac{1}{\pi i}\sum_{\substack{j\in\mathbf{Z}\\ j\neq k}} \frac{u(t_j^{(n)})}{t_j^{(n)} - t_k^{(n)}}\,\frac{1}{n} + \frac{1}{\pi i}\,\frac{1}{n}\,u'(t_k^{(n)}). \tag{0.9}$$

Replacing the integrals of (0.4) by (0.5) and (0.9) and neglecting the small term $\frac{1}{\pi i n}\,u'(t_k^{(n)})$ we obtain

$$\begin{aligned}&a(t_k^{(n)})\,u_n(t_k^{(n)}) + \frac{1}{\pi i}\sum_{\substack{j\in\mathbf{Z}\\ j\neq k}}\frac{u_n(t_j^{(n)})}{t_j^{(n)} - t_k^{(n)}}\,\frac{1}{n} + \sum_{j\in\mathbf{Z}} k(t_k^{(n)}, t_j^{(n)})\,u_n(t_j^{(n)})\,\frac{1}{n}\\ &= f(t_k^{(n)}), \qquad k\in\mathbf{Z}.\end{aligned} \tag{0.10}$$

The corresponding quadrature methods to (0.8) and (0.10) on smooth curves has been considered in [19, 23]. In the present paper we extend the analysis to the case of curves with corners.

Now let us consider a quadrature method which is completely new, even in the case of smooth curves. Therefore, in the discretization of $(A_{\mathbf{R}}u)(t_k^{(n)}) = f(t_k^{(n)})$ we use

$$\int_{\mathbf{R}} g(t)\,dt \sim \sum_{\substack{j\in\mathbf{Z}\\ j\equiv k+1 \bmod 2}} g(t_j^{(n)})\,\frac{2}{n}.$$

Analogously to the derivation of (0.10) we get

$$\begin{aligned}&a(t_k^{(n)})\,u_n(t_k^{(n)}) + \frac{1}{\pi i}\sum_{\substack{j\in\mathbf{Z}\\ j\equiv k+1 \bmod 2}}\frac{u_n(t_j^{(n)})}{t_j^{(n)} - t_k^{(n)}}\,\frac{2}{n} + \sum_{\substack{j\in\mathbf{Z}\\ j\equiv k+1 \bmod 2}} k(t_k^{(n)}, t_j^{(n)})\,u_n(t_j^{(n)})\,\frac{2}{n}\\ &= f(t_k^{(n)}), \qquad k\in\mathbf{Z}.\end{aligned} \tag{0.11}$$

Since no substitution $u(\tau_k^{(n)}) \approx u(t_k^{(n)})$ and no neglect of $1/(\pi i\, n)\,u'(t_k^{(n)})$ is needed, this method converges faster then (0.8) and (0.10) in the case of smooth curves. Furthermore, the invertibility of operator A_Γ will be enough to secure the stability of (0.11). For the unit circle, method (0.11) and the method of trigonometric collocation coincide.

All the quadrature and collocation methods of this paper have one thing in common. The equation $A_\Gamma u = f$ is replaced by a discrete operator equation of the type

$A_n u_n = f_n$, where A_n is an approximate operator of A acting in the space X_n of spline functions and $f_n \in X_n$ is an interpolation of f. A numerical method of this kind is said to be *stable* if A_n is invertible for n large enough and $\sup \|A_n^{-1}\| < \infty$. If the method in consideration is stable, f is Riemann integrable and A_n converges strongly to A, then the approximate solutions u_n converge to u (see, e.g., [15: p. 432]). Furthermore, the stability implies the condition number of the finite linear system of equations $A_n u_n = f_n$ to be bounded as $n \to \infty$. Thus the main point is the proof of the stability.

For every approximation method under consideration, the problem of stability will be reduced to that one of the corresponding method for a model problem on an angle utilizing a localization principle. Moreover, Mellin techniques are applied in order to handle the model problems. These arguments are generalizations of those used in the case of smooth curves (see, e.g., [2, 18, 25, 26]). In comparison with proof techniques based on strong ellipticity (see, e.g., [1]) they are more complicated. However, in many situations strong ellipticity arguments do not work. Furthermore, contrary to the strong ellipticity techniques our proofs yield not only the sufficiency, but also the necessity of the stability conditions.

0.3. We conclude this section by introducing some notations:

$\mathbf{T}$	— unit circle $\{z \in \mathbf{C}: \lvert z\rvert = 1\}$;
$R(\Gamma)$	— class of bounded Riemann integrable functions on Γ;
$PC(\Gamma)$	— class of piecewise continuous functions on Γ (i.e., for $f \in PC(\Gamma)$, there exist the finite limits $f(t \pm 0)$ and $f(t + 0) = f(t - 0)$ except a finite number of points $t \in \Gamma$);
l^2	— Hilbert space of sequences $\{\xi_n\}_{n=0}^{\infty}$, $\xi_n \in \mathbf{C}$;
$\tilde{l}^2$	— Hilbert space of sequences $\{\xi_n\}_{n=-\infty}^{\infty}$, $\xi_n \in \mathbf{C}$;
X	— an abstract Banach space;
X_n	— linear space of column vectors of length n with entries from X;
$X_{n \times n}$	— linear space of $n \times n$-matrices with entries from X;
$T(a)$	— Toeplitz operator generated by $a \in PC(\mathbf{T})$ (i.e. $T(a) = (a_{j-k})_{j,k=0}^{\infty}$, where a_l ($l \in \mathbf{Z}$) is the l-th Fourier coefficient of the function a);
$\mathcal{L}(X)$	— Banach space of continuous linear operators on X.

1. Quadrature methods for singular integral equations on curves with corners

1.1. Quadrature methods on an angle

Let us consider quadrature methods for the approximate solution of singular integral equations on curves with corner points. To this end, we shall use simple quadrature methods which are similar to those ones used in the case of smooth curves (see [19, 22, 23]). Our aim is to establish necessary and sufficient conditions for the stability. Since these conditions will be shown to be of local nature, we start with the simplest situation of an angle. After that we attribute the general case to that one of an angle by using localization techniques.

Let Γ_ω ($0 < \omega < 2\pi$) denote the angle $\{t\, e^{i\omega}, 0 \leq t < \infty\} \cup \{t, 0 \leq t < \infty\}$. Suppose the singular integral operator $A = cI + dS_{\Gamma_\omega}$ with $c, d \in \mathbf{C}$ to be invertible in $L^2(\Gamma_\omega)$, i.e., $c \pm d \neq 0$. If we seek an approximation u_n for the solution $u \in L^2(\Gamma_\omega)$ of the equation $Au = f$, $f \in R(\Gamma_\omega) \cap L^2(\Gamma_\omega)$, then [19, 22, 23] suggest the following quadrature methods: Choose two different numbers ε, δ $(0 < \varepsilon, \delta < 1)$ and set

$$t_k^{(n)} = \begin{cases} \dfrac{k+\delta}{n} & \text{if } k \geqq 0, \\ -\dfrac{k+\delta}{n}\, e^{i\omega} & \text{if } k < 0, \end{cases} \qquad \tau_k^{(n)} = \begin{cases} \dfrac{k+\varepsilon}{n} & \text{if } k \geqq 0, \\ -\dfrac{k+\varepsilon}{n}\, e^{i\omega} & \text{if } k < 0. \end{cases}$$

Determine approximate values $\xi_j^{(n)}$ of $u(t_j^{(n)})$ $(j \in \mathbf{Z})$ by solving one of the systems

$$\{c - i \cot(\pi(\varepsilon - \delta))\, d\}\, \xi_k^{(n)} + d\, \frac{1}{\pi i} \left\{ \sum_{j=0}^{\infty} \frac{\xi_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \frac{1}{n} + \sum_{j=-\infty}^{-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \frac{-e^{i\omega}}{n} \right\} = f(\tau_k^{(n)}), \qquad k \in \mathbf{Z}, \tag{1.1}$$

$$c\xi_k^{(n)} + d\, \frac{1}{\pi i} \left\{ \sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \frac{1}{n} + \sum_{\substack{j=-\infty \\ j \neq k}}^{-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \frac{-e^{i\omega}}{n} \right\} = f(t_k^{(n)}), \qquad k \in \mathbf{Z}, \tag{1.2}$$

$$c\xi_k^{(n)} + d\, \frac{1}{\pi i} \left\{ \sum_{\substack{j=1 \\ j \equiv k+1 \bmod 2}}^{\infty} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \frac{2}{n} + \sum_{\substack{j=-\infty \\ j \equiv k+1 \bmod 2}}^{-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \frac{-2e^{i\omega}}{n} + \sum_{\substack{j=0 \\ j \equiv k+1 \bmod 2}} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \frac{1}{n} (1 - e^{i\omega}) \right\} = f(t_k^{(n)}), \qquad k \in \mathbf{Z}. \tag{1.3}$$

If there exists a unique solution $(\xi_k^{(n)})_{k \in \mathbf{Z}}$, then we obtain an approximate solution u_n by setting

$$u_n = \sum_{k \in \mathbf{Z}} \xi_k^{(n)} \chi_k^{(n)},$$

$$\chi_k^{(n)}(t) = \begin{cases} 1 & \text{if } k/n \leqq t < (k+1)/n, \\ 0 & \text{else}, \end{cases} \quad k = 0, 1, 2, \ldots,$$

$$\chi_k^{(n)}(t) = \begin{cases} 1 & \text{if } k/n \leqq -e^{-i\omega} t < (k+1)/n, \\ 0 & \text{else}. \end{cases} \quad k = -1, -2, \ldots.$$

If the methods (1.1), (1.2) or (1.3) are stable, then it is not hard to prove the convergence of u_n to the exact solution u of the equation $Au = f$. However, we consider the quadrature methods (1.1)–(1.3) as model schemes for adequate numerical procedures on general curves with corners. From this point of view, it suffices to establish necessary and sufficient conditions for the methods (1.1), (1.2) and (1.3) to be stable.

Let A_n denote the matrix of the system (1.1), (1.2) or (1.3), respectively. We define the interpolation projection K_n by $K_n y = \sum_{k \in \mathbf{Z}} y(\tau_k^{(n)})\, \chi_k^{(n)}$ $(y \in R(\Gamma_\omega))$ and denote the orthogonal projection onto $\operatorname{im} K_n \cap L^2(\Gamma_\omega)$ by L_n. In what follows, we shall identify the operators of $\mathcal{L}(\operatorname{im} L_n)$ with their matrices corresponding to the base $\{\chi_k^{(n)}, k \in \mathbf{Z}\}$. Due to

$$\left\| \sum_{k \in \mathbf{Z}} \xi_k \chi_k^{(n)} \right\|_{L^2(\Gamma_\omega)} = n^{-1/2} \, \|\{\xi_k\}_{k \in \mathbf{Z}}\|_{l^2},$$

these matrices are considered to be operators in $\tilde{l}^2$. In particular, since the matrices $A_n \in \mathcal{L}(\tilde{l}^2)$ are independent of n, the sequence $\{A_n\}$ $(A_n \in \mathcal{L}(\operatorname{im} L_n))$ is stable if and only if $A_1 \in \mathcal{L}(\tilde{l}^2)$ is invertible.

Theorem 1.1: *The following assertions are valid.*

a) *The operator* $A_1 \in \mathcal{L}(\tilde{l}^2)$ *is a Fredholm operator with index* 0 *if and only if* $\frac{c+d}{c-d} \notin \Omega$, *where* $\Omega := \{0\}$ *for* (1.3), $\Omega := (-\infty, 0]$ *for* (1.2) *and* $\Omega := \{e^{-i\pi(\varepsilon-\delta)}t, -\infty < t \leqq 0\}$ *for* (1.1).

b) *The operator* A_1 *is invertible in* $\tilde{l}^2$ *if and only if* $\frac{c+d}{c-d} \notin \Omega \cup \Phi$. *Here* Φ *denotes an at most countable subset of* $\mathbf{C} \setminus \Omega$ *whose accumulation points belong to* Ω.

c) *If* $\omega = \pi$, *then* $\Phi = \emptyset$.

Assertion b) of Theorem 1.1 is an easy consequence of assertion a). To see this, we set $B(\lambda) = \frac{1}{c-d} A_1$ for $\lambda = \frac{c+d}{c-d}$. Obviously, the function $\mathbf{C} \setminus \Omega \ni \lambda \to B(\lambda)$ is analytic (even linear) and its values are Fredholm operators with index 0. Since $\mathbf{C} \setminus \Omega$ is connected and $B(1) = I$, the points of $\Phi := \{\lambda \in \mathbf{C} \setminus \Omega, B(\lambda) \text{ is not invertible}\}$ must be isolated and b) follows.

To show a) we need some results on Toeplitz operators which are due to Gohberg and Krupnik (see [11, 13]). Let $\mathfrak{A} \subseteq \mathcal{L}(l^2)$ denote the smallest algebra containing all Toeplitz operators $T(a)$ with $a \in PC(\mathbf{T})$. Then $\mathfrak{A}_{n\times n} \subseteq \mathcal{L}(l^2)_{n\times n}$ $(n \in \mathbf{N})$ is an algebra of continuous operators in l_n^2. There exists a multiplicative linear mapping $\mathfrak{A}_{n\times n} \ni B \to \mathcal{A}_B$ into the algebra of bounded $n \times n$-matrix functions over $\mathbf{T} \times [0, 1]$. The symbol $\mathcal{A}_B$ of $B = (B_{k,j})_{k,j=1}^n$, $B_{k,j} \in \mathfrak{A}$, is equal to $(\mathcal{A}_{B_{k,j}})_{k,j=1}^n$ and the symbol $\mathcal{A}_{T(a)}$ of $T(a)$ with $a \in PC(\mathbf{T})$ is given by $\mathcal{A}_{T(a)}(\tau, \mu) := \mu a(\tau + 0) + (1 - \mu)\, a(\tau - 0)$, where $(\tau, \mu) \in \mathbf{T} \times [0, 1]$. Furthermore, $B \in \mathfrak{A}_{n\times n}$ is a Fredholm operator if and only if $\det \mathcal{A}_B(\tau, \mu) \neq 0$ for all $\tau \in \mathbf{T}$ and $0 \leqq \mu \leqq 1$. Suppose $B \in \mathfrak{A}_{n\times n}$ is a Fredholm operator and there exist $\omega_j \in (0, 2\pi)$ $(j = 1, \ldots, k)$, $\omega_0 := 0$, $\omega_{k+1} := 2\pi$ such that $\mathcal{A}_B(\tau, \mu) = \mathcal{A}_B(\tau, \mu')$ for $\tau \neq e^{i\omega_j}$ $(j = 0, \ldots, k)$ and $0 \leqq \mu, \mu' \leqq 1$. Then the index of B is equal to $-\operatorname{ind} \det \mathcal{A}_B$, i.e., to the negative index of the curve $\Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k$, $\Gamma_j := \{\det \mathcal{A}_B(e^{ix}, 0), \ \omega_j \leqq x \leqq \omega_{j+1}\} \cup \{\det \mathcal{A}_B(e^{i\omega_{j+1}}, \mu), 0 \leqq \mu \leqq 1\}$. Finally, the algebra $\mathfrak{A}_{n\times n}$ contains all compact operators and, moreover, $B \in \mathfrak{A}_{n\times n}$ is compact if and only if $\mathcal{A}_B \equiv 0$.

By virtue of $l^2 \oplus l^2 = \tilde{l}^2$, we can identify $\mathcal{L}(\tilde{l}^2)$ with $\mathcal{L}(l^2)_{2\times 2}$ and obtain $\mathfrak{A}_{2\times 2} \subseteq \mathcal{L}(\tilde{l}^2)$. Assertion a) of Theorem 1.1 will be proved if we show $A_1 \in \mathfrak{A}_{2\times 2}$ and $\operatorname{ind} \det \mathcal{A}_{A_1} = 0$. To do this, let us start with (see [24: Lemma 3.1 and Lemma 3.2])

Lemma 1.1: *Let* $z \in \mathbf{C}$, $-1/2 < \operatorname{Re} z < 1/2$, $\Lambda^z := \left((k+1)^z \delta_{k,j}\right)_{k,j=0}^\infty$ *and* $a \in PC(\mathbf{T})$. *Suppose that there exist* $\omega_j \in (0, 2\pi)$, $\omega_0 := 0$, $\omega_{k+1} := 2\pi$ *such that the restriction of* a *to* $\{e^{ix}, \omega_j \leqq x \leqq \omega_{j+1}\}$ $(j = 1, \ldots, k)$ *is twice differentiable. Then the following assertions are valid.*

(i) *The matrix* $\Lambda^{-z}T(a)\Lambda^z$ *belongs to* $\mathfrak{A}$ *and*

$$\mathcal{A}_{\Lambda^{-z}T(a)\Lambda^z}(\tau, \mu) = \begin{cases} a(\tau) & \text{if } \tau \neq e^{i\omega_j},\ j = 0, \ldots, k, \\ \dfrac{\mu a(\tau+0) + (1-\mu)\, a(\tau-0)\, e^{-i2\pi z}}{\mu + (1-\mu)\, e^{-i2\pi z}} & \text{if } \tau = e^{i\omega_j},\ j = 0, \ldots, k. \end{cases} \tag{1.4}$$

(ii) *The function* $z \to \Lambda^{-z}T(a)\Lambda^z$ *is continuous on* $\{z, -1/2 < \operatorname{Re} z < 1/2\}$.

Now let us consider the method (1.1). The operator $A_1 \in \mathcal{L}(l^2)_{2\times 2}$ takes the form

$$A_1 = \begin{Bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{Bmatrix},$$

where

$$A_{1,1} = \big(c - \mathrm{i}\cot\big(\pi(\varepsilon - \delta)\big)\, d\big)\, I + d\left(\frac{1}{\pi \mathrm{i}}\, \frac{1}{-(k-j) - (\varepsilon - \delta)}\right)_{k,j=0}^{\infty},$$

$$A_{2,1} = d\left(\frac{1}{\pi \mathrm{i}}\, \frac{1}{(j+\delta) + (-k-1+\varepsilon)\,\mathrm{e}^{\mathrm{i}\omega}}\right)_{k,j=0}^{\infty},$$

$$A_{1,2} = d\left(\frac{1}{\pi \mathrm{i}}\, \frac{-\mathrm{e}^{\mathrm{i}\omega}}{-(-j-1+\delta)\,\mathrm{e}^{\mathrm{i}\omega} - (k+\varepsilon)}\right)_{k,j=0}^{\infty},$$

$$A_{2,2} = \big(c - \mathrm{i}\cot\big(\pi(\varepsilon - \delta)\big)\, d\big)\, I + d\left(\frac{1}{\pi \mathrm{i}}\, \frac{1}{(k-j) + (\delta - \varepsilon)}\right)_{k,j=0}^{\infty}.$$

For $-1 < \nu < 1$, $\nu \neq 0$, we set

$$f^{\nu}(\mathrm{e}^{\mathrm{i}2\pi x}) = 2\left\{\mathrm{e}^{-\mathrm{i}\pi\nu(x-1)}\, \frac{\sin(-\pi\nu x)}{\sin(-\pi\nu)}\right\} - 1, \qquad 0 \leqq x < 1.$$

Then a straightforward computation shows $f^{\nu} = \sum_{k\in\mathbf{Z}} f_k^{\nu} t^k$, where $f_k^{\nu} = \frac{1}{\pi \mathrm{i}}\, \frac{1}{-k-\nu} - \mathrm{i}\cot(\pi\nu)\,\delta_{k,0}$. Thus we obtain $A_{1,1} = T(c + df^{(\varepsilon-\delta)})$ and $A_{2,2} = T(c - df^{(\delta-\varepsilon)})$.

Now let us prove $A_{2,1} \in \mathfrak{A}$. The residue theorem together with the well-known formula (see [5, 6])

$$\frac{1}{\pi \mathrm{i}}\, \frac{1}{1 - \mathrm{e}^{\mathrm{i}\omega} x} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = 1/2} x^{-z} \left\{-\mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}(\omega-\pi)z}}{\sin(\pi z)}\right\} dz$$

gives

$$\frac{1}{\pi \mathrm{i}}\, \frac{1}{1 - \mathrm{e}^{\mathrm{i}\omega} x} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = 1/4} x^{-z} \left\{-\mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}(\omega-\pi)z}}{\sin(\pi z)}\right\} dz,$$

$$\frac{1}{\pi \mathrm{i}}\, \frac{x}{1 - \mathrm{e}^{\mathrm{i}\omega} x} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = 1/2} x^{-(z-1)} \left\{-\mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}(\omega-\pi)z}}{\sin(\pi z)}\right\} dz$$

$$= \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = 5/4} x^{-(z-1)} \left\{-\mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}(\omega-\pi)z}}{\sin(\pi z)}\right\} dz - \mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}\omega}}{\pi}$$

$$\frac{1 - x}{1 - \mathrm{e}^{\mathrm{i}\omega} x} = \frac{1 - \mathrm{e}^{-\mathrm{i}\omega}}{2} \int\limits_{\mathrm{Re}\, z = 1/4} x^{-z} \left\{-\mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}(\omega-\pi)z}}{\sin(\pi z)}\right\} dz - \mathrm{e}^{-\mathrm{i}\omega}. \tag{1.5}$$

Hence

$$A_{2,1} = d\left(\frac{1-\dfrac{k+1-\varepsilon}{j+\delta}}{1-\dfrac{k+1-\varepsilon}{j+\delta}\,e^{i\omega}}\,\frac{1}{\pi i}\,\frac{1}{(j+\delta)-(k+1-\varepsilon)}\right)_{k,j=0}^{\infty}$$

$$= d\left(\frac{1-e^{-i\omega}}{2}\int\limits_{\mathrm{Re}\,z=1/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\frac{1}{\pi i}\,\frac{\left(\dfrac{k+1-\varepsilon}{j+\delta}\right)^{-z}}{-(k-j)-(1-\varepsilon-\delta)}\,dz\right)_{k,j=0}^{\infty}$$

$$- d\,e^{-i\omega}\left(\frac{1}{\pi i}\,\frac{1}{-(k-j)-(1-\varepsilon-\delta)}\right)_{k,j=0}^{\infty}.$$

The last relation and (1.5) with $x = 1$ imply

$$A_{2,1} = d\,\frac{1-e^{-i\omega}}{2}\int\limits_{\mathrm{Re}\,z=1/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\left((k+1-\varepsilon)^{-z}\,\delta_{k,j}\right)_{k,j=0}^{\infty}\,T(f^{(1-\varepsilon-\delta)})$$

$$\otimes\left((j+\delta)^{z}\,\delta_{k,j}\right)_{k,j=0}^{\infty}\,dz - d\,e^{-i\omega}T(f^{(1-\varepsilon-\delta)}).$$

Since the operator function

$$z \to \left\{\Lambda^{-z}T(f^{(1-\varepsilon-\delta)})\,\Lambda^{z} - \left((k+1-\varepsilon)^{-z}\,\delta_{k,j}\right)_{k,j}\right.$$

$$\left.\otimes\, T(f^{(1-\varepsilon-\delta)})\left((j+\delta)^{z}\,\delta_{k,j}\right)_{k,j}\right\}$$

is continuous and bounded on $\{z, \mathrm{Re}\,z = 1/4\}$ and takes compact values only, there exists a compact operator $T \in \mathcal{L}(l^2)$ such that

$$A_{2,1} = T + d\,\frac{1-e^{-i\omega}}{2}\int\limits_{\mathrm{Re}\,z=1/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\Lambda^{-z}T(f^{(1-\varepsilon-\delta)})\,\Lambda^{z}\,dz$$

$$- d\,e^{-i\omega}T(f^{(1-\varepsilon-\delta)}).$$

Thus, by Lemma 1.1 we obtain $A_{2,1} \in \mathfrak{A}$ and

$$\mathcal{A}_{A_{2,1}} = d\,\frac{1-e^{-i\omega}}{2}\int\limits_{\mathrm{Re}\,z=1/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\mathcal{A}^{z}\,dz - d\,e^{-i\omega}\mathcal{A}^{0},$$

where $\mathcal{A}^{z} = \mathcal{A}_{\Lambda^{-z}T(f^{(1-\varepsilon-\delta)})\Lambda^{z}}$. Extending $z \to \mathcal{A}^{z}$ to a 1-periodic analytic function, we get

$$\mathcal{A}_{A_{2,1}} = -d\,e^{-i\omega}\mathcal{A}^{1} + \frac{d}{2}\left[\int\limits_{\mathrm{Re}\,z=1/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\mathcal{A}^{z}\,dz\right.$$

$$\left.- \int\limits_{\mathrm{Re}\,z=5/4}\left\{-i\,\frac{e^{-i(\omega-\pi)z}}{\sin(\pi z)}\right\}\mathcal{A}^{z}\,dz\right].$$

In the strip $\{z, 1/4 < \mathrm{Re}\,z < 5/4\}$, the function $z \to \mathcal{A}^{z}(\tau, \mu)$ is constant if $\tau \neq 1$ and has a pole at $z_0 = \dfrac{1}{2} + i\,\dfrac{1}{2\pi}\log\left(\dfrac{\mu}{1-\mu}\right)$ if $\tau = 1$. Consequently, the residue theorem implies

$$\mathcal{A}_{A_{1,1}}(1,\mu) = -d\,e^{-i\omega}\mathcal{A}^1(1,\mu) - 2\pi i\,\frac{d}{2}\left[\frac{-i\,e^{-i(\omega-\pi)}}{\pi}\,\mathcal{A}^1(1,\mu)\right.$$

$$\left. + \left(-\frac{1}{\pi i}\right)(-i)\,\frac{e^{-i(\omega-\pi)\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)}\right],$$

$$\mathcal{A}_{A_{1,1}}(\tau,\mu) = -d\,e^{-i\omega}\mathcal{A}^1(\tau,\mu) - 2\pi i\,\frac{d}{2}\left[\frac{-i\,e^{-i(\omega-\pi)}}{\pi}\,\mathcal{A}^1(\tau,\mu)\right], \qquad \tau \neq 1,$$

$$\mathcal{A}_{A_{2,1}}(\tau,\mu) = \begin{cases} 0 & \text{if } \tau \neq 1, \\ d(-i)\,\dfrac{e^{-i(\omega-\pi)\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)} & \text{if } \tau = 1,\ 0 \leqq \mu \leqq 1. \end{cases}$$

In a similar manner we can prove $A_{1,2} \in \mathfrak{A}$ and compute $\mathcal{A}_{A_{1,2}}$. Finally, we obtain

$$\mathcal{A}_{A_1}(\tau,\mu)$$

$$= \begin{cases} \begin{pmatrix} \{c + df^{(\varepsilon-\delta)}(\tau)\} & 0 \\ 0 & \{c - df^{(\delta-\varepsilon)}(\tau)\} \end{pmatrix} & \text{if } \tau \neq 1,\ 0 \leqq \mu \leqq 1, \\ \begin{pmatrix} \{c + d(-\mu + (1-\mu))\} & \left\{-d(-i)\,\dfrac{e^{-i(\pi-\omega)\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)}\right\} \\ \left\{d(-i)\,\dfrac{e^{-i(\omega-\pi)\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2}+\frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)}\right\} & \{c - d(-\mu + (1-\mu))\} \end{pmatrix} & \text{if } \tau = 1,\ 0 \leqq \mu \leqq 1. \end{cases} \tag{1.6}$$

Since $\det \mathcal{A}_{A_1}$ is independent of ω, we may suppose $\omega = \pi$. In this case the operator $A_1 \in \mathcal{L}(l^2)_{2\times 2}$ is a Fredholm operator with index 0 if and only if the convolution operator $A_1 = cI + d(f^{(\varepsilon-\delta)}_{k-j})_{k,j\in\mathbb{Z}} \in \mathcal{L}(l^2)$ is Fredholm and its index vanishes, i.e., if and only if $c + df^{(\varepsilon-\delta)}(\tau) \neq 0$ for all $\tau \in \mathbf{T}$. A simple computation shows that the last condition is equivalent to $\dfrac{c+d}{c-d} \notin \Omega$.

The operator A_1 corresponding to the methods (1.2) and (1.3) can be treated analogously. We omit the details and remark only that in these cases $f^{(\varepsilon-\delta)}$ has to be replaced by the functions f^0 and $f^{\#}$, respectively, where

$$f^0(e^{i2\pi x}) = 2x - 1, \qquad 0 \leqq x < 1, \text{ and}$$

$$f^{\#}(e^{i2\pi x}) = \begin{cases} -1, & 0 \leqq x < 1/2, \\ 1, & -1/2 \leqq x < 1. \end{cases}$$

If $\omega = \pi$, then $\tilde{A}_1 \in \mathcal{L}(\tilde{l}^2)$ is a discrete convolution operator. Since the Fredholm property of a convolution operator implies its invertibility, assertion c) is obvious. This completes the proof of Theorem 1.1.

1.2. Quadrature methods on curves with corners

Let the simple closed curve Γ be given by the 1-periodic continuous parametrization $\gamma\colon \mathbf{R} \to \mathbf{C}$. For a finite subset M of $[0, 1)$, we suppose that γ is twice continuously differentiable on $[0, 1) \setminus M$, that γ' and γ'' have finite limits at the points of M and that $\gamma'(s+0) \neq -\gamma'(s-0)$, $s \in M$. Let $c, d \in C(\Gamma)$, $k \in C(\Gamma \times \Gamma)$ and define S_Γ, T, $A \in \mathcal{L}(L^2(\Gamma))$ by

$$(S_\Gamma x)(t) = \frac{1}{\pi \mathrm{i}} \int_\Gamma \frac{x(\tau)}{\tau - t} \, d\tau, \qquad (Tx)(t) = \int_\Gamma k(t, \tau) \, x(\tau) \, d\tau,$$

$$A = cI + dS_\Gamma + T.$$

We seek an approximate solution of the equation $Au = f$, $f \in R(\Gamma)$.

For the sake of simplicity, let us assume that M is contained in $\{k/N_0, k = 0, \ldots, N_0 - 1\}$ ($N_0 \in \mathbf{N}$) and choose n to be a multiple of N_0. The quadrature methods will be defined as follows: Let $t_k^{(n)} := \gamma\left(\frac{k+\delta}{n}\right)$, $\tau_k^{(n)} := \gamma\left(\frac{k+\varepsilon}{n}\right)$ $(0 < \varepsilon, \delta < 1$, $\varepsilon \neq \delta$, $k \in \mathbf{Z})$ and determine approximate values $\xi_k^{(n)}$ of $u(t_k^{(n)})$ by solving one of the systems

$$\left\{c(\tau_k^{(n)}) - \mathrm{i} \cot\left(\pi(\varepsilon - \delta)\right) d(\tau_k^{(n)})\right\} \xi_k^{(n)} + d(\tau_k^{(n)}) \frac{1}{\pi \mathrm{i}} \sum_{j=0}^{n-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \Delta t_j^{(n)}$$
$$+ \sum_{j=0}^{n-1} k(\tau_k^{(n)}, t_j^{(n)}) \, \xi_j^{(n)} \Delta t_j^{(n)} = f(\tau_k^{(n)}), \qquad k = 0, \ldots, n-1, \tag{1.7}$$

$$c(t_k^{(n)}) \, \xi_k^{(n)} + d(t_k^{(n)}) \frac{1}{\pi \mathrm{i}} \sum_{\substack{j=0 \\ j \neq k}}^{n-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \Delta t_j^{(n)}$$
$$+ \sum_{j=0}^{n-1} k(t_k^{(n)}, t_j^{(n)}) \, \xi_j^{(n)} \Delta t_j^{(n)} = f(t_k^{(n)}), \qquad k = 0, \ldots, n-1, \tag{1.8}$$

$$c(t_k^{(n)}) \, \xi_k^{(n)} + d(t_k^{(n)}) \frac{1}{\pi \mathrm{i}} \sum_{\substack{j=0 \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - t_k^{(n)}} \Delta t_j^{(n)}$$
$$+ \sum_{\substack{j=0 \\ j \equiv k+1 \bmod 2}}^{n-1} k(t_k^{(n)}, t_j^{(n)}) \, \xi_j^{(n)} \Delta t_j^{(n)} = f(t_k^{(n)}), \qquad k = 0, \ldots, n-1, \tag{1.9}$$

where $\Delta t_j^{(n)} = \gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j}{n}\right)$ for (1.7) and (1.8) and $\Delta t_j^{(n)} = \gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j-1}{n}\right)$ for (1.9). The number n appearing in (1.9) is supposed to be even. If $\chi_j^{(n)}$ denotes the characteristic function of the arc $\left[\gamma\left(\frac{j}{n}\right), \gamma\left(\frac{j+1}{n}\right)\right)$, then the approximate solution will be defined by $u_n = \sum_{j=0}^{n-1} \xi_j^{(n)} \chi_j^{(n)}$.

Before formulating the stability theorem, let us introduce some notation. We set $\tilde{A} = cI - dS_\Gamma - T$. Analogously to the method of freezing the coefficients in the theory of partial differential equations, we shall consider certain model problems. For $\tau \in \Gamma$, let us define $\omega_\tau \in (0, 2\pi)$ by $\omega_\tau = \arg\left(-\frac{\gamma'(\tau - 0)}{\gamma'(\tau + 0)}\right)$ and set $A^\tau = c(\tau) + d(\tau) S_{\Gamma_{\omega_\tau}}$. The model problem for the quadrature method (1.7), (1.8) or (1.9), respectively, is the method (1.1), (1.2) or (1.3), respectively, applied to the operator $A^\tau \in \mathcal{L}(L^2(\Gamma_{\omega_\tau}))$. The matrix of the corresponding system of equations will be denoted by A_1^τ. In the proof of Theorem 1.1 we have shown that $A_1^\tau \in \mathfrak{A}_{2\times 2}$.

Theorem 1.2: *The following assertions are valid.*

a) *The method* (1.7) *or* (1.8), *respectively, is stable if and only if the operators* $A \in \mathcal{L}(L^2(\Gamma))$ *and* $A_1^\tau \in \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ *are invertible. The method* (1.9) *is stable if and only if the operators* $A, \tilde{A} \in \mathcal{L}(L^2(\Gamma))$ *and* $A_{\tilde{1}}^\tau \in \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ *are invertible.*

b) *If the quadrature method is stable and* $f \in R(\Gamma)$, *then the systems* (1.7), (1.8) or (1.9), *respectively, are uniquely solvable for n large enough and the approximate solutions* u_n *converge to* $u = A^{-1}f$ *as* $n \to \infty$.

This theorem will be proved in Section 1.4.

Combining Theorem 1.1 and Theorem 1.2 we get necessary and sufficient conditions for the quadrature methods (1.7)—(1.9) to be stable. In general, the only trouble is that the set Φ in Theorem 1.1 is unknown. We conjecture that it is void in nearly all cases. But now suppose we are out of luck and have the following situation. The operators A and, for (1.9), also $\tilde{A}$ are invertible and the operators A_1^τ, $\tau \in \Gamma$, are at least Fredholm operators with index 0. Moreover, let us assume that in one or more corner points the operators A_1^τ have nontrivial null spaces. Then the quadrature methods only need a little modification in the neighbourhood of these points in order to become stable (cf. [22]).

1.3. A local principle for spline approximation methods

The aim of this section is to establish a local principle which reduces the stability of approximation methods for an operator $A \in \mathcal{L}(L^2(\Gamma))$ to the stability of corresponding methods for certain model operators. Let us suppose that there is given a sequence $\{A_n\}$ of approximate operators $A_n \in \mathcal{L}(\text{im } L_n)$, where L_n is the $L^2(\Gamma)$-orthogonal projection onto the subspace span $\{\chi_k^{(n)}, k = 0, \ldots, n-1\}$ and the scalar product in $L^2(\Gamma)$ is given by

$$(f, g) = \int_0^1 f(\gamma(t))\, \overline{g(\gamma(t))}\, dt. \tag{1.10}$$

Furthermore, let there exist certain model operators $A_1^\tau \in \mathfrak{A}_{2\times 2} \subseteq \mathcal{L}(\tilde{l}^2)$. In order to describe the connection between $\{A_n\}$ and A_1^τ, $\tau \in \Gamma$, we need some notation.

Let the projections $K_n^\varepsilon, K_n^\delta \colon R(\Gamma) \to L^2(\Gamma)$ and $P_n \in \mathcal{L}(\tilde{l}^2)$ be defined by

$$K_n^\varepsilon f = \sum_{k=0}^{n-1} f(\tau_k^{(n)})\, \chi_k^{(n)}, \qquad K_n^\delta f = \sum_{k=0}^{n-1} f(t_k^{(n)})\, \chi_k^{(n)},$$

$$P_n\{\xi_k\}_{k\in\mathbf{Z}} = \{\eta_k\}_{k\in\mathbf{Z}}, \qquad \eta_k = \begin{cases} \xi_k & \text{if } -n/2 < k \leq n/2, \\ 0 & \text{else.} \end{cases}$$

For given $\tau \in \Gamma$ and $n \in \mathbf{N}$, we introduce $E_n^\tau \colon \text{im } P_n \to \text{im } L_n$ by

$$\text{im } P_n \ni \{\delta_{k,j}\}_{k\in\mathbf{Z}} \to \chi_{j(\tau,n)+j},$$

where $j(\tau, n) \in \{0, \ldots, n-1\}$ is defined by $\chi_{j(\tau,n)}(\tau) = 1$. Since E_n^τ is bijective, the mapping $\mathcal{L}(\text{im } L_n) \ni B_n \to B_n^E := E_n^{\tau\,-1} B_n E_n^\tau \in \mathcal{L}(\text{im } P_n)$ is an isomorphism. Moreover, a sequence $\{B_n\}$, $B_n \in \mathcal{L}(\mathrm{im}\, L_n)$, is uniformly bounded (stable) if and only if $\{B_n^E\}$ has the same property.

Let $M(\Gamma)$ denote the set of Lipschitz-continuous functions χ on Γ satisfying $0 \leqq \chi \leqq 1$. For $\chi \in C(\Gamma)$, we set $\chi_n \doteq K_n{}^\varepsilon \chi \mid \mathrm{im}\, L_n$. We shall say that $\{A_n\}$ is *equivalent* to $A_1{}^\tau$ at τ if, for any $\varepsilon' > 0$, there exist $n_0 \in \mathbb{N}$ and a neighbourhood U of τ such that $\chi \in M(\Gamma)$, $\mathrm{supp}\, \chi \subseteqq U$ and $n \geqq n_0$ imply $\|\chi_n{}^E(A_n{}^E - A_1{}^\tau)\, \chi_n{}^E\| < \varepsilon'$.

Theorem 1.3: *Suppose $A \in \mathcal{L}(L^2(\Gamma))$ and $\{A_n\}$, $A_n \in \mathcal{L}(\mathrm{im}\, L_n)$, satisfy the following conditions.*

(i) *There exists a finite subset $\Gamma' \subseteqq \Gamma$ such that $\chi_n A_n \chi_n L_n \to \chi A \chi$ and $\chi_n A_n{}^* \chi_n L_n \to \chi A^* \chi$ for all functions $\chi \in C(\Gamma)$ satisfying* $\mathrm{supp}\, \chi \cap \Gamma' = \emptyset$.

(ii) *The operator $\chi A - A\chi$ is compact in $L^2(\Gamma)$ for any $\chi \in C(\Gamma)$.*

(iii) *The norm $\|\chi_n A_n - A_n \chi_n - L_n(\chi A - A\chi) \mid \mathrm{im}\, L_n\|$ converges to 0 for any $\chi \in C(\Gamma)$ and $n \to \infty$.*

(iv) *There exist operators $A_1{}^\tau \in \mathfrak{A}_{2\times 2} \subseteqq \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ such that $\{A_n\}$ is equivalent to $A_1{}^\tau$ at τ.*

Then $A_n L_n$ converges strongly to A. Moreover, $\{A_n\}$ is stable if and only if the operators $A \in \mathcal{L}(L^2(\Gamma))$ and $A_1{}^\tau \in \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ are invertible.

This local principle will be used in order to prove the stability of the methods (1.7) and (1.8). For the proof of the stability of (1.9), we need the following slight modification.

Define $W_n \in \mathcal{L}(\mathrm{im}\, L_n)$ by $W_n \chi_j{}^{(n)} = (-1)^j \chi_j{}^{(n)}$, $j = 0, \ldots, n-1$, and set $\tilde{B}_n = W_n B_n W_n$ for $B_n \in \mathcal{L}(\mathrm{im}\, L_n)$.

Theorem 1.4: *Suppose $A \in \mathcal{L}(L^2(\Gamma))$ and $\{A_n\}$, $A_n \in \mathcal{L}(\mathrm{im}\, L_n)$ satisfy the assumptions* (i), (ii) *and* (iv) *of Theorem* 1.3. *Assume that, additionally, there hold the following properties.*

(i)′ *There exists an $\tilde{A} \in \mathcal{L}(L^2(\Gamma))$ such that $\chi_n \tilde{A}_n \chi_n L_n \to \chi \tilde{A} \chi$ and $\chi_n \tilde{A}_n{}^* \chi_n L_n \to \chi \tilde{A}^* \chi$ for all $\chi \in C(\Gamma)$ satisfying* $\mathrm{supp}\, \chi \cap \Gamma' = \emptyset$.

(ii)′ *The operator $\chi \tilde{A} - \tilde{A}\chi$ is compact for any $\chi \in C(\Gamma)$.*

(iii)′ *For each $\chi \in C(\Gamma)$,*

$$\|(\chi_n A_n - A_n \chi_n) - L_n(\chi A - A\chi) \mid \mathrm{im}\, L_n - W_n L_n(\chi \tilde{A} - \tilde{A}\chi) \mid \mathrm{im}\, L_n W_n\| \to 0.$$

Then $A_n L_n$ converges strongly to A and $\{A_n\}$ is stable if and only if the operators A, $\tilde{A} \in \mathcal{L}(L^2(\Gamma))$ and $A_1{}^\tau \in \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ are invertible.

Since the proof of Theorem 1.3 runs analogously to that one of Theorem 1.4, we only prove Theorem 1.4. First, let us recall some results on an algebra of approximate operators (see [27: § 2]). Let $\mathfrak{B}$ denote the algebra of all sequences $\{B_n\}$, $B_n \in \mathcal{L}(\mathrm{im}\, L_n)$, such that there exist operators $B, \tilde{B} \in \mathcal{L}(L^2(\Gamma))$ with $B_n L_n \to B$, $B_n{}^* L_n \to B^*$, $\tilde{B}_n L_n \to \tilde{B}$ and $\tilde{B}_n{}^* L_n \to \tilde{B}^*$. If $T \in \mathcal{L}(L^2(\Gamma))$ is compact, then $\{L_n T \mid \mathrm{im}\, L_n\}$, $\{W_n L_n T \mid \mathrm{im}\, L_n W_n\} \in \mathfrak{B}$ and $W_n L_n T \mid \mathrm{im}\, L_n W_n L_n \to 0$. Define

$$J_0 = \{\{L_n T_1 \mid \mathrm{im}\, L_n + W_n L_n T_2 \mid \mathrm{im}\, L_n W_n + C_n\},\; T_1, T_2 \in \mathcal{L}(L^2(\Gamma)) \text{ compact}, \|C_n\| \to 0\}$$

and denote the closure of J_0 by J. Then J forms a two-sided ideal in $\mathfrak{B}$. We set $\mathfrak{B}^0 = \mathfrak{B}/J$ and $\{B_n\}^0 = \{B_n\} + J$. It has been proved in [27: § 2] that a sequence $\{B_n\} \in \mathfrak{B}$ is stable if and only if $B, \tilde{B} \in \mathcal{L}(L^2(\Gamma))$ and $\{B_n\}^0 \in \mathfrak{B}^0$ are invertible.

Now we show that the sequence $\{A_n\}$ of Theorem 1.4 belongs to the algebra $\mathfrak{B}$.

Lemma 1.2: *If* $\{A_n\}$ *satisfies the assumptions of Theorem* 1.4, *then* $A_nL_n \to A$, $A_n^*L_n \to A^*$, $\tilde{A}_nL_n \to \tilde{A}$ *and* $\tilde{A}_n^*L_n \to \tilde{A}^*$.

Proof: First of all, let us show that $\{A_n\}$ is uniformly bounded. In view of assumption (iv) of Theorem 1.3 there exist points $\tau_1, \ldots, \tau_k \in \Gamma$ and functions $\chi^1, \ldots, \chi^k$, $\psi^1, \ldots, \psi^k \in M(\Gamma)$ such that

$$\chi^j\psi^j = \psi^j, \qquad \|\chi_n^{jE}(A_n^E - A_1^{\tau_j})\,\chi_n^{jE}\| < 1,$$

$$j = 1, \ldots, k, \qquad \sum_{j=1}^{k} \psi^j = 1.$$

Since $A_n = \sum_{j=1}^{k} \psi_n^jA_n(I - \chi_n^j) + \sum_{j=1}^{k} \psi_n^jA_n\chi_n^j$, it suffices to show the uniform boundedness of $\psi_n^jA_n\chi_n^j$ and $\psi_n^jA_n(I - \chi_n^j)$. Obviously,

$$\psi_n^jA_n\chi_n^j = \psi_n^jE_n^{\tau_j}\{\chi_n^{jE}A_1^{\tau_j}\chi_n^{jE} + \chi_n^{jE}(A_n^E - A_1^{\tau_j})\,\chi_n^{jE}\}\,(E_n^{\tau_j})^{-1}$$

implies that $\psi_n^jA_n\chi_n^j$ is uniformly bounded. From (iii)′ and $\psi_n^jA_n(I - \chi_n^j) = \psi_n^j(\chi_n^jA_n - A_n\chi_n^j)$ we observe the uniform boundedness of $\psi_n^jA(I - \chi_n^j)$.

Now, for $\chi', \chi \in M(\Gamma)$ and $\chi'\chi = \chi$, we get

$$\begin{aligned}(\chi_n' - I)\,\tilde{A}_n^*\chi_n &= (\chi_n'\tilde{A}_n^* - \tilde{A}_n^*\chi_n')\,\chi_n \\ &= W_n\{(A_n\chi_n' - \chi_n'A_n) - L_n(A\chi' - \chi'A) \mid \operatorname{im} L_n \\ &\quad - W_nL_n(\tilde{A}\chi' - \chi'\tilde{A}) \mid \operatorname{im} L_nW_n\}^*\, W_n\chi_n \\ &\quad + W_nL_n(A\chi' - \chi'A)^* \mid \operatorname{Im} L_nW_n\chi_n \\ &\quad + L_n(\tilde{A}\chi' - \chi'\tilde{A})^* \mid \operatorname{Im} L_n\chi_n.\end{aligned}$$

Assumptions (ii), (ii)′ and (iii)′ yield $(\chi_n' - I)\,\tilde{A}_n^*\chi_nL_n \to (\chi' - I)\,\tilde{A}^*\chi$. If, additionally, $\operatorname{supp}\chi' \cap \Gamma = \emptyset$, then assumption (i)′ gives $\tilde{A}_n^*\chi_nL_n \to \tilde{A}^*\chi \in \mathcal{L}(L^2(\Gamma))$. Since $\sup\|A_n\| = \sup\|\tilde{A}_n^*\| < \infty$, we obtain $\tilde{A}_n^*L_n \to \tilde{A}^*$. The other strong convergences can be derived analogously ∎

In order to prove the sufficiency of the stability conditions in Theorem 1.4 we only have to show the invertibility of $\{A_n\}^0$. To this end we shall use the local principle of Gohberg and Krupnik (see [10: XII, § 1]). For $\tau \in \Gamma$, the set $M_\tau = \{\{\chi_n\}^0, \chi \in M(\Gamma), \chi \equiv 1$ in a neighbourhood of $\tau\}$ is a localizing class in $\mathfrak{B}^0$ and $\{M_\tau, \tau \in \Gamma\}$ forms a covering system (cf. [12, Lemma 2.6]). By virtue of (iii)′, the elements of $U\{M_\tau, \tau \in \Gamma\}$ commute with $\{A_n\}^0$. Hence $\{A_n\}^0$ is invertible if and only if $\{A_n\}^0$ is M_τ-invertible from the right for all $\tau \in \Gamma$.

Lemma 1.3: *If* $A_1^\tau \in \mathcal{L}(\tilde{l}^2)$ *is invertible, then* $\{A_n\}^0$ *is* M_τ — *invertible from the right.*

The proof of this lemma is based on the following two lemmas.

Lemma 1.4: *If* $B^\tau \in \mathfrak{A}_{2\times 2} \subseteq \mathcal{L}(\tilde{l}^2)$, *then the sequence* $\{E_n^\tau B^\tau(E_n^\tau)^{-1}\}$ *belongs to* $\mathfrak{B}$.

Proof: a) Let $B_n := E_n^\tau B^\tau(E_n^\tau)^{-1}$ and $W := \left(\delta_{j,k}(-1)^j\right)_{j,k\in\mathbb{Z}} \in \mathcal{L}(\tilde{l}^2)$. Then $B^\tau \in \mathfrak{A}_{2\times 2}$ implies $WB^\tau W \in \mathfrak{A}_{2\times 2}$ and we have $\tilde{B}_n = E_n^\tau WB^\tau W(E_n^\tau)^{-1}$. Hence it suffices to prove that there exists an operator $B \in \mathcal{L}(L^2(\Gamma))$ such that $B_nL_n \to B$ and $B_n^*L_n \to B^*$.

b) On the real axis, we denote the characteristic function of the interval $[j/n, (j+1)/n]$ by $\varphi_j^{(n)}$ and the orthogonal projection onto span $\{\varphi_j^{(n)}, j \in \mathbf{Z}\}$ by $L_n^{\mathbf{R}}$. Let us identify the operators of $\mathcal{L}(\operatorname{im} L_n^{\mathbf{R}})$ with their matrices corresponding to the base $\{\varphi_j^{(n)}, j \in \mathbf{Z}\}$. Thus the convolution operator $C(a) = (a_{k-j})_{k,j\in\mathbf{Z}}$ $(a \in PC(\mathbf{T}))$ can be considered to operate in im $L_n^{\mathbf{R}}$. We shall show the strong convergence of $\{C(a) L_n^{\mathbf{R}}\}$.

For $a(t) = 1 - t$, the convergence $C(a) L_n^{\mathbf{R}} \to 0$ is easily verified. If $S_{\mathbf{R}}$ is the Cauchy singular operator on $\mathbf{R}$ and φ denotes the function (see [20])

$$\varphi(e^{i2\pi x}) = -\frac{\sin^2(\pi x)}{\pi^2} \sum_{k\in\mathbf{Z}} \frac{\operatorname{sign}(k+1/2)}{(x+k)^2}, \quad 0 < x < 1,$$

then $L_n^{\mathbf{R}} S_{\mathbf{R}} \mid \operatorname{im} L_n^{\mathbf{R}} = C(\varphi)$ and $C(\varphi) L_n^{\mathbf{R}} \to S_{\mathbf{R}} = \varphi(1+0) Q_{\mathbf{R}} + \varphi(1-0) P_{\mathbf{R}}$, $P_{\mathbf{R}} := 2^{-1}(I + S_{\mathbf{R}})$, $Q_{\mathbf{R}} := I - P_{\mathbf{R}}$. Consequently, for $a(t) := a(1+0)\left(\frac{1-\varphi(t)}{2}\right) + a(1-0)\left(\frac{1+\varphi(t)}{2}\right) + b(t)(1-t)$, the sequence $\{C(a) L_n^{\mathbf{R}}\}$ converges strongly to $a(1+0) Q_{\mathbf{R}} + a(1-0) P_{\mathbf{R}}$. By a density argument we conclude $C(a) L_n^{\mathbf{R}} \to a(1+0) \times Q_{\mathbf{R}} + a(1-0) P_{\mathbf{R}}$ for any $a \in PC(\mathbf{T})$.

c) Now we consider the case of the half axis $\mathbf{R}^+$. Let $L_n^+ \in \mathcal{L}(L^2(\mathbf{R}^+))$ denote the orthogonal projection onto span $\{\varphi_j^{(n)}, j = 0, 1, \ldots\}$ and let us identify im L_n^+ with l^2. From b) we conclude $T(a) L_n^+ \to a(1+0) Q_{\mathbf{R}^+} + a(1-0) P_{\mathbf{R}^+}$, where $P_{\mathbf{R}^+} := 2^{-1}(I + S_{\mathbf{R}^+})$, $Q_{\mathbf{R}^+} := I - P_{\mathbf{R}^+}$ and $S_{\mathbf{R}^+}$ is the Cauchy singular operator on $\mathbf{R}^+$. If we define the Mellin transform $M: L^2(\mathbf{R}^+) \to L^2(\{z, \operatorname{Re} z = 1/2\})$ by $Mf(z) = \int_0^\infty t^{z-1} f(t)\, dt$ and the Mellin convolution operator $g(\partial) \in \mathcal{L}(L^2(\mathbf{R}^+))$ $(g \in PC(\{z, \operatorname{Re} z = 1/2\}))$ by $g(\partial) f = M^{-1}(gMf)$, then (see [10])

$$\begin{aligned} a(1+0) Q_{\mathbf{R}^+} + a(1-0) P_{\mathbf{R}^+} &= a(1+0) \frac{1 + i\cot(\pi\partial)}{2} \\ &\quad + a(1-0) \frac{1 - i\cot(\pi\partial)}{2} \\ &= \mathcal{A}_{T(a)}\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right). \end{aligned}$$

Therefore, the mapping $T(a) \to a(1+0) Q_{\mathbf{R}^+} + a(1-0) P_{\mathbf{R}^+}$ extends to a multiplicative linear mapping $\mathfrak{A} \ni A \to \mathcal{A}_A\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right) \in \mathcal{L}(L^2(\mathbf{R}^+))$ and AL_n^+ converges strongly to $\mathcal{A}_A\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right)$.

d) Let $\mathbf{I} := [0, 1]$ and let $L_n^{\mathbf{I}} \in \mathcal{L}(L^2(\mathbf{I}))$ denote the orthogonal projection onto span $\{\varphi_j^{(n)}, j = 0, \ldots, n-1\}$. If π is the projection of $L^2(\mathbf{R}^+)$ onto $L^2(\mathbf{I})$ and $\pi_n \in \mathcal{L}(l^2)$ the projection defined by

$$\pi_n\{\xi_k\}_{k=0}^\infty = \{\eta_k\}_{k=0}^\infty, \qquad \eta_k = \begin{cases} \xi_k & \text{if } k < n, \\ 0 & \text{else,} \end{cases}$$

then $A \in \mathfrak{A}$ and part c) of this proof imply $\pi_n A \pi_n L_n^{\mathbf{I}} \to \pi\mathcal{A}_A\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right)\Big|$ im $\pi \in \mathcal{L}(L^2(\mathbf{I}))$. Furthermore, $\pi_n A^* \pi_n L_n^{\mathbf{I}} \to \pi\mathcal{A}_{A^*}\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right)\Big|$ im π $= \left\{\pi\mathcal{A}_A\left(1, \frac{1 + i\cot(\pi\partial)}{2}\right)\Big| \operatorname{im} \pi\right\}^*$. Transforming the interval to a subarc of Γ, we obtain the strong convergences of $\{B_n L_n\}$ and $\{B_n^* L_n\}$ ∎

Lemma 1.5: *Let $\tau \in \Gamma$ be fixed and $B^\tau \in \mathfrak{A}_{2\times 2} \in \mathcal{L}(\tilde{l}^2)$. For each $\chi' \in M(\Gamma)$ which is identically equal to 1 in a neighbourhood of τ and for any $\varepsilon' > 0$, there exists a smaller neighbourhood U of τ such that $\chi \in M(\Gamma)$ and* supp $\chi \subseteqq U$ *imply* $\|\chi_n{}^E B^\tau (I - \chi_n{}'^E)\| < \varepsilon'$.

Proof: a) Let $\mathfrak{C}$ denote the set of all $B^\tau \in \mathcal{L}(\tilde{l}^2)$ such that the assertion of the lemma holds. From

$$\chi_n{}^E B^\tau C^\tau (I - \chi_n{}'^E) = [\chi_n{}^E B^\tau (I - \chi_n{}''^E)]\, C^\tau (I - \chi_n{}'^E) + \chi_n{}^E B^\tau [\chi_n{}''^E C^\tau (I - \chi_n{}'^E)]$$

we observe that $\mathfrak{C}$ is an algebra. It is not hard to show $\mathfrak{C}$ to be closed with respect to the operator norm, i.e. $\mathfrak{C}$ is a closed subalgebra of $\mathcal{L}(\tilde{l}^2)$.

b) Now consider $B^\tau = (b_{j,k})_{j,k\in\mathbf{Z}}$ which satisfies $b_{j,k} = 0$ for $j \neq \pm k$. Choosing $U = \{\tau \in \Gamma, \chi'(\tau) = 1\}$, we obtain $\chi_n{}^E B^\tau (I - \chi_n{}'^E) = 0$. Thus $B^\tau \in \mathfrak{C}$.

c) Let $B^\tau := C(a) = (a_{k-j})_{k,j\in\mathbf{Z}}$, where a is piecewise continuous. Furthermore, suppose a is twice differentiable at the points of continuity and these derivatives are piecewise continuous. We shall show $B^\tau \in \mathfrak{C}$. Let $\tau = \gamma(\sigma)$ and assume $\chi'(\gamma(s)) = 1$ for $\sigma - \delta_1 < s < \sigma + \delta_1$. We choose $U = \{\gamma(s), \sigma - \delta_2 < s < \sigma + \delta_2\}$ for a suitable $\delta_2 < \delta_1$. Then the element in the j-th row and k-th column of $(I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E$ is smaller than Cc_{k-j}, where $c_k = |k|^{-1}$ if $|k| \geqq (\delta_1 - \delta_2)\, n$ and $c_k = 0$ if $|k| < (\delta_1 - \delta_2)\, n$. For $\{\xi_j\} \in \tilde{l}^2$, define $\{\eta_j\} \in \tilde{l}^2$ by $\eta_j = 0$ if $j \geqq \delta_2 n$ and $\eta_j = |\xi_j|$ if $j < \delta_2 n$. Then we get

$$\left\|\left((I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E\right) \{\xi_j\}\right\|_{\tilde{l}^2} = \left\|\left\{\sum_{k\in\mathbf{Z}} \left((I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E\right)_{j,k} \xi_k\right\}_{j\in\mathbf{Z}}\right\|_{\tilde{l}^2} \leqq C \left\|\left\{\sum_{k\in\mathbf{Z}} c_{j-k} \eta_k\right\}_{j\in\mathbf{Z}}\right\|_{\tilde{l}^2}.$$

Young's inequality yields $\left\|\left((I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E\right) \{\xi_j\}\right\|_{\tilde{l}^2} \leqq C\, \|\{\eta_k\}\|_{\tilde{l}^1}\, \|\{c_j\}\|_{\tilde{l}^2}$. Using

$$\|\{\eta_k\}\|_{\tilde{l}^1} \leqq \sqrt{\delta_2 n}\, \|\{\eta_k\}\|_{\tilde{l}^2} \leqq \sqrt{\delta_2 n}\, \|\{\xi_k\}\|_{\tilde{l}^2},$$

$$\|\{c_j\}\|_{\tilde{l}^2} \leqq \Bigg(\sum_{\substack{k\in\mathbf{Z}\\ |k| < (\delta_1 - \delta_2) n}} 1/k^2\Bigg)^{1/2} \leqq C/\sqrt{(\delta_1 - \delta_2)\, n}$$

we conclude $\left\|\left((I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E)\right) \{\xi_j\}\right\|_{\tilde{l}^2} \leqq C \sqrt{\dfrac{\delta_2}{\delta_1 - \delta_2}}\, \|\{\xi_j\}\|_{\tilde{l}^2}$. If we choose δ_2 small enough, then

$$\|\chi_n{}^E B^\tau (I - \chi_n{}'^E)\|_{\mathcal{L}(\tilde{l}^2)} = \|(I - \chi_n{}'^E)\, B^{\tau*} \chi_n{}^E\|_{\mathcal{L}(\tilde{l}^2)} \leqq C \sqrt{\frac{\delta_2}{\delta_1 - \delta_2}} < \varepsilon'.$$

d) Now $\mathfrak{A}_{2\times 2} \subseteqq \mathfrak{C}$ follows by the fact that $\mathfrak{A}_{2\times 2}$ is in the smallest closed subalgebra of $\mathcal{L}(\tilde{l}^2)$ containing the operators B^τ of parts b) and c) of this proof. This completes the proof of Lemma 1.5 ∎

Proof of Lemma 1.3: Let $\tau \in \Gamma$ be fixed and choose $\chi, \chi', \chi'' \in M(\Gamma)$ such that $\chi \equiv 1$ in a neighbourhood of τ and supp $\chi \subseteqq \{t \in \Gamma, \chi''(t) = 1\} \subseteqq$ supp χ'' $\subseteqq \{t, \chi'(t) = 1\}$. Then we get $\chi_n{}''\chi_n = \chi_n$, $\chi_n{}''\chi_n{}' = \chi_n{}''$ and

$$\chi_n{}^E A_n{}^E \chi_n{}'^E = \chi_n{}^E F_n + \chi_n{}^E A_1{}^\tau,$$

$$F_n := \chi_n{}''^E (A_n{}^E - A_1{}^\tau)\, \chi_n{}'^E - \chi_n{}''^E A_1{}^\tau (I - \chi_n{}'^E),$$

$$\chi_n{}^E A_n{}^E \chi_n{}'^E (A_1{}^\tau)^{-1} = \chi_n{}^E \{I + F_n (A_1{}^\tau)^{-1}\}.$$

In view of assumption (iv) and Lemma 1.5, we may choose χ, χ', χ'' in such a manner that $\|F_n\| < 2^{-1} \|(A_1^\tau)^{-1}\|^{-1}$. Hence we obtain $\|\{I + F_n(A_1^\tau)^{-1}\}^{-1}\| < 2$ and

$$\chi_n^E A_n^E \chi_n'^E (A_1^\tau)^{-1} \{I + F_n(A_1^\tau)^{-1}\}^{-1} = \chi_n^E,$$

$$\chi_n A_n R_n = \chi_n, \qquad R_n := E_n^\tau(\chi_n'^E (A_1^\tau)^{-1}\{I + F_n(A_1^\tau)^{-1}\}^{-1}) (E_n^\tau)^{-1}.$$

If $\{R_n\} \in \mathfrak{B}$, then $\{\chi_n\}^0 \{A_n\}^0 \{R_n\}^0 = \{\chi_n\}^0$, and $\{A_n\}^0$ is M_τ-invertible from the right.

It remains to show $E_n^\tau(A_1^\tau)^{-1} \{I + F_n(A_1^\tau)^{-1}\}^{-1} (E_n^\tau)^{-1} \in \mathfrak{B}$. Since the Neumann series $\{I + F_n(A_1^\tau)^{-1}\}^{-1}$ converges with respect to the operator norm, it suffices to prove $E_n^\tau(A_1^\tau)^{-1} \{F_n(A_1^\tau)^{-1}\}^j (E_n^\tau)^{-1} \in \mathfrak{B}$ for $j = 0, 1, \ldots$. Now the latter term is the sum of certain products whose factors are of the form χ_n'', χ_n', A_n or $E_n^\tau B^\tau (E_n^\tau)^{-1}$ ($B^\tau \in \mathfrak{A}_{2\times 2}$). Thus $\{R_n\} \in \mathfrak{B}$ follows by Lemma 1.4. This completes the proof ∎

Now let us assume $\{A_n\}$ to be stable. We shall prove the necessity of the conditions in Theorem 1.4. The invertibility of $A, \tilde{A} \in \mathcal{L}(L^2(\Gamma))$ follows by $\{A_n\} \in \mathfrak{B}$ (see the properties of $\mathfrak{B}$ listed above). We fix $\tau \in \Gamma$ and show $A_n^E P_n \to A_1^\tau$, $A_n^{E*} P_n \to A_1^{\tau *}$. If this will be done, then $\{\xi_k\} \in \tilde{l}^2$ and the stability of $\{A_n\}$ imply

$$\|A_n^E P_n\{\xi_k\}\|_{\tilde{l}^2} \geqq \frac{1}{C} \|P_n\{\xi_k\}\|_{\tilde{l}^2}, \qquad \|A_n^{E*} P_n\{\xi_k\}\|_{\tilde{l}^2} \geqq \frac{1}{C} \|P_n\{\xi_k\}\|_{\tilde{l}^2}.$$

Passing to the limit as $n \to \infty$, we get

$$\|A_1^\tau\{\xi_k\}\|_{\tilde{l}^2} \geqq \frac{1}{C} \|\{\xi_k\}\|_{\tilde{l}^2}, \qquad \|A_1^{\tau *}(\{\xi_k\}\|_{\tilde{l}^2} \geqq \frac{1}{C} \|\{\xi_k\}\|_{\tilde{l}^2}.$$

Here the first inequality proves A_1^τ to be injective and im A_1^τ to be closed. The second inequality shows im A_1^τ to be dense. Hence $A_1^\tau \in \mathcal{L}(\tilde{l}^2)$ is invertible.

Since $\{A_n^{E*}\}$ is uniformly bounded (see Lemma 1.2), the strong convergence $A_n^{E*} P_n \to A_1^{\tau *}$ follows from $A_n^{E*} P_n \{\delta_{j,k}\}_{k\in\mathbb{Z}} \to A_1^{\tau *}\{\delta_{j,k}\}_{k\in\mathbb{Z}}$. To show this, let $\chi, \chi' \in M(\Gamma)$ satisfy $\chi \equiv 1$, $\chi' \equiv 1$ in a neighbourhood of τ and suppose $\chi'\chi = \chi$. Then

$$A_n^{E*} P_n\{\delta_{j,k}\}_{k\in\mathbb{Z}} = A_1^{\tau *}\{\delta_{jk,}\}_{k\in\mathbb{Z}} + t_1 + t_2 + t_3 + t_4 + t_5,$$

$$t_1 := A_1^{\tau *}[\chi_n^E - I] \{\delta_{j,k}\}_{k\in\mathbb{Z}}, \qquad t_2 := [(\chi_n'^E - I) A_1^{\tau *} \chi_n^E] \{\delta_{j,k}\}_{k\in\mathbb{Z}},$$

$$t_3 := [\chi_n^E (A_n^E - A_1^\tau) \chi_n'^E]^* \{\delta_{j,k}\}_{k\in\mathbb{Z}}, \qquad t_4 := (I - \chi_n'^E) A_n^{E*} \chi_n^E \{\delta_{j,k}\}_{k\in\mathbb{Z}},$$

$$t_5 := A_n^{E*}[I - \chi_n^E] \{\delta_{j,k}\}_{k\in\mathbb{Z}}.$$

If we choose χ, χ' by Lemma 1.5 and assumption (iv), then the terms t_2 and t_3 become small. For j fixed and n large enough, t_1 and t_5 vanish. Now we rewrite

$$t_4 = [A_n^{E*} \chi_n'^E - \chi_n'^E A_n^{E*}] \chi_n^E \{\delta_{j,k}\}_{k\in\mathbb{Z}} = t_6 + t_7 + t_8;$$

$$t_6 := (E_n^\tau)^{-1} \{\chi_n[\chi_n' A_n - A_n \chi_n' - L_n(\chi' A - A\chi') \mid \text{im } L_n$$
$$- W_n L_n(\chi' \tilde{A} - \tilde{A}\chi') \mid \text{im } L_n W_n]\}^* E_n^\tau \{\delta_{j,k}\}_{k\in\mathbb{Z}},$$

$$t_7 := (E_n^\tau)^{-1} \{\chi_n L_n(\chi' A - A\chi') \mid \text{im } L_n\}^* E_n^\tau \{\delta_{j,k}\}_{k\in\mathbb{Z}},$$

$$t_8 := (E_n^\tau)^{-1} W_n \{\chi_n L_n(\chi' \tilde{A} - \tilde{A}\chi') \mid \text{im } L_n\}^* W_n E_n^\tau \{\delta_{j,k}\}_{k\in\mathbb{Z}}.$$

The term t_6 is small by assumption (iii)' and we have

$$\chi_n L_n(\chi' A - A\chi') = [\chi_n L_n - \chi] (\chi' A - A\chi') + \chi(\chi' A - A\chi').$$

Here the second term on the right-hand side becomes small for a suitable χ, whereas the norm of the first term tends to zero as $n \to \infty$. Thus t_7 becomes small. An analogous consideration for t_8 yields $A_n^{E*} P_n\{\delta_{j,k}\}_{k\in\mathbb{Z}} \to A_1^{\tau *}\{\delta_{j,k}\}_{k\in\mathbb{Z}}$, i.e., $A_n^{E*} P_n \to A_1^{\tau *}$. Similarly, one shows $A_n^E \to A_1^\tau$. This completes the proof of Theorem 1.4. ∎

1.4. The proof of Theorem 1.2

1.4.1. Let us consider the quadrature method (1.7). We identify the operators of $\mathcal{L}(\text{im } L_n)$ with their matrices corresponding to the base $\{\chi_j^{(n)}, j = 0, \ldots, n-1\}$ and denote the matrix of the system (1.7) by A_n. For the proof of Theorem 1.2 in the case of the method (1.7), it suffices to prove the assumptions (i)—(iv) of Theorem 1.3. The validity of (ii) is well known. Let us denote the set of all corner points of Γ by Γ'. Then, while proving assumption (i), the curve Γ can be assumed to be smooth. For smooth curves, the convergence $\chi_n A_n \chi_n L_n \to \chi A \chi$ has already been proved (see [23, 22]). Using this result and $\Delta t_k^{(n)} = \gamma'(\tau_k^{(n)})\, \gamma'(t_j^{(n)})^{-1}\, \Delta t_j^{(n)} + O(n^{-2})$, we easily obtain $\chi_n A_n^* \chi_n L_n \to \chi A^* \chi$. (Note that the adjoint of the integral operator T with kernel $k(t, \tau)$ $(t, \tau \in \Gamma)$ is the integral operator with kernel $\gamma'(t)\, \overline{k(t, \tau)/\gamma'(\tau)}$. This follows by (1.10).) Thus assumption (i) of Theorem 1.3 is satisfied.

1.4.2. Now we shall investigate the validity of (iii) in Theorem 1.3. Setting

$$F : L^2(\Gamma) \to \mathbf{C}, \qquad Fx = \int_\Gamma x(\tau)\, d\tau,$$

we get $(Tx)(\tau) = F(k(\tau, \cdot)\, x)$. The approximate operator $T_n = \left(k(\tau_k^{(n)}, t_j^{(n)})\, \Delta t_j^{(n)}\right)_{k,j=0}^{n-1} \in \mathcal{L}(\text{im } L_n)$ takes the form $(T_n x_n)(\tau) = K_{n,\tau}^\varepsilon F K_n^\delta (k(\tau, \cdot)\, x_n)$. Thus we obtain

$$(T_n - K_n^\varepsilon T \mid \text{im } L_n)\, x_n(\tau) = K_{n,\tau}^\varepsilon F(I - K_n^\delta)\, k(\tau, \cdot)\, L_n x_n.$$

If ω denotes the moduls of continuity $\omega(\delta') = \sup \{|k(\tau, t_1) - k(\tau, t_2)|, \tau \in \Gamma, t_1, t_2 \in \Gamma, |t_1 - t_2| < \delta'$ and there is no corner point between t_1 and $t_2\}$, then

$$\|(I - K_n^\delta)\, k(\tau, \cdot)\, L_n\|_{\mathcal{L}(L^2(\Gamma))} \leqq C\omega(1/n),$$

$$\left|F\left((I - K_n^\delta)\, k(\tau, \cdot)\, L_n x_n\right)\right| \leqq C\omega(1/n)\, \|x_n\|_{L^2(\Gamma)}$$

(see [21, Lemma 4.1]). The latter inequalities imply

$$\|T_n - K_n^\varepsilon T \mid \text{im } L_n\|_{L^2(\Gamma) \to L^\infty(\Gamma)} \leqq C\omega(1/n)$$

$$\|T_n - K_n^\varepsilon T \mid \text{im } L_n\|_{\mathcal{L}(L^2(\Gamma))} \to 0 \qquad (n \to \infty).$$

Since $T : L^2(\Gamma) \to C(\Gamma)$ is compact, we get $\|(K_n^\varepsilon - L_n)\, T\| \to 0$ and $\|T_n - L_n T \mid \text{im } L_n\| \to 0$ $(n \to \infty)$. Replacing T by χT or $T\chi$, respectively, and T_n by $\chi_n T_n$ or $T_n K_n^\delta \chi \mid \text{im } L_n$, respectively, we arrive at

$$\|\chi_n T_n - L_n \chi T \mid \text{im } L_n\| \to 0,$$

$$\|T_n \chi_n - L_n T \chi \mid \text{im } L_n\| \leqq \|T_n\|\, \|\chi_n - K_n^\delta \chi \mid \text{im } L_n\| + \|T_n K_n^\delta \chi \mid \text{im } L_n - L_n T \chi \mid \text{im } L_n\| \to 0,$$

$$\|\chi_n T_n - T_n \chi_n - L_n(\chi T - T\chi) \mid \text{im } L_n\| \to 0.$$

Since $\chi_n c_n = c_n \chi_n$ and $\chi c = c\chi$ imply $\chi_n c_n - c_n \chi_n - (\chi c - c\chi) = 0$, it remains to show (iii) for the singular operator S_Γ and

$$S_n := -\mathrm{i} \cot\left(\pi(\varepsilon - \delta)\right) I + \left(\frac{1}{\pi \mathrm{i}}\, \frac{\Delta t_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}}\right)_{k,j=0}^{n-1}.$$

Without loss of generality, we suppose that $\chi \circ \gamma$ is continuously differentiable and set $k'(t, \tau) = \left(\chi(t) - \chi(\tau)\right)/(t - \tau)$. Thus $k'(t, \cdot)$ is continuous on $\Gamma \setminus \Gamma'$ and piecewise continuous on Γ. Consequently,

$$\chi_n S_n - S_n \chi_n = M_n^1 + M_n^2, \qquad M_n^1 := \left(k'(\tau_k^{(n)}, t_j^{(n)}) \, \Delta t_j^{(n)}\right)_{k,j=0}^{n-1},$$

$$M_n^2 := \left(\frac{1}{\pi \mathrm{i}} \frac{\Delta t_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \left(\chi(\tau_j^{(n)}) - \chi(t_j^{(n)})\right)\right)_{k,j=0}^{n-1},$$

where $\|M_n^1 - L_n(\chi S_\Gamma - S_\Gamma \chi) \mid \operatorname{im} L_n\| \to 0$ can be shown analogously to $\|T_n - L_n T \mid \operatorname{im} L_n\| \to 0$. The obvious estimate

$$\left|\frac{1}{\pi \mathrm{i}} \frac{\Delta t_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}} \left(\chi(\tau_j^{(n)}) - \chi(t_j^{(n)})\right)\right| < C \frac{1}{n} \frac{1}{|j \dot{-} k| + 1}$$

with

$$j \dot{-} k = \begin{cases} j - k & \text{if } -n/2 < j - k \leqq n/2; \\ j - k + n & \text{if } -3n/2 < j - k \leqq -n/2, \\ j - k - n & \text{if } n/2 < j - k < 3n/2 \end{cases}$$

implies $\|M_n^2\| \leqq C n^{-1} \log n \to 0$ $(n \to \infty)$. Thus we obtain $\|\chi_n S_n - S_n \chi_n - L_n(\chi S_\Gamma - S_\Gamma \chi) \mid \operatorname{im} L_n\| \to 0$ and assumption (iii) of Theorem 1.3 is fulfilled.

1.4.3. Now we prove assumption (iv) of Theorem 1.3. Let us fix $\tau \in \Gamma$ and $\varepsilon' > 0$. The elements of T_n satisfy $|k(\tau_k^{(n)}, t_j^{(n)}) \, \Delta t_j^{(n)}| < C/n$. If we choose $U = \{\gamma(s), \sigma - \varepsilon'/2C < s < \sigma + \varepsilon'/2C\}$ and $\chi \in M(\Gamma)$ with $\operatorname{supp} \chi \subseteqq U$, then simple estimates show $\|\chi_n^E T_n^E \chi_n^E\| < \varepsilon'$. Thus (iv) is proved for T instead of A. If we choose U in such a manner that $t \in U$ implies $|c(t) - c(\tau)| < \varepsilon'$, then $\|\chi_n^E(c_n^E - c(\tau) I)\| < \varepsilon'$ holds and (iv) is satisfied for c instead of A. It remains to consider the case $A = S_\Gamma$, $A_n = S_n$.

Without loss of generality, let $\tau = \gamma(0)$ be a corner point and set $\omega := \omega_\tau := \arg\left(-\gamma'(1-0)/\gamma'(1+0)\right) \in (0, 2\pi)$. Choose $\chi' \in M(\Gamma)$ such that the only corner point of $\operatorname{supp} \chi'$ is τ and $\chi' \equiv 1$ in a neighbourhood of τ. We define $v\colon \mathbf{R} \to \Gamma_\omega$ and $\psi\colon \Gamma \to \Gamma_\omega$ by

$$\psi(\gamma(s)) = v(s) \quad \text{if } -\frac{1}{2} < s \leqq \frac{1}{2}, \qquad v(s) = \begin{cases} s & \text{if } s \geqq 0, \\ -\mathrm{e}^{\mathrm{i}\omega} s & \text{if } s \leqq 0, \end{cases}$$

and set $S'x = \left(S_{\Gamma_\omega}[(\chi' x) \circ \psi^{-1}]\right) \circ \psi$. Then $T' = \chi'(S_\Gamma \chi' - S')$ is a compact integral operator and its kernel k' satisfies (see e.g. [15: p. 58])

$$k'(\tau, t) = \chi'(\tau) \frac{1}{\pi \mathrm{i}} \left\{\frac{1}{t - \tau} - \frac{\frac{d}{dt} \psi(t)}{\psi(t) - \psi(\tau)}\right\} \chi'(t), \qquad k'(\tau, \tau) = -\chi'(\tau)^2 \frac{1}{\pi \mathrm{i}} \frac{\frac{d^2}{dt^2} \psi(\tau)}{\frac{d}{dt} \psi(\tau)}.$$

Setting $T_n' = \left(k'(\tau_k^{(n)}, t_j^{(n)}) \, \Delta t_j^{(n)}\right)_{k,j=0}^{n-1}$, for $\chi \in M(\Gamma)$ and $\chi \chi' = \chi$, we obtain

$$\chi_n S_n \chi_n = \chi_n U_n \chi_n + \chi_n T_n' \chi_n,$$

$$U_n := -\mathrm{i} \cot\left(\pi(\varepsilon - \delta)\right) I_n + \left(\frac{1}{\pi \mathrm{i}} \frac{\frac{d}{dt} \psi(t_j^{(n)})}{\psi(t_j^{(n)}) - \psi(\tau_j^{(n)})} \Delta t_j^{(n)}\right)_{k,j=0}^{n-1}.$$

As we have shown above, the operator $\chi_n^E T_n'^E \chi_n^E$ becomes smaller than any prescribed $\varepsilon' > 0$ if $\operatorname{supp} \chi$ is contained in a suitable small neighbourhood of τ. Therefore, it remains to show that $G_n = \chi_n^E(U_n^E - B_1^\tau) \chi_n^E$ is small, where

$$B_1^\tau := -\mathrm{i} \cot\left(\pi(\varepsilon - \delta)\right) I + \left(\frac{1}{\pi \mathrm{i}} \frac{v\left(\frac{j+1}{n}\right) - v\left(\frac{j}{n}\right)}{v\left(\frac{j+\delta}{n}\right) - v\left(\frac{k+\varepsilon}{n}\right)}\right)_{k,j \in \mathbf{Z}} \in \mathfrak{A}_{2\times 2}.$$

By

$$\frac{\frac{d}{dt}\,\psi(t_j^{(n)})}{\psi(t_j^{(n)}) - \psi(\tau_k^{(n)})}\,\Delta t_j^{(n)} = \frac{v\left(\frac{j+1}{n}\right) - v\left(\frac{j}{n}\right)}{v\left(\frac{j+\delta}{n}\right) - v\left(\frac{k+\varepsilon}{n}\right)}\;\frac{\frac{dv}{ds}\left(\frac{j+\delta}{n}\right)\frac{1}{n}}{v\left(\frac{j+1}{n}\right) - v\left(\frac{j}{n}\right)}\;\frac{\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j}{n}\right)}{\gamma'\left(\frac{j+\delta}{n}\right)\frac{1}{n}},$$

$$\frac{\frac{dv}{ds}\left(\frac{j+\delta}{n}\right)\frac{1}{n}}{v\left(\frac{j+1}{n}\right) - v\left(\frac{j}{n}\right)} = 1$$

we get $G_n = \chi_n{}^E B_1{}^\tau \chi_n{}^E D_n$, where $D_n := (\delta_{j,k} d_j{}^n)_{j,k\in\mathbf{Z}}$ and

$$d_j{}^n := \begin{cases} 0 & \text{if } |j| \geqq \frac{n}{2}, \\ \dfrac{\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j}{n}\right)}{\gamma'\left(\frac{j+\delta}{n}\right)\frac{1}{n}} - 1 & \text{if } |j| < \frac{n}{2}. \end{cases}$$

Since γ' is piecewise Hölder continuous, we conclude $\|\chi_n{}'^E D_n\| \to 0$. Consequently, if $\varepsilon' > 0$ is prescribed, then there exists a number n_0 such that $n \geqq n_0$ implies $\|\chi_n{}'^E D_n\| < \varepsilon' \|B_1{}^\tau\|^{-1}$ and $\|G_n\| < \varepsilon'$.

1.4.4. The method (1.8) can be treated analogously. Let us remark only that $M_n{}^2$ has to be replaced by

$$\left(\frac{1}{\pi i}\,\Delta t_j^{(n)}\,\frac{d\chi}{dt}\,(t_j^{(n)})\,\delta_{j,k}\right)_{k,j=0}^{n-1},$$

where the norm of the latter term tends to 0 as $n \to \infty$. The verification of (i), (ii), (ii)' and (iv) (see Theorems 1.3 and 1.4) for the method (1.9) is also similar to the preceding proof. To show (i)', we consider $\tilde{A} := cI - dS_\Gamma - T$ and the corresponding quadrature method for $\tilde{A}$. If $(\tilde{A})_n$ denotes the corresponding approximate operator, then $(\tilde{A})_n = \tilde{A}_n := W_n A_n W_n$. Thus (i)' follows from (i). It remains to show (iii)'.

For T and S_Γ, define T_n, T_n', S_n and S_n' by

$$T_n = \left(k(t_k^{(n)}, t_j^{(n)})\left(\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j-1}{n}\right)\right)\tilde{\delta}_{k,j}\right)_{k,j=0}^{n-1},$$

$$T_n' = \left(k(t_k^{(n)}, t_j^{(n)})\left(\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j}{n}\right)\right)\right)_{k,j=0}^{n-1},$$

$$S_n = \left(\frac{1}{\pi i}\,\frac{1}{t_j^{(n)} - t_k^{(n)}}\left(\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j-1}{n}\right)\right)\tilde{\delta}_{k,j}\right)_{k,j=0}^{n-1},$$

$$S_n' = \left(\frac{1}{\pi i}\,\frac{1}{t_j^{(n)} - t_k^{(n)}}\left(\gamma\left(\frac{j+1}{n}\right) - \gamma\left(\frac{j}{n}\right)\right)\right)_{k,j=0}^{n-1},$$

where $1/(t_j^{(n)} - t_j^{(n)}) := 0$ and $\tilde{\delta}_{k,j} = 0$ for $k - j$ even and $\tilde{\delta}_{k,j} = 1$ for $k - j$ odd. Then it is easy to prove that $\|T_n - (T_n' - W_n T_n' W_n)\| \to 0$ $(n \to \infty)$. Thus we obtain

$$\|\chi_n T_n - T_n \chi_n - \{(\chi_n T_n' - T_n'\chi_n) - W_n(\chi_n T_n' - T_n'\chi_n) W_n\}\| \to 0,$$
$$\|\chi_n S_n - S_n \chi_n - \{(\chi_n S_n' - S_n'\chi_n) - W_n(\chi_n S_n' - S_n'\chi_n) W_n\}\| \to 0.$$

Since T_n', S_n' are the approximate operators corresponding to the method (1.8) and (iii) is fulfilled for (1.8), we get

$$\|\chi_n T_n - T_n\chi_n - \{L_n(\chi T - T\chi) \mid \mathrm{im}\, L_n - W_n L_n(\chi T - T\chi) \mid \mathrm{im}\, L_n W_n\}\| \to 0,$$
$$\|\chi_n S_n - S_n\chi_n - \{L_n(\chi S_\Gamma - S_\Gamma\chi) \mid \mathrm{im}\, L_n - W_n L_n(\chi S_\Gamma - S_\Gamma\chi) \mid \mathrm{im}\, L_n W_n\}\| \to 0.$$

This completes the proof of Theorem 1.2 ∎

2. Collocation methods for singular integral equations on curves with corners. Piecewise constant trial functions

2.1. Collocation methods on an angle

Similarly to the quadrature methods, one can treat other spline approximation methods, i.e., collocation methods and Galerkin-Petrov methods using splines as test or trial functions. For simplicity, we shall restrict our considerations to the collocation with piecewise constant trial functions. In this section, we establish the stability of the model problem, more precisely, the stability of the collocation for singular integral equations with constant coefficients on an angle. Using these results in the next section, we extend our analysis to collocation for equations with continuous coefficients on general curves with corners.

Let us retain the notation of Section 1.1. For the ε-collocation method $(0 < \varepsilon < 1)$, we seek an approximate solution $u_n = \sum_{k \in \mathbf{Z}} \xi_k^{(n)} \chi_k^{(n)} \in \mathrm{im}\, L_n \subseteq L^2(\Gamma_\omega)$ satisfying the equations $(Au_n)(\tau_k^{(n)}) = f(\tau_k^{(n)})$, $k \in \mathbf{Z}$. The latter system can be written as $A_n u_n = K_n f$, where $A_n := K_n A \mid \mathrm{im}\, L_n \in \mathcal{L}(\mathrm{im}\, L_n)$. Here again, A_n can be considered to belong to $\mathcal{L}(\tilde{l}^2)$ and these operators do not depend on n. Thus, the sequence $\{A_n\}$ $\big(A_n \in \mathcal{L}(\mathrm{im}\, L_n)\big)$ is stable if and only if $A_1 \in \mathcal{L}(\tilde{l})^2$ is invertible.

Theorem 2.1: *The following assertions are valid.*

a) *The operator $A_1 \in \mathcal{L}(\tilde{l}^2)$ is Fredholm of index zero if and only if $\dfrac{c+d}{c-d} \notin \Omega$, where*

$$\Omega := \left\{\frac{\psi_\varepsilon(\mu) - 1}{\psi_\varepsilon(\mu)},\ 0 \le \mu \le 1\right\} \quad and \quad \psi_\varepsilon(\mu) := \int_0^1 \mathrm{e}^{-\mathrm{i}\pi(\varepsilon-\delta)(\mu-1)} \frac{\sin\big(-\pi(\varepsilon-\delta)\,\mu\big)}{\sin\big(-\pi(\varepsilon-\delta)\big)}\, d\delta$$
$$= \frac{1}{2}\left\{1 + \int_0^1 \frac{2\mathrm{e}^{-\mathrm{i}2\pi(\varepsilon-\delta)\mu}}{\mathrm{e}^{-\mathrm{i}2\pi(\varepsilon-\delta)} - 1}\, d\delta\right\}.$$

In particular, $\Omega = (-\infty, 0]$ for $\varepsilon = 1/2$.

b) *The operator A_1 is invertible in $\tilde{l}^2$ if and only if $\dfrac{c+d}{c-d} \notin \Omega \cup \Phi$. Here Φ denotes an at most countable subset of $\mathbf{C} \setminus \Omega$ whose accumulation points belong to Ω.*

c) *If $\omega = \pi$, then $\Phi = \emptyset$.*

Proof: Assertions b) and c) can be derived analogously to the corresponding assertions of Theorem 1.1. In order to verify a), we shall prove $A_1 \in \mathfrak{A}_{2\times 2}$ and show $\det \mathcal{A}_{A_1}$ to be independent of ω. Thus it suffices to establish a) for $\omega = \pi$. In this case, A_1 becomes a discrete convolution operator and a) will follow easily.

For the sake of brevity, we shall restrict ourselves to the case $\varepsilon = 1/2$. Then

$$A_1 = cI + d\left(\frac{1}{\pi i}\int_{\Gamma_\omega} \frac{\chi_j^{(1)}(\tau_k)}{\tau - \tau_k^{(1)}}\,d\tau\right)_{k,j\in\mathbf{Z}}. \tag{2.1}$$

If t_j^δ ($j \in \mathbf{Z}$, $0 < \delta < 1$) denotes the point $(k+\delta)$ for $k \geqq 0$ and $-(k+\delta)\,e^{i\omega}$ for $k < 0$, then

$$\frac{1}{\pi i}\int_{\Gamma_\omega} \frac{\chi_j^{(1)}(\tau)}{\tau - \tau_k^{(1)}}\,d\tau = \frac{1}{\pi i}\int_0^{1/2}\left\{\frac{1}{t_j^\delta - \tau_k^{(1)}} + \frac{1}{t_j^{1-\delta} - \tau_k^{(1)}}\right\}d\delta \begin{cases} 1 & \text{if } j \geqq 0, \\ -e^{i\omega} & \text{if } j < 0. \end{cases} \tag{2.2}$$

Let us set

$$A_1^\delta = \left[c - i\cot\left(\pi\left(\frac{1}{2} - \delta\right)\right)d\right]I + d\left(\frac{1}{\pi i}\,\frac{1}{t_j^\delta - \tau_k^{(1)}}\,\Delta t_j^{(n)}\right)_{k,j\in\mathbf{Z}},$$

$$\Delta t_j = \begin{cases} 1 & \text{if } j \geqq 0, \\ -e^{i\omega} & \text{if } j < 0 \end{cases}$$

and consider the operator-valued function $\delta \to A(\delta) := A_1^\delta + A_1^{1-\delta}$ defined on $[0, 1/2]$. The proof of Theorem 1.1 shows $A(\delta) \in \mathfrak{A}_{2\times 2} \subseteqq \mathcal{L}(l^2)$. Moreover, the obvious estimates

$$\left|\frac{1}{t_j^\delta - \tau_k^{(1)}} - \frac{1}{t_j^{\delta'} - \tau_k^{(1)}}\right| \leqq |\delta' - \delta|\,\frac{1}{|j-k|^2}, \qquad j \neq k,\ j, k \in \mathbf{Z},$$

$$\frac{1}{t_k^\delta - \tau_k^{(1)}} + \frac{1}{t_k^{1-\delta} - \tau_k^{(1)}} = 0, \qquad k \in \mathbf{Z},$$

imply the continuity of the function $\delta \to A(\delta)$. The equations (2.1) and (2.2) yield

$$A_1 = \int_0^{1/2} \{A_1^\delta + A_1^{1-\delta}\}\,d\delta \in \mathfrak{A}_{2\times 2}, \qquad \mathcal{A}_{A_1} = \int_0^{1/2} \{\mathcal{A}_{A_1\delta} + \mathcal{A}_{A_1^{1-\delta}}\}\,d\delta.$$

By (1.6) we conclude

$$\mathcal{A}_{A_1}(\tau,\mu)$$

$$= \begin{cases} \begin{pmatrix} \{c + d(2\psi_\varepsilon(\lambda) - 1)\} & 0 \\ 0 & \{c - d(\overline{2\psi_\varepsilon(\lambda)} - 1)\} \end{pmatrix} & \text{if } \tau = e^{i2\pi\lambda},\ 0 < \lambda < 1,\ 0 \leqq \mu \leqq 1, \\[2ex] \begin{pmatrix} \{c + d(-\mu + (1-\mu))\} & -d(-i)\left\{\dfrac{e^{-i(\pi-\omega)\left(\frac{1}{2} + \frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2} + \frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)}\right\} \\ \left\{d(-i)\dfrac{e^{-i(\omega-\pi)\left(\frac{1}{2} + \frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)}}{\sin\left(\pi\left(\frac{1}{2} + \frac{i}{2\pi}\log\frac{\mu}{1-\mu}\right)\right)}\right\} & \{c - d(-\mu + (1-\mu))\} \end{pmatrix} & \\ & \text{if } \tau = 1,\ 0 \leqq \mu \leqq 1. \end{cases}$$

Thus $\det \mathcal{A}_{A_1}$ is independent of ω. For $\omega = \pi$, A_1 takes the form $A_1 = cI + d(f_{k-j})_{k,j\in\mathbf{Z}}$, where f_j denotes the j-th Fourier coefficient of the function $f(e^{i2\pi\lambda}) := 2\psi_\varepsilon(\lambda) - 1$, $0 < \lambda < 1$. This convolution operator is a Fredholm operator with index 0 if and only if $c + df(\tau) \neq 0$ for all $\tau \in T$, i.e., if $\frac{c+d}{c-d} \notin \Omega$. This completes the proof of the theorem ∎

2.2. Collocation on curves with corners

Let us retain the notation introduced in Sections 1.2—1.3. The ε-collocation method determins an approximate solution $u_n = \sum_{k=0}^{n-1} \xi_k^{(n)} \chi_k^{(n)} \in \operatorname{im} L_n \subseteq L^2(\Gamma)$ by solving the equations

$$(Au_n)(\tau_k^{(n)}) = f(\tau_k^{(n)}), \qquad k = 0, \ldots, n-1. \tag{2.3}$$

This system can be written as $A_n u_n = K_n^\varepsilon f$, where $A_n := K_n^\varepsilon A \mid \operatorname{im} L_n \in \mathcal{L}(\operatorname{im} L_n)$. If we fix $\tau \in \Gamma$, then the model problem of the ε-collocation for the operator $A \in \mathcal{L}(L^2(\Gamma))$ is the ε-collocation for $A^\tau \in \mathcal{L}(L^2(\Gamma_{\omega_\tau}))$ (cf. Section 1.2) described in Section 2.1. The matrix of the corresponding system will be denoted by A_1^τ. By the proof of Theorem 2.1 we get $A_1^\tau \in \mathfrak{A}_{2\times 2}$.

Theorem 2.2: *The following assertions are valid.*

a) *The ε-collocation $(0 < \varepsilon < 1)$ for the operator A is stable if and only if the operators $A \in \mathcal{L}(L^2(\Gamma))$ and $A_1^\tau \in \mathcal{L}(\tilde{l}^2)$ $(\tau \in \Gamma)$ are invertible.*

b) *If the collocation method is stable and f is Riemann integrable, then the system (2.3) is uniquely solvable for n large enough and the approximate solutions u_n converge to $u = A^{-1}f$ as $n \to \infty$.*

Combining Theorems 2.1 and 2.2 we obtain necessary and sufficient conditions for the stability of the collocation method.

Proof: It suffices to show that the assumptions of Theorem 1.3 are fulfilled. The validity of (i) and (ii) can be derived analogously to Subsection 1.4.1. Now let us verify property (iii) of Theorem 1.3. Without loss of generality, we suppose $\chi \circ \gamma$ to be continuously differentiable and obtain

$$\begin{aligned}
&\chi_n A_n - A_n \chi_n - L_n(\chi A - A\chi) \mid \operatorname{im} L_n \\
&= K_n^\varepsilon \chi L_n K_n^\varepsilon A \mid \operatorname{im} L_n - K_n^\varepsilon A L_n K_n^\varepsilon \chi \mid \operatorname{im} L_n - L_n(\chi A - A\chi) \mid \operatorname{im} L_n \\
&= K_n^\varepsilon A(I - K_n^\varepsilon)\, \chi \mid \operatorname{im} L_n + (K_n^\varepsilon - L_n)(\chi A - A\chi) \mid \operatorname{im} L_n.
\end{aligned}$$

Since $\chi A - A\chi: L^2(\Gamma) \to C(\Gamma)$ is compact and $(K_n^\varepsilon - L_n): C(\Gamma) \to L^2(\Gamma)$ converges strongly to 0, we get $\|(K_n^\varepsilon - L_n)(\chi A - A\chi) \mid \operatorname{im} L_n\| \to 0$. By virtue of $K_n^\varepsilon A(I - K_n^\varepsilon)\chi \mid \operatorname{im} L_n = K_n^\varepsilon b L_n K_n^\varepsilon S_\Gamma (I - K_n^\varepsilon)\chi \mid \operatorname{im} L_n$, it remains to show $\|K_n^\varepsilon S_\Gamma(I - K_n^\varepsilon)\chi \mid \operatorname{im} L_n\| \to 0$ (compare [21]). The latter relation is an immediate consequence of

$$K_n^\varepsilon S_\Gamma (I - K_n^\varepsilon)\, \chi \mid \operatorname{im} L_n = \left(\frac{1}{\pi i} \int_\Gamma \frac{\chi(\tau) - \chi(\tau_j^{(n)})}{\tau - \tau_k^{(n)}} \chi_j^{(n)}(\tau)\, d\tau \right)_{k,j=0}^{n-1}$$

and of the obvious estimate

$$\left| \frac{1}{\pi i} \int_\Gamma \frac{\chi(\tau) - \chi(\tau_j^{(n)})}{\tau - \tau_k^{(n)}} \chi_j^{(n)}(\tau)\, d\tau \right| \le C \frac{1}{n} \frac{1}{|k - j| + 1}.$$

Thus assumption (iii) is satisfied.

Now we consider property (iv) of Theorem 1.3 retaining the notation of S', T', ψ and v introduced in Subsection 1.4.3. Repeating the argumentation from this subsection we get the validity of (iv) for $A = T$, $A_n = K_n{}^\varepsilon T \mid \operatorname{im} L_n$ and for $A = c$, $A_n = K_n{}^\varepsilon c \mid \operatorname{im} L_n$. Therefore, we can assume $A = S_\Gamma$ and $A_n = K_n{}^\varepsilon S_\Gamma \mid \operatorname{im} L_n$. In this case, $A_1{}^\tau$ takes the form

$$A_1{}^\tau := \left(\frac{1}{\pi i} \int\limits_j^{j+1} \frac{1}{v(s) - v(k+\varepsilon)} \, dv(s)\right)_{k,j\in\mathbf{Z}}$$

By

$$\int\limits_j^{j+1} \frac{1}{v(s) - v(k+\varepsilon)} \, dv(s) = \int\limits_{j/n}^{(j+1)/n} \frac{1}{v(s) - v\left(\dfrac{k+\varepsilon}{n}\right)} \, dv(s)$$

$$= \int\limits_{j/n}^{(j+1)/n} \frac{\dfrac{d\psi}{dt}\left(\gamma(s)\right)}{\psi\circ\gamma(s) - \psi\circ\gamma\left(\dfrac{k+\varepsilon}{n}\right)} \, d\gamma(s) = \int\limits_\Gamma \frac{\dfrac{d\psi}{dt}(t)}{\psi(t) - \psi(\tau_k{}^{(n)})} \chi_j{}^{(n)}(t) \, dt$$

and

$$\chi_n(K_n{}^\varepsilon S' \mid \operatorname{im} L_n) \chi_n = \chi_n \left(\frac{1}{\pi i} \int\limits_\Gamma \frac{\dfrac{d\psi}{dt}(t)}{\psi(t) - \psi(\tau_k{}^{(n)})} \chi_j{}^{(n)}(t) \, dt\right)_{k,j} \chi_n,$$

$$\chi_n{}^E(K_n{}^\varepsilon S' \mid \operatorname{im} L_n)^E \chi_n{}^E = \chi_n{}^E A_1{}^\tau \chi_n{}^E$$

we obtain $\chi_n{}^E\{(K_n{}^\varepsilon S_\Gamma \mid \operatorname{im} L_n)^E - A_1{}^\tau\} \chi_n{}^E = \chi_n{}^E(K_n{}^\varepsilon T' \mid \operatorname{im} L_n)^E \chi_n{}^\varepsilon$. Since assertion (iv) is true for the case A replaced by the compact operator T' and $A_1{}^\tau$ replaced by 0, the last expression becomes smaller than any prescribed $\varepsilon' > 0$. This complete the proof of the theorem ∎

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