Zeitschrift für Analysis und ihre Anwendungen
Bd. 8 (3) 1989, S. 237 – 260

On Spectral Properties of Elliptic Pseudo-Differential Operators Far from Self-Adjoint Ones

M. S. AGRANOVICH and A. S. MARKUS

Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Wir erhalten unter gewissen Voraussetzungen grobe Asymptotik für die Eigenwerte elliptischer Operatorèn; eine der Voraussetzungen besteht darin, daß die Werte des Hauptsymbols nicht die ganze komplexe Ebene ausfüllen. Wir betrachten auch eine Reihe von Beispielen nichtselbstadjungierter elliptischer Operatoren; insbesondere solche mit ungewöhnlichen Formeln für die Asymptotik der Eigenwerte; mit regulärer Asymptotik der Eigenwerte, aber ohne Vollständigkeit der verallgemeinerten Eigenfunktionen; mit vollständigem System von Eigenfunktionen, das keine Basis bildet.

Выводится грубая асимптотика собственных значений для эллиптических операторов при некоторых предположениях; одно из них состоит в том, что значения главного символа не заполняют всей комплексной плоскости. Рассмотрен ряд примеров несамосопряженных эллиптических операторов, в частности: с необычными формулами для асимптотики собственных значений; с правильной асимптотикой собственных значений, но без полноты корневых функций; с полной системой собственных функций, не являющейся базисом.

We establish a rough asymptotics for eigenvalues of elliptic operators under some assumptions; one of them is that the values of the principal symbol do not cover the whole complex plane. We consider also a collection of some examples of non-self-adjoint elliptic operators: in particular, with unusual formulas for asymptotics of eigenvalues; with a regular asymptotics of eigenvalues but without the completeness of root functions; with a complete system of eigenfunctions which is not a basis.

0. Introduction

Let M be an *n*-dimensional closed C^{∞} -manifold, provided with a positive density dx , and let A be a classical (i.e. polyhomogeneous) elliptic pseudo-differential operator of order $t > 0$ on M with the principal symbol $a_0(x, \xi)$. At first we suppose A to be a scalar operator. Denote by $H_s(M)$ ($s \in \mathbb{R}$) the Sobolev space of order s on M; $H_0(M)$ $= L^2(M)$. We may regard A as a closed operator in $H_0(M)$ with the dense domain $H_t(M)$. If its spectrum $\sigma(A)$ does not cover the whole plane, A has the compact resolvent $R_A(\lambda) = (A - \lambda I)^{-1}$ and $\sigma(A)$ consists of eigenvalues of finite multiplicity with the only possible limit point at infinity.

By spectral properties of A we mean first of all asymptotic properties of the counting function of modules of its eigenvalues and geometric properties of the system of its root functions, i.e. generalized eigenfunctions (to be complete, to form the basis etc.) in $H_s(M)$. If $a_0(x, \xi) > 0$ on non-zero cotangent vectors, then A is near the selfadjoint pseudo-differential operator $A_0 = (A + A^*)/2$ in the following sense: the order of $A - A_0$ is not greater than $t - 1$. (Here and below we denote by A^* the pseudo-differential operator formally adjoint to A , with respect to the natural scalar product $(u, v) = \int u(x) v(x) dx$ on M, as well as the adjoint to A as an operator in

238 M. S. **AGRANOVICH** and A. S. **MARKS**

 $L^2(M)$.) The case just indicated has been-well studied (see e.g. [2, 18] and references there). In this case, in particular, the eigenvalues $\lambda_i(A)$ of A are known to be contained $L^2(M)$.) The case just indicated has been well studied (see e.g. [2, 18] and references
there). In this case, in particular, the eigenvalues $\lambda_j(A)$ of A are known to be contained
in some "parabolic" neighborhood of the *A} M.* S. AGRANOVICH and A. S. MARKUS
A} (h). The case just indicated has been with
the regular asymptotics
some "parabolic" neighborhood of the
 λ } has the regular asymptotics
 $N(\lambda) = d_0 \lambda^{n/l} + O(\lambda^{(n-1)/l})$ $(\lambda \rightarrow \infty$ 238 I
 $2^2(M)$.) I

here). In

n some
 $\leq \lambda$ } has *M. S. AGRANOVICH and A. S. MARKUS*

² ^{*2*} *Agrical Example 1 <i>A. S. MARKUS*

² *Athis case, in particular, the eigenvalues* $\lambda_j(A)$ of *A* are know

² *Parabolic*² *neighborhood* of the half-axis **R**₊ and *f* and references
 \circ be contained

card $\{j : |\lambda_j(A)|$
 $\downarrow d x d \xi.$ (0.1)
 \vdots $L^2(M)$.) The case just indicated has been well studied (see e.g. [2, 18] and refer
there). In this case, in particular, the eigenvalues $\lambda_i(A)$ of A are known to be con
in some "parabolic" reighborhood of the half-axis

$$
\leq \lambda \} \text{ has the regular asymptotics}
$$

\n
$$
N(\lambda) = d_0 \lambda^{n/l} + O(\lambda^{(n-1)/l}) \quad (\lambda \to \infty) \quad \text{with} \quad d_0 = \frac{1}{(2\pi)^n} \int_{a_0(x,\xi)} dx \, d\xi. \tag{0.1}
$$

\nMoreover, in this case one can construct a complete minimal system of root functions

of *A*, and this system is a good "basis with parentheses" if the order of $A - A_0$ is smallenough.

The case when $a_0(x, \xi)$ has a non-constant argument has been studied far less (below we list the corresponding papers known to us). It is just the case to be studied in

In Section 1 we assume that the values of the principal symbol do not cover the whole plane: $|\arg a_0(x, \xi)| \leq \theta$ where $\theta < \pi$. Our aim is to obtain some lower and the present paper.
In Section 1 we assume that the values of the principal symbol do not cover the whole plane: $|\arg a_0(x,\xi)| \leq \theta$ where $\theta < \pi$. Our aim is to obtain some lower and upper bounds for lower and upper limits $\leq \lambda$) has the regular asymptotics
 $N(\lambda) = d_0 \lambda^{n/l} + O(\lambda^{(n-1)/l})$ $(\lambda \to \infty)$

Moreover, in this case one can construct a com

of A, and this system is a good "basis with \bf{r}

small enough.

The case when $a_0(x, \xi)$ h The case when $a_0(x, \xi)$ has a non-constant argument has been
 v we list the corresponding papers known to us). It is just the

e present paper.

In Section 1 we assume that the values of the principal sym

iole plane:

Moreover, in this case one can construct a complete minimal system of root functions
of A, and this system is a good "basis with parentheses" if the order of
$$
A - A_0
$$
 is
small enough.
The case when $a_0(x, \xi)$ has a non-constant argument has been studied far less (be-
low we list the corresponding papers known to us). It is just the case to be studied in
the present paper.
In Section 1 we assume that the values of the principal symbol do not cover the
whole plane: $|\arg a_0(x, \xi)| \leq \theta$ where $\theta < \pi$. Our aim is to obtain some lower and
upper bounds for lower and upper limits l_- and l_+ of the function $\lambda^{-n/l}N(\lambda)$ as $\lambda \to \infty$.
Let us introduce two quantities
 $d = \frac{1}{(2\pi)^n n} \int_{M} dx \int_{|\xi|=1} [a_0(x, \xi)]^{-n/l} dS_{\xi}$,
 $d = \frac{1}{(2\pi)^n n} \int_{M} dx \int_{|\xi|=1} [a_0(x, \xi)]^{-n/l} dS_{\xi}$.
(Here they are put down roughly, without using local coordinates; the exact expres-
sions are presented below in (1.30)–(1.33); If $a_0(x, \xi) > 0$, we have $d = \Delta = d_0$.
The main results of Section 1 are as follows:
 $l_- > 0$ if $d \neq 0$; $l_- \leq \Delta$; $|d| \leq l_+ \leq \epsilon \Delta$.
When $d \neq 0$, we obtain from (0.3) the rough asymptotics
 $N(\lambda) \times \lambda^{n/l}$, i.e. $C_1 \leq \lambda^{-n/l} N(\lambda) \leq C_2$ $(\lambda \geq C_3)$ (0.4)
with positive constants C_1, C_2, C_3 . We do not know if the case $l_- < l_+$ is possible (an
interesting question, in our opinion). If $l_- = l_+ = l$, we obtain from (0.3) that

(Here they are put down roughly, without using local coordinates; the exact expressions are presented below in (1.30)–(1.33). If $a_0(x, \xi) > 0$, we have $d = \Delta = d_0$. *•* Figure they are put down roughly, without using local coord

ons are presented below in $(1.30) - (1.33)$; If $a_0(x, \xi) >$

the main results of Section 1 are as follows:
 $l > 0$ if $d + 0$; $l \leq d$; $|d| \leq l_+ \leq eJ$.
 l *dian dian dia*

$$
l > 0 \quad \text{if} \quad d \neq 0; \qquad l \leq \Delta; \qquad |d| \leq l_{+} \leq e\Delta. \tag{0.3}
$$

$$
l > 0 \quad \text{if} \quad d + 0; \qquad l \leq \Delta; \qquad |d| \leq l_+ \leq e\Delta. \tag{0.3}
$$
\n
$$
\neq 0, \text{ we obtain from (0.3) the rough asymptotics}
$$
\n
$$
N(\lambda) \times \lambda^{n/l}, \qquad \text{i.e.} \quad C_1 \leq \lambda^{-n/l} N(\lambda) \leq C_2 \qquad (\lambda \geq C_3)
$$
\n
$$
\dots \qquad (0.4)
$$

with positive constants C_1 , C_2 , C_3 . We do not know if the case $l_- < l_+$ is possible (an $N(\lambda) \times \lambda^{n/l}$, i.e. $C_1 \leq \lambda^{-n/l} N(\lambda) \leq C_2$ $(\lambda \geq C_3)$
with positive constants C_1 , C_2 , C_3 . We do not know if the case $l < l_+$ is possi-
interesting question, in our opinion). If $l_- = l_+ = l$, we obtain from (0.3 (0.4)
 $\begin{array}{c}\n 0.5\n \end{array}$
 $\begin{array}{c}\n 0.5\n \end{array}$

$$
|d| \le l \le \Delta. \tag{0.5}
$$

The inequalities (0.3) with some corollaries are proved in Subsection 1.3. In Subsections 1.1 and 1.2 some preliminary- material is contained. In Subsection 1.1 we establish a certain Tauberian inequality. Namely we prove that if $N(\lambda)$ is a nondecreasing function on \mathbf{R}_{+} and if its Stieltjes transform of order q, $\mathcal{B} = 0$, we obtain from (0.3) the rough asymptotics
 $N(\lambda) \times \lambda^{n/t}$, i.e. $C_1 \leq \lambda^{-n/t} N(\lambda) \leq C_2$ ($\lambda \geq C_3$) (0.4)

tive constants C_1 , C_2 , C_3 . We do not know if the case $l < l$, is possible (an

g questio

$$
S_q(\mu) = \int_0^\infty (\lambda + \mu)^{-q} dN(\lambda), \tag{0.6}
$$

has the rough asymptotics $S_q(\mu) \times \mu^{d-q}$ ($0 < \delta < q$), then $N(\lambda)$ has the rough asymptotics $N(\lambda) \times \lambda^s$. The last statement is analogous to the well-known Tauberian Hardy-Littlewood theorem (see Subsection 1.1). In Subsection 1.2 we establish inequalities analogous to $l_- \leq \Delta$, $l_+ \leq \epsilon \Delta$ for compact operators in the abstract Hilbert space. Note that-the constant *e* in the second inequality turns out to be exact. In Subsection

1.4' the main results of Subsection 13 are extended to matrix elliptic pseudo-differential operators with the spectrum of the principal symbol lying in two closed sectors A_1 and A_2 which have the unique common point 0. We investigate separately the behaviour of counting functions $N_1(\lambda)$ and $N_2(\lambda)$ for modules of eigenvalues of *A* in slightly broader sectors $A_1(\varepsilon)$ and $A_2(\varepsilon)$. In Subsection 1.5 we consider briefly operators corresponding to elliptic boundary problems with homogeneous boundary 1.4 the main results of Sub
tial operators with the spe
 Λ_1 and Λ_2 which have the
behaviour of counting fun
slightly broader sectors Λ_1
tors corresponding to ell
conditions and outline the
- The quantity d is o conditions and outline the proofs of assertions analogous to the main results.'

The quantity *d* is obviously non-zero if θ *n* $\lt \pi$ *t*/2 (when *n* > 1 and *M* is connected, it is true if $\theta n \leq \pi t/2$). In this case one can see from (0.4) that there are "sufficiently many" eigenvalues. Another well-known indication of such situation is the completeness of the system of root functions whiëh has been established exactly under the condition $\theta n < \pi t/2$ (cf. [1]). Such condition is only sufficient both for the completeness of root functions and for the presence of the rough asymptotics for $N(\lambda)$. Indeed these two properties of *A* are preserved when we pass to A^k (with positive integer *k*), while • the sector free from values of the principal symbol can disappear. On'the other hand, if the values of the principal symbol cover the whole plane, we cannot point out any sufficient conditions for the completeness or for the presence of the rough asymptotics' for $N(\lambda)$ (and even conditions, under which the spectrum of A is non-empty or discrete).

We examine these problems in Section 2 on some examples. First of all we give very simple examples of elliptic operators on the torus either with the empty spectrum or with the spectrum filling the whole plane (each point is an eigenvalue). Then we discuss in detail (in Subsections 2.2 and 2.3) the example of the elliptic differential operator of first order on the circle. As it has turned out, this example has been considered by *SEELEY* before us. In his note [26] he indicates the conditions under which the spectrum of the operator is empty or covers the whole plane. He also points out that if neither of the two degenerate cases takes place, then $N(\lambda)$ has the regular asymptotics with somewhat unusually defined coefficient *d⁰ .* We recall these calcilations, and in addition we.obtain' in Subsection 2.2 the exact condition for the completeness of eigenfunctions of this operator. **(In** the non-degenerate case all its root functions are eigenfunctions.) This condition deals only with the principal symbol and is non-local. The counting function $N(\lambda)$ has the regular asymptotics in the non-degenerate case even if these is no completeness. Assuming the completeness, we deduce in Subsection 2.3 the exact condition, under which the system of eigenfunctions is a basis, and obtain an example of an elliptic operator whose system of eigenfunctions is complete but is not a basis. We construct also such examples of operators on the torus using the separation of variables. In Subsection 2.4 we consider another example of an elliptic differential operator on the torus, admitting the separation of variables, in order to demonstrate the possibility in (0.5) of all three cases *If* the example of the elliptic differential sturned out, this example has been con-
J he indicates the conditions under which vers the whole plane. He also points out takes place, then $N(\lambda)$ has the regulared coefficie

 $|d|=l<\varDelta$, $|d|< l=\varDelta$,

$$
|d|
$$

In Subsection 2.5 we establish the existence of the regular asymptotics of $N(\lambda)$ for elliptic differential operators on the unit circle whose coefficients admit continuous extentions in the unit disk holomorphic in its interior.

Now we list the results known to us and more or less close to the subject of the present article. In the paper[6] of BOJMATOV some abstract test for validity of (0.4) has been formulated. For differential operators this test yields (0.4) if $\partial n \leq \pi t/4$. Koževnikov [14] has considered a matrix elliptic pseudo-differential operator'A with spectrum of the principal symbol lying on several half-lines. He has obtained asymptotics of eigenvalues of A close to one of the halflines. An extension of this result to the case when in addition to the half-line under considera. tion there is a sector covered by eigenvalues $\lambda_i(x, \xi)$ $((x, \xi) \in T^*M \setminus 0)$ of the principal symbol,

240 M. S. AGRANOVICH and A. S. MARKUS

has been obtained by AGRANOVICH [3]. ROSENBLOOM [21, 9] has obtained an asymptotic formula for the modules of eigcnvalues for normal elliptic operators and elliptic operators very. close to normal, in a sector, whose bounds may even contain the values of the principal symbol.

The main results of this paper have been reported at the 10th Session of the Petrowskii Seminar on differential equations and mathematical problems of physics and Moscow Mathematical Society in January of 1987 [4]. M. S. AGRANOVICH and A. S. MARKUS

has been obtained by AGRANOVICH [3]. ROSENBLOOM [21, 9] has obtained an asymptotic for-

mula for the modules of eigenvalues for normal elliptic operators and elliptic operators very.

c

We wish to express our sincere gratitude to V. I. Matsaev and F. L. Friedlander

1. Rough asymptotics for counting functions

1.1. An analogue of the Tauberian Hardy-Littlewood theorem. Let $N(\lambda)$ be a nonnegative non-decreasing function on the non-negative half-axis $\mathbf{\bar{R}}_{+}$, with $N(0) = 0$. If $q>0$ and *f2-* **o** *f2-* **o** *f2 f2 f2 <i>f2 f2 <i>f2 f2 f2 <i>f2 f2 f2 f2 <i>f2 f2 f2 <i>f2 f2 f2 f2 <i>f2 f2 f2 f2 f2 f2 <i>f2* *****f2 f2 f2 f2*

$$
\int_{1}^{\infty} \lambda^{-q} dN(\lambda) < \infty, \tag{1.1}
$$

then the *Stielijes transform of* order *q* of *N(2)* is defined by (0.6). We shall only need the case when $N(\lambda)$ is the *counting function* for some non-decreasing sequence $\{a_i\}_i^{\infty}$ of positive numbers σ_j with $\sigma_j \to \infty$ as $j \to \infty$: $N(\lambda)$ is the number of the σ_j not exceeding λ . In this case the condition (1.1) means that $\sum \sigma_j^{-q} < \infty$ and we have $\int_{1}^{\infty} \lambda^{-q} dN(\lambda) < \infty$, φ

then the *Stieltjes transform* of order *q* of *N(2)* is def

the case when *N(2)* is the *counting function* for som

of positive numbers *σ_i* with *σ_i* $\rightarrow \infty$ as $j \rightarrow \infty$: *N*

ex

$$
S_q(\mu) = \sum_{j=1}^{\infty} (\sigma_j + \mu)^{-q}.
$$

\n(a) and $N_2(\lambda)$ be positive functions for $\lambda \ge \lambda_0$. We shall write
\n
$$
N_1(\lambda) \sim N_2(\lambda) \quad \text{if} \quad \lim_{\lambda \to \infty} N_1(\lambda)/N_2(\lambda) = 1.
$$

Let $N_1(\lambda)$ and $N_2(\lambda)$ be positive functions for $\lambda \geq \lambda_0$. We shall write

$$
N_1(\lambda) \sim N_2(\lambda) \quad \text{if} \quad \lim_{\lambda \to \infty} N_1(\lambda)/N_2(\lambda) = 1
$$

\n A) and
$$
N_2(\lambda)
$$
 be positive functions for $\lambda \geq \lambda_0$. We shall write\n
$$
N_1(\lambda) \sim N_2(\lambda) \quad \text{if} \quad \lim_{\lambda \to \infty} N_1(\lambda)/N_2(\lambda) = 1
$$
\n and\n
$$
N_1(\lambda) \times N_2(\lambda) \quad \text{if} \quad \lim_{\lambda \to \infty} N_1(\lambda)/N_2(\lambda) > 0, \quad \lim_{\lambda \to \infty} N_1(\lambda)/N_2(\lambda) < \infty
$$
\n

(the *weak equivalence).* Since the classical works of Carleman, the following Tau-' berian Hardy-Littlewood theorem has been successfully applied in the study of of positive numbers σ_j with $\sigma_j \to \infty$ as $j \to \infty$: $N(\lambda)$ is the number exceeding λ . In this case the condition (1:1) means that $\sum \sigma_j^{-q} <$
 $S_q(\mu) = \sum_{j=1}^{\infty} (\sigma_j + \mu)^{-q}$.

Let $N_1(\lambda)$ and $N_2(\lambda)$ be positive f *Suppose (1.1) holds and* $S_q(\mu) \sim \rho \mu^{\delta-q}$ ($\mu \to \infty$) *for some of equivalence*). Since the classical works of Carleman, the following Taurinan Hardy-Littlewood theorem has been successfully applied in the study of ect Fince the classical works of
 11 (theorem has been success

11 $S_q(\mu) \sim q\mu^{5-q}$ ($\mu \to \infty$)
 nand $S_q(\mu) \sim q\mu^{5-q}$ ($\mu \to \infty$)
 ne
 11 $\mu + t$)^{-*q*} dt = $\delta B(\delta, q - \delta)$. $\lim_{\lambda \to \infty} N_1(\lambda)/2$
Carleman,
fully appli
for some δ
ivivalence in (the *weak equivalence*). Since
berian Hardy-Littlewood the
spectral asymptotics (see e.g.
 $\mathcal{S}upppose$ (1.1) holds and S_i
 $\sim b_{\delta,q}\varrho \lambda^{\delta} (\lambda \to \infty)$, where
 $b_{\delta,q} = \delta \int_{0}^{\infty} t^{\delta-1}(1+t)$
We shall need an analogous
one:

 $\sim b_{\delta,q}$ *Q* λ^{δ} ($\lambda \rightarrow \infty$), where

$$
b_{\delta,q} = \delta \int\limits_{0}^{\infty} t^{\delta-1} (1+t)^{-q} dt = \delta B(\delta, q-\delta). \tag{1.3}
$$

We shall need an analogous result for the weak equivalence instead of the strong one:

Proof: Since

than Hardy-Luttewood theorem has been successfully append in the study of
\nectral asymptotics (see e.g. [29: Chapter V]):

\nSuppose (1.1) holds and
$$
S_q(\mu) \sim \rho \mu^{\delta-q} (\mu \to \infty)
$$
 for some $\delta \in (0, q)$. Then $N(\lambda)$

\n $b_{\delta,q} \rho \lambda^{\delta} (\lambda \to \infty)$, where

\n $b_{\delta,q} = \delta \int_{0}^{\infty} t^{\delta-1} (1 + t)^{-q} dt = \delta B(\delta, q - \delta)$.

\nWe shall need an analogous result for the weak equivalence instead of the strong

\ne:

\nTheorem 1.1: Let (1.1) hold and $S_q(\mu) \times \mu^{\delta-q} (\mu \to \infty)$ for some $\delta \in (0, q)$. Then

\n $\lambda \geq \lambda^{\delta} (\lambda \to \infty)$.

\nProof: Since

\n $\int_{0}^{t} (\lambda + \mu)^{-q} dN(\lambda) = (\lambda_0 + \mu)^{-q} N(\lambda_0) + q \int_{0}^{t} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda$, (1.4)

from (1.1) it follows that

Spectral Properties of Elliptic Pseudo-Diff. Op. 241
\nfrom (1.1) it follows that
\n
$$
\int_{0}^{\infty} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda < \infty.
$$
\nHence the limit of $N(\lambda_0) (\lambda_0 + \mu)^{-q}$ as $\lambda \to \infty$ exists, and from (1.5) we see that it is equal to 0. So (1.4) implies
\n
$$
S_q(\mu) = q \int_{0}^{\infty} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda.
$$
\nBy the assumption,
\n(1.6)

Hence the limit of $N(\lambda_0) (\lambda_0 + \mu)^{-q}$ as $\lambda \to \infty$ exists, and from (1.5) we see that it is equal to 0. So (1.4) implies
 $S(u) = a \int_{0}^{\infty} (\lambda + u)^{-q-1} N(\lambda) d\lambda$. (1.6)

$$
S_q(\mu) = q \int_{0}^{\infty} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda.
$$
 (1.6)

Spectral Properties of Elliptic Pseudo-Diff. Op. 241
\nfrom (1.1) it follows that
\n
$$
\int_{0}^{2\pi} (2 + \mu)^{-q-1} N(\lambda) d\lambda < \infty.
$$
\n(1.5)
\nHence the limit of $N(\lambda_0) (\lambda_0 + \mu)^{-q}$ as $\lambda \to \infty$ exists, and from (1.5) we see that it is
\nequal to 0. So (1.4) implies
\n
$$
S_q(\mu) = q \int_{0}^{\infty} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda.
$$
\nBy the assumption,
\n
$$
\lim_{\mu \to \infty} \mu^{\alpha-\delta} S_q(\mu) = \varrho_1 > 0,
$$
\n
$$
\lim_{\mu \to \infty} \mu^{\alpha-\delta} S_q(\mu) = \varrho_2 < \infty.
$$
\n(1.8)
\nObviously
\n
$$
\int_{0}^{\infty} \int_{0}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \ge \int_{\mu}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \ge N(\mu) \int_{\mu}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\mu)}{q(2\mu)^{q}}.
$$
\nFrom this relation, (1.6) and (1.8) we have
\n
$$
\lim_{\mu \to \infty} \mu^{-s} N(\mu) \le 2^q \varrho_2.
$$
\nNow we want to estimate $\lim_{\mu \to \infty} \mu^{-\delta} N(\mu)$. Evidently for $\gamma > 0$.
\n
$$
\int_{0}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \le N(\gamma\mu) \int_{0}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\gamma\mu)}{q\mu^q}.
$$

-

V

V

V **V**

V

V

 \mathbf{V}

ously

\n
$$
\int_{0}^{\frac{\infty}{\pi}} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \geq \int_{\mu}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \geq N(\mu) \int_{\mu}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\mu)}{q(2\mu)}
$$
\nthis relation (1.6) and (1.8) we have

nis relation, (1.6) and (1.8) we have
\n
$$
\lim_{\mu \to \infty} \mu^{-\delta} N(\mu) \leq 2^q \varrho_2.
$$
\n(1.9)

Diviously

\n
$$
\int_{0}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \geq \int_{\mu}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \geq N(\mu) \int_{\mu}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\mu)}{q(2\mu)^{q}}.
$$
\nfrom this relation, (1.6) and (1.8) we have

\n
$$
\lim_{\mu \to \infty} \mu^{-\delta} N(\mu) \leq 2^{q} Q_{2}.
$$
\nNow we want to estimate

\n
$$
\lim_{\nu \to \infty} \mu^{-\delta} N(\mu) \leq N(\mu).
$$
\nEvidently for $\gamma > 0$.

\n
$$
\int_{0}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \leq N(\gamma \mu) \int_{0}^{\gamma \mu} \frac{d\lambda}{(\lambda + \mu)^{q+1}} \leq N(\gamma \mu) \int_{0}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\gamma \mu)}{q \mu^{q}}.
$$
\n(1.10)

\nIn the other hand, by (1.9) we have for any $\varepsilon > 0$, if $\gamma \mu$ is sufficiently large,

On the other hand, by (1.9) we have for any $\varepsilon > 0$, if $\gamma \mu$ is sufficiently large,

$$
\lim_{\mu \to \infty} \mu^{-\delta} N(\mu) \leq 2^{q} Q_{2}.
$$
\nNow we want to estimate $\lim_{\mu \to \infty} \mu^{-\delta} N(\mu)$. Evidently for $\gamma > 0$.
\n
$$
\int_{0}^{\mu} \frac{N(\lambda) d\lambda}{(\lambda + \mu)^{q+1}} \leq N(\gamma \mu) \int_{0}^{\gamma} \frac{d\lambda}{(\lambda + \mu)^{q+1}} \leq N(\gamma \mu) \int_{0}^{\infty} \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\gamma \mu)}{q\mu^{q}}.
$$
\n(1.10)
\nOn the other hand, by (1.9) we have for any $\epsilon > 0$, if $\gamma \mu$ is sufficiently large,
\n
$$
\int_{\alpha}^{\alpha} (\lambda + \mu)^{-q-1} N(\lambda) d\lambda \leq (Q_{2} + \epsilon) 2^{q} \int_{\mu}^{\infty} (\lambda + \mu)^{-q-1} \lambda^{q} d\lambda
$$
\n
$$
\leq 2^{q} (Q_{2} + \epsilon) \int_{\mu}^{\infty} \lambda^{\delta - q - 1} d\lambda = 2^{q} (Q_{2} + \epsilon) (q - \delta)^{-1} (\gamma \mu)^{\delta - q}.
$$
\n(1.11)
\nFrom (1.6), (1.10), (1.11) it follows that
\n
$$
\mu^{q - \delta} S_{q}(\mu) - q 2^{q} (Q_{2} + \epsilon) (q - \delta)^{-1} \gamma^{\delta - q} \leq \mu^{-\delta} N(\gamma \mu).
$$
\nChoose μ_0 so large that $\mu^{q - \delta} S_{q}(\mu) - q 2^{q} (Q_{2} + \epsilon) (q - \delta)^{-1} \gamma^{\delta - q} \leq \mu^{-\delta} N(\gamma \mu).$
\n(1.12)
\nChoose μ_0 so large that $\mu^{q - \delta} S_{q}(\mu) > \rho_1 - \epsilon$ for $\mu > \mu_0$ (see (1.7)) and γ so large that
\nthe second term in the left-hand side of (1.12) is less than ϵ . Then $N(\gamma \mu) > (\rho_1 - 2\epsilon)\mu^{\delta}$
\

3), (1.10), (1.11) it follows that
\n
$$
\mu^{q-\delta}S_q(\mu) - q2^q(\varrho_2 + \varepsilon) (q-\delta)^{-1} \gamma^{\delta-q} \leq \mu^{-\delta}N(\gamma\mu).
$$
\n(1.12)

From (1.6), (1.10), (1.11) it follows that
 $\mu^{q-\delta}S_q(\mu) - q^{2q}(\varrho_2 + \varepsilon) (q-\delta)^{-1} \gamma^{\delta-q} \leq \mu^{-\delta} N(\gamma \mu)$. (1.12)

Choose μ_0 so large that $\mu^{q-\delta}S_q(\mu) > \varrho_1 - \varepsilon$ for $\mu > \mu_0$ (see (1.7)) and γ so large that

th Choose μ_0 so large that $\mu^{q-\delta}S_q(\mu) > \varrho_1 - \varepsilon$ for $\mu > \mu_0$ (see (1.7)) and γ so large that $(\mu > \mu_0)$, and therefore so large that μ^q ⁻
d term in the left-h
and therefore
 $\lim_{n \to \infty} \lambda^{-\delta} N(\lambda) \geq \gamma^{-\delta}$ $\mathcal{P}(\mathcal{P}(-\mathcal{$ $\begin{align} | \psi \rangle &> (\varrho_1 - 2 \varepsilon) \mu^3 \ \langle \psi \rangle &\; , \end{align}$ $\mu^{q-\delta}S_q(\mu) - q2^q(\varrho_2 + \varepsilon) (q-\delta)^{-1} \gamma^{\delta-q} \leq \mu^{-\delta}N(\gamma\mu).$

Choose μ_0 so large that $\mu^{q-\delta}S_q(\mu) > \varrho_1 - \varepsilon$ for $\mu > \mu_0$ (see (1.7)) and γ

the second term in the left-hand side of (1.12) is less than ε . T

$$
\underline{\lim_{\lambda \to \infty}} \lambda^{-\delta} N(\lambda) \ge \gamma^{-\delta}(\varrho_1 - 2\varepsilon).
$$
\n(1.13)
\n(1.13) we obtain the conclusion of the theorem.

V

. -

^V V -

V

242 M. S. AGRANOVICH and A. S. MARKUS

Remark 1.2: In the proof of (1.9) only (1.8) has been used, whereas in the proof of (1.13) we have used both (1.7) and (1.8) .

Remark 1.3: We do not try to obtain the best estimates for lim $\lambda^{-\delta}N(\lambda)$ and $\lim_{\lambda \to \infty} \lambda^{-\delta}N(\lambda)$ in terms of ρ_1 and ρ_2 .

Remark 1.4: In the proof of Theorem 1.1 we have shown that (1.7) and (1.9) imply $\lim \lambda^{-\delta} N(\lambda) > 0.$

We shall need also the following statement which is inverse to Theorem 1.1 (Abelian theorem). It is valid in a sharper form, and the requirement that $N(\lambda)$ should have a Remark 1.2: In the proof of (1)

have used both (1.7) and (1.8).
 Remark 1.3: We do not try to α
 Lemark 1.4: In the proof of
 Lemark 1.4: In the proof of
 Lemark 1.4: In the proof of
 Lemark 1.4: In the proo

Let
$$
(1.1)
$$
 hold. Then

finite variation on each finite segment instead of monotonicity-is sufficient *bô.qll* • A-+oo - *),- 6N(2),* (1.14) *^lim b65 Ui2N(2),* (1.15) S S -

where $b_{\delta,q}$ is defined in (1.3) .

The proof is elementary; see e.g. $[29: Chapter V]$.

1.2. Asymptotic estimates for eigenvalues of a compact operator by its singular values. In this subsection *K* is a compact operator in a Hilbert space. Let $\{X_n(K)\}\$ be
the sequence of its eigenvalues, counted according to their multiplicities $(i.e.,$
the dimensions of the corresponding root subspaces) and ar the sequence of its eigenvalues, counted according to their multiplicities (i.e. the dimensions of the corresponding root subspaces) and arranged so that $|\lambda_1(K)|$ where $b_{\delta,q}$ is defined in (1.3).

The proof is elementary; see e.g. [29: Chapter V]:

1.2. Asymptotic estimates for eigenvalues of a compact operator by its singular values

In this subsection K is a compact operator i $\mathcal{L} \geq |\lambda_2(K)| \geq \ldots$ If K has only a finite number of non-zero eigenvalues, we complete the sequence by zeros. The numbers $s_n(K) = \lambda_n((K^*K)^{1/2})$ are called the *singular values* of *K*. The eigenvalues and the singular values are connected by the well-known $\lim_{\mu\to\infty}\mu^{q-5}S_q(\mu) \leq b_{\delta,q}\lim_{\lambda\to\infty} \lambda^{-s}N(\lambda),$ (1.1

where $b_{\delta,q}$ is defined in (1.3).

The proof is elementary; see e.g. [29: Chapter V].

1.2. Asymptotic estimates for eigenvalues of a compact operator by its si r V]:

mpact opera

in a Hilbert

cording to

spaces) and
 $(n-2e^x)$ are come
 $\frac{1}{2}$

s are come

.

.

.

.

.

. totic estimates for eigenvalues of a compact operator by its singular values.

besection *K* is a compact operator in a Hilbert space. Let $\{\lambda_n(K)\}$ \approx be

cec of its eigenvalues, counted according to their multipliciti

$$
\prod_{j=1}^{n} |\lambda_j(K)| \leq \prod_{j=1}^{n} s_j(K) \qquad (n = 1, 2, \ldots).
$$
 (1.16)

They have many consequences; in particular,

$$
\sum_{j=1}^{n} |\lambda_j(K)|^p \leq \sum_{j=1}^{n} s_j^p(K) \qquad (n = 1, 2, \dots, p > 0).
$$
 (1.17)

Denote by $n(t)$ (respectively by $\nu(t)$) the counting function for $|\lambda_i^{-1}(K)|$ (respectively for $s_n^{-1}(K)$. As it is pointed out in [19], (1.16) can be rewritten in the form

$$
\prod_{j=1}^{n} |\lambda_j(K)| \leq \prod_{j=1}^{n} s_j(K) \qquad (n = 1, 2, ...).
$$
\n(1.16)

\nThey have many consequences; in particular,

\n
$$
\sum_{j=1}^{n} |\lambda_j(K)|^p \leq \sum_{j=1}^{n} s_j^p(K) \qquad (n = 1, 2, ..., p > 0).
$$
\n(1.17)

\nDenote by $n(t)$ (respectively by $v(t)$) the counting function for $|\lambda_j^{-1}(K)|$ (respectively for $s_n^{-1}(K)$). As it is pointed out in [19], (1.16) can be rewritten in the form

\n
$$
\int_{0}^{1} t^{-1} n(t) dt \leq \int_{0}^{\lambda} t^{-1} v(t) dt \qquad (\lambda > 0).
$$
\n(1.18)

\nHere we want to establish some connections between the asymptotic behaviour of

\n $n(t)$ and that of $v(t)$ following from (1.17) and (1.18).\nTheorem 1.5: For any $\delta > 0$,

\n
$$
\lim_{\lambda \to \infty} \lambda^{-\delta} n(\lambda) \leq \lim_{\lambda \to \infty} \lambda^{-\delta} v(\lambda).
$$
\n(1.19)

\nProof: If K has only a finite number of non-zero eigenvalues, then the left-hand

Here we want to establish some connections between the asymptotic behaviour of $n(t)$ and that of $\nu(t)$ following from (1.17) and (1.18).

$$
\underline{\lim_{\lambda \to \infty}} \lambda^{-\delta} n(\lambda) \leq \overline{\lim_{\lambda \to \infty}} \lambda^{-\delta} \nu(\lambda).
$$
 (1.19)

Emany consequences; in particular,
 $\sum_{j=1}^{n} |\lambda_j(K)|^p \leq \sum_{j=1}^{n} s_j^p(K)$ $(n = 1, 2, ..., p > 0)$. (1.17)
 $\sum_{j=1}^{n} |\lambda_j(K)|^p \leq \sum_{j=1}^{n} s_j^p(K)$ $(n = 1, 2, ..., p > 0)$. (1.17)

(1.18)

(1.18)

(2.1) As it is pointed out in [19], (Proof: If K has only a finite number of non-zero eigenvalues, then the left-hand side of (1.19) is equal to 0. Therefore we can assume that all the $\lambda_n(K)$ (and by (1.16) all the $s_n(K)$ are distinct from 0. Set $\mu_n = |\lambda_n^{-1}(K)|$ and $\sigma_n = s_n^{-1}(K)$. Obviously, *li*). As it is pointed out in [19], (1.16) can be rewr
 $\int_{0}^{1} t^{-1}n(t) dt \leq \int_{0}^{1} t^{-1}v(t) dt$ ($\lambda > 0$).

want to establish some connections between the

that of $v(t)$ following from (1.17) and (1.18).

em 1.5: For any (1.19)

alues, then the left-hand
 l the $\lambda_n(K)$ (and by (1.16)
 $\sigma_n = s_n^{-1}(K)$. Obviously,

(1.20)

$$
\overline{\lim}_{\lambda\to\infty}\lambda^{-\delta}\nu(\lambda)\geq \overline{\lim}_{n\to\infty}\sigma_n^{-\delta}n
$$

Spectral Properties of Elliptic Pseudo-Diff. Op.

and for each $\varepsilon > 0$ we have

$$
\lim_{\lambda \to \infty} \lambda^{-\delta} n(\lambda) \leq \lim_{n \to \infty} (\mu_n - \varepsilon)^{-\delta} n = \lim_{n \to \infty} \mu_n^{-\delta} n.
$$
\n(1.21)

Suppose that (1.19) is false. Then (1.20) and (1.21) imply $\lim_{n \to \infty} \mu_n^{-\delta} n > \lim_{n \to \infty} \sigma_n^{-\delta} n$, and therefore, with some $a > 0$, $\mu_n^{-\delta} - \sigma_n^{-\delta} > an^{-1}$ $(n \ge n_0)$. Hence $\sum_{n=-\infty}^{m} \mu_n^{-\delta} - \sum_{n=-\infty}^{m} \sigma_n^{-\delta}$ $> a \sum_{n=n_0}^{m} n^{-1} \to \infty$ as $m \to \infty$, which contradicts (1.17) with $p = \delta$.

Theorem 1.6: For any $\delta > 0$.

$$
\overline{\lim}_{\lambda \to \infty} \lambda^{-\delta} n(\lambda) \leq e \overline{\lim}_{\lambda \to \infty} \lambda^{-\delta} \nu(\lambda). \tag{1.22}
$$

Proof: It follows from (1.18) that for any $\gamma > 0$

$$
\int_{0}^{\lambda \gamma} t^{-1} \nu(t) dt \geq \int_{0}^{\lambda \gamma} t^{-1} n(t) dt \geq \int_{\lambda}^{\lambda \gamma} t^{-1} n(t) dt \geq n(\lambda) \ln \gamma.
$$
 (1.23)

Let d_2 be the upper limit in the right-hand side of (1.22) (we assume it to be finite) and ε be an arbitrary positive number. Then $v(\lambda) \leq (d_2 + \varepsilon) \lambda^{\delta}$ for $\lambda \geq \lambda_0$, and therefore we obtain from (1.23)

$$
n(\lambda) \leq (\ln \gamma)^{-1} \left(\int_0^{\lambda_0} t^{-1} \nu(t) dt + (d_2 + \varepsilon) \int_0^{\lambda_0} t^{\delta - 1} dt \right)
$$

= $(\ln \gamma)^{-1} (\text{Const} + (d_2 + \varepsilon) \delta^{-1} \lambda^{\delta} \gamma^{\delta}),$
 $\lambda^{-\delta} n(\lambda) \leq (\ln \gamma)^{-1} (\lambda^{-\delta} \text{Const} + (d_2 + \varepsilon) \delta^{-1} \gamma^{\delta}).$

So lim

$$
\lambda^{-\delta} n(\lambda) \leq (\ln \gamma)^{-1} \left(\lambda^{-\delta} \text{Const} + (d_2 + \varepsilon) \delta^{-1} \gamma^{\delta} \right).
$$

Setting here $\gamma = e^{i/\delta}$, we obtain the inequality

$$
\lambda^{-\delta} n(\lambda) \leq \delta(\lambda^{-\delta} \text{Const} + (d_2 + \varepsilon) \, \delta^{-1} \, e),
$$

from which (1.22) follows \blacksquare

Remark 1.7: The constant e in (1.22) is the best possible. Indeed, by Horn's theorem [11], for any integers l and m ($0 \le l < m$) there exists an operator $A_{l,m}$, acting in a Hilbert space of for any integers i and m ($0 \le i < m$) there exists an operator $A_{l,m}$, acting in a millet space of
finite dimension $m^* - l$, such that $\lambda_i(A_{l,m}) = (l!/m!)^{1/(m-l)}$ and $s_j(A_{l,m}) = (l + j)^{-1}$ $(j = 1, ..., m - l)$. Choose an increasing seq formula, we obtain

$$
\overline{\lim}_{t \to \infty} \frac{n(t)}{t} \geqq \lim_{q \to \infty} \left(\frac{m_{q-1}!}{m_q!} \right)^{\frac{1}{m_q - m_{q-1}}} = e \lim_{q \to \infty} \left(\frac{m_{q-1}}{m_q} \right)^{\frac{m_{q-1}+1/2}{m_q - m_{q-1}}} = e.
$$

 $t^{-1}n(t) \geqq e = e \lim_{t \to \infty} t^{-1} \nu(t)$

1.3. Theorems on rough asymptotics for modules of eigenvalues of elliptic operators. Let M be an *n*-dimensional C^{∞} -manifold provided with a positive C^{∞} -density dx. We consider a classical (i.e. polyhomogeneous) scalar pseudo-differential operator A of order $t > 0$. Let $a_0(x, \xi)$ be its principal symbol. It is a C^{∞} -function on $T^*M \setminus 0$,

_i 244 M. S. AORANOVICH and A. S. MARKUS

<u> </u>

-

/

244 M. S. AGRANOVICH and A. S. MARKUS

positively homogeneous of order *t* in ξ (see e.g. [27]). The values of $a_0(x, \xi)$ ($(x, \xi) \in T^*M$) cover a sector with vertex at the origin. We assume that the sector does not

co positively homogeneous of order t in ξ (see e.g. [27]). The values of $a_0(x, \xi)$ $((x, \xi) \in T^*M)$ cover a sector with vertex at the origin. We assume that the sector does not coincide with the whole plane C. We may also suppose that its bisectrix is \mathbf{R}_{+} , and then our assumption is as follows: M. S. AGRANOVICH and A. S. MARKUS

vely homogeneous of order t in ξ (see e.g. [27]). The values of $a_0(x, \xi)$ ((x, ξ)

4) cover a sector with vertex at the origin. We assume that the sector does not

de with the whole $\in T^*M$
coincide
then ou
This co M. S. AGRANOVICH and A. S. MARKUS

ely homogeneous of order t in ξ (see e.g. [27]). The values of $a_0(x, \xi)$ ((x, ξ))

over a sector with vertex at the origin. We assume that the sector does not

e with the whole plan

$$
|\arg a_0(x,\xi)| \leq \theta \quad (\theta < \pi). \tag{1.24}
$$

0

This condition means that A is elliptic with a parameter in any sector

$$
\{\lambda \colon |\arg \lambda - \pi| \leq \pi - \theta - \varepsilon\} \qquad (0 < \varepsilon < \pi - \theta). \tag{1.25}
$$

It follows (see [23]) that A (as an operator in $L^2(M) = H_0(M)$ with the domain $H_i(M)$) has the compact resolvent $R_A(\lambda) = (A - \lambda I)^{-1}$ and that in any sector (1.25) *A* may have only a finite number of eigenvalues. Moreover each half-line $\{\lambda : \arg \lambda = \varphi\}$ positively homogeneous of order t
 $\in T^*M$ cover a sector with vertex a

coincide with the whole plane C. We

then our assumption is as follows:
 $|\arg a_0(x, \xi)| \leq \theta - (\theta < \pi)$

This condition means that A is ellipt
 $\{ \lambda : |\arg \$ **lying** outside the sector $\{\lambda : \text{arg } \lambda \leq \theta\}$ is a ray of *maximal decrease* (by the terminology in [1], a ray of minimal growth) of the resolvent, i.e. $||R_A(\lambda)|| = O(|\lambda|^{-1})$ as $\lambda \to \infty$ along such a half-line. Replacing if necessary λ by $\lambda - c$ with an appropriate c, we assume (without loss of generality) that all the points in some sector (1.25), including O , are regular for A . have only a finite number of eigenvalues. Moreover each half-line $\{\lambda : \arg \lambda = \varphi\}$
lying outside the sector $\{\lambda : |\arg \lambda| \leq \theta\}$ is a ray of maximal decrease (by the termino-
logy in [1], a ray of minimal growth) of the res *if* α is a minimization of eigentside the sector $\{\lambda : |\arg \lambda| \leq 1\}$, a ray of minimal growth)

ich a half-line. Replacing if

(without loss of generality) the

egular for A .
 $> n$ ($l \in N$); then $[R_A(\lambda)]^l$ be
 $[R_A(\lambda)]$ For (ε) is the parameter in
 (ε) is $(0 \leq \varepsilon \leq \pi - \theta)$
 (ε) ator in $L^2(M) = H_0(M)$
 $(A - \lambda I)^{-1}$ and that in
 ε is a ray of maximal
 θ is a ray of maximal

of the resolvent, i.e. $\|R\|$
 θ is a ray This condition means that A is elliptic with a parameter in any sector
 $\{ \lambda : |\arg \lambda - \pi| \leq \pi - \theta - \epsilon \}$ $(0 \leq \epsilon \leq \pi - \theta)$.

It follows (see [23]) that A (as an operator in $L^2(M) = H_0(M)$ with the domain $H_i(M)$)

has the comp $\frac{1}{2}$
 $\log y$
 $\log y$
 $\log y$
 $\log y$
 $\log x$
 r of minimal growth) of the resolvent, i.e. $||R_A(\lambda)|| = O(|\lambda|^{-1})$ as λ
If-line. Replacing if necessary λ by $\lambda - c$ with an appropriate
loss of generality) that all the points in some sector (1.25), incl
 $\lambda \in \mathbb{N}$; **(2)**
 (2) $\int_{R}^{R} f(x) dx$ $\int_{R}^{R} f(x) dx$ $\int_{R}^{R} f(x) dx$ $\int_{R}^{R} f(x) dx$ $\int_{R}^{R} f(x) dx$ (a) $\int_{R}^{R} f(x) dx$ (a) $\int_{R}^{R} f(x) dx$ (a) $\int_{R}^{R} f(x) dx$ (b) $\int_{R}^{R} f(x) dx$ (a) $\int_{R}^{R} f(x) dx$ (b) $\int_{R}^{R} f(x) dx$ (a) $\int_{R}^{R} f(x) dx$

trace tr $[R_A(\lambda)]^l$. It is well known (see e.g. [14]) that

$$
\operatorname{tr}\left[R_A(-\mu)\right]^{l} \sim c_l \mu^{n/l-l} \qquad (\mu \to +\infty),\tag{1.26}
$$

$$
c_l = (2\pi)^{-n} \int_{T^*M} \left(a_0(x,\xi) + 1 \right)^{-l} dx \, d\xi \, . \tag{1.27}
$$

The coefficient c_1 may be expressed also, using a sufficiently small partition of unity $\{\varphi_k(x)\}\$ ^m on *M* and values $a_0^{(k)}(x,\xi)$ of the principal symbol in corresponding local coordinates, in the form where

where
 $\frac{1}{\sqrt{p_k(x)}}$

The coefficial
 $\frac{1}{\sqrt{p_k(x)}}$
 $\frac{1}{\sqrt{p_k(x)}}$ -

The coefficient
$$
c_l
$$
 may be expressed also, using a sufficiently small partition of unity $\{\varphi_k(x)\}_1^m$ on M and values $a_0^{(k)}(x,\xi)$ of the principal symbol in corresponding local coordinates, in the form\n
$$
c_l = \frac{b_{n|l,l}}{(2\pi)^n n} \int_{-k}^{\infty} \sum_{k=1}^{m} \varphi_k(x) \int_{|k|=1}^{n} \left(a_0^{(k)}(x,\xi) \right)^{-n|l} dS_k \, dx.
$$
\n(1.28)\nHere $d\xi = e^{n-1} d\varphi \, dS_k$, $\varrho = |\xi|$ in local coordinates, and by $\left(a_0^{(k)}(x,\xi) \right)^{-n|l}$ we mean the main value of the function $z^{-n|l|}$ for $z = a_0^{(k)}(x,\xi)$ (if $z = r e^{i\psi}, -\pi < \psi \leq \pi$, $z = a_0^{(k)}(x,\xi)$ (if $z = r e^{i\psi}, -\pi < \psi \leq \pi$).

Here $d\xi = e^{n-1} d\theta dS_i$, $\theta = |\xi|$ in local coordinates, and by $(a_0^{(k)}(x, \xi))^{-n/t}$ we mean The coefficient c_l may be expressed also, using a sufficiently small partition of unity $\langle \varphi_k(x) \rangle_1^m$ on *M* and values $a_0^{(k)}(x, \xi)$ of the principal symbol in corresponding local coordinates, in the form
 $c_l = \frac{$ For $\text{tr }[R_A(-\mu)]^l \sim c_l \mu^{n/l-l} \quad (\mu \to +\infty)$,

where
 $c_l = (2\pi)^{-n} \int_{\mathbb{T}^*M} (a_0(x,\xi) + 1)^{-l} dx d\xi$.

The coefficient c_l may be expressed also, using a sufficiently
 $\{\varphi_k(x)\}_1^m$ on *M* and values $a_0^{(k)}(x,\xi)$ of the pr $e^{\frac{1}{2}} = \frac{1}{(2\pi)^n n} \int_{M} \frac{1}{k}$
 $e^{n-1} d\rho dS_i$; $\rho =$

ralue of the fun
 $e^{n} = r^{-n/l} e^{-1\psi n/l}$. For $\frac{e^{n-1} d\rho}{(a\rho + 1)^\mu} = \nu$ *t*). From
= $v^{-1}a^{-1}$ $\begin{aligned} \n= e^{n-1} \, d \, e^{n-1} \n\end{aligned}$
 $\begin{aligned} \n\frac{d}{dx} \cdot \frac{d}{dx} \n\end{aligned}$
 $\int_{0}^{\infty} \frac{e^{n-1}}{(a e^{n})^2} \, d \, e^{n-1} \n\end{aligned}$ $\sum_{M} (2\pi)^{n} n \int_{M} k=1 \lim_{|k|=1} (2\pi)^{n} n \int_{|k|=1} k=1$
 $= \varrho^{n-1} d\varrho dS_{i}; \varrho = |\xi| \text{ in local coordinates, and by } (a_{0}^{(k)}(x, \xi))^{-n/k} \text{ we meet}$
 1 value of the function $z^{-n/k}$ for $z = a_{0}^{(k)}(x, \xi)$ (if $z = r e^{i\psi}, -\pi < \psi \leq t = r^{-n/k} e^{-i\psi n/k}$). From The coefficient c_t may be expressed also, using a sufficiently small partition of $\left\{ \varphi_k(x) \right\}$ ^m on *M* and values $a_0^{(k)}(x, \xi)$ of the principal symbol in corresponding coordinates, in the form
 $c_t = \frac{b_{n\lfloor t \rf$

$$
\iint_{\frac{1}{2}} \frac{1}{z} \, dz = \int_{0}^{\infty} \frac{1}{z} \, dz
$$
\n
$$
\int_{0}^{\infty} \frac{e^{n - 1} \, d\varrho}{(a\varrho' + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n|v, \mu - n|v) \quad (1 \le n < \nu\mu, a \in (-\infty, 0]) \quad (1.29)
$$
\n
$$
[9: p. 299]). \text{ Introduce two quantities}
$$
\n
$$
d = b_{n|l,l}^{-1} (2\pi)^{-n} \int_{0}^{\infty} (a_0(x, \xi) + 1)^{-l} \, dx \, d\xi,
$$
\n
$$
d = b_{n|l,l}^{-1} (2\pi)^{-n} \int_{0}^{\infty} |a_0(x, \xi)| + 1)^{-l} \, dx \, d\xi.
$$
\n(1.30)

$$
\begin{aligned}\n&= \int_{0}^{\infty} \int_{-\infty}^{\infty} u \, d\omega \, d\omega, \, \xi \, d\omega = |\xi| \text{ in local coordinates, and by } \left(u_0 - (x, \xi) \right) \quad \text{we mean} \\
&= r^{-\frac{n}{2}} t = r^{-\frac{n}{2}} t e^{-1\frac{1}{2}n/2}, \text{ From (1.27) to (1.28) one may pass applying the well-known function, we have:\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1} d\omega}{(a \omega + 1)^{\mu}} = \nu^{-1} a^{-n/r} B(n/r, \mu - n/r) \cdot (1 \leq n < \nu \mu, a \in (-\infty, 0]) \quad (1.29)\n\]\n\[\n\int_{0}^{\infty} \frac{e^{n-1
$$

then
$$
z^{-n/l} = r^{-n/l} e^{-1\psi n/l}
$$
. From (1.27) to (1.28) one may pass applying the well-known formula
\nformula
\n
$$
\int_{0}^{\infty} \frac{\varrho^{n-1} d\varrho}{(a\varrho^2 + 1)^{\mu}} = v^{-1} a^{-n/r} B(n/v, \mu - n/v) \cdot (1 \leq n < \nu\mu, a \in (-\infty, 0]) \quad (1.29)
$$
\n(see e.g. [9: p. 299]). Introduce two quantities
\n
$$
d = b_{n/l,l}^{-1}(2\pi)^{-n} \int_{T^*M} (a_0(x, \xi) + 1)^{-l} dx d\xi,
$$
\n
$$
A = b_{n/l,l}^{-1}(2\pi)^{-n} \int_{T^*M} |a_0(x, \xi)| + 1)^{-l} dx d\xi.
$$
\nUsing (1.29), we can rewrite (1.30), (1.31) in the form
\n
$$
d = \frac{1}{(2\pi)^n n} \int_{M}^{\infty} \sum_{k=1}^{m} \varphi_k(x) \int_{|\xi|=1}^{\infty} (a_0^{(k)}(x, \xi))^{-n/t} dS_{\xi} dx,
$$
\n
$$
A = \frac{1}{(2\pi)^n n} \int_{M}^{\infty} \sum_{k=1}^{m} \varphi_k(x) \int_{|\xi|=1}^{\infty} |a_0^{(k)}(x, \xi)|^{-n/l} dS_{\xi} dx.
$$
\n(1.32)

From this we see that $|d| \leq \varDelta$ and that d , \varDelta do not depend on l . Since (1.24) implies

s we see that
$$
|d| \leq \Delta
$$
 and that d, Δ do not depe
Re $[(a_0^{(k)}(x,\xi))^{-n/l}] \geq |a_0^{(k)}(x,\xi)|^{-n/l} \cos{(\theta n/l)},$

we obtain from (1.32), (1.33) also

$$
|d| \geq \Delta \cos(\theta n/t). \tag{1.34}
$$

Spect

we see that $|d| \leq \Delta$ and that
 $Re\left[\left(a_0^{(k)}(x,\xi)\right)^{-n/l}\right] \geq |a_0^{(k)}(x,\xi)|$

from (1.32), (1.33) also
 $d| \geq \Delta \cos(\theta n|t)$.

by $N(\lambda)$ the counting funct

f them in the circle $\{z: |z| \leq 1\}$

to its multiplicity. Denote by $N(\lambda)$ the counting function for modules of eigenvalues of A , i.e. the number of them in the circle $\{z: |z| \leq \lambda\}$. Recall that each eigenvalue is counted mumber of them in the circulated
according to its multiplicity.
Theorem 1.8: Under the example $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n/\mu} N(\lambda)$ $\left[\frac{I_1}{2}, \frac{I_2}{2}\right]^{-n/l} \geq |a_0^{(k)}(x, \xi)|^{-n/l} \cos(\theta n|t),$
 $\left[\frac{I_1}{2}, \frac{I_2}{2}\right] \geq |a_0^{(k)}(x, \xi)|^{-n/l} \cos(\theta n|t),$
 $\left[\frac{I_2}{2}, \frac{I_2}{2}\right] \leq \lambda}.$ (1.34)
 $\left[\frac{I_1}{2}, \frac{I_2}{2}\right] \leq \lambda.$ Recall that each eigenvalue is counte *•* **•** *•* $\{a_i \geq 0 : (1.36) \}$ *•• ••*

S
S

Theorem 1.8: Under the above assumptions,
\n
$$
|d| \leq \overline{\lim}_{\lambda \to \infty} \lambda^{-n} N(\lambda) \leq e\Delta.
$$
\n(1.35)

Proof: Let us verify that in the proof of the inequality

$$
|d| \leq \lim_{\lambda \to \infty} \lambda^{-n/t} N(\lambda) \tag{1.36}
$$

 t may be replaced by t/p , where p is an arbitrary positive integer. By this replacement $a_0(x, \xi)$ turns into $(a_0(x, \xi))^{1/p}$ while *d*, as it is seen from (1.32), remains the same. Further, if $N^{(p)}(\lambda)$ is the counting function for modules of new eigenvalues λ , $^{1/p}$, then $\mu^{n/l} N(\mu) = \lambda^{npl} N^{(p)}(\lambda)$ for $\mu = \lambda^p$. Therefore the right-hand side of (1.36) also does not change. Hence we may assume that all the eigenvalues λ , of A lie in $\{\lambda : |\arg \lambda| \leq \varphi\}$ where $\varphi < \pi/2$. We shall first establish the inequality of the type (1.36) for the count-
ing function $N_n(\lambda)$ of the real parts of λ . ing function $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n/\ell} N(\lambda) \leq e\Lambda$.

Proof: Let us verify that in the proof of the ine
 $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n/\ell} N(\lambda)$

i may be replaced by t/p , where *p* is an arbitrary pos
 $a_0(x, \xi)$ turns into $(a_0(x,$ *•* $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n} N(\lambda) \leq \epsilon \Delta$. (1.35)
 Proof: Let us verify that in the proof of the inequality
 $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n} N(\lambda)$ (1.36)
 t may be replaced by t/p , where *p* is an arbitrary positive integer. By $\mu^{n/l} N(\mu) = \lambda^{np/l} N^{(p)}(\lambda)$ for $\mu = \lambda^p$. The
not change. Hence we may assume that
where $\varphi < \pi/2$. We shall first establish
ing function $N_R(\lambda)$ of the real parts of
 $|d| \leq \lim_{\lambda \to \infty} \lambda^{-n/l} N_R(\lambda)$.
It is obvious that, arbitrary positive integer. By this replacement
as it is seen from (1.32), remains the same.
tion for modules of new eigenvalues $\lambda_r^{1/p}$, then
erefore the right-hand side of (1.36) also does
all the eigenvalues λ , of

$$
|d| \leq \overline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N_R(\lambda). \tag{1.37}
$$

-and consequently

$$
\text{Equation 1: } \begin{aligned} \text{Equation 2: } \text{Equation 3: } \text{Equation 4: } \text{Equation 5: } \text{Equation 5: } \text{Equation 6: } \text{Equation 7: } \text{Equation 7: } \text{Equation 8: } \text{Equation 8: } \text{Equation 8: } \text{Equation 9: } \text{Equation 9: } \text{Equation 9: } \text{Equation 1: } \text{Equation 1:
$$

to *cjj,* and we.may rewrite (1.38) in the form

$$
\overline{\lim}_{\mu\to\infty}\mu^{t-n/t}\int\limits_{0}^{\infty}(\lambda+\mu)^{-t}dN_{R}(\lambda)\geq|c_{t}|\,.
$$

Since tr $[R_A(-\mu)]^{-l} = \sum_{i} (\lambda_i + \mu)^{-l}$, by (1.26) the right-hand side of (1.38) is equal
to $|c_l|$, and we may rewrite (1.38) in the form
 $\lim_{\mu \to \infty} \mu^{l-n/l} \int_{0}^{\infty} (\lambda + \mu)^{-l} dN_R(\lambda) \ge |c_l|$.
Now (1.15) and the equality $c_l =$ $\lim_{\mu \to \infty} \mu^{t-n/l} \int_{0}^{\infty} (\lambda + \mu)^{-l} dN_R(\lambda) \geq |c_l|.$

Now (1.15) and the equality $c_l = b_{n/l} d$ imply (1.37). Since $|\arg \lambda_r| \leq$

have $N(\lambda) \geq N_R(\lambda \cos \varphi)$, so from (1.37) it follows that $|d| (\cos \varphi)^{n/l} \leq$ $\overline{\lim}$ $\lambda^{-n/t}N(\lambda)$. Replace here *I* by t/p : $|d|$ $(\cos{(\varphi/p)})^{npl} \leq \lim_{n \to \infty} \lambda^{-n/l} N(\lambda)$. Evidently $(\cos{(\varphi/p)})^{npl} \to 1$ as $p \rightarrow \infty$, so in the limit we obtain (1.36).

Now pass to the proof of the right inequality in (1.35) . Without loss of generality, Now pass to the proof of the right inequality in (1.35). Without loss of generality,
assume that $0 \notin \sigma(A)$. If $K = A^{-1}$, then obviously $N(\lambda) = n(\lambda)$, where $n(\lambda)$ is the
counting function for modules of the characteristic counting function for modules of the characteristic values of the compact operator *K*. By Theorem 1.6 Replace here *t* by $t/p: |d| \left(\cos{(\varphi/p)})\right)^{npl} \le \lim_{\lambda \to \infty} \lambda^{-n/l} N(\lambda)$. Evidently (coordination (1.36) .

Now pass to the proof of the right inequality in (1.35). Without loss

assume that $0 \notin \sigma(A)$. If $K = A^{-1}$, then obvi

$$
\overline{\lim}_{\lambda\to\infty}\lambda^{-n/t}n(\lambda)\leq e\overline{\lim}_{\lambda\to\infty}\lambda^{-n/t}(\lambda).
$$

246 M. S. AGRANOVICH and A. S. MARKUS

It is easily seen that $v(t)$ coincides with the counting function for eigenvalues of $B = (A^*A)^{1/2}$. It is a positive elliptic operator of order t, and its principal symbol is equal $|a_0(x,\xi)|$. Therefore

$$
\nu(\lambda) \sim \Delta \tilde{\lambda}^{n/t}, \qquad \lambda \to \infty \tag{1.40}
$$

(see e.g. $[27: \S 15]$). Thus (1.39) yields the right inequality in (1.35) \blacksquare

Using (1.40) and Theorem 1.5, we obtain the following assertion.

Theorem 1.9: Under the above assumptions,

$$
\lim_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \leq \Delta. \tag{1.41}
$$

Remark 1.10: As it is seen from the proofs, the right inequality in (1.34) and (1.41) are both valid for each elliptic pseudo-differential operator with a discrete spectrum (the condition (1.24) is not necessary).

Theorem 1.11: Let
$$
d \neq 0
$$
. Then
\n
$$
\lim_{l \to \infty} \lambda^{-n/l} N(\lambda) > 0.
$$

Proof: Since, according to our assumption, all the eigenvalues λ , lie in some sector $\{\lambda : |\arg \lambda| \leq \theta_1\}$ $(\theta_1 < \pi)$, we have for $\mu > 0$

$$
|\lambda_{\mathbf{r}} + \mu|^2 = |\lambda_{\mathbf{r}}|^2 + \mu^2 + 2 |\lambda_{\mathbf{r}}| \mu \cos \arg \lambda_{\mathbf{r}} \geq (|\lambda_{\mathbf{r}}| + \mu)^2 \cos^2 \frac{\sigma_1}{2}, \qquad (1.43)
$$

 an

$$
\sum_{\nu=1}^{\infty}(|\lambda_{\nu}|+\mu)^{-1}\geq C\sum_{\nu=1}^{\infty}|\lambda_{\nu}+\mu|^{-1}\geq C\left|\sum_{\nu=1}^{\infty}(\lambda_{\nu}+\mu)^{-1}\right|,
$$

where $C = (\cos{(\theta_1/2)})^{-1}$. From this we obtain

$$
\lim_{\mu\to\infty}\mu^{l-n/l}\sum_{\nu=1}^{\infty}(|\lambda_{\nu}|+\mu)^{-l}\geq C\lim_{\mu\to\infty}\mu^{l-n/l}\left|\sum_{\nu=1}^{\infty}(\lambda_{\nu}^{\alpha}+\mu)^{-l}\right|.
$$

By (1.26) the latter limit is equal to $|c_i| = |b_{n/i}, d|$. So

$$
\lim_{\mu\to\infty}\mu^{l-n/l}\int\limits_{0}^{\infty}(\lambda+\mu)^{-l}\,dN(\lambda)>0.\tag{1.44}
$$

On the other hand, by Theorem 1.8

$$
\overline{\lim}\,\lambda^{-n/l}N(\lambda)<\infty.
$$

According to Remark 1.4, (1.44) and (1.45) imply (1.42)

From Theorems 1.8 and 1.11 follows

Corollary 1.12: If $d \neq 0$, then $N(\lambda) \times \lambda^{n/l}$.

Remark 1.13: If

 $n\theta < \pi t/2$.

 (1.46)

 (1.45) ⁻

where θ is the same as in (1.24), then $d \neq 0$. This follows from (1.34). If M is connected and $n > 1$, the sign \lt in (1.46) may be replaced by \leq . Indeed, in this case the real part of the integrand in (1.32) is non-negative, and if $d = 0$, we have Re $(a_0^{(k)}(x, \xi))^{-n} = 0$ on $U_k = \{(x, \xi):$

247

 $\varphi_k(x) > 0$, $|\xi| = 1$ $(k = 1, ..., m)$. But then, since M and the unit sphere in \mathbb{R}^n $(n > 1)$ is $\varphi_k(x) > 0$, $|\xi| = 1$ ($k = 1, ..., m$). But then, since *M* and the unit sphere in K^n ($n > 1$) is connected, there exist a number k and a point $(x_0, \xi_0) \in U_k$ such that Im $(a_0^{(k)}(x_0, \xi_0))^{-n/k} = 0$, so that $a_0^{(k)}(x_0, \xi_0) = 0$, which contradicts the ellipticity of *A*. Spectral Properties of Elliptic Pseudo-Diff. Op.
 $\varphi_k(x) > 0$, $|\xi| = 1$ $(k = 1, ..., m)$. But then, since M and the unit sphere in \mathbb{R}^n (*n* connected, there exist a number *k* and a point $(x_0, \xi_0) \in U_1$ such that Im

Remark 1.14: Corollary 1.12 shows that *A* has "many" eigenvalues if $d \neq 0$. As it has been mentioned in the Introduction, the condition (1.46) assures the completeness of root functions

Theorems 1.8 and 1.9 imply

Corollary 1.15: If the limit of $\lambda^{-n/I}N(\lambda)$ as $\lambda \to \infty$ exists, it belongs to the segment $[|d|, \Delta].$

Remark 1.16: If $|d| = \Delta$, then the limit of $\lambda^{-n/k}N(\lambda)$ as $\lambda \to \infty$ exists (and obviously coincides with $|d| = 4$). Indeed, from (1.32) and (1.33) it follows that in this case arg $a_0(x, \xi)$ cides with $|d| = \Delta$). Indeed, from (1.32) and (1.33) it follows that in this case arg $a_0(x, \xi) =$
const. But then $A = \alpha(A_0 + B)$, where $\alpha \in \mathbb{C}$, A_0 is a selfadjoint elliptic pseudo-differential operator (with the principal symbol $\alpha^{-1}a_0(x, \xi) > 0$), and *B* is a pseudo-differential operator of order $\leq t-1$. Therefore for A the formula of the type (0.1) is valid with $\alpha^{-1}a_0$ instead of a_0 mark 1.16: If $|d| = \Delta$, then the limit of $\lambda^{-n/\ell} N(\lambda)$ as $\lambda \to \infty$ exists (and obviously coin-
with $|d| = \Delta$). Indeed, from (1.32) and (1.33) it follows that in this case arg $a_0(x, \xi) =$
. But then $A = \alpha(A_0 + B)$, where (see e.g. [18]).

1.4. Generalizations to matrix elliptic operators. Let *A* be a $(r \times r)$ -matrix elliptic pseudo-differential operator of order $t > 0$ with the principal symbol $a_0(x, \xi)$. Denote by $\lambda_j(x, \xi)$ $(j = 1, ..., r)$ the eigenvalues of the matrix $a_0(x, \xi)$. Under the condition $|\arg \lambda_i(x, \xi)| \leq \theta < \pi$ ($j = 1, ..., r$), Theorems 1.8, 1.9, 1.11 and Corollaries 1.12, 1.15 can be easily extended to the matrix case, with the replacement of $(a_0 + 1)^{-l}$ and $(|a|_0 + 1)^{-l}$ in (1,30) and (1.31) by tr $(a_0 + E)^{-l}$ and tr $((a_0 * a_0)^{1/2} + E)^{-l}$, respectively. Assume now that the eigenvalues of $a_0(x, \xi)$ lie in two closed sectors A_1 and A_2 with vertex at the origin and without any other common points. For definiteness assume that izations to matrix elliptime.

intential operator of order
 $= 1, ..., r$ the eiger
 $\leq \theta < \pi$ ($j = 1, ...,$

easily extended to the
 $1)^{-l}$ in (1.30) and

Assume now that the

vertex at the origin

intertance that
 $= \{\zeta : |\arg \zeta$ *A₂* = { ζ arg ζ } are $(a_0 + b)$
A₂ = { ζ : | arg ζ | $\geq \theta_2$ },

$$
\Lambda_1 = \{\zeta : |\arg \zeta| \leq \theta_1\}, \qquad \Lambda_2 = \{\zeta : |\arg \zeta| \geq \theta_2\},\
$$

where $0 \leq \ell_1 < \ell_2 \leq \pi$. For arbitrary small $\epsilon > 0$, A is elliptic with a parameter in respectively. Assume now that the eigenvalues of $a_0(x, \xi)$ lie in two closed sectors A_1
and A_2 with vertex at the origin and without any other common points. For defini-
teness assume that
 $A_1 = {\xi : |\arg \zeta| \leq \theta_1}, \qquad$ $A_{\epsilon}^{\pm} = \langle \zeta : \theta_1 + \epsilon \leq \pm \arg \zeta \leq \theta_2 - \epsilon \rangle \; \left(\epsilon \langle \theta_2 - \theta_1 \rangle / 2 \right)$, and these sectors can concounting function for modules of those $\lambda_r(A)$ which lie in $\Lambda_1(\varepsilon) = \langle \zeta : | \arg \zeta | \leq \theta_1 + \varepsilon \rangle$. Generalizations we are going to obtain concern $N_1(\lambda)$ (instead of $N(\lambda)$). Since $N_1(\lambda)$ $\leq N(\lambda)$, some upper bounds for $N_1 (\lambda)$ come from appropriate bounds for $N(\lambda)$, so we shall deal only with lower bounds. Set. where $0 \leq q_1 < q_2 \leq n$, For arouslay small $\varepsilon > 0$, A is empire with a parameter in $A_{\varepsilon}^{\pm} = \{\xi : \theta_1 + \varepsilon \leq \pm \arg \xi \leq \theta_2 - \varepsilon\}$ $(\varepsilon < (\theta_2 - \theta_1)/2)$, and these sectors can contain only a finite number of eigenva $\zeta \geq \theta_2$,
 $\zeta \geq \theta_2$,
 ζ , $\geq \theta_3$,
 ζ , ζ is elliptic v
 θ_1 , β), and the

Fixing $\varepsilon > 0$, d

lie in $A_1(\varepsilon) = \{ \zeta \}$
 $\zeta_1(\lambda)$ (instead of
 ζ_1 a appropriate b
 $d\xi$, $d^{(1)} = b$
 $d\xi$,

some upper bounds for
$$
N_1(\lambda)
$$
 come from appropriate bounds for $N(\lambda)$, so
deal only with lower bounds. Set

$$
c_t^{(1)} = (2\pi)^{-n} \int \sum_{\mathcal{I}^*M} \left(\lambda_i(x,\xi) + 1\right)^{-1} dx d\xi, \quad d^{(1)} = b_{n\ell,\ell}^{-1}c_t^{(1)}, \quad (1.47)
$$

$$
T^*M \cdot \lambda_i \epsilon A_1
$$

$$
S_{1,\ell}(\zeta) = \sum_{\lambda_r \in A_1(\epsilon)} \left(\lambda_r(A) - \zeta\right)^{-1}, \quad (1.48)
$$
is an arbitrary positive integer greater than n/t (one can verify easily that $d^{(1)}$ defined on l). The desired generalizations will be derived from the following
rem 1.17: If $l - 1 < n/t < l$, then

$$
S_{1,l}(\zeta) = c_l^{(1)}(-\zeta)^{n\ell-1} + o(|\zeta|^{n\ell-1}) \quad (\zeta \to \infty)
$$

$$
y \in \mathbb{Z}
$$
is l (1.49)
by $in \{\zeta : \text{large } \zeta | \geq \theta_1 + \varepsilon\}$.
if: We begin with the known formula (see e.g. [14]).
for $(R_A(\zeta))^l = (2\pi)^{-n} \int \text{tr} [a_0(x, \zeta) - \zeta E]^{-1} dx d\xi + O\left(|\zeta|^{\frac{n-1}{l}-l}\right),$

$$
L_2(\zeta) = \sum_{\lambda_{\nu} \in A_1(\epsilon)} (\lambda_{\nu}(A) - \zeta)^{-1}, \tag{1.48}
$$

where *l* is an arbitrary positive integer greater than n/t (one can verify easily that $d^{(1)}$ $S_{1,l}(\zeta) = \sum_{\lambda_r \in \Lambda_1(\epsilon)} (\lambda_r(A) - \zeta)^{-l}$, (1.48)
where *l* is an arbitrary positive integer greater than *n/t* (one can verify easily that $d^{(1)}$ does not depend on *l*). The desired generalizations will be derived from where *l* is an arbitrary positive indoes not depend on *l*). The desire
 $\text{Theorem 1.17:} \quad If \quad l - 1 < n/l$
 $\hat{S}_{1,l}(\zeta) = c_l^{(1)}(-\zeta)^{n/l-1} +$

uniformly in $\langle \zeta : |\arg \zeta| \geq \theta_1 + \varepsilon \rangle$.

Proof: We begin with the kno

Theorem 1.17: *If* $l-1 < n/t < l$ *, then*

$$
\hat{S}_{1,l}(\zeta) = c_l^{(1)}(-\zeta)^{n/l-1} + o(|\zeta|^{n/l-1}) \qquad (\zeta \to \infty)
$$
\n(1.49)

uniformly in $\{\zeta : |\arg \zeta| \geq \theta_1 + \varepsilon\}.$
Proof: We begin with the known formula (see e.g. [14]).

$$
S_{1,l}(\zeta) = \sum_{\lambda_r \in A_1(t)} (\lambda_r(A) - \zeta)^{-l},
$$

where *l* is an arbitrary positive integer greater than *n*/*l* (one can verify e
does not depend on *l*). The desired generalizations will be derived from
Theorem 1.17: If $l - 1 < n/l < l$, then

$$
\hat{S}_{1,l}(\zeta) = c_l^{(1)}(-\zeta)^{n}l^{l-1} + o(|\zeta|^{n/l-1}) \qquad (\zeta \to \infty)
$$

uniformly in $|\zeta|$: $|\arg \zeta| \geq \theta_1 + \varepsilon$.
Proof: We begin with the known formula (see e.g. [14]).
tr $(R_A(\zeta))^l = (2\pi)^{-n} \int \text{tr} [a_0(x, \zeta) - \zeta E]^{-l} dx d\zeta + O(|\zeta|^{\frac{n-1}{l}-l})$
17*

which is valid in A_t^{\pm} . Setting $S_{2,i}(\zeta) = \text{tr} (R_A(\zeta))^i - S_{1,i}(\zeta)$, we put down this formula in the form

$$
S_{1,1}(\zeta)+S_{2,1}(\zeta)=(2\pi)^{-n}\int\limits_{T^*M}\sum_{\lambda_j\in\Lambda_1}\left(\lambda_j(x,\xi)-\zeta\right)^{-1}dxd\xi
$$

$$
+ (2\pi)^{-n} \int\limits_{T^*M} \sum\limits_{\lambda_j\in\Lambda_1} (\lambda_j(x,\xi)-\zeta)^{-l} dx \,d\xi + O\left(|\zeta|^{\frac{n-1}{l}-l}\right).
$$

Here the first integral makes sense if $|\arg \zeta| > \theta_1$ and we can transform it by setting $\xi = \mu^{1/t}$ for $\zeta = -\mu < 0$ and using the holomorphic extension in ζ . The second integral makes sense if $|\arg \zeta| < \theta$, and admits a similar transformation: we set $\xi = \mu^{1/t} \eta$ for $\zeta = \mu > 0$ and then use the holomorphic extension in ζ . So we obtain for $\zeta \in \varLambda_{\epsilon}^{\pm}$

$$
S_{1,l}(\zeta) + S_{2,l}(\zeta) = c_l^{(1)}(-\zeta)^{n/l-l} + c_l^{(2)}\zeta^{n/l-l} + O\left(|\zeta|^{\frac{n-1}{l}-l}\right), \qquad (1.50)
$$

where $c_1^{(1)}$ is defined by the first equality in (1.47) and $c_1^{(2)}$ by the analogous equality with $\lambda_i(x, \xi) \in A_2$.

Let us estimate the growth of $S_{1,l}(\zeta)$ when $|\arg \zeta| \geq \theta_1 + \varepsilon$. Since $|\arg \lambda_i| \leq \theta_1 + \varepsilon/2$ in each term in (1.48) with sufficiently large ν , we have, by the inequality analogous to (1.43) ,

$$
|S_{1,l}(\zeta)| \leq \sum_{\lambda_{\nu} \in \Lambda_1(\epsilon)} |\lambda_{\nu}(A) - \zeta|^{-l} \leq C_1 \sum_{\lambda_{\nu} \in \Lambda_1(\epsilon)} (|\lambda_{\nu}(A)| + |\zeta|)^{-l}
$$

for sufficiently large $|\zeta|$. Since $N(\lambda) = O(\lambda^{n/t})$ (see (1.35)), we have $|\lambda - 1(A)| = O(\nu^{-t/n})$, and therefore

$$
|S_{1,l}(\zeta)| \leq C_1 \sum_{\nu=1}^{\infty} (\nu^{t/n} + |\zeta|)^{-l} \leq C_1 \int_0^{\infty} \frac{dx}{(x^{t/n} + |\zeta|)^l} = C_1 |\zeta|^{n/l-l} \int_0^{\infty} \frac{dy}{(y^{t/n} + 1)^l}
$$

(we use the substitution $x = |\zeta|^{n/t} y$). Thus

$$
S_{1,l}(\zeta) = O(|\zeta|^{n/l-l}) \qquad (\zeta \to \infty, |\arg \zeta| \geq \theta_1 + \varepsilon). \tag{1.51}
$$

Similarly we can verify that

$$
S_{2,l}(\zeta) = O(|\zeta|^{n/l-1}) \qquad (\zeta \to \infty, |\arg \zeta| \leq \theta_2 - \varepsilon). \tag{1.52}
$$

The formula (1.49) we shall derive from (1.50) by "separating" the asymptotics of $S_{1,l}(\zeta)$. To do this, take the contour Γ consisting of two half-lines ζ : arg $\zeta = \pm 0$. $\{+\varepsilon\}$, passing from ∞ to 0 on the lower and from 0 to ∞ on the upper half-line. If some $\lambda_r(A)$ are found on Γ (there is at most a finite number of such $\lambda_r(A)$), we slightly. deform Γ near such points so as to avoid them, but make this to that all the eigenvalues of A, contained in the sector $\langle \zeta : | \arg \zeta | > \theta_1 + \varepsilon \rangle$, remain at the left of Γ and so that the origin remains the unique point common to Γ and R. Divide (1.50) by $2\pi i(\zeta - z)$ and integrate along Γ , assuming z lies at the left of Γ :

$$
\frac{1}{2\pi i} \int_{\Gamma} \frac{S_{1,l}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Gamma} \frac{S_{2,l}(\zeta)}{\zeta - z} d\zeta
$$
\n
$$
= \frac{c_l^{(1)}}{2\pi i} \int_{\Gamma} \frac{(-\zeta)^{n/l-1}}{\zeta - z} d\zeta + \frac{c_l^{(2)}}{2\pi i} \int_{\Gamma} \frac{\zeta^{n/l-1}}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Gamma} \frac{R(\zeta)}{\zeta - z} d\zeta, \qquad (1.53)
$$

-

where $R(\zeta)$ is the remainder in where $R(\zeta)$ is the remainder in (1.50) (all the integrals converge absolutely in virtue of the condition $n/l < l$ and relations (1.51) and (1.52)). In the first term of the righthand side we can replace Γ by a closed contour surrounding z and lying at the left of Γ . From the Cauchy integral formula it follows immediately that this term is equal to $c_1(1)(-z)^n l^{t-1}$. Analogously, in the second term of the right-hand side we can replace *I* by a closed contour lying at the right of Γ ; so this term is equal to 0. Similar arguments permit to calculate easily the terms in the left-hand side of (1.53). The first of them is equal to $S_{1,1}(z)$, the second is equal to 0. It remains to estimate the third term in the right-hand side of (1.53). Let r be so large that the part \varGamma'' of \varGamma lying outside T by a closed contour lying at the right of Γ ; so this term is equal to 0. Similar arguments permit to calculate easily the terms in the left-hand side of (1.53). The first of them is equal to $S_{1,t}(z)$, the second is Spectral Properties of Elliptic Pseudo-Diff

where $R(\zeta)$ is the remainder in (1.50) (all the integrals converge absolut

of the condition $n|t| < l$ and relations (1.51) and (1.52)). In the first term

hand side we can re (all the integrals converg

1.51) and (1.52)). In the i

1 contour surrounding z a
 t follows immediately that
 d to *d derm* of the right-har
 derm of the right-har
 derm is equal to 0. It remains to es
 de s Spectral Properties of Elliptical

Spectral Properties of Elliptical

of the condition $n/l < l$ and relations (1.50) (all the integrals com

and side we can replace P by a closed contour surrounding

Properties ($\alpha^{(1)}(-z$

0•

then is equal to
$$
S_1(tz)
$$
, the second is equal to 0. It remains to estimate the third central in the right-hand side of (1.53). Let r be so large that the part I'' of I lying outside the disk $\langle \zeta : |\zeta| \leq r$ consists of half-lines, and let I'' be the part of I lying inside the disk. Then\n
$$
\left| \int \frac{R(\zeta)}{\zeta - z} d\zeta \right| \leq \int \frac{|R(\zeta)|}{|\zeta - z|} |d\zeta| + C_3 \int \frac{|\zeta|^{-\frac{1}{t}} - l}{|\zeta| + |z|} |d\zeta|
$$
\nfor $|\arg z| \geq \theta_1 + \varepsilon$ (we again apply the inequality analogous to (1.43)). The first term on the right-hand side obviously has the order $O(|z|^{-1})$. The second term is not greater than\n
$$
C_3 |z|^{\frac{n-1}{t}} - t \left(\int_{r|z|^{-1}}^{1} \frac{n-1}{r} \frac{1}{t} dt + \int_{1}^{\infty} \frac{n-1}{t} \frac{1}{t} - t \frac{1}{(1 + \tau)^{-1}} dt \right)
$$
\n(here we use the substitution $|\zeta| = \tau |z|$ and the inequality $(1 + \tau)^{-1} < 1$ for $\tau \in (0, 1)$). The order of this quantity is $O(|z|^{-\frac{n-1}{t}} - t)$, when $(n-1)/t - l > -1$, $O(|z|^{-1} \ln z)$, when

for $\left|\int_{r} \right|$
for $\left|\arg z\right| \ge$
on the right- $0 \ge \theta_1 + \varepsilon$ (we again apply the inequality analogous to (1.43)). The first term that hand side obviously has the order $O(|z|^{-1})$. The second term is not greater
 $C_3 |z|^{\frac{n-1}{l}-l} \left(\int_{r|z|^{-1}}^1 \frac{r^{n-1}-l}{r^{l-1}} \, dr + \int_{$ on the right-hand side obviously has the order $O(|z|^{-1})$. The second term is not greater than them is equal to S_1 ,

in the right-hand site

the disk $\langle \zeta : |\zeta| \leq r$

disk. Then
 $\left| \int \frac{R(\zeta)}{\zeta - z} \right|$

for $|\arg z| \geq \theta_1 + \varepsilon$

on the right-hand sithan
 $C_3 |z|$

$$
\left| \int_{\Gamma} \frac{\zeta - z}{z - z} d\zeta \right| \leq \int_{\Gamma'} \frac{|d\zeta| + C_3}{\sqrt{|\zeta - z|}} d\zeta
$$
\n
$$
|z| \geq \theta_1 + \varepsilon \text{ (we again apply the inequality analogous to } \frac{1}{2} \text{ that } \frac{1}{2} \text{ and } \frac{1}{2} \text{ and
$$

(here we use the substitution $|\zeta| = \tau |z|$ and the inequality $(1 + \tau)^{-1} < 1$ for $\tau \in (0, 1)$). Let r be so large that the thalf-lines, and let T' be thalf-lines, and let T' be that
 $\frac{R(\zeta)}{|z|} |d\zeta| + C_3 \int \frac{|\zeta|^{-\frac{n-1}{t}}}{|\zeta| + \frac{n-1}{t}}$

pply the inequality analog has the order $O(|z|^{-1})$. The $\int_{1}^{\infty} \frac{1}{$ $C_3 |z|^{-t} \left(\int_{|z|=1}^{\infty} \tau^{-t} \frac{z^t}{t} dt + \int_{1}^{\infty} \tau^{-t} (1+\tau)^{-1} d\tau \right)$
(here we use the substitution $|\zeta| = \tau |z|$ and the inequality $(1 + \tau)^{-1} < 1$ for $\tau \in (0, 1)$).
The order of this quantity is $O(|z|^{-t-1})$, when $(n-1$ $\mathcal{L}(n-1)/t - l = -1$, and $O(|z|^{-1})$, when $(n-1)/t - l < -1$. Since $l - 1 < n/t < l$, in all the cases we obtain the estimate $o(|z|^{n/t-l})$ for the third integral in the right-hand (here we use the substitution $|\zeta| = \tau |z|$ and the inequality $(1 + \tau)^{-1} < 1$ for $\tau \in (0, 1)$).

The order of this quantity is $O(|z|^{-1})$, when $(n - 1)/t - l > -1$, $O(|z|^{-1} \ln z)$, when
 $(n - 1)/t - l = -1$, and $O(|z|^{-1})$, when $(n - 1)/t$ side of (1.53) I all the cases we obtain the estima
le of (1.53) **I**
The main result of the present si
Theorem $1.18: If d^{(1)} \neq 0, then$ $\begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \end{array}} \end{array}}}} \end{array}}} \end{array}}} \end{array}}} \end{array}}} \end{array}} \end{array}} \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \end{array}}}} \end{array}} \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \end{$

The main result of the present subsection is

 $\mathcal{O}(\frac{1}{\epsilon})$

e cases we obtain the estimate
$$
\delta(|z|^{n/t-1})
$$
 for the third integral in the right-hand
1.53) **ii**
nain result of the present subsection is
rem. 1.18: If $d^{(1)} \neq 0$, then

$$
\lim_{\lambda \to \infty} \lambda^{-n/t} N_1(\lambda) > 0, \qquad \lim_{\lambda \to \infty} \lambda^{-n/t} N_1(\lambda) \ge |d^{(1)}|.
$$
 (1.54)

to that of Theorem To prove the second inequality, it is convenient to change the notations and assume that the bisectrix of one of the sectors separating A_1 and A_2 coincides with R₋. For this, A is to be replaced by $e^{i\varphi}A$ with an appropriate ψ . Theorem 1.17 gives the asymptotics of $S_{1,1}(\zeta)$ outside the angular neighbourhood of A_1 in its new position and, in particular, along R_ if *n/i* is not an integer. Now we note that it is sufficient to obtain the desired result for $A_{\alpha} = A^{\alpha}$ with an arbitrary $\alpha \in (0, 1)$ (taking into account that the first formula in (1.47) can be rewritten in a form analogous to (1.32) with $\sum \lambda_i$ ^{-n/t} instead of $a_0^{-n/l}$. Therefore we may assume that *t* is irrational and that we have a formula of type (1.49) for $A^{1/p}$ along $R_$. Now we can prove the second inequality in $\lim_{\lambda \to \infty} \lambda^{-n/\mu} N_1(\lambda) > 0$, $\lim_{\lambda \to \infty} \lambda^{-n/\mu} N_1(\lambda) \ge |d^{\lambda/1}|$.

Proof: The proof of the first inequality is quite similar To prove the second inequality, it is convenient to change that the bisectrix of one of the *Au* fin 0, *B1U6* = 0 *(j* = 1, ...,rn) - • (1.55)

Note that if $\theta_1 = 0$, i.e. if A_1 is reduced to \mathbf{R}_+ , one of the authors [3] has obtained the regular asymptotics $N_1(\lambda) \sim d^{(1)}\lambda^{n/\ell}$ ($\lambda \to \infty$). Here we have used the way of reasoning employed in [3].

1.5. Results for elliptic boundary value problems. Let G be a bounded domain in \mathbb{R}^n with a C^{∞} -boundary ∂G . Consider an elliptic boundary value problem

$$
Au = f \text{ in } G, \qquad B_i u|_{\partial G} = 0 \qquad (i = 1, ..., m) \tag{1.55}
$$

S

(see e.g. $[15]$) with homogeneous boundary conditions. Here u, f are scalar (for simplicity) functions, A is a differential operator of order $t = 2m$, elliptic in \overline{G} , B_j are differential-operators of order $t_i < t$ and all the operators have C^{∞} -coefficients. Denote by A_B the corresponding closed operator in $L^2(G)$; its domain is the subspace in $H_{2m}(G)$ defined by the boundary conditions $B_i u|_{\partial G} = 0$ ($j = 1, ..., m$). Suppose the problem, obtained from (1.55) by replacing *A* with $A - \lambda I$, is elliptic with a parameter in a (closed) sector \mathcal{L} with the bisectrix **R**₋. Then the boundary operators form a normal system (see e.g. [24]), the resolvent $R_{A_{\mathcal{B}}}(\lambda)$ exists for $\lambda \in \mathscr{L}$ with sufficiently large $|\lambda|$ and satisfies the estimate $||R_{A_B}(\lambda)|| = O(|\lambda|^{-1})$ (see [1]). As in Subsection 1.3, we may assume that $R_{A_B}(\lambda)$ exists for all $\lambda \in \mathcal{L}$. be e.g. [15]) with homogeneous boundary conditions. Here it is a differential operator of order $t =$
fferential operators of order $t_i < t$ and all the operators of
te by A_B the corresponding closed operator in $L^2(G)$; i

Define *d* and Δ by (0.2) (where $a_0(x, \xi)$ is the principal symbol of *A* and $t = 2m$)
th *G* instead of *M*. Let *N*(λ) be the counting function for modules of eigenvalues
 A_B).
Theorem 1.19: Under the above assu with *G* instead of *M*. Let *N(2)* be the counting function for modules of eigenvalues $\lambda_*(A_B)$.

Theorem 1.19: Under the above assumptions,
 $|d| \le \lim_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \le e\Lambda$, $\lim_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \le \Lambda$;

furthermore $\lambda_r(A_B)$.

$$
|d| \leqq \overline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \leqq e \Delta, \qquad \underline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \leqq \Delta;
$$

furthermore, if $d \neq 0$, *then* $\lim_{n \to \infty} \lambda^{-n} N(\lambda) > 0$.

The proof is similar in the main to the proofs of Theorems 1.8, 1.9, 1.11, and we restrict ourselves to the following explanations. First of all, a formula for A_B of the form (1.40) is valid. It comes from the fact that the composition $A_B^*A_B$ corresponds to the self-adjoint elliptic boundary value problem in G for the differential operator A^*A with the principal symbol $|a_0(x,\xi)|^2$ (see e.g. [10]). Secondly, one can define the powers A_B^{α} of A_B , $0 < \alpha < 1$ [25]. Set $R_{\alpha,q}(\lambda) = (A_{\beta}^{\dagger} - \lambda I)^{-q} (q \in \mathbb{N}).$ If $2m\alpha q > n$, this operator belongs to the trace class and the following lemma is valid. $|d| \leq \overline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) \leq e\Lambda, \qquad \underline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N(\lambda)$

ore, if $d \neq 0$, then $\overline{\lim}_{\lambda \to \infty} \lambda^{-n/l} N(\lambda) > 0$.

oof is similar in the main to the proofs of Th

to the following explanations. First of all, a form ms 1.8, 1.9, 1.11, and
for A_B of the form (1.
ponds to the self-adjo
 A^*A with the princis
 A_B^* of A_B , $0 < \alpha <$
eelongs to the trace cl
($\lambda \rightarrow -\infty$). ful \equiv into Λ is $N(\lambda) \ge \epsilon \Delta$, $\frac{\ln \lambda}{\lambda} \sim N(\lambda) \ge 2$,
furthermore, if $d \ne 0$, then $\lim_{\lambda \to \infty} \lambda^{-n} N(\lambda) > 0$.
The proof is similar in the main to the proofs of Theorems 1
ourselves to the following explanations. Fi

Lemma 1.20: *Under the above assumptions,*

The initial is valid.
\n(a 1.20: Under the above assumptions,
\n
$$
\text{tr } R_{\alpha,q}(\lambda) = b_{n/2m\alpha,q}d(-\lambda)^{\frac{n}{2m\alpha}-q} + O(|\lambda|^{\frac{n-1}{2m\alpha}-q}) \qquad (\lambda \to -\infty
$$
\nthis formula one must apply the equality
\n
$$
R_{\alpha,q}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} (\mu^{\alpha} - \lambda)^{-q} R_{A_{B}}(\mu) d\mu
$$

To prove this formula one must apply the equality

$$
R_{\alpha,q}(\lambda) = \frac{1}{2\pi i} \int\limits_{\Gamma} (\mu^{\alpha} - \lambda)^{-q} R_{A_B}(\mu) d\mu
$$

To prove this formula one must apply the equality
 $R_{\alpha,q}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} (\mu^{\alpha} - \lambda)^{-q} R_{A_B}(\mu) d\mu$

and SEELEY's formulas [24] for the parametrix, which approximates R_{A_B} in *Y*. Here *I*' is the contour consisting of two half-lines $\{\mu: \arg \mu = \pm \psi, |\mu| > \delta\}$ and the arc $\{\mu: |\mu| = \delta, |\arg \mu| \leq \psi\}$; δ is a small positive number; $0 < \psi < \pi$ and ψ is sufficiently close to π ; the direction of by the equality
 $R_{\alpha,q}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} (\mu^{\alpha} - \lambda)^{-q} R_{A_B}(\mu) d\mu$

SEELEY's formulas [24] for the parametrix, which approximates R_{A_B} in \mathcal{L} . Here Γ is the our consisting of two half-lines $\{\mu : \arg \mu = \pm \psi, |\mu$ band χ value problem in G for the dimensional $|a_0(x,\xi)|^2$ (see e.g. [10]). Secondly, one can define the $R_{\alpha,q}(\lambda) = (A_{\beta} \cdot \lambda I)^{-q}$ ($q \in \mathbb{N}$). If $2m\alpha q > n$, this operable following lemma is valid.

Le m m a 1.20

One can take a compact manifold with boundary instead of *0* and consider effiptic boundary value problems for vector functions, including the case of two sectors of ellipticity with a parameter. But we shall not dwell on that.

2. Examples and counterexamples

2.1. Elliptic operators with empty spectrum and with spectrum tilling the whole plane. - Consider an elliptic differential operator of-the form

$$
A = e^{i \beta \cdot x} P(D)
$$

on the *n*-dimensional torus \mathbf{T}^n . Here $P(\xi)$ is a polynomial and β is a non-zero multiindex. (We identify functions on \mathbf{T}^n with appropriate functions of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

2*x*-periodic in every x_i .) Assume first that $P(x) = 0$ for each $\alpha \in \mathbb{Z}^n$. Suppose $Au = \lambda u$ for some $\lambda \in \mathbb{C}$ and some function $u \in L^2(\mathbb{T}^n)$. Substituting here the Fourier expansion

$$
2\pi
$$
-periodic in every x_j .) Assume first that $P(\alpha) \neq 0$ for each $\alpha \in \mathbb{Z}^n$. Suppose $Au = \lambda u$
for some $\lambda \in \mathbb{C}$ and some function $u \in L^2(\mathbb{T}^n)$. Substituting here the Fourier expansion

$$
u(x) = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha} e^{i\alpha \cdot x}
$$
(2.1)
We obtain

$$
P(\alpha) c_{\alpha} = \lambda c_{\alpha+\beta}.
$$
(2.2)
If $\lambda = 0$, from (2.2) it follows that $c_{\alpha} = 0$ for all α , so that $u(x) \equiv 0$. Now let $\lambda \neq 0$.
If $c_{\alpha} = 0$ for some α_0 , then by (2.2)

$$
c_{\alpha_0+k\beta} = \lambda^{-k} P(\alpha_0) P(\alpha_0 + \beta) ... P(\alpha_0 + (k-1)\beta) c_{\alpha_0} \qquad (k \in \mathbb{N}),
$$
so that $c_{\alpha_0+k\beta} \to \infty$ as $\bar{k} \to \infty$. This contradicts the condition $u \in L^2(\mathbb{T}^n)$. Hence $c_{\alpha} = 0$
for all α i.e. $u(x) = 0$. Thus A has no eigenvalues. The same fact can be similarly

•

$$
P(\alpha) c_{\alpha} = \lambda c_{\alpha+\beta}.
$$

 (2.2)

,

If $\lambda = 0$, from $(2,$ If $c_{\alpha} \neq 0$ for some α_0 , then by (2.2) . \overline{z}^n

= $\lambda c_{\alpha+\beta}$. (2.2)

2) it follows that $c_{\alpha} = 0$ for all α , so that $u(x) \equiv 0$. Now let $\lambda \neq 0$.

e α_0 , then by (2.2) $P(\alpha) c_{\alpha} = \lambda c_{\alpha+\beta}$.

from (2.2) it follows that c_{α}
 *C*_{$\alpha_{\alpha}+k\beta} = \lambda^{-k}P(\alpha_0) P(\alpha_0 + \beta)$
 $\lambda + k\beta \rightarrow \infty$ as $k \rightarrow \infty$. This containst} *P(* α *)* $c_a = \lambda c_{a+\beta}$. (2.2)
 If $\lambda = 0$, from (2.2) it follows that $c_a = 0$ for all α , so that $u(x) \equiv 0$. Now let $\lambda = 0$.
 If $c_{a_0} + 0$ for some α_0 , then by (2.2)
 $c_{a_0+k\beta} = \lambda^{-k} P(\alpha_0) P(\alpha_0 + \beta) ... P(\alpha_0 + (k$

$$
c_{\alpha_0+k\beta}=\lambda^{-k}P(\alpha_0) P(\alpha_0+\beta)\ldots P(\alpha_0+(k-1)\beta) c_{\alpha_0} \qquad (k\in\mathbb{N}),
$$

for all α , i.e. $u(x) \equiv 0$. Thus *A* has no eigenvalues. The same fact can be similarly established for A^* . So the spectrum of *A* is empty.
Now consider the case when $P(\alpha_0) = 0$ for some $\alpha_0 \in \mathbb{Z}^n$. Since *A* is elliptic, $P(\xi) \neq 0$

 $P(\alpha) c_{\alpha} = \lambda c_{\alpha+\beta}$.

(2.2)

If $\lambda = 0$, from (2.2) it follows that $c_{\alpha} = 0$ for all α , so that $u(x) = 0$. Now let $\lambda = 0$.

If $c_{\alpha_0} + 0$ for some α_0 , then by (2.2)
 $c_{\alpha_0+k\beta} = \lambda^{-k}P(\alpha_0) P(\alpha_0 + \beta) ... P(\alpha_0 + (k$ for sufficiently large $|\xi|$. Hence we may assume that $P(\alpha_0 - l\beta) \neq 0$ for $l \in N$. For an for all other α . Evidently $Au = \lambda u$ where u ($\in L^2(\mathbf{T}^n)$) is given by (2.1). Thus in this case the eigenvalues of *A* cover the whole plane. If $\lambda = 0$, from (2.2) it follows that $c_a = 0$ for all α , so that $u(x) \equiv 0$. Now let λ

If $c_{a_*} + 0$ for some α_0 , then by (2.2)
 $c_{a_*+k\beta} = \lambda^{-k}P(\alpha_0) P(\alpha_0 + \beta) \dots P(\alpha_0 + (k-1)\beta) c_{a_*}$ $(k \in \mathbb{N})$,

so that c_{a_*+ $\mathcal{L}_{\alpha_0+k\beta} = \lambda^{-k} P(\alpha_0) P(\alpha_0 + \beta) \dots P(\alpha_0 + (k-1) \beta)$
 $\mathcal{L}_{n+k\beta} \to \infty$ as $k \to \infty$. This contradicts the condition

i.e. $u(x) \equiv 0$. Thus *A* has no eigenvalues. The sa

ad for A^* . So the spectrum of *A* is empty.

I Spectral Properties of Elliptic Pacudo-Diff. Op.

251

27. periodic in every z_i .) Assume first that $P(\alpha) \neq 0$ for each $\alpha \in \mathbb{Z}^d$. Suppose $Au = \lambda u$.

for some $\lambda \in \mathbb{G}$ and some function $u \in L^2(\mathbb{T}^*)$. Substit *u*(*x*) *u c*_a = *u*_(*x*) *c*_a - *p*) *i c*_a - *g*) *i c*_a - *g*) *i c*_a - *g*) *i c*_a - *g*) *i y d c*_a = *0 e c*_{*x*} = *i f C*_{*a*} - *g* = *i*^{*f*} $P(\alpha_0 - \beta)$ *i* \rightarrow $P(\alpha_0 -$

2.2. Elliptic operator with an incomplete system of eigenfunctions. Consider the differential operator
 $A = a_0(x) D + a_1(x)$ ($D = -i d/dx$) (2.3

$$
A = a_0(x) D + a_1(x) \qquad (D = -i d/dx) \tag{2.3}
$$

we identify functions on T with appropriate 2π -periodic functions on R. Assume also that *A* is elliptic: $a_0(x) \neq 0$ for all x. Each solution of $Au = \lambda u$ has the form $\begin{array}{ll}\n 2.2. & \text{Ellip}\n \end{array}$

Ellifferenti

on the cilenti

hat A is

. For C is C , set $C_a = 1$, $c_a - u_B = 2$. If $(c_0 - \mu)$ is $c_i = 1$, $c_i = 0$ and c_a .

For all other α . Evidently $Au = \lambda u$ where u $(\epsilon L^2(\mathbf{T}^n))$ is given by (2.1). Thus in

case the eigenvalues of A cover the whole differential operator
 $A = a_0(x) D + a_1(x)$ $(D = -i d/dx)$

on the circle T with complex functions a_k $(k = 1, 2)$. For

we identify functions on T with appropriate 2π -periodic

that A is elliptic: $a_0(x) \neq 0$ for all x. Each

If y functions on T with appropriate
$$
2\pi
$$
-periodic functions on R. Assume also
\ne elliptic: $a_0(x) \neq 0$ for all x. Each solution of $Au = \lambda u$ has the form
\n
$$
u(x) = C \exp \left[i \left(\lambda \int_0^x a_0^{-1}(t) dt - \int_0^x a_1(t) a_0^{-1}(t) dt \right) \right].
$$
\n(2.4)
\n0 this function is 2π -periodic if and only if
\n
$$
\lambda \int_0^{2\pi} a_0^{-1}(t) dt - \int_0^{2\pi} a_1(t) a_0^{-1}(t) dt \in 2\pi \mathbb{Z}.
$$
\nis it follows that in the case when
\n
$$
\int_0^{2\pi} a_0^{-1}(t) dt = 0,
$$
\n(2.5)
\n
$$
\int_0^{2\pi} a_0^{-1}(t) dt = 0,
$$
\n(2.5)
\n
$$
\int_0^{2\pi} a_0^{-1}(t) dt = 0,
$$

$$
\lambda \int\limits_{0}^{2\pi} a_0^{-1}(t) dt - \int\limits_{0}^{2\pi} a_1(t) a_0^{-1}(t) dt \in 2\pi \mathbb{Z}.
$$

From this it follows that in the case when
\n
$$
\int_{0}^{2\pi} a_0^{-1}(t) dt = 0,
$$
\n
$$
\int_{0}^{2\pi} a_0^{-1}(t) dt = 0,
$$
\n
$$
\int_{0}^{2\pi} a_1(t) a_0^{-1}(t) dt \in 2\pi \mathbb{Z}
$$
\n(2.5)

\n(i) The condition is not fulfilled). In the case when

the spectrum of *A* either covers the whole plane (if $\int_{0}^{2\pi} a_1(t) a_0^{-1}(t) dt \in 2\pi \mathbb{Z}$) or is empty (if the condition is not fulfilled). In the case when $^{\circ}$

\n
$$
\int_{0}^{2\pi} a_0(t) \, dt = 0,
$$
\n

\n\n The first two terms of A is not fulfilled. In the case when 0 is a function of t is not fulfilled. In the case when 0 is a function of t is not provided. The result is:\n
$$
\int_{0}^{2\pi} a_0^{-1}(t) \, dt = 0,
$$
\n (2.6)\n

 (2.5)
upty (2.6)
here the spectrum of *A* consists of the eigenvalues $\lambda_k = a(k + c)$ ($k \in \mathbb{Z}$), where the spectrum of *A* either covers the whole plane (if $\int_{a_1}^{2\pi} a_1(t) a_0^{-1}(t) dt \in 2\pi \mathbb{Z}$) or is empty

(if the condition is not fulfilled). In the case when ⁰
 $\int_{0}^{2\pi} a_0^{-1}(t) dt \neq 0$, (2.6)

the spectrum of *A f* $a_0^{-1}(t) dt + 0$,

the spectrum of *A* consists of the eigenvalues $\lambda_k = a(k + c)$ $(k \in \mathbb{Z})$, where
 $a = \left(\frac{1}{2\pi} \int_0^{2\pi} a_0^{-1}(t) dt\right)^{-1}$, $c = \frac{1}{2\pi} \int_0^{2\pi} a_1(t) a_0^{-1}(t) dt$. The λ_k approach the line $z = at$
 $(t \in \math$

V - *^I*

252 \sim M. S. AORANOVICH and A. S. MARKUS

asymptotics $N(\lambda) = 2 |a|^{-1} \lambda + O(1) (\lambda \to \infty)$. By (2.4) the asymptotics $N(\lambda) = 2 |a|^{-1} \lambda + O(1) (\lambda \to \infty)$. By (2.4) the eigenfunction corresponding to the eigenvalue λ_k has the form 252 M. S. AGRANOVICH and A. S. MAREUS

asymptotics $N(\lambda) = 2 |a|^{-1} \lambda + O(1) (\lambda \to \infty)$. By (2.4) the

ing to the eigenvalue λ_k has the form
 $\varphi_k(x) = g^k(x) h(x)$ ($k \in \mathbb{Z}$),

$$
\varphi_k(x) = g^k(x) h(x) \qquad (k \in \mathbf{Z}), \qquad (2.7)
$$

252 • M. S. AORANOVICH and A. S. MARKUS
\nasymptotics
$$
N(\lambda) = 2 |a|^{-1} \lambda + O(1) (\lambda \to \infty)
$$
. By (2.4) the eigenfunction correspond-
\ning to the eigenvalue λ_k has the form
\n
$$
\varphi_k(x) = g^k(x) h(x) \qquad (k \in \mathbb{Z}),
$$
\nwhere
\n
$$
g(x) = \exp\left(ia \int_0^x a_0^{-1}(t) dt\right),
$$
\n
$$
h(x) = \exp\left[i \left(ac \int_0^x a_0^{-1}(t) dt - \int_0^x a_1(t) a_0^{-1}(t) dt\right)\right].
$$
\n(2.8)
\nTheise assertions are contained in [26].
\nLet us now discuss the properties of the system of eigenfunctions (2.7) of A (we assume that (2.6) holds unless otherwise is specified). One can easily deduce from
\n(2.6) that the equation $Au - \lambda_k u = \varphi_k$ has no 2π -periodic solutions. This means that
\nall the root functions of A are eigenfunctions and that the multiplicity of each eigen-

Let us now discuss the properties of the system of eigenfunctions (2.7) of *A* (we
assume that (2.6) holds unless otherwise is specified). One can easily deduce from
(2.6) that the equation $Au - \lambda_k u = \varphi_k$ has no 2π -peri all the root functions of *A* are eigenfunctions and that the multiplicity of each eigenvalue is equal to 1. Denote by Γ the closed curve given by the equation $z = g(x)$ $(0 \le x \le 2\pi)$. Since $g'(x) \ne 0$, Γ is smooth. It does not pass through the origin. assume that (2.6) holds unless otherwise is specified). One can easily deduce from
(2.6) that the equation $Au - \lambda_k u = \varphi_k$ has no 2π -periodic solutions. This means that
all the root functions of A are eigenfunctions and Since Re $\left(a \int a_0^{-1}(t) dt\right)$ is a continuously depending on x value of arg $g(x)$, *F* goes around the origin once in the positive direction while x goes from 0 to 2π (the index.
of $g(x)$ is equal to 1). ²(2.8)

¹ $h(x) = \exp\left[i\left(ac\int^2 a_0^{-1}(t) dt - \int^2 a_1(t) a_0^{-1}(t) dt\right)\right]$.

These assertions are contained in [26].

Let us now discuss the properties of the system of eigenfunctions (2.7) of *A* (we

assume that (2.6) holds unle

Proposition 2.1: The system $\{\varphi_k\}_{-\infty}^{\infty}$ of eigenfunctions of (2.3) is complete in $L^2(\mathbf{T})$ if and only if Γ has no points of self-intersection. If this condition is not satisfied, then *the system has an infinite defect.*

.Proof; If $g(x_1) + g(x_2)$ for $0 \le x_1 < x_2 < 2\pi$, then the function $z = g(x)$ defines a mapping of the segment $[0, 2\pi]$ with identified endpoints onto Γ which is one-to-one and continuous and has continuous inverse. It generates the mapping $f(z) \rightarrow f[g(x)]$ of $L^2(\Gamma)$ onto $L^2(\Gamma)$ which is a continuous (in both directions) isomorphism. Hence the study of geometric properties of ${g^k(x)}_{-\infty}^{\infty}$ (and by the inequality $h(x) \neq 0$ also of $\{\varphi_k(x)\}_{-\infty}^{\infty}$) in $L^2(\mathbb{T})$ is reduced to the study of appropriate properties of $\{z^k\}_{-\infty}^{\infty}$ in $L^2(\Gamma)$. In the case under consideration the system $\{z^k\}$ is complete in $C(\Gamma)$ (see e.g. [28: Chapter II, Theorem 7]) and hence in $L^2(\Gamma)$. Therefore $\{\varphi_k(x)\}$ is complete in $L^2(\mathbf{T})$. of $g(x)$ is equal to 1).

Proposition 2.1: 1

if and only if Γ has no

the system has an infin

Proof: If $g(x_1) \neq g(x_2)$

mapping of the segment

and continuous and has

of $L^2(\Gamma)$ onto $L^2(\Gamma)$ wl

the study of geom Since $\text{Ke}\left(a_j \ a_0^{-1}(t) dt\right)$ is a continuously depending on x value of $g(x)$ is equal to 1). The system $\{p_k\}_{-\infty}^{\infty}$ of eigenfunctions of (2.3) is \mathcal{H} and only if Γ has no points of $\text{sech}\left[\frac{p_0}{2}\right]_{-\infty}^$

Now assume *P* to have at least one point of self-intersection. Since *P* has no cusps and goes around the origin exactly once, the set $C \setminus \Gamma$ has at least one bounded connected component G not containing the origin. The functions z^k ($k \in \mathbb{Z}$) are holomorphic in \bar{C} ; and if some sequence of their linear combinations converges to a function $f(z)$ in $L^2(\Gamma)$, then evidently $f(z)$ must belong to Smirnov's class $E^2(G)$ (see e.g. [20: Chapter III, Section 17.2]). Therefore $\{z^k\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\overline{\Gamma})$; thus $\{\varphi_k(x)\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\overline{\Gamma})$ \blacksquare *:* Let us continuous and ass continuous inverse. It generates the map
 $L^2(I)$ onto $L^2(I)$ which is a continuous (in both directions) is
 $e^{\pm} \sin(\sqrt{p\sqrt{q}})$ (geometric properties of $\left\{\varphi'(\sqrt{q})\right\}_{\infty}^{\infty}$ (and by the

3: Chapter III, Section 17.2]). Therefore $\{z^k\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\Gamma)$;

us $\{\gamma_k(x)\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\Gamma)$ **i**

Let us consider a particular example.

Example 2.2: Let $a_0(x$ Let us consider a particular example.

Let us consider a particular example.

Example 2.2: Let $a_0(x) = (1 + ib e^{ix}) (b \in \mathbb{R}, b \neq \pm 1)$. Then the function (2.8)

has the form $g(x) = \exp[i(x + b e^{ix} - b)]$. By Proposition 2.1, $\langle \varphi_k(x) \$ in $L^2(T)$ if and only if for some $k \in \mathbb{Z}$ the system of equations $x_1 - x_2 + b(\cos x_1 - \cos x_2) = 2k\pi$, sin $x_1 = \sin x_2$ has a solution (x_1, x_2) with $0 \le x_1 < x_2 < 2\pi$. It is easily seen that, if $|b| > \pi/2$, such a solutio $-cos x_2 = 2k\pi$, $sin x_1 = sin x_2$ has a solution (x_1, x_2) with $0 \le x_1 < x_2 < 2\pi$. It is easily seen that, if $|b| > \pi/2$, such a solution exists for $k = -1$ and, if $|b| \in (1, \pi/2]$, for $k = 0$. If $b \in (-1, 1)$, then the function arg $g(x) = x + b(\cos x - 1)$ is increasing, 。
is: $-\cos x_2 = 2k\pi$, $\sin x_1 = \sin x_2$ has a solution (x_1, x_2) with $0 \le x_1 < x_2 < 2\pi$. It is
easily seen that, if $|b| > \pi/2$, such a solution exists for $k = -1$ and, if $|b| \in (1, \pi/2]$,
for $k = 0$. If $b \in (-1, 1)$, then the functio

Remark 2.3: The completeness of $\{\varphi_k(x)\}_{-\infty}^{\infty}$ in $L^2(\mathbb{T})$ yields its completeness in Sobolev's space $H_l(\mathbf{T})$ for each $t \in \mathbf{R}$. Indeed, if λ_0 is a regular point of *A*, then $B_n = (A - \lambda_0 I)^{-n}$ maps Spectral Properties of Elliptic Pseudo-Diff.

Remark 2.3: The completeness of $\langle \varphi_1(x) \rangle_{-\infty}^{\infty}$ in $L^2(\mathbb{T})$ yields its completeness

space $H_l(\mathbb{T})$ for each $t \in \mathbb{R}$. Indeed, if λ_0 is a regular point of A, s comp

en B_n :

ion). Tl
 $J^{-n} \varphi_k$.

easily
 2^n
 2^n
 $\ell = \int \ell$ Spe

Remark 2.3: The completeness of $\{ \text{space } H_i(\mathbf{T}) \text{ for each } t \in \mathbf{R} \text{. Indeed, if } \lambda_0 \mathbf{Z}^2(\mathbf{T}) \text{ onto } H_n(\mathbf{T}) \text{ isomorphically and co-
is complete in } H_n(\mathbf{T}) \text{, and it remains to
Remark 2.4: By means of Levy's t
points of self-intersection if and only
implies that $x_2 - x_1 = 2\pi$.
 \therefore Remark 2.5: If (2.5) holds and the s
there corresponds the infinite chain of ro
 $$$ es of Elliptic Pseudo-Diff. Op. 25.

²(T) yields its completeness in Sobolev'

²(T) yields its completeness in Sobolev'

oint of A, then $B_n = (A - \lambda_0 I)^{-n}$ map.

1 both direction). Therefore $\{B_n \varphi_k\}_{k=-\infty}^{\infty}$
 $\$

Remark 2.4: By means of Levy's theorem [16: $\S 34$] one can easily show that Γ has no points of self-intersection if and only if the equality $\int a_0^{-1}(t) dt = \int a_0^{-1}(t) dt'(x_1, x_2 \in \mathbb{R})$

 ℓ Remark 2.5: If (2.5) holds and the spectrum of (2.3) covers the whole plane, to any $\lambda \in \mathbb{C}$ there corresponds the infinite chain of root functions $u_{\lambda,k}(x) = d^k u_1(x) / d\lambda^k$ ($k = 0, 1, \ldots$), where $u_{\lambda,0}(x) = u_{\lambda}(x)$ is an eigenfunction. In accordance with (2.4), $u_1(x)$ can be put down in the form

$$
u_1(x) = v(x) \exp (i\lambda \delta^{-1} w(x)), \quad w(x) = \delta \int_0^x a_0^{-1}(t) dt,
$$

Example 2.4: By means of Levy's theorem [16: § 34] one can easily show that Γ has no

points of self-intersection if and only if the equality $\int_{z_1}^{z_0-1}(t) dt = \int_{z_1}^{z_0-1}(t) dt'(x_1, x_2 \in \mathbb{R})$

implies that $x_2 - x$ where $\delta > 0$ has been chosen so small that $|\text{Re } w(x)| < \pi/2$ ($0 \le x \le 2\pi$). Then the curve $\gamma = \{z = \exp(iw(x)) : 0 \le x \le 2\pi\}$ lies in the open right half-plane and the functions $d^kz^{l/\delta}/$ $d\lambda^k$ ($\lambda \in \mathbb{C}$; $k = 0, 1, \ldots$) are holomorphic (in *z*) in each bounded component of the complement of y. It follows immediately that the closed linear span of the root functioris of *A* has an infinite defect in $L^2(\mathbf{T})$ (cf. with the second part of the proof of Proposition-2.1).

Now *we* shall give some examples of operators on a two-dimensional manifold with 'incomplete systems of eigenfunetions.

Now we shall give some examples of operators on a two-dimensional manifold with
incomplete systems of eigenfunctions.
Example 2.6: Consider the elliptic differential operator $A = a_0(x) (D_x + i\overline{D}_y)$ on
T² (we write $(x, \hat$ $\int a_0^{-1}(t) dt = 2\pi$. Let $u(x, y)$ be an eigenfunction of *A*. Expand. it in Fourier series in $y: u(x, y) = \sum_{-\infty}^{\infty} v_i(x) e^{i l y}$. Substituting this into the equation $Au = \lambda u$, we obtain $a_0(x) (D_x + i l) v_l(x) = \lambda v_l(x)$ for every $l \in \mathbb{Z}$, from which $v_l(x) = \exp\left(i \int_0^{\infty} a_0^{-1}(l) dl + lx\right)$ (up to a numerical multiplier). The • ' $a_0(x) (D_x + i\vec{l}) v_l(x) = \lambda v_l(x)$ for every $l \in \mathbb{Z}$, from which $v_l(x) = \exp \left(i \lambda \int a_0^{-1}(t) dt\right)$ + lx) (up to a numerical multiplier). The condition of 2π -periodicity of this function
gives $2\pi i\lambda + 2\pi l = 2\pi i k$, so we obtain the set of eigenvalues
 $\lambda_{k,l} = k - i\bar{l}$ (k, $l \in \mathbb{Z}$) - (2.9) 3: Consider the elliptic differential operator $A = a_0(x) (D_x + iD_y)$ on \hat{y}) instead of (x_1, x_2)). The C^{∞} -function $a_0(x)$ is normalized so that π . Let $u(x, y)$ be an eigenfunction of A. Expand it in Fourier ser *Uk = 2n.* Let $u(x, y)$ be an eigenfunction of *A*. Expand it in Fourier series in
 $=\sum_{-\infty}^{\infty} v_i(x) e^{i l y}$. Substituting this into the equation $Au = \lambda u$, we obtain
 $+i\lambda v_i(x) = \lambda v_i(x)$ for every $l \in \mathbb{Z}$, from which $v_i(x)$ Spectral Properties of Elliptic Foundable Diff. Operators with the properties of the properties

$$
k_{i} = k - i l \qquad (k, l \in \mathbf{Z}) \tag{2.9}
$$

and the set of eigenfunetions

•

• -

$$
u_{k,l}(x,y) = \exp\left[i(k - il)\int_{0}^{x} a_0^{-1}(t) dt + l(x + iy)\right].
$$
 (2.10)

One can easily verify that $\{\bar{\lambda}_{k,l}\}$ is the set of all eigenvalues of A^* and that there are no root functions of *A* except eigenfunctions. So the spectrum of *A* coincides with the set (2.9) of its eigenvalues and all of them are simple. It is not difficult to verify also that the system (2.10) of eigenfunctions of *A* is complete in $L^2(\mathbb{T}^2)$ if and only if the system of eigenfunctions of $a_0(x) D_x$ is complete in $L^2(T)$. Using Proposition 2.1, we obtain examples of two-dimensional elliptic differential operators with incomplete systems of eigenfunctions. Note that the modules of the eigenvalues (2.9) are equal to $(k^2 + l^2)^{1/2}$ and coincide with the eigenvalues of the self-adjoint pseudo-differential operator $(D_x^2 + D_y^2)^{1/2}$. They clearly have the regular asymptotics. and the set of eigenfunctions
 u_k , $\vec{l}(x, y) = \exp\left[i(k - il)\int_0^x a_0^{-1}(t) dt + l(x + iy)\right]$. (2.10)

One can easily verify that $\{\vec{l}_{k,l}\}\$ is the set of all eigenvalues of A^* and that there are

no root functions of A except ei

2.3. Elliptic operators, with complete systems of eigenfunctions which are not bases.

Proposition 2.7: Let the system of eigenfunctions of the operator (2.3) be complete in $L^2(\mathbb{T})$. This system is a basis in $L^2(\mathbb{T})$ if and only if arg $a_0(x) \equiv \text{const.}$

Proof: For $z \in \Gamma$, $|z| = \exp \left(-\text{Im}\left(a \int_{0}^{z} a_0^{-1}(t) dt\right)\right]$, hence Γ is a circle with the

center at the origin if and only if $\text{Im}\left(a\int_a^a a_0^{-1}(t) dt\right) \equiv \text{const, i.e. if } \text{Im}\left(aa_0^{-1}(x)\right) \equiv 0,$ which is equivalent to $\arg a_0(x) \equiv \text{const.}$ So it remains to show that $\{z^k\}_{k=1}^{\infty}$ is a basis in $L^2(\Gamma)$ if and only if Γ is the circle with the center in the origin. The sufficiency is obvious (and the basis in this case is orthogonal); we must verify the necessity. Let $x = x(z)$ be the function inverse to $z = q(x)$. Then obviously the system

$$
u_k(z) = i \bar{z}^{-k-1} \, \overline{g'(x(z))} \, \big(2\pi \, |g'(x(z))| \big)^{-1} \qquad (k \in \mathbb{Z})
$$

is biorthogonal to $\{z^k\}_{-\infty}^{\infty}$. Set $r = \min \{|z| : z \in \Gamma\}$, $R = \max \{|z| : z \in \Gamma\}$ and suppose $r < R$. Fix numbers r_0 , R_0 with $r < r_0 < R_0 < R$ and set $E_1 = \{z \in \Gamma : |z| > R_0\}$, $E_2 = \{z \in \Gamma : |z| < r_0\}.$ It is easily seen that, for $k \in \mathbb{N}$, $||z^k||_{L^1(\Gamma)} \ge R_0^k \delta_1^{1/2}$, $||u_k||_{L^1(\Gamma)}$ $\geq (2\pi)^{-1} r_0^{-k-1} \delta_2^{1/2}$, where $\delta_k = \text{mes } E_k$. Hence

$$
|z^k||_{L^1(\Gamma)}||u_k||_{L^1(\Gamma)} \to \infty \qquad (k \to +\infty).
$$

It follows (see e.g. [17: Chapter III, §6, p. 170]) that $\langle z^k \rangle_{-\infty}^{\infty}$ is not a basis in $L^2(\Gamma)$ (and no permutation can make it a basis) |

Remark 2.8: It is easily seen that in case arg $a_0(x) \equiv$ const the system $\{\varphi_k\}_{-\infty}^{\infty}$ is an unconditional basis in $H_t(\mathbf{T})$ for each $t \in \mathbf{R}$.

Remark 2.9: In case arg $a_0(x) \neq \text{const}$ the system $\langle \varphi_k \rangle_{-\infty}^{\infty}$ is also not a basis with parentheses. For, suppose the contrary. Then $\{z^k\}_{-\infty}^{\infty}$ is a basis with parentheses in $L^2(\Gamma)$, i.e. there exist increasing sequences $\{m_k\}_1^{\infty}$, $\{n_k\}_1^{\infty}$ of positive integers such that

$$
\left\|\sum_{j=-m_k}^{n_k-1}c_jz^j-\tilde{f}(z)\right\|_{L^1(\Gamma)}\to 0 \qquad (k\to\infty) \text{ for any } f\in L^2(\tilde{\Gamma}),
$$

where c_i are the Fourier coefficients of $f(z)$ with respect to $\{z^j\}$. Let P be the natural projector to the corresponding Smirnov's space $E²(G)$. Since it is bounded, we obtain

$$
\left\|\sum_{i=k}^{\infty}\sum_{j=n_i}^{n_{i+1}-1}c_jz^j\right\|_{L^1(\Gamma)}\to 0,\quad \text{i.e.}\quad \|z^{n_k}P(z^{-n_k}f\|_{L^2(\Gamma)}\to 0\qquad (k\to\infty).
$$

Therefore the norms of the operators $z^{n_k}P(z^{-n_k}\cdot)$ in $L^2(\Gamma)$ are uniformly bounded. It follows that $\sup ||P||_{L^{4}(\Gamma, |z|^{2n_{k}})} < \infty$, where $L^{2}(\Gamma, |z|^{2n_{k}})$ is the L^{2} -space with appropriate weight. By the Stein-Weiss theorem (see e.g. [5: Section 5.4]), we obtain sup $\{||P||_{L^1(\Gamma, |z|^{1n}: n \geq n_1\}} < \infty$. So we may conclude that the norms of $z^n P(z^{-n})$ ($n \ge 1$) in $L^2(\Gamma)$ are uniformly bounded. Since $c_n z^n$ $= z^n P(z^{-n}) - z^{n-1} P(z^{1-n})$, we have $|c_n| ||z^n||_{L^1(\Gamma)} \le c ||f||_{L^1(\Gamma)}$ $(n \ge 1)$. This contradicts (2.11) because of $c_n = \int f(z) \overline{u_n(z)} |dz|$.

Remark 2.10: Assume that $|\arg a_0(x)| \leq \theta < \pi/2$. Then, for any $\alpha > 0$, the Fourier series of $u \in H_r(\mathbb{T})$ with respect to the eigenfunctions of (2.3) is summable by Abel's method of order α if α is greater than 1 and close enough to 1 (see [2: § 35]). In the present case there is no need of parenthesis, which generally one puts into the series with Abelian factors $\exp(-\lambda_k^{\alpha}t)$ for convergence, since the series $\sum \exp(-\lambda_k t) c_k \varphi_k(x)$ converges for all $t > 0$ (here c_k are the Fourier coefficients of $u(x)$ with respect to $\{\varphi_k\}$).

 (2.11)

Remark 2.11: Returning to Example 2.6, we can easily verify that the system of eigenfunctions of $a_0(x)$ $(D_x + iD_y)$ on \mathbf{T}^2 is complete bat is not a basis in $H_t(\mathbf{T}^2)$ $(t \ge 0)$ if and only if the same is true for the system of eigenfunctions of $a_0(x) D_x$ on T.

2.4 Examples for theorems on rough asytnptoties from Subsection 1.3. L'et *A* be an elliptic pseudo-differential operator of order $t > 0$ on the *n*-dimensional torus Tⁿ. If *A* is a *normal* operator (i.e. $A^*A = AA^*$; for example if *A* is a differential-operator with constant coefficients), then the modules of its eigenvalues coincide with eigenvalues of the elliptic pseudo-differential operator $(A^*A)^{1/2}$, and in virtue of (1.40) Spectral Properties of Elliptic Pseud
 N 2.11: Returning to Example 2.6, we can easily verify that

of $a_0(x) (D_x + iD_y)$ on T² is complete but is not a basis in $H_t(T$

is true for the system of eigenfunctions of $a_0(x) D$ Let A be an letter of \mathbf{r} . If it is the set of (1.40) Remark 2.11: Returning to Example 2.6, we can easily verify that the system of eigentuations of $a_0(x) (D_x + iD_y)$ on \mathbf{T}^2 is complete but is not a basis in $H_l(\mathbf{T}^2)$ $(l \ge 0)$ if and if the same is true for the syste *A* is true for the system of eigenfunctions of
 A and the system of eigenfunctions of
 A and operator (i.e. $A^*A = AA^*$; for ex
 A stant coefficients), then the modules of

the elliptic pseudo-differential operator
 casily verify that the system of eigen-

not a basis in $H_t(\mathbb{T}^2)$ ($t \ge 0$) if and only
 $a_0(x) D_x$ on **T**.

ics from Subsection 1.3. Let A be an

0 on the *n*-dimensional torus \mathbb{T}^n . If

ample if A is a differenti mal operator (i.e. $A^*A = AA^*$; for example if A is a differential-operator
tant coefficients), then the modules of its eigenvalues coincide with eigen-
he elliptic pseudo-differential operator $(A^*A)^{1/2}$, and in virtue

$$
N(\lambda) \sim \Delta \lambda^{n/t} \qquad (\lambda \to \infty), \tag{2.12}
$$

where Δ is defined by (1.33).

- Assume for simplicity that $n = 2$ and consider the following differential operator on T^2 which admits the separation of variables:

$$
A = [a_0(x) D_x]^2 + [b_0(y) D_y]^2; \qquad (2.13)
$$

here the functions $a_0(x)$, $b_0(y)$ are C^{∞} , 2π -periodic and non-zero everywhere. Assume that

$$
|\arg a_0(x)| \leq \theta_1, \qquad |\arg b_0(y)| \leq \theta_2, \qquad \theta_1 + \theta_2 < \pi/2. \tag{2.14}
$$

This provides the ellipticity of *A* and even its ellipticity with a parameter in some angular neighbourhood of R₋. Hence the spectrum of *A* does not cover the whole

with constant coefficients), then the modules of its eigenvalues coincide with eigenvalues of the elliptic pseudo-differential operator
$$
(A^*A)^{1/2}
$$
, and in virtue of (1.40)
\n $N(\lambda) \sim \Delta \lambda^{n/l}$ $(\lambda \to \infty)$, (2.12)
\nwhere Δ is defined by (1.33).
\nAssume for simplicity that $n = 2$ and consider the following differential operator
\non T² which admits the separation of variables:
\n $A = [a_0(x) D_x]^2 + [b_0(y) D_y]^2$;
\nhere the functions $a_0(x)$, $b_0(y)$ are C^{∞} , 2π -periodic and non-zero everywhere. Assume
\nthat
\n $|\arg a_0(x)| \leq \theta_1$, $|\arg b_0(y)| \leq \theta_2$, $\theta_1 + \theta_2 < \pi/2$. (2.14)
\nThis provides the ellipticity of A and even its ellipticity with a parameter in some
\nangular neighbourhood of R. Hence the spectrum of A does not cover the whole
\nplane.
\nRecall that the spectra of $a_0(x) D_x$ and $b_0(y) D_y$ consist of eigenvalues ak. $(k \in \mathbb{Z})$ and
\n bl $(l \in \mathbb{Z})$, respectively, where $a = \left(\frac{1}{2\pi} \int_0^{2\pi} a_0^{-1}(x) dx\right)^{-1}$, $b = \left(\frac{1}{2\pi} \int_0^{2\pi} b_0^{-1}(y) dy\right)^{-1}$,
\nand that the systems $\{\varphi_k(x)\}_{-\infty}^{\infty}$ and $\{\psi_l(y)\}_{-\infty}^{\infty}$ of corresponding eigenfunctions are

complete in $L^2(T)$. This follows, for instance, from asserted in Subsection 2.2. Indeed, by (2.14) Re $a_0^{-1}(x) > 0$ and Re $b_0^{-1}(y) > 0$, therefore $a_0(x)$ and $b_0(y)$ satisfy (2.6) and the arguments of the corresponding functions (2.8) are monotonic. d that the systems $\{\varphi_k(x)\}_{-\infty}^{\infty}$ and $\{\psi_l(y)\}_{-\infty}^{\infty}$ of complete in $L^2(\mathbf{T})$. This follows, for instance, from ass $r(2.14)$ Re $a_0^{-1}(x) > 0$ and Re $b_0^{-1}(y) > 0$, therefore e arguments of the corresponding fun

numbers of the corresponding functions (2.8) are monotonic.

\nently A has the eigenvalues

\n
$$
a^2k^2 + b^2l^2 \qquad (k, l \in \mathbb{Z}),
$$
\n(2.15)

corresponding to the eigenfunctions $\varphi_k(x)$ $\psi_l(y)$. Since the system $\{\varphi_k(x)$ $\psi_l(y)\}$ is comcorresponding to the eigenfunctions $\varphi_k(x) \psi_l(y)$. Since the system $\{\varphi_k(x) \psi_l(y)\}\$ is complete in $L^2(\mathbb{T}^2)$, the set of all eigenvalues of *A* (repeated according to their multiplicities) coincides with (2.15). The n ties) coincides with (2.15). The normal differential operator $a^2D_x^2 + b^2D_y^2$ has the of A has the asymptotics $N(\lambda) \sim d_0 \lambda^{n/t}$, where that the system
plete in $L^2(\mathbf{T})$. The $a_0^{-1}(x)$:
2.14) Re $a_0^{-1}(x)$:
inguments of the
idently A has t
 $a^2k^2 + b^2l^2$
esponding to the
in $L^2(\mathbf{T}^2)$, the securious with
exigenvalues, and
has the asympt
 $d_0 = \frac{1}{2$

same eigenvalues, and by (2.12) the counting function
$$
N(\lambda)
$$
 for modules of eigenvalues
of A has the asymptotics $N(\lambda) \sim d_0 \lambda^{n/l}$, where

$$
d_0 = \frac{1}{2} \int_{0}^{2\pi} |a^2 \cos^2 \theta + b^2 \sin^2 \theta|^{-1} d\theta.
$$
 (2.16)
Write down the quantities d and Δ for (2.13). By (1.32)

$$
d = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{2\pi} du \int_{0}^{n/2} (a^2(x) \cos^2 \theta + b^2(u) \sin^2 \theta)^{-1} d\theta
$$

Write down the quantities d and Δ for (2.13). By (1.32)

$$
d_0 = \frac{1}{2} \int_0^{2\pi} |a^2 \cos^2 \theta + b^2 \sin^2 \theta|^{-1} d\theta.
$$
 (2.16)
\n
$$
\text{Write down the quantities } d \text{ and } \Delta \text{ for (2.13). By (1.32)}
$$
\n
$$
d = \frac{1}{2\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\pi/2} (a_0^2(x) \cos^2 \theta + b_0^2(y) \sin^2 \theta)^{-1} d\theta
$$
\n
$$
= \frac{1}{2\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\infty} (a_0^2(x) + b_0^2(y) \tau^2)^{-1} d\tau = \frac{1}{4\pi} \int_0^{2\pi} \frac{dx}{a_0(x)} \int_0^{2\pi} \frac{dy}{b_0(y)}.
$$
\n(2.17)

 $255\,$

256 M. S. AORANOVICH and A. S. MARKUS
\nThus
\n
$$
d = \frac{\pi}{ab} = \frac{1}{2} \int_{0}^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{-1} d\theta.
$$
\n(2.18)
\nComparing (2.16) with (2.18), we see that $|d| = d_0$ if and only if arg $a = \arg b$. Further,
\ncomparing the initial expression for d in (2.17) with the equality

Comparing (2.16) with (2.18), we see that $|d| = d_0$ if and only if $\arg a = \arg b$. Further, comparing the initial expression for *d* in (2.17) with the equality

$$
d = \frac{\pi}{ab} = \frac{1}{2} \int_{0}^{a} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{-1} d\theta.
$$

ng (2.16) with (2.18), we see that $|d| = d_0$ if and only if arg $a =$ ng the initial expression for d in (2.17) with the equality

$$
d = \frac{1}{2\pi^2} \int_{0}^{2\pi} dx \int_{0}^{2\pi} dy \int_{0}^{\pi/2} |a_0^2(x) \cos^2 \theta + b_0^2(y) \sin^2 \theta|^{-1} d\theta,
$$

which follows from (1.33), we conclude that $|d| \ll \Delta$ if at least one of the functions $a_0(x)$, $b_0(y)$ has a non-constant argument. Take $a_0(x)$ with a non-constant argument. Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < 1$, while if $b_0(x) = a_0(x) e^{i\epsilon}$ we have (for which follows from (1.33), we conclude that $|d| < \Delta$ if at least one of the functions $a_0(x)$, $b_0(y)$ has a non-constant argument. Take $a_0(x)$ with a non-constant argument.
Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < \Delta$, wh which follows from (1.33), we conclude that $|d| < \Delta$ if at least one of the functions $a_0(x)$, $b_0(y)$ has a non-constant argument. Take $a_0(x)$ with a non-constant argument.
Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < \Delta$, wh $b_0(y) \equiv b$ with arg $a + \arg b$. Then $|d| < d_0 = \angle l$. The last equality holds because in this case A is normal. So we have shown that for (2.13) all the cases (0.7) are possible. $a_0(x)$, $b_0(y)$ has a non-constant argument. Take $a_0(x)$ with a non-constant argument.

Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < \Delta$, while if $b_0(x) = a_0(x) e^{i\epsilon}$ we have (for sufficiently small $\epsilon > 0$) $|d| < d_0 < \Delta$. Now Comparing (2.1b) with (2.18), we see that $|d| = d_0$ if and only i
comparing the initial expression for d in (2.17) with the equal
comparing the initial expression for d in (2.17) with the equal
 $\Delta = \frac{1}{2\pi^2} \int_0^{2\pi} dx \int$ 256 M. S.

Thus
 $d =$
 \therefore Thus
 $d =$
 \therefore Comparing (2

comparing th
 \therefore $\$ $\frac{1}{2\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\pi/2} |a_0^2(x) \cos^2 \theta + b_0^2(y) \sin^2 \theta|^{-1} d\theta$,

from (1.33), we conclude that $|d| < \Delta$ if at least one of the functions

is a non-constant argument. Take $a_0(x)$ with a non-constant argume Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < 1$, while if $b_0(x) = a_0(x)$ even have $[(for
sufficiently small $\varepsilon > 0)$ $|d| < d_0 < 1$. Now take constant functions $a_0(x) \equiv a$ and $b_0(y) \equiv b$ with arg $a \neq \arg b$. Then $|d| < d_0 = 1$. The last equality holds because in
this case A is normal. So we have shown that for (2.13) all the cases (0.7) are possible.
2.5. Another class of elliptic$ which follows from (1.33), we conclude that $|d| \le a_0(x), b_0(y)$ has a non-constant argument. Take a_0
Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < \Delta$, whil
sufficiently small $\varepsilon > 0$ $|d| < d_0 < \Delta$. Now take
 $b_0(y) \equiv b$ with arg

$$
A = \sum_{k=0}^{\infty} a_k (e^{ix}) D_x^{n-k}
$$
 (2.19)

on the circle **T**. Assume that $a_k(\zeta)$ belongs to $C^{\infty}(\mathbf{T})$ and admit holomorphic extensions (y) $\equiv b$ with arg $a + \arg b$. Then $|d| < d_0 = A$. The last equality holds because in
is case *A* is formal. So we have shown that for (2.13) all the cases (0.7) are possible
5. Another class of elliptic operators with a regul Ther class of elliptic operators with a regular behaviour of eigenvalifferential operator
 $A = \sum_{k=0}^{\infty} a_k(e^{iz}) D_x^{n-k}$

Cle T. Assume that $a_k(\zeta)$ belongs to C^{∞} (T) and admit holomorphic

disk $\{\zeta : |\zeta| < 1\}$, con into the disk $\langle \zeta : |\zeta| < 1$, continuous up to the boundary, and that $a_0(\zeta) \neq 0$ for $|\zeta| \leq 1$.
Proposition 2.12: The set of eigenvalues of (2.19) coincides with the set Finally shown that for (2.13) all the cases (0.7) are possible.
 c operators with a regular behaviour of eigenvalues. Concretion
 $a_k(\zeta)$ belongs to $C^{\infty}(\mathbf{T})$ and admit holomorphic extensions

continuous up to the 2.5. Another class of elliptic operators with a regular behaviour of eigenva

sider the differential operator
 $A = \sum_{k=0}^{\infty} a_k(e^{iz}) D_x^{n-k}$

on the circle **T**. Assume that $a_k(\zeta)$ belongs to $C^{\infty}(\mathbf{T})$ and admit holom cole **T**. Assume that $a_k(\zeta)$ belongs to $C^{\infty}(\mathbf{T})$ and admit holomorphic extensions

disk $|\zeta: |\zeta| < 1$, continuous up to the boundary, and that $a_0(\zeta) \neq 0$ for

sittion 2.12: The set of eigenvalues of (2.19) coin

$$
\sum_{k=0}^n a_k(0) \; m^{n-k} \qquad (m \in \mathbb{Z});
$$

moreover, the multiplicity of each eigenvalue λ_0 *is equal to the number of such m* \in **Z** *that* λ_0 *can be written down in the form* (2.20).

To prove this, we shall need two lemmas.

 $k \cdot$ Lemma 2.13: Let the numbers $a_{n,k}$ $(k = 1, ..., n; n = 1, 2, ...)$ *satisfy the recurrence relations*

$$
a_{n,k} = a_{n-1,k} + (n-k+1) a_{n-1,k-1} \qquad (1 < k < n) \qquad (2.21)
$$

and $a_{n,1} = a_{n,n} = 1$. *Then, for all* ϱ ,

• -

$$
\sum_{k=1}^{n} a_{n,k} \varrho(\varrho-1) \dots (\varrho-n+k) = \varrho^{n}.
$$
\n(2.22)

Proposition 2.12: The set of eigenvalues of (2.19) coincides with the set
 $\sum_{k=0}^{n} a_k(0) m^{n-k}$ ($m \in \mathbb{Z}$);
 $\sum_{k=0}^{n} a_k(0) m^{n-k}$ ($m \in \mathbb{Z}$);
 \therefore (2.20)
 \therefore are b *a a be written down in the form* and $a_{n,1} = a_{n,n} = 1$. Then, for all ϱ ,
 $\sum_{k=1}^{n} a_{n,k} \varrho(\varrho - 1) \dots (\varrho - n + k) = \varrho^n$.

Próof: We shall verify (2.22) by induction with respect to *n*. As a preliminary, we

note that (2.22) is obviously valid for $\varrho =$ $\sum_{k=1}^{\infty} a_{n,k} \varrho(\varrho - 1) \dots (\varrho - n + k) = \varrho^n.$

Proof: We shall verify (2.22) by induction with respect

note that (2.22) is obviously valid for $\varrho = 0$, and that dividenting $\varrho = k + 1$ ($k = 0, ..., n - 1$) we obtain, for these and $a_{n,1}$
 \vdots *a*, *x* **c** 2.22 is obviously valid for $\rho = 0$, and that dividing both the sides by ρ and $\Rightarrow k + 1$ ($k = 0, ..., n - 1$) we obtain, for these k , $a_{n,n} + ka_{n,n-1} + k(k - 1) a_{n,n-2} + ... + k! a_{n,n-k} = (k + 1)^{n-1}$. (2.23)

$$
a_{n,n} + ka_{n,n-1} + k(k-1) a_{n,n-2} + \ldots + k! a_{n,n-k} = (k+1)^{n-1}.
$$
 (2.23)

On the other hand, since both the sides of (2.22) are polynomials of degree *n* in ρ , setting $\rho = k + 1$ ($k = 0, ..., n - 1$) we obtain, for these k ,
 $a_{n,n} + ka_{n,n-1} + k(k-1) a_{n,n-2} + ... + k! a_{n,n-k} = (k+1)^{n-1}$. (2.23)

On the other hand, since both the sides of (2.22) are polynomials of degree *n* in ρ ,

(2.22) foll obviously holds. Suppose now that it holds for some positive integer n , and verify

obviously holds. Suppose now that it holds for some positive integer n, and verify
\nthat it remains true after replacing n by
$$
n + 1
$$
. From (2.21) it follows that, for
\n $k = 0, ..., n - 1$,
\n $a_{n+1,n+1} + ka_{n+1,n} + k(k-1) a_{n+1,n-1} + ... + k! a_{n+1,n+1-k}$
\n $= a_{n,n} + k(a_{n,n} + 2a_{n,n-1}) + k(k-1)(a_{n,n-1} + 3a_{n,n-2}) + ...$
\n $+ k! (a_{n,n+1-k} + (k+1) a_{n,n-k})$
\n $= (k + 1) a_{n,n} + (k + 1) ka_{n,n-1} + (k + 1) k(k-1) a_{n,n-2} + ...$
\n $+ (k + 1)! a_{n,n-k}$
\n $= (k + 1) (a_{n,n} + ka_{n,n-1} + k(k-1) a_{n,n-2} + ... + k! a_{n,n-k})$
\n $= (k + 1) (k + 1)^{n-1} = (k + 1)^n$
\n(in the next to the last equality we have used (2.23) with $k = 0, ..., n - 1$). Thus
\nwe have proved (2.23) with the replacement of n by $n + 1$, for $k = 0, ..., n - 1$.
\nThis yields (2.22) with $n + 1$ instead of n
\nSet $\partial = \partial_c = d/d\zeta$. Consider the equation
\n $\zeta^n \partial^n y + \sum_{k=0}^{n-1} \zeta^k r_k(\zeta) \partial^k y = 0$.
\nLemma 2.14: Let $r_k(\zeta)$ be holomorphic in $\{\zeta : |\zeta| < 1\}$ and continuous in $\{\zeta : |\zeta| \leq 1\}$.
\nIf the equation (2.14) has a non-trivial solution on $\{\zeta : |\zeta| = 1\}$, then at least one root

(in the next to the last equality we have used (2.23) with $k = 0, ..., n - 1$). Thus we have proved (2.23) with the replacement of *n* by $n + 1$, for $k = 0, ..., n - 1$. **-**
 ., <u>*.*_{*1*}
 ., *2***1**</u> This yields (2.22) with $n + 1$ instead of $n \in \mathbb{R}$
Set $\partial = \partial_{\zeta} = d/d\zeta$. Consider the equation . -S -

$$
\zeta^n \partial^n y + \sum_{k=0}^{n-1} \zeta^k r_k(\zeta) \; \partial^k y = 0. \tag{2.24}
$$

is yields (2.22) with $n + 1$ instead of $n \in \mathbb{R}$

Set $\partial = \partial_{\zeta} = d/d\zeta$. Consider the equation
 $\zeta^{n} \partial^{n} y + \sum_{k=0}^{n-1} \zeta^{k} r_{k}(\zeta) \partial^{k} y = 0$. (2.24)

Lemma 2.14: Let $r_{k}(\zeta)$ be holomorphic in $\{\zeta : |\zeta| < 1\}$ *1f the equation* (2.14) has a non-trivial solution on $\{\zeta : |\zeta| = 1\}$, then at least one root we have proved (2.23) with *n* -
This yields (2.22) with *n* -
Set $\partial = \partial_{\zeta} = d/d\zeta$. Cons
 $\zeta^n \partial^n y + \sum_{k=0}^{n-1} \zeta^k r_k(\zeta)$
Lemma 2.14: Let $r_k(\zeta)$
If the equation (2.14) has
of the equation
 $\varrho(\varrho - 1) \dots (\varrho -$

$$
\varrho(\varrho-1)\ldots(\varrho-n+1)+\sum_{k=1}^{n-1}r_k(0)\varrho(\varrho-1)\ldots(\varrho-k+1)+r_0(0)=0
$$
\n(2.25)

is an integer.

•

•

Proof: Let $y(\zeta) \neq 0$ be a solution of (2.24) on $\zeta : |\zeta| = 1$. Since the coefficients of the equation are holomorphic for $0 < |\zeta| < 1$ and continuous for $0 < |\zeta| \leq 1$, is an integer.

Proof: Let $y(\zeta)$ ($\neq 0$) be a solution of (2.24) on $\zeta : |\zeta| = 1$. Since the coefficients

of the equation are holomorphic for $0 < |\zeta| < 1$ and continuous for $0 < |\zeta| \leq 1$,

the function $y(\zeta)$ can be Proof: Let $y(\zeta) (\not\equiv 0)$ be a solution of (2.24) on $\{\zeta : |\zeta| = 1\}$. Since the coefficients
of the equation are holomorphic for $0 < |\zeta| < 1$ and continuous for $0 < |\zeta| \le 1$,
the function $y(\zeta)$ can be extended on $\{\z$ is an integer.

Proof: Let $y(\zeta)$ ($\equiv 0$) be a solution of (2.24) on $\{\zeta : |\zeta| = 1\}$. Since the coefficients

of the equation are holomorphic for $0 < |\zeta| < 1$ and continuous for $0 < |\zeta| \le 1$,

the function $y(\zeta)$ can b each ζ_0 with $|\zeta_0|=1$, and by the uniqueness theorem [7] $y_0(\zeta)=y_1(\zeta)$. So (2.24) has the holomorphic solution in $\langle \zeta : 0 \rangle \langle \zeta | \zeta |$.

The equation (2.24) has a regular singular point at $\zeta = 0$, and (2.25) is called the *indicial equation* of (2.24). If ρ_k ($k = 1, ..., n$) are all the roots of (2.25), then (2.24) has the following fundamental system of solutions in $\zeta: 0 < |\zeta| < 1$ (see e.g. [12: Chapter 1, Section 18.2]). If ρ_k is such a root that no difference $\rho_k - \rho_j$ the holomorphic solution in $\{\xi : 0 < |\xi| < 1\}$.

The equation (2.24) has a regular singular point at $\zeta = 0$, and (2.25) is called

the *indicial equation* of (2.24). If ϱ_k ($k = 1, ..., n$) are all the roots of (2.25), (2.24) has the following fundamental system of solutions in $\{\zeta:$

(2.24) has the following fundamental system of solutions in $\{\zeta:$

e.g. [12: Chapter 1, Section 18.2]). If ϱ_k is such a root that no ζ

($j \neq$ (2.24) where $\varphi_k(\zeta)$ is holomorphic for $|\zeta| < 1$. Further, if $\varrho_1, \ldots, \varrho_{l+m}$ is a set of such • value integer, then $y_0(k)$ is holomorphic for $|\zeta| < 1$. Further, if $\varrho_1, \ldots, \varrho_{1+n}$ is a set of such roots that all its differences are integers and, moreover, $\varrho_k - \varrho_{k+1} \geq 0$ ($k = 1, \ldots, n$) then $\varrho_k(k)$ ((4) where $\varphi_k(\zeta)$ is holomorphic for $|\zeta| < 1$. Further, if $\varrho_l, ..., \varrho_{l+n}$
ts that all its differences are integers and, moreover, $\varrho_k - \varrho_{k+n}$
 $m-1$, then to this set there corresponds the set of solutions of
 $y_k(\z$ *k*, $\{2.24\}$ has a regular singular point at $\zeta = 0$, and (2.25) is call 2.24) has a regular singular point at $\zeta = 0$, and (2.25) is call 2.24). If ϱ_k $(k = 1, ..., n)$ are all the roots of (2.25) , the owing **Example is solution in** $\langle \zeta : 0 \rangle \langle \zeta | \rangle \langle 1 \rangle$.

• equation (2.24) has a regular singular point at $\zeta = 0$, and (2.25) is

icial equation of (2.24). If ϱ_k $(k = 1, ..., n)$ are all the roots of (2.25)

as the following e.g. [12: Chapter 1, Section 18.2]). If θ_k is such a root that no difference $\theta_k - (j \neq k)$ is an integer, then to θ_k there corresponds the solution $y_k(\zeta) = \zeta^{e_k} \varphi_k(\zeta)$ (2.24) where $\varphi_k(\zeta)$ is holomorphic for

$$
l + m - 1
$$
, then to this set there corresponds the set of solutions of the form

$$
y_k(\zeta) = \sum_{j=l}^k \zeta^{e_j} \varphi_{kj}(\zeta) \ln^{k-j} \zeta \qquad (k = l, ..., l + m),
$$

258 M. S. AGRANOVICH and A. S. MARKUS

258 M. S. AGRANOVICH and A. S. MARKUS
where $\varphi_{kj}(\zeta)$ are holomorphic for $|\zeta| < 1$. It is easily seen that if no root ϱ_k is an
integer, then the fundamental system of solutions of (2.24) just indicated contains
no where $\varphi_{kj}(\zeta)$ are holomorphic for $|\zeta| < 1$. It is easily seen that if no root ϱ_k is an integer, then the fundamental system of solutions of (2.24) just indicated contains 258 M. S. AGRANOVICH and A. S. MARKUS

where $\varphi_{kj}(\zeta)$ are holomorphic for $|\zeta| < 1$. It is easily seen that if no root ϱ_k is an

integer, then the fundamental system of solutions of (2.24) just indicated contains
 no function holomorphic for $0 < |\zeta| < 1$; moreover, no non-trivial combination of these solutions is holomorphic for $0 < |\zeta| < 1$ EXANOVICH and A.S. MARKUS

SEANOVICH and A.S. MARKUS

Solutions of (2.24) just in

domorphic for $0 < |\zeta| < 1$. It is easily seen that if

domorphic for $0 < |\zeta| < 1$; moreover, no non-triv

is is holomorphic for $0 < |\zeta| < 1$
 258 M. S. AGRANOVICH and A. S. MARKUS
 • • *• •* EXANOVIET and A. S. MARKUS
 k= **c** holomorphic for $|\zeta| < 1$. It is easily
 e fundamental system of solutions of

lomorphic for $0 < |\zeta| < 1$; moreover

is is holomorphic for $0 < |\zeta| < 1$
 l
 k= **c** position 2.12: Di

Proof of Proposition 2.12: Divide the equation $Ay = \lambda y$ by $a_0(e^{ix})$ and set $a_0^{-1}(e^{ix}) = q(e^{ix}), a_k(e^{ix})/a_0(e^{ix}) = p_k(e^{ix}) \ (k = 1, ..., n):$ *k* is holomorphic
 h position 2.12
 $a_k(e^{ix})/a_0(e^{ix}) =$
 $\sum_{n=0}^{k} p_{n-k}(e^{ix}) D^k y$

ution $e^{ix} = \zeta$. A
 k
 $a_k, \zeta^{k-j+1} \partial_{\zeta}^{k-j}$ **Proof** of Proposition 2.12: Divide the equation $Ay = iy$ by $a_0(e^{ix})$ and set
 $a_0^{-1}(e^{ix}) = q(e^{ix}), a_k(e^{ix})/a_0(e^{ix}) = p_k(e^{ix})$ $(k = 1, ..., n)$:
 $D^ny + \sum_{k=0}^{5} p_{n-k}(e^{ix}) D^ky = \lambda q(e^{ix})y$.

(2.26)

Make the substitution $e^{ix} = \zeta$. A simple

$$
D^{n}y+\sum_{k=0}^{n-1}p_{n-k}(e^{kx}) D^{k}y=\lambda q(e^{kx}) y.
$$

 $\frac{1}{2}$

Make the substitution
$$
e^{ix} = \zeta
$$
. A simple induction shows that
\n
$$
D_x^k = \sum_{j=1}^k a_{k,j} \zeta^{k-j+1} \partial_{\zeta}^{k-j+1},
$$

- $D_x^k = \sum_{j=1}^k a_{k,j} \zeta^{k-j+1} \partial_{\zeta}^{k-j+1}$,
where $a_{k,j}$ satisfy the conditions of Lemma 2.13. After
(2.26) looks as follows (for convenience we set $p_0(\zeta) =$
 $\sum_{k=1}^n a_{k,j} \zeta^k$ (2.26) looks as follows (for convenience we set $p_0(\zeta) \equiv 1$): $D^{n}y + \sum_{k=0}^{n-1} p_{n-k}(e^{ix}) D^{k}y = \lambda q(e^{ix}) y$.

Make the substitution $e^{ix} = \zeta$. A simple induction shows that
 $D_{x}^{k} = \sum_{j=1}^{k} a_{k,j} \zeta^{k-j+1} \partial_{\zeta}^{k-j+1}$,

where $a_{k,j}$ satisfy the conditions of Lemma 2.13. After th

$$
\sum_{k=1}^n p_{n-k}(\zeta) \sum_{j=1}^k a_{k,j} \zeta^{k-j+1} \partial_{\zeta}^{k-j+1} y + (p_n(\zeta) - \lambda q(\zeta)) y = 0.
$$

If λ is an eigenvalue of A , the latter equation has a non-trivial solution in the unit circle. By Lemma 2.14 at least one root of the equation

$$
k,j
$$
 satisfy the conditions of Lemma 2.13. After the substitution the
\nobss as follows (for convenience we set $p_0(\zeta) \equiv 1$):
\n
$$
\sum_{k=1}^n p_{n-k}(\zeta) \sum_{j=1}^k a_{k,j} \zeta^{k-j+1} \partial_{\zeta}^{k-j+1} y + (p_n(\zeta) - \lambda q(\zeta)) y = 0.
$$
\nIn eigenvalue of A, the latter equation has a non-trivial solution in
\n
$$
y \text{ Lemma 2.14 at least one root of the equation}
$$
\n
$$
\sum_{k=1}^n p_{n-k}(0) \sum_{j=1}^k a_{k,j} \varrho(\varrho-1) \dots (\varrho - k + 1) + p_n(0) - \lambda q(0) = 0
$$
\neger. By Lemma 2.13 this equation may be rewritten in the form
\n
$$
\sum_{k=1}^n p_{n-k}(0) \varrho^k + p_n(0) - \lambda q(0) = 0.
$$

• is an integer. By Lemma 2.13 this equation may be rewritten in the form

$$
\sum_{k=1}^{n} p_{n-k}(0) \, \varrho^{k} + p_{n}(0) - \lambda q(0) = 0
$$

So every eigenvalue *A* of (2.19) is given by the formula

$$
\lambda = q^{-1}(0) \sum_{k=0}^{n} p_{n-k}(0) \, m^k = \sum_{k=0}^{n} a_{n-k}(0) \, m^k
$$

for some $m \in \mathbb{Z}$.

 $\sum_{k=1}^{n} p_{n-k}(0) \sum_{j=1}^{n} a_{k,j} \varrho(\varrho)$

is an integer. By Lemma 2.1
 $\sum_{k=1}^{n} p_{n-k}(0) \varrho^k + p_n(0)$

So every eigenvalue λ of (2.1)
 $\lambda = q^{-1}(0) \sum_{k=0}^{n} p_{n-k}(0)$

for some $m \in \mathbb{Z}$.

Since $a_0(\zeta) \neq 0$ for $|\$ $a_0(\zeta) = 0$ for $|\zeta| \leq 1$, we can select a one-valued branch of its logarithm.

words, there exists a function $u(\zeta)$ holomorphic for $|\zeta| < 1$, continuous for

and such that $a_0 = \exp u$. Set
 $a_0^{(t)}(\zeta) = \exp \left(\tau u(\zeta) +$ $\mathbf{r}^{\prime}=\mathbf{r}^{\prime}+\mathbf{r}^{\prime}$ In other words, there exists a function $u(\zeta)$ holomorphic for $|\zeta| < 1$, continuous for $\sum_{k=1}^{n} p_{n-k}(0) \sum_{j=1}^{k} a_{k,j} \varrho(\varrho - 1) \dots (\varrho - k + 1) + p_n(0) - \lambda q(0) = 0$

in integer. By Lemma 2.13 this equation may be rewritten in the f
 $\sum_{k=1}^{n} p_{n-k}(0) \varrho^k + p_n(0) - \lambda q(0) = 0$.

every eigenvalue λ of (2.19) is giv $\frac{1}{2}$ or some Since α
b Since α
c α other
 $\zeta \leq 1$, a

$$
a_0^{(t)}(\zeta) = \exp \left(\tau u(\zeta) + (1 - \tau) u(0) \right),
$$

\n
$$
a_k^{(t)}(\zeta) = \tau a_k(\zeta) + (1 - \tau) a_k(0) \qquad (k > 0),
$$

\n
$$
A^{(t)} = \sum_{k=0}^n a_k^{(t)}(e^{iz}) D^{n-k}.
$$

By the result above the eigenvalues of $A^{(r)}$, for each $\tau \in [0, 1]$, are contained in (2.20). On the other hand, the set of eigenvalues of the operator $A^{(0)} = \sum a_k(0) D^{n-k}$ coincides with the set (2.20), the multiplicity of each eigenvalue λ_0 being equal to the number of $m \in \mathbb{Z}$ such that λ_0 admits the representation (2.20). Applying the theorem mumber of $m \in \mathbb{Z}$ such that λ_0 admits the representation (2.20). Applying the theorem
about the stability of root multiplicity (see e.g. [13: Chapter IV, Theorem 3.18]), we
conclude that these assertions are valid for some $m \in \mathbb{Z}$.

Since $a_0(\zeta) = 0$ for $|\zeta| \leq 1$, we can select a one-valued branch of its logari

In other words, there exists a function $u(\zeta)$ holomorphic for $|\zeta| < 1$, continuou
 $|\zeta| \leq 1$, and such that

S
S
S
S

S
S
S

Corollary 2.15: The counting function for modules of eigenvalues of the operator (2.19) has the regular asymptotics

$$
N(\lambda) = 2 |a_0(0)|^{-1/n} \lambda^{1/n} + O(1) \qquad (\lambda \to \infty).
$$

REFERENCES

- [1] AGMON, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math. 15 (1962), 119-147.
- [2] Агранович, М. С.: Спектральные свойства задач дифракции. Дополнение к книге: Войтович, Н. Н., Каценеленбаум, Б. З., и А. Н. Сивов: Обобщенный метод собственных колебаний в теории дифракции. Москва: Изд-во Наука 1977, 289. - 416.
- [3] Агранович, М. С.: Некоторые асимтотические формулы для эллиптических псевдодифференциальных операторов. Функц. анализ и его прил. 21 (1987), 63-65.
- [4] Агранович, М. С., и А. С. Маркус: Замечания о спектре несамосопряженных эллиптических операторов. Успехи мат. наук 42 (1987) 4, 142.
- [5] BERGH, J., and J. LÖFSTRÖM: Interpolation spaces. Berlin-Heidelberg-New York: Springer-Verlag. 1976.
- [6] Боймлтов, К. Х.: Асимптотическое поведение собственных значений несамосопряженных операторов. Функц. анализ и его прил. 11 (1977) 4, 74-75.
- [7] CODDINGTON, E. A., and N. LEVINSON: Theory of ordinary differential equations. New York-Toronto-London: McGraw-Hill Book Co. 1955.
- [8] Гохверг, И. Ц., и М. Г. Крейн: Введение в теорию несамосопряженных операторов. Москва: Изд-во Наука 1965.
- [9] Геадштейн, И. С., и И. М: Рыжик: Таблицы интегралов, сумм, рядов и произведений. Москва: Физматгиз 1962.
- 10] GRUBB, G.: Functional calculus of pseudo-differential boundary problems. Boston-Basel-Stuttgart: Birkhäuser Verlag 1986.
- [11] HORN, A.: On the eigenvalues of a matrix with prescribed singular values. Proc. Amer. Math. Soc. $5(1954)$, $4-7$.
- [12] KAMKE, E.: Differentialgleichungen. Lösungsmethoden und Lösungen. I. Leipzig: Akad. Verlagsges. Geest & Portig 1946.
- [13] KATO, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer-Verlag 1966.
- [14] Кожевников, А. Н.: Об асимптотике собственных значений эллиптических систем. Функц. анализ и его прил. 11 (1977) 4, 84-85.
- [15] LIONS, J.-L., et E. MAGENES: Problèmes aux limites non homogènes et applications. Vol. I. Paris: Dunod 1968.
- [16] Люстерник, Л. А.: Выпуклые фигуры и многогранники. Москва: Гостехиздат 1956.
- [17] Люстерник, Л. А., и В. И. Соболев: Элементы функционального анализа. Москва: Изд-во Наука 1965.
- [18] Менкус, А. С.: Введение в спектральную теорию полиномиальных операторных пучков. Кишинев: Изд-во Штиница 1986.
- [19] МАРКУС, А. С., и В. И. ПАРАСКА: Об оценке числа собственных значений линейного оператора. Изв. Акад. наук МолдССР (1965) 7, 101-104.
- [20] Привллов, И. И.: Граничные свойства аналитических функций. Москва-Ленинград: Гостехиздат 1950.
- [21] Розенвлюм, Г. В.: Спректральная асимптотика нормальных операторов. Функц. анализ и его прил. 16 (1982) 2, 82-83.
- [22] Розвивлюм, Г. В.: Угловая асимптотика спектра операторов, близких к нормальным: В кн.: Линейные и нелинейные краевые задачи. Спектральная теория. Ленинград: Изд-во Ленингр. ун-та 1986, 180-195.
- [23] SEELEY, R. T.: Complex powers of an elliptic operator. In: Singular Integrals (Proc. Symp. Pure Math. 10). Providence, R. I.: Amer. Math. Soc. 1967, 288-307.
- [24] SEELEY, R. T.: The resolvent of an elliptic boundary problem. Amer. J. Math. 91 (1969), $889 - 920.$

[25] SEELEY, R. T.: Analytic extension of the trace associated with elliptic boundary problems. Amer. J. Math. 91 (1969), 963-983.

[26] SEELEY, R. T.: A simple example of spectral pathology for differential operators. Comm. Part. Diff. Eq. 11 (1986), 595-598.

- [27] Шубин, М. А.: Псевдодифференциальные операторы и спектральная теория. Москва: Изд-во Наука 1978.
- [28] WALSH, J. L.: Interpolation and approximation by rational functions in the complex domain. Providence, R. I.: Amer. Math. Soc. 1960.

[29] WIDDER, D. V.: The Laplace transform. Princeton: Princeton Univ. Press 1946.

Manuskripteingang: 25.03.1988

VERFASSER:

Проф. М. С. Агранович Московский институт электронного машиностроения Б. Вузовский пер. 3/12 СССР - 109028 Москва

Ст. н. с. А. С. Маркус

Институт математики с ВЦ АН Молдавской ССР ул. Гросула 5. СССР-277028 Кишинев