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On the Singular Values of Toeplitz Matrices

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Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Im Jahre 1920 bewies G. Szegö ein grundlegendes Resultat über die asymptotische Verteilung der Eigenwerte selbstadjungierter Toeplitzscher Matrizen $T_n(f)$. Ein analoges Resultat gilt für die singulären Werte, und zwar auch, falls $T_n(f)$ nicht notwendig selbstadjungiert ist. In dieser Arbeit geben wir eine Präzisierung des letzteren Resultats an (eine Formel „zweiter Ordnung") und bestimmen außerdem die „Grenzmenge" der singulären Werte, beides unter geeigneten Voraussetzungen an die Funktion f.

В 1920 году G. Szegö доказал основной результат о асимптотическом распределении собственных чисел самосопряженных матриц Тёплица $T_n(f)$. Аналогичный результат справедлив для сингулярных чисел даже в случае когда $T_n(f)$ не обязательно самосопряженная. В этой работе мы представляем уточнение последнего результата (формулу „второго порядка") и определяем „предельное множество" сингулярных чисел, и то и другое при подходящих дополнительных условиях для функции f.

In 1920 G. Szegö proved the basic result concerning the asymptotic distribution of the eigenvalues of selfadjoint Toeplitz matrices $T_n(f)$. An analogous result holds for the singular values in case $T_n(f)$ is not necessarily selfadjoint. In this paper we present a refinement of this (a second-order result) and also determine the limiting set of the singular values, both under appropriate hypotheses in the function f.

Introduction. A classical theorem of Szegö states that if $\lambda_1^{(n)} \geqq \dots \geqq \lambda_n^{(n)}$ are the eigenvalues of the Toeplitz matrix $T_n(f) = (\hat{f}_{i-j})_{i,j=0}^{n-1}$ associated with a bounded real-valued function f on the unit circle, then for any continuous (indeed, Riemann integrable) function F one has

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(\lambda_k^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\bigl(f(\theta)\bigr)\, d\theta. \tag{1}$$

An analogous result holds for the singular values $s_1^{(n)} \geqq \cdots \geqq s_n^{(n)}$ of not necessarily selfadjoint Toeplitz matrices. (Recall that the singular values of a matrix A are the eigenvalues of $(A^*A)^{1/2}$). The analogue of (1) is that

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(s_k^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\bigl(|f(\theta)|\bigr)\, d\theta. \tag{2}$$

As with (1) nowadays, (2) can be proved in different ways. In [9] PARTER established it for bounded f satisfying a certain subsidiary condition. The result for general bounded f can be deduced easily from a theorem of AVRAM [1] on the trace of a product of Toeplitz matrices.

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The first purpose of this paper is to prove a refinement of (2) under appropriate conditions on F and f. If we set $M = \operatorname{ess\,sup} f$, $m = \operatorname{dist}\big(0, \operatorname{co} R(f)\big)$, where $R(f)$ denotes the essential range of f and "co" denotes convex hull, then it is easy to see that each singular value belongs to the interval $[m, M]$. (See Lemma I.2 below.) Thus the function F need only be defined on this interval for (2) to make sense. The refinement is a little simpler to state in terms of the squares of the singular values $t_k^{(n)} = (s_k^{(n)})^2$ rather than the singular values themselves. We shall assume about f that it belongs to the algebra K of all functions on the circle such that

$$f \in L_\infty \quad \text{and} \quad |||f||| := \left\{\sum_{k=-\infty}^{\infty} |k|\, |\hat{f}_k|^2\right\}^{1/2} < \infty.$$

The point about K, observed in [8], is that if $f \in K$, then the Hankel matrices $H(f)$ and $H(\tilde{f})$ both represent Hilbert-Schmidt operators on l_2 of the nonnegative integers. Here

$$H(f) = (\hat{f}_{i+j+1})_{i,j=0,1,\dots} \tag{3}$$

and $\tilde{f}$ is defined by $\tilde{f}(\theta) = f(-\theta)$. Given our function f we set $M = \operatorname{ess\,sup} |f|$, and we denote by $T(f)$ the infinite Toeplitz matrix $(\hat{f}_{i-j})_{i,j=0,1,\dots}$ thought of also as an operator on l_2 (of the nonnegative integers).

Theorem I: *Assume $f \in K$ and $G \in C^3([m^2, M^2])$. Then*

$$\lim_{n\to\infty} \left\{\sum_{k=1}^{n} G(t_k^{(n)}) - \frac{n}{2\pi} \int_{-\pi}^{\pi} G\big(|f(\theta)|^2\big)\, d\theta\right\}$$
$$= \operatorname{tr}\left[G\big(T(\tilde{f})\, T(f)\big) + G\big(T(f)\, T(\tilde{f})\big) - 2T\big(G(|f|^2)\big)\right]. \tag{4}$$

Here is an explanation of the ingredients of the right side of (4). Since $T(\tilde{f}) = T(f)^*$, the operators $T(\tilde{f})\, T(f)$ and $T(f)\, T(\tilde{f})$ are selfadjoint, and both their spectra will be shown to be contained in the interval $[m^2, M^2]$. (Their spectra can differ only when $m > 0$ and then the only possible difference is that 0 may belong to one but not the other.) The operators $G\big(T(\tilde{f})\, T(f)\big)$ and $G\big(T(f)\, T(\tilde{f})\big)$ are then defined using the spectral theorem. It will transpire that the operator in brackets is trace class, and "tr" denotes its trace. As the proof will show, the condition $G \in C^3$ can be relaxed somewhat. The form of the right side of (4) is a little unfortunate and it would be nice to have a more concrete expression for it. Such an expression exists if f is real-valued but an extension to general f eludes us.

Formula (4) with $G(\lambda) = F(\lambda^{1/2})$ is clearly a refinement of (2). An analogous refinement of (1) was obtained in [14]. (Actually that paper was about the real line analogue of the Toeplitz matrices, the Wiener-Hopf operators. The two cases are quite similar.) The proof of Theorem I uses the method, and in fact one of the results, of [14]. For the little bit of the theory of trace class (= nuclear) and Hilbert-Schmidt operators that will be needed we refer the reader to [6].

Second will be the determination of the "limiting set" of the singular values. This is a set Λ with the following properties:

(i) If $\lambda \in \Lambda$, then there exists a sequence $\{k_n\}$ such that $\lim_{n\to\infty} s_{k_n}^{(n)} = \lambda$.

(ii) If for some sequences $\{k_m\}$ and $\{n_m\}$ (with $n_1 < n_2 < \dots$) we have $\lim_{m\to\infty} s_{k_m}^{(n_m)} = \lambda$, then $\lambda \in \Lambda$.

Of course there is no *a priori* reason why such a limiting set should exist. Nevertheless we shall show that for a large class of f's it does exist and that in fact

$$\Lambda = \sigma\big((T(\tilde{f})\, T(f))^{1/2}\big) \cup \sigma\big((T(f)\, T(\tilde{f}))^{1/2}\big). \tag{5}$$

Here "σ" denotes "spectrum". This set Λ will be seen to have property (i) for all $f \in L_\infty$ but property (ii) requires more. We shall describe two conditions, either of which suffices. The first is that f belongs to the algebra PC, the closure in L_∞ of the algebra of piecewise continuous functions. Equivalently f has limits from left and right at each point. The other condition involves another subalgebra of L_∞ denoted by QC (for "quasicontinuous"). A function $w \in L_\infty$ belongs to QC if and only if it has the representation $w = u + Cv$ where u and v are continuous and Cv denotes the conjugate function of v, defined in terms of Fourier coefficients by $\widehat{Cv_k} = -\mathrm{i}\,(\operatorname{sgn} k)\,\hat{v}_k$. Alternatively $w \in QC$ if and only if the two Hankel operators $H(w)$ and $H(\tilde{w})$ are compact. (These and other facts about QC can be found in [11].)

Theorem II: *If either $f \in PC$ or f is the product of a bounded real-valued function and a function in QC, then the limiting set Λ exists and is given by* (5).

The two sufficient conditions are overlapping but neither contains the other, as can be shown by examples. The assumption $f \in L_\infty$ is not sufficient for the conclusion of the theorem to hold. It is a question of the applicability of the so-called "projection method" or "finite section method" in this context. (See Part 4 of the last section.)

Thanks are due to Don Sarason for some very helpful remarks on questions that arose in the course of this work.

Proof of Theorem I. As is usual we denote by $\|A\|_1, \|A\|_2, \|A\|$ the trace norm, Hilbert-Schmidt norm, and operator norms of A, respectively. We write P_n for the projection operator, defined by

$$P_n(x_0, x_1, \ldots) = (x_0, \ldots, x_{n-1}, 0, \ldots),$$

from l_2 to the subspace of l_2 on which $T_n(f)$ may be thought of as acting. We identify $T_n(f)$ with $P_n T(f) P_n$ in the obvious way. We define the operator Q_n on l_2 by

$$Q_n(x_0, x_1, \ldots) = (x_{n-1}, \ldots, x_0, 0, \ldots).$$

Finally we recall the definition (3) and the notation $\tilde{f}(\theta) = f(-\theta)$.

Lemma I.1: *For any $f, g \in L_\infty$ we have*

$$T(fg) - T(f)\,T(g) = H(f)\,H(\tilde{g}), \tag{6}$$

$$T_n(fg) - T_n(f)\,T_n(g) = P_n H(f)\,H(\tilde{g})\,P_n + Q_n H(\tilde{f})\,H(g)\,Q_n. \tag{7}$$

Proof: Routine computation. (Or see [2, Props. 2.7 and 3.6].) ∎

Next we prove the assertions made in the introduction concerning the location of the singular values of $T_n(f)$ and the spectra of $T(\bar{f})\,T(f)$ and $T(f)\,T(\bar{f})$. The numbers m and M are as before.

Lemma I.2: *The spectra of the operators $\left(T_n(\bar{f})\,T_n(f)\right)^{1/2}$ and $\left(T(\bar{f})\,T(f)\right)^{1/2}$ lie in the interval* $[m, M]$.

Proof: Consider the first operator. (The second is similar.) If $x = (x_0, \ldots, x_{n-1}, 0, \ldots)$ is a vector of norm 1, then

$$\left(T_n(\bar{f})\,T_n(f)\,x, x\right) = \|T_n(f)\,x\|^2 = \frac{1}{2\pi} \int |f(\theta)|^2 \left|\sum_{k=0}^{n-1} x_k\, e^{ik\theta}\right|^2 d\theta$$

which is clearly $\leqq M^2$. On the other hand

$$\|T_n(f)\,x\| \geqq \left|\left(T_n(f)\,x, x\right)\right| = \left|\left(P_n T(f)\,x, x\right)\right| = \left|\left(T(f)\,x, x\right)\right|$$

since $P_n x = x$, and this equals

$$\left| \frac{1}{2\pi} \int f(\theta) \left| \sum_{k=0}^{n-1} x_k \, e^{ik\theta} \right|^2 d\theta \right|. \tag{8}$$

For some complex number α of absolute value 1 we have $\operatorname{Re} \alpha f(\theta) \geqq m$ a.e. and from this it follows that (8) is $\geqq m$. We have shown $\big(T_n(\bar f)\, T_n(f)\, x, x\big) \subset [m^2, M^2]$ for all x with $\|x\| = 1$ and this implies that $\sigma\big(T_n(\bar f)\, T_n(f)\big) \subset [m^2, M^2]$, whence the first assertion of the lemma ∎

Passing to the proof of the theorem we consider first the special case of the function $G(\lambda) = e^{it\lambda}$ where t is a real parameter. For any operators A and B we have $d(e^{isA}\, e^{-isB})/dt = ie^{isA}(A - B)\, e^{-isB}$. Integrating with respect to s from 0 to t and right-multiplying by e^{itB} gives

$$e^{itA} - e^{itB} = i \int_0^t e^{isA}(A - B)\, e^{i(t-s)B}\, ds. \tag{9}$$

We apply this first with $A = A_n = T_n(\bar f)\, T_n(f)$, $B = B_n = T_n(|f|^2)$ and apply (7) to obtain, with these A_n and B_n,

$$e^{itA_n} - e^{itB_n} = -i \int_0^t e^{isA_n}[P_n H(\bar f)\, H(\tilde f)\, P_n + Q_n H(\tilde{\bar f})\, H(f)\, Q_n]\, e^{i(t-s)B_n}\, ds. \tag{10}$$

We also apply (9) with $A = T(\bar f)\, T(f)$, $B = T(|f|^2)$ and identity (6) to obtain, with these A and B,

$$e^{itA} - e^{itB} = -i \int_0^t e^{isA} H(\bar f)\, H(\tilde f)\, e^{i(t-s)B}\, ds. \tag{11}$$

The right side of (10) consists of two parts, one involving P_n and one involving Q_n. The one involving P_n is

$$-i \int_0^t e^{isA_n} P_n H(\bar f)\, H(\tilde f)\, P_n\, e^{i(t-s)B_n}\, ds. \tag{12}$$

Now this approaches the right side of (11), formally, as $n \to \infty$. Here is why it actually converges to it in trace norm. First, it is easy to see that $e^{isA_n} P_n$ converges strongly to e^{isA}, uniformly in s. Second, and this is crucial,

$$\|H(f)\|_2^2 = \sum_{i,j \geqq 0} |\hat f_{i+j+1}|^2 = \sum_{k=1}^{\infty} k\, |\hat f_k|^2 < \infty \tag{13}$$

so all the Hankel operators appearing in (10) are Hilbert-Schmidt. We use the general fact that if $C_n \to C$ strongly and H is Hilbert-Schmidt, then $C_n H \to CH$ in Hilbert-Schmidt norm (this is trivial if H has finite rank and follows in general by the density of the finite rank operators in the Hilbert-Schmidt operators) to deduce that $e^{isA_n} P_n H(\bar f) \to e^{isA} H(\bar f)$ in Hilbert-Schmidt norm. Similarly $H(\tilde f)\, P_n\, e^{i(t-s)B_n} \to H(\tilde f)\, e^{i(t-s)B}$ in Hilbert-Schmidt norm and so the integrand in (12) converges in trace norm to the integrand in (11), uniformly in s. We deduce that the trace of (12) has limit tr $[e^{itT(\bar f)T(f)} - e^{itT(|f|^2)}]$.

Next we use the identity $Q_n T_n(f)\, Q_n = T_n(\tilde f)$ and the fact that Q_n equals P_n times a commuting unitary operator to deduce, by the same argument, that the trace of the part of (10) involving Q_n converges as $n \to \infty$ to tr $[e^{itT(\tilde{\bar f})T(\tilde f)} - e^{itT(|\tilde f|^2)}]$. The transpose of the matrix in brackets here is the matrix $e^{itT(f)T(\bar f)} - e^{itT(|f|^2)}$ and so the two have the same trace. We have therefore shown that

$$\lim_{n\to\infty} \operatorname{tr}\left[e^{itT_n(\bar f)T_n(f)} - e^{itT_n(|f|^2)}\right] = \operatorname{tr}\left[e^{itT(\bar f)T(f)} + e^{itT(f)T(\bar f)} - 2e^{itT(|f|^2)}\right]. \quad (14)$$

Next we extend this from the exponential function to any function $G \in C^3([m^2, M^2])$. We extend G to a C^3 function with compact support on all of $(-\infty, \infty)$ and apply the Fourier inversion formula

$$G(\lambda) = \int \hat G(t)\, e^{it\lambda}\, dt \qquad \text{where} \qquad \hat G(t) = \frac{1}{2\pi}\int G(\lambda)\, e^{-it\lambda}\, d\lambda.$$

The operator version of this gives

$$G\big(T_n(\bar f)\, T_n(f)\big) - G\big(T_n(|f|^2)\big) = \int \hat G(t)\left[e^{itT_n(\bar f)T_n(f)} - e^{itT_n(|f|^2)}\right] dt. \quad (15)$$

Now it follows from (10) that the operator in brackets here has trace norm $O(|t|)$ as $t \to \pm\infty$. The reason is that since A_n and B_n are selfadjoint the exponential factors have operator norm 1 and so (13) shows the operator in brackets in (10) has trace norm at most $2\,|||f|||^2$. Moreover, $t^3\hat G(t)$ is bounded since G belongs to C^3 and has compact support. It follows that we can take $\lim_{n\to\infty}$ of the trace under the integral sign in (15), and then apply (14) and the Fourier inversion formula once again, to obtain

$$\begin{aligned}&\lim_{n\to\infty} \operatorname{tr}\left[G\big(T_n(\bar f)\, T_n(f)\big) - G\big(T_n(|f|^2)\big)\right]\\ &= \operatorname{tr}\left[G\big(T(\bar f)\, T(f)\big) + G\big(T(f)\, T(\bar f)\big) - 2G\big(T(|f|^2)\big)\right].\end{aligned}$$

What we are interested in, of course, is not this but

$$\lim_{n\to\infty} \operatorname{tr}\left[G\big(T_n(\bar f)\, T_n(f)\big) - T_n\big(G(|f|^2)\big)\right].$$

So it remains to evaluate $\lim_{n\to\infty} \operatorname{tr}\left[G\big(T_n(|f|^2)\big) - T_n\big(G(|f|^2)\big)\right]$. But it is precisely limits of this sort (for the continuous, i.e., Wiener-Hopf, analogue) that were obtained in [14], by methods very much like those used above. The result, not surprisingly, is that the limit is equal to $2\operatorname{tr}\left[G\big(T(|f|^2)\big) - T\big(G(|f|^2)\big)\right]$. The requirement on G is that $t^2\hat G(t) \in L_1$. To see that this holds note that since $G''' \in L_2$ we have $t^3\hat G(t) \in L_2$, and we need only apply Schwarz's inequality. Putting these things together shows that the proof of Theorem I is complete ∎

Proof of Theorem II. Here also we work with the squares $t_k^{(n)}$ of the singular values, and set

$$\Delta = \sigma\big(T(\bar f)\, T(f)\big) \cup \sigma\big(T(f)\, T(\bar f)\big)$$

so that $\Delta = \Lambda^2$. A number t is not equal to any $t_k^{(n)}$ $(k = 1, \dots, n)$ if and only if the operator

$$T_n(\bar f)\, T_n(f) - tI_n \quad (16)$$

(where I_n is the identity operator on the range of P_n) is invertible. In fact

$$\min_k |t_k^{(n)} - t| = \left\|\big(T_n(\bar f)\, T_n(f) - tI_n\big)^{-1}\right\|^{-1}.$$

A sequence of operators $\{A_n\}$ is called "uniformly invertible" if the operators are invertible for sufficiently large n and the norms $\|A_n^{-1}\|$ are bounded as $n \to \infty$. By the above remarks we see that the properties (i) and (ii) required of Λ can be rephrased in this terminology as follows:

(i′) If $t \in \Delta$, then no subsequence of (16) is uniformly invertible.
(ii′) If $t \notin \Delta$, then the sequence (16) is uniformly invertible.

Lemma II.1: *For any $f \in L_\infty$ the set Δ has property* (i′).

Proof: Write $A_n = T_n(\bar{f})\, \tilde{T}_n(f) - tI_n$, $A = T(\bar{f})\, T(f) - tI$. If for any sequence of n's tending to infinity we had $\|A_n^{-1}\| \leqq \mu$, then we would have $\|A_n \tilde{P}_n x\| \geqq \mu^{-1} \|P_n x\|$ for all $x \in l_2$ and so also $\|Ax\| \geqq \mu \|x\|$. Since A is selfadjoint, this implies its invertibility. Similarly, since the $t_k^{(n)}$ are also the eigenvalues of $T_n(f)\, T_n(\bar{f})$, we find that $T(f)\, T(\bar{f}) - tI$ is invertible also, contradicting the assumption $t \in \Delta$ ∎

Suppose A_n is a (noncommutative) polynomial in the Toeplitz matrices $T_n(f_i)$ $(i = 1, \ldots, r)$,

$$A_n = p\big(T_n(f_1), \ldots, T_n(f_r)\big). \tag{17}$$

One says that the "projection method" or "finite section method" applies to this sequence if the A_n are uniformly invertible. (The reason for the terminology is that then the inverse of the strong limit of A_n is equal to the strong limit of A_n^{-1}.) A necessary condition for this is that the operators

$$A = p\big(T(f_1), \ldots, T(f_r)\big), \qquad \tilde{A} = p\big(T(\bar{f}_1), \ldots, T(\bar{f}_r)\big)$$

are both invertible. The argument for this is very similar to the proof of Lemma II.1 which is equivalent to this assertion in case A_n is given by (16). (Note that the complex conjugates of the matrix entries of $\tilde{A}$ in this case are equal to the matrix entries of $T(f)\, T(\bar{f}) - tI$ so the operators are simultaneously invertible or not.)

Property (ii′) is of course just the assertion that the finite section method applies to (16) when $t \notin \Delta$. Our proof of this under the stated hypotheses on f relies on a theorem of Silbermann [12] (or [2, Th. 3.16]) which gives a necessary and sufficient condition for the uniform invertibility of sequences $\{A_n\}$ including all those given by (17). Here is the result.

For a sequence of operators A_n acting in the range of P_n define $\tilde{A}_n = Q_n A_n Q_n$, where Q_n is as before. Define $\mathcal{A}$ to be the set of all sequences $\{A_n\}$ for which there are operators A and $\tilde{A}$ on l_2 such that

$$A_n P_n \to A\,, \qquad A_n^* P_n \to A^*, \qquad \tilde{A}_n P_n \to \tilde{A}\,, \qquad \tilde{A}_n^* P_n \to \tilde{A}^*$$

strongly. This is a Banach algebra under the norm $\|\{A_n\}\|_{\mathcal{A}} = \sup_n \|A_n\|$ and it contains all sequences $\{T_n(f)\}$ with $f \in L_\infty$. Inside $\mathcal{A}$ there is the closed ideal $\mathcal{J}$ of all sequences $P_n K P_n + Q_n L Q_n + C_n$ with K and L compact operators on l_2 and $\|C_n\| \to 0$.

Theorem (Silbermann): *A sequence $\{A_n\} \in \mathcal{A}$ is uniformly invertible if and only if the operators A and $\tilde{A}$ are invertible and the image of $\{A_n\}$ under the quotient mapping $\mathcal{A} \to \mathcal{A}/\mathcal{J}$ is invertible.*

In our case of the sequence (16) the operators A and $\tilde{A}$ are invertible. This is precisely the assumption $t \notin \Delta$. What we must show then can be restated $\sigma\big(\{T_n(\bar{f})\, T_n(f)\}\big) \subset \Delta$ where "σ" denotes the spectrum of the image of the sequence $\{T_n(\bar{f})\, T_n(f)\}$ in $\mathcal{A}/\mathcal{J}$.

We consider first the (easier) case where f satisfies the second condition, $f = gh$ where g is real-valued and $h \in QC$. Since the Hankel operators $H(h)$ and $H(\tilde{h})$ are both compact, it follows from identity (6) that for any $\varphi \in L_\infty$

$$\{T_n(\varphi h) - T_n(\varphi)\, T_n(h)\} \in \mathcal{J}\,, \qquad \{T_n(\varphi h) - T_n(h)\, T_n(\varphi)\} \in \mathcal{J} \tag{18}$$

and it follows easily from this that

$$\{T_n(\bar{f})\, T_n(f) - T_n(g)^2\, T_n(|h|^2)\} \in \mathcal{J}. \tag{19}$$

Now $h \in QC$ implies $|h| \in QC$. (Being a closed subalgebra of L_∞ closed under complex conjugation QC is a C^*-algebra and so by the Gelfand-Naimark theorem [4, Th. 4.29] $F(h) \in QC$ for any continuous function F.) It follows therefore from (18) and (19) that $\{T_n(\bar{f})\, T_n(f) - T_n(g\,|h|)^2\} \in \mathcal{J}$ and so $\sigma(\{T_n(\bar{f})\, T_n(f)\}) = \sigma(\{T_n(g\,|h|)^2\})$.

Now in analogy with what we have just done, using identity (10) rather than (6), we find that $\sigma_e(T(\bar{f})\, T(f)) = \sigma_e(T(g\,|h|)^2) = \sigma_e(T(g\,|h|))^2$ where "σ_e" denotes "essential spectrum", the spectrum of the image of the operator under the quotient map from the algebra of bounded operators to its quotient by the ideal of compact operators. Since $g\,|h|$ is real-valued, we have $\sigma(T(g\,|h|)) = [\text{ess inf}\, g\,|h|, \text{ess sup}\, g\,|h|]$ by a theorem of Hartman and Wintner (see [4, Th. 7.20] or [2, Sec. 2.12]). This is also equal to $\sigma_e(T(g\,|h|))$ since for selfadjoint operators the essential spectrum is obtained from the spectrum by removing the isolated eigenvalues of finite multiplicity, and the spectrum has no such points in this case. Thus, writing $J = [\text{ess inf}\, g\,|h|, \text{ess sup}\, g\,|h|]$, we find that

$$\Delta \supset \sigma(T(\bar{f})\, T(f)) \supset J^2. \tag{20}$$

But the spectra of all $T_n(g\,|h|)$ lie in the interval J, so the spectra of all $T_n(g\,|h|)^2$ lie in J^2 and so our assumption $t \notin \Delta$ and (20) imply that the operators $T_n(g\,|h|)^2 - tI_n$ are uniformly invertible. (We use here, of course, the selfadjointness of $T_n(g\,|h|)$.) And this in turn implies $t \notin \sigma(\{T_n(g\,|h|)^2\}) = \sigma(\{T_n(\bar{f})\, T_n(f)\})$ as desired.

The proof of sufficiency of the first condition on f relies on a theorem of Gohberg and Krupnik [7] which determines the essential spectrum of any operator from the algebra generated by all $T(f)$ with $f \in PC$. Given such an f one defines a function f^* on the product of the circle with $[0, 1]$ by $f^*(\theta, \mu) = (1 - \mu)\, f(\theta-) + \mu f(\theta+)$. The theorem of Gohberg and Krupnik is that for any $f_1, \ldots, f_r \in PC$ and any (noncommutative) polynomial p one has

$$\sigma_e(p(T(f_1), \ldots, T(f_r))) = \text{range}\, p(f_1^*, \ldots, f_r^*).$$

To apply this in our situation we shall use a lemma on the representation of so-called "locally sectorial" functions. Given a function f we set

$$m_0 = \inf_\theta \lim_{\delta \to 0} \text{dist}\,(0, \text{co}\,[f(\theta - \delta, \theta + \delta)]).$$

Here $f(\theta - \delta, \theta + \delta)$ denotes the range of the restriction of f to the interval $(\theta - \delta, \theta + \delta)$. Clearly $m_0 \geqq m$.

Lemma II.2: *Assume $m_0 > 0$. Then for any $\varepsilon > 0$ we can write $f = gh$ where g is continuous and satisfies $|g| \geqq 1$ everywhere and h satisfies* $\text{Re}\, h \geqq m_0 - \varepsilon$ *everywhere.*

Proof: For any locally sectorial function f there exists a continuous function φ of absolute value 1 such that $\text{Re}\, \varphi f > 0$ everywhere [5]. Since replacing f by φf does not change the value of m_0 we may assume to begin with that $\text{Re}\, f > 0$. We shall then define $\log f$ as the principal value of the logarithm. It follows from our assumption, and the compactness of the circle, that we can find a finite open covering $\{U_i\}$ of the circle, and for each i a constant α_i of absolute value 1, such that $\text{Re}\, f/\alpha_i = m_0 - \varepsilon$ on U_i. Let E be the image of $\{z : \text{Re}\, z \geqq m_0 - \varepsilon\}$ under the (principal value) logarithm function. Then we have $\log f - \log \alpha_i \in E$ on U_i for some determinations of $\log \alpha_i$. Let $\{\psi_i\}$ be a partition of unity, subordinate to the covering $\{U_i\}$, consisting of nonnegative continuous functions. Since E is convex (a fact which is easily checked), we have $\log f - \sum (\log \alpha_i)\, \psi_i \in E$ everywhere. The desired functions g and h are given by $g = \exp\{\sum (\log \alpha_i)\, \psi_i\}$, $h = f/g$ ∎

To complete the proof of Theorem II in this case we shall show that $f \in PC$ implies

$$\sigma(\{T_n(\bar f)\, T_n(f)\}) \subset \sigma_e(T(\bar f)\, T(f)), \tag{22}$$

which will give the result. (This is actually a special case of Theorem 7.33(d) in the forthcoming book [3] of BÖTTCHER and SILBERMANN. We present our alternative proof here since it is fairly easy, given the ideas already introduced.) We may assume that f is, say, right continuous. It follows from the result of Gohberg and Krupnik that the set on the right in (22) is precisely the interval $[m_0^2, M^2]$ where m_0 is as in the statement of the lemma and $M = \operatorname{ess\,sup} |f|$ as before. The reason is that, as can easily be shown, the range of f^* is compact and connected. Of course, if $t \notin [0, M^2]$, then the operators (16) are trivially uniformly invertible, and so it suffices to show that $0 \leqq t < m_0^2$ implies

$$t \notin \sigma(\{T_n(\bar f)\, T_n(f)\}). \tag{23}$$

Let g and h be as in the statement of the lemma. Since g is continuous, we have

$$\{T_n(\bar f)\, T_n(f) - T_n(|g|^2)\, T_n(\bar h)\, T_n(h)\} \in \mathcal{J}.$$

Since $\operatorname{Re} h \geqq m_0 - \varepsilon$ the operators $T_n(h)$ and $T_n(\bar h)$ are invertible for all n and satisfy

$$\|T_n(h)^{-1}\| \leqq (m_0 - \varepsilon)^{-1}, \qquad \|T_n(\bar h)^{-1}\| \leqq (m_0 - \varepsilon)^{-1}.$$

Moreover it follows from (6) with f, g replaced by $|g|^2, |g|^{-2}$ that the image of $\{T_n(|g|^2)\}$ in $\mathcal{A}/\mathcal{J}$ is invertible and the norm of the inverse is at most 1. It follows that the image of $\{T_n(\bar f)\, T_n(f)\}$ is invertible and the norm of the inverse is at most $(m_0 - \varepsilon)^{-2}$. Letting $\varepsilon \to 0$ we see that the norm of this inverse is at most m_0^{-2}. It follows from this (since the spectrum of an inverse is the inverse of the spectrum) that $t \in \sigma(\{T_n(\bar f)\, T_n(f)\})$ implies $|t| \geqq m_0^2$ and so $0 \leqq t < m_0^2$ implies (23) as desired ∎

Remarks and conjectures. 1. The function G in Theorem I is given in terms of the function F in relation (2) by $G(\lambda) = F(\lambda^{1/2})$. For G to belong to C^3 it is not enough that F belongs to C^3 but we also must have $F'(0) = F''(0) = F'''(0) = 0$. We conjecture that $F \in C^3$, $F'(0) = 0$ is enough to imply the conclusion of Theorem I and that $F'(0) = 0$ is necessary, at least if all we assume about f is that it belongs to K. However, we conjecture that even this is unnecessary if we assume that f is sufficiently nice. The question is interesting because consideration of even the simplest quantity $\sum s_k^{(n)}$ corresponds to the case $F(\lambda) = \lambda$. See also the next remark.

2. Suppose f is real-valued. Then $T(\bar f)\, T(f) = T(f)\, T(\bar f) = |T(f)|^2$ and the right side of (4) can be written

$$2\operatorname{tr}\left[F_1(T(f)) - T(F_1(f))\right] \tag{24}$$

where $F_1(\lambda) = F(|\lambda|) = G(\lambda^2)$. Traces such as these have explicit integral representations if F_1 is smooth enough. (The Wiener-Hopf analogue is in [14].) Unfortunately $F_1(\lambda) = |\lambda|$ does not meet the smoothness criterion but we conjecture that if f is sufficiently well-behaved then the operator in (24) is trace class and the formula alluded to holds. It would be very interesting to find an analogue of the formula for the traces that appear in (4) in the nonselfadjoint case, even under severe conditions on f and G.

3. In the two cases in which Theorem II was proved the set Δ was shown to contain the essential spectrum of $T(\bar f)\, T(f)$, which was an interval. The set Δ, though, can be larger than this set (although not by more than a discrete subset of its complement).

For assume that f is continuous and has constant absolute value, say 1. Then the essential spectrum in question consists of just the point 1. We claim, however, that Δ is infinite unless $f(\theta)$ is a rational function of $e^{i\theta}$. For we have in this case $T(f)\, T(\bar{f}) = I - H(f)\, H(f)^*$ and so $\sigma\big(T(f)\, T(\bar{f})\big)$ is infinite unless $\sigma\big(H(f)\, H(f)^*\big)$ is finite. Since $H(f)$ is compact, this can occur only if it has finite rank. By a theorem of Kronecker [10, Ch. 7, Prob. 27] a necessary and sufficient condition for this is that $\sum_{k=1}^{\infty} \hat{f}_k\, e^{ik\theta}$ is a rational function of $e^{i\theta}$. Similarly $\sigma\big(T(\bar{f})\, T(f)\big)$ is finite (if and) only if $\sum_{k=-\infty}^{-1} \hat{f}_k\, e^{ik\theta}$ is a rational function of $e^{i\theta}$ and our claim is established.

4. Some assumption beyond $f \in L_\infty$ is necessary for the conclusion of Theorem II to hold. In fact examples were found by Treil [13] of bounded functions f for which $T(f)$ and $T(\bar{f})$ are both invertible but $\{T_n(f)\}$ is not uniformly invertible. For such an f we have $0 \notin \Delta$ but property (ii′) is violated for $t = 0$.

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