Zeitschrift für Analysis
und ihre Anwendungen
Bd. 8 (3) 1989, S. 237–260

On Spectral Properties of Elliptic Pseudo-Differential Operators Far from Self-Adjoint Ones

M. S. AGRANOVICH and A. S. MARKUS

Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Wir erhalten unter gewissen Voraussetzungen grobe Asymptotik für die Eigenwerte elliptischer Operatoren; eine der Voraussetzungen besteht darin, daß die Werte des Hauptsymbols nicht die ganze komplexe Ebene ausfüllen. Wir betrachten auch eine Reihe von Beispielen nichtselbstadjungierter elliptischer Operatoren; insbesondere solche mit ungewöhnlichen Formeln für die Asymptotik der Eigenwerte; mit regulärer Asymptotik der Eigenwerte, aber ohne Vollständigkeit der verallgemeinerten Eigenfunktionen; mit vollständigem System von Eigenfunktionen, das keine Basis bildet.

Выводится грубая асимптотика собственных значений для эллиптических операторов при некоторых предположениях; одно из них состоит в том, что значения главного символа не заполняют всей комплексной плоскости. Рассмотрен ряд примеров несамосопряженных эллиптических операторов, в частности: с необычными формулами для асимптотики собственных значений; с правильной асимптотикой собственных значений, но без полноты корневых функций; с полной системой собственных функций, не являющейся базисом.

We establish a rough asymptotics for eigenvalues of elliptic operators under some assumptions; one of them is that the values of the principal symbol do not cover the whole complex plane. We consider also a collection of some examples of non-self-adjoint elliptic operators: in particular, with unusual formulas for asymptotics of eigenvalues; with a regular asymptotics of eigenvalues but without the completeness of root functions; with a complete system of eigenfunctions which is not a basis.

0. Introduction

Let M be an n-dimensional closed C^∞-manifold, provided with a positive density dx, and let A be a classical (i.e. polyhomogeneous) elliptic pseudo-differential operator of order $t > 0$ on M with the principal symbol $a_0(x, \xi)$. At first we suppose A to be a scalar operator. Denote by $H_s(M)$ ($s \in \mathbf{R}$) the Sobolev space of order s on M; $H_0(M) = L^2(M)$. We may regard A as a closed operator in $H_0(M)$ with the dense domain $H_t(M)$. If its spectrum $\sigma(A)$ does not cover the whole plane, A has the compact resolvent $R_A(\lambda) = (A - \lambda I)^{-1}$ and $\sigma(A)$ consists of eigenvalues of finite multiplicity with the only possible limit point at infinity.

By spectral properties of A we mean first of all asymptotic properties of the counting function of modules of its eigenvalues and geometric properties of the system of its root functions, i.e. generalized eigenfunctions (to be complete, to form the basis etc.) in $H_s(M)$. If $a_0(x, \xi) > 0$ on non-zero cotangent vectors, then A is near the self-adjoint pseudo-differential operator $A_0 = (A + A^*)/2$ in the following sense: the order of $A - A_0$ is not greater than $t - 1$. (Here and below we denote by A^* the pseudo-differential operator formally adjoint to A, with respect to the natural scalar product $(u, v) = \int_M u(x)\, \overline{v(x)}\, dx$ on M, as well as the adjoint to A as an operator in

$L^2(M)$.) The case just indicated has been well studied (see e.g. [2, 18] and references there). In this case, in particular, the eigenvalues $\lambda_j(A)$ of A are known to be contained in some "parabolic" neighborhood of the half-axis $\mathbf{R}_+$ and $N(\lambda) = \operatorname{card}\{j: |\lambda_j(A)| \leq \lambda\}$ has the regular asymptotics

$$N(\lambda) = d_0\lambda^{n/t} + O(\lambda^{(n-1)/t}) \ (\lambda \to \infty) \quad \text{with} \quad d_0 = \frac{1}{(2\pi)^n} \int\limits_{a_0(x,\xi)\leq 1} dx\, d\xi. \tag{0.1}$$

Moreover, in this case one can construct a complete minimal system of root functions of A, and this system is a good "basis with parentheses" if the order of $A - A_0$ is small enough.

The case when $a_0(x, \xi)$ has a non-constant argument has been studied far less (below we list the corresponding papers known to us). It is just the case to be studied in the present paper.

In Section 1 we assume that the values of the principal symbol do not cover the whole plane: $|\arg a_0(x, \xi)| \leq \theta$ where $\theta < \pi$. Our aim is to obtain some lower and upper bounds for lower and upper limits l_- and l_+ of the function $\lambda^{-n/t}N(\lambda)$ as $\lambda \to \infty$. Let us introduce two quantities

$$\begin{aligned} d &= \frac{1}{(2\pi)^n\, n} \int\limits_M dx \int\limits_{|\xi|=1} [a_0(x, \xi)]^{-n/t}\, dS_\xi, \\ \Delta &= \frac{1}{(2\pi)^n\, n} \int\limits_M dx \int\limits_{|\xi|=1} |a_0(x, \xi)|^{-n/t}\, dS_\xi. \end{aligned} \tag{0.2}$$

(Here they are put down roughly, without using local coordinates; the exact expressions are presented below in (1.30)–(1.33).) If $a_0(x, \xi) > 0$, we have $d = \Delta = d_0$. The main results of Section 1 are as follows:

$$l_- > 0 \quad \text{if} \quad d \neq 0; \qquad l_- \leq \Delta; \qquad |d| \leq l_+ \leq e\Delta. \tag{0.3}$$

When $d \neq 0$, we obtain from (0.3) the rough asymptotics

$$N(\lambda) \asymp \lambda^{n/t}, \quad \text{i.e.} \quad C_1 \leq \lambda^{-n/t}N(\lambda) \leq C_2 \qquad (\lambda \geq C_3) \tag{0.4}$$

with positive constants C_1, C_2, C_3. We do not know if the case $l_- < l_+$ is possible (an interesting question, in our opinion). If $l_- = l_+ = l$, we obtain from (0.3) that

$$|d| \leq l \leq \Delta. \tag{0.5}$$

The inequalities (0.3) with some corollaries are proved in Subsection 1.3. In Subsections 1.1 and 1.2 some preliminary material is contained. In Subsection 1.1 we establish a certain Tauberian inequality. Namely we prove that if $N(\lambda)$ is a non-decreasing function on $\mathbf{R}_+$ and if its Stieltjes transform of order q,

$$S_q(\mu) = \int_0^\infty (\lambda + \mu)^{-q}\, dN(\lambda), \tag{0.6}$$

has the rough asymptotics $S_q(\mu) \asymp \mu^{\delta-q}$ $(0 < \delta < q)$, then $N(\lambda)$ has the rough asymptotics $N(\lambda) \asymp \lambda^\delta$. The last statement is analogous to the well-known Tauberian Hardy-Littlewood theorem (see Subsection 1.1). In Subsection 1.2 we establish inequalities analogous to $l_- \leq \Delta$, $l_+ \leq e\Delta$ for compact operators in the abstract Hilbert space. Note that the constant e in the second inequality turns out to be exact. In Subsection

1.4 the main results of Subsection 1.3 are extended to matrix elliptic pseudo-differential operators with the spectrum of the principal symbol lying in two closed sectors Λ_1 and Λ_2 which have the unique common point 0. We investigate separately the behaviour of counting functions $N_1(\lambda)$ and $N_2(\lambda)$ for modules of eigenvalues of A in slightly broader sectors $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$. In Subsection 1.5 we consider briefly operators corresponding to elliptic boundary problems with homogeneous boundary conditions and outline the proofs of assertions analogous to the main results.

The quantity d is obviously non-zero if $\theta n < \pi t/2$ (when $n > 1$ and M is connected, it is true if $\theta n \leqq \pi t/2$). In this case one can see from (0.4) that there are "sufficiently many" eigenvalues. Another well-known indication of such situation is the completeness of the system of root functions which has been established exactly under the condition $\theta n < \pi t/2$ (cf. [1]). Such condition is only sufficient both for the completeness of root functions and for the presence of the rough asymptotics for $N(\lambda)$. Indeed these two properties of A are preserved when we pass to A^k (with positive integer k), while the sector free from values of the principal symbol can disappear. On the other hand, if the values of the principal symbol cover the whole plane, we cannot point out any sufficient conditions for the completeness or for the presence of the rough asymptotics for $N(\lambda)$ (and even conditions under which the spectrum of A is non-empty or discrete).

We examine these problems in Section 2 on some examples. First of all we give very simple examples of elliptic operators on the torus either with the empty spectrum or with the spectrum filling the whole plane (each point is an eigenvalue). Then we discuss in detail (in Subsections 2.2 and 2.3) the example of the elliptic differential operator of first order on the circle. As it has turned out, this example has been considered by SEELEY before us. In his note [26] he indicates the conditions under which the spectrum of the operator is empty or covers the whole plane. He also points out that if neither of the two degenerate cases takes place, then $N(\lambda)$ has the regular asymptotics with somewhat unusually defined coefficient d_0. We recall these calculations, and in addition we obtain in Subsection 2.2 the exact condition for the completeness of eigenfunctions of this operator. (In the non-degenerate case all its root functions are eigenfunctions.) This condition deals only with the principal symbol and is non-local. The counting function $N(\lambda)$ has the regular asymptotics in the non-degenerate case even if these is no completeness. Assuming the completeness, we deduce in Subsection 2.3 the exact condition, under which the system of eigenfunctions is a basis, and obtain an example of an elliptic operator whose system of eigenfunctions is complete but is not a basis. We construct also such examples of operators on the torus using the separation of variables. In Subsection 2.4 we consider another example of an elliptic differential operator on the torus, admitting the separation of variables, in order to demonstrate the possibility in (0.5) of all three cases

$$|d| = l < \Delta, \qquad |d| < l = \Delta, \qquad |d| < l < \Delta. \tag{0.7}$$

In Subsection 2.5 we establish the existence of the regular asymptotics of $N(\lambda)$ for elliptic differential operators on the unit circle whose coefficients admit continuous extentions in the unit disk holomorphic in its interior.

Now we list the results known to us and more or less close to the subject of the present article. In the paper [6] of BOJMATOV some abstract test for validity of (0.4) has been formulated. For differential operators this test yields (0.4) if $\theta n < \pi t/4$. KOŽEVNIKOV [14] has considered a matrix elliptic pseudo-differential operator A with spectrum of the principal symbol lying on several half-lines. He has obtained asymptotics of eigenvalues of A close to one of the half-lines. An extension of this result to the case when in addition to the half-line under consideration there is a sector covered by eigenvalues $\lambda_j(x, \xi)$ $((x, \xi) \in T^*M \setminus 0)$ of the principal symbol, has been obtained by Agranovich [3]. Rosenbloom [21, 9] has obtained an asymptotic formula for the modules of eigenvalues for normal elliptic operators and elliptic operators very close to normal, in a sector, whose bounds may even contain the values of the principal symbol.

The main results of this paper have been reported at the 10^{th} Session of the Petrowskii Seminar on differential equations and mathematical problems of physics and Moscow Mathematical Society in January of 1987 [4].

We wish to express our sincere gratitude to V. I. Matsaev and F. L. Friedlander for valuable discussions and to V. I. Matov for the help in translation.

1. Rough asymptotics for counting functions

1.1. An analogue of the Tauberian Hardy-Littlewood theorem. Let $N(\lambda)$ be a non-negative non-decreasing function on the non-negative half-axis $\overline{\mathbb{R}}_+$, with $N(0) = 0$. If $q > 0$ and

$$\int_1^\infty \lambda^{-q}\, dN(\lambda) < \infty, \tag{1.1}$$

then the *Stieltjes transform* of order q of $N(\lambda)$ is defined by (0.6). We shall only need the case when $N(\lambda)$ is the *counting function* for some non-decreasing sequence $\{\sigma_j\}_1^\infty$ of positive numbers σ_j with $\sigma_j \to \infty$ as $j \to \infty$: $N(\lambda)$ is the number of the σ_j not exceeding λ. In this case the condition (1.1) means that $\sum \sigma_j^{-q} < \infty$ and we have

$$S_q(\mu) = \sum_{j=1}^\infty (\sigma_j + \mu)^{-q}. \tag{1.2}$$

Let $N_1(\lambda)$ and $N_2(\lambda)$ be positive functions for $\lambda \geqq \lambda_0$. We shall write

$$N_1(\lambda) \sim N_2(\lambda) \quad \text{if} \quad \lim_{\lambda\to\infty} N_1(\lambda)/N_2(\lambda) = 1$$

(the *strong equivalence*), and

$$N_1(\lambda) \asymp N_2(\lambda) \quad \text{if} \quad \varliminf_{\lambda\to\infty} N_1(\lambda)/N_2(\lambda) > 0; \qquad \varlimsup_{\lambda\to\infty} N_1(\lambda)/N_2(\lambda) < \infty$$

(the *weak equivalence*). Since the classical works of Carleman, the following Tauberian Hardy-Littlewood theorem has been successfully applied in the study of spectral asymptotics (see e.g. [29: Chapter V]):

Suppose (1.1) *holds and* $S_q(\mu) \sim \varrho\mu^{\delta-q}$ $(\mu \to \infty)$ *for some* $\delta \in (0, q)$. *Then* $N(\lambda) \sim b_{\delta,q}\varrho\lambda^\delta$ $(\lambda \to \infty)$, *where*

$$b_{\delta,q} = \delta \int_0^\infty t^{\delta-1}(1 + t)^{-q}\, dt = \delta B(\delta, q - \delta). \tag{1.3}$$

We shall need an analogous result for the weak equivalence instead of the strong one:

Theorem 1.1: *Let* (1.1) *hold and* $S_q(\mu) \asymp \mu^{\delta-q}$ $(\mu \to \infty)$ *for some* $\delta \in (0, q)$. *Then* $N(\lambda) \asymp \lambda^\delta$ $(\lambda \to \infty)$.

Proof: Since

$$\int_0^{\lambda_0} (\lambda + \mu)^{-q}\, dN(\lambda) = (\lambda_0 + \mu)^{-q} N(\lambda_0) + q \int_0^{\lambda_0} (\lambda + \mu)^{-q-1} N(\lambda)\, d\lambda, \tag{1.4}$$

from (1.1) it follows that

$$\int_0^\infty (\lambda + \mu)^{-q-1} N(\lambda)\, d\lambda < \infty. \tag{1.5}$$

Hence the limit of $N(\lambda_0)(\lambda_0 + \mu)^{-q}$ as $\lambda \to \infty$ exists, and from (1.5) we see that it is equal to 0. So (1.4) implies

$$S_q(\mu) = q \int_0^\infty (\lambda + \mu)^{-q-1} N(\lambda)\, d\lambda. \tag{1.6}$$

By the assumption,

$$\underline{\lim}_{\mu\to\infty} \mu^{q-\delta} S_q(\mu) = \varrho_1 > 0, \tag{1.7}$$

$$\overline{\lim}_{\mu\to\infty} \mu^{q-\delta} S_q(\mu) = \varrho_2 < \infty. \tag{1.8}$$

Obviously

$$\int_0^\infty \frac{N(\lambda)\, d\lambda}{(\lambda + \mu)^{q+1}} \geqq \int_\mu^\infty \frac{N(\lambda)\, d\lambda}{(\lambda + \mu)^{q+1}} \geqq N(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\mu)}{q(2\mu)^q}.$$

From this relation, (1.6) and (1.8) we have

$$\overline{\lim}_{\mu\to\infty} \mu^{-\delta} N(\mu) \leqq 2^q \varrho_2. \tag{1.9}$$

Now we want to estimate $\underline{\lim}\, \mu^{-\delta} N(\mu)$. Evidently for $\gamma > 0$

$$\int_0^{\gamma\mu} \frac{N(\lambda)\, d\lambda}{(\lambda + \mu)^{q+1}} \leqq N(\gamma\mu) \int_0^{\gamma\mu} \frac{d\lambda}{(\lambda + \mu)^{q+1}} \leqq N(\gamma\mu) \int_0^\infty \frac{d\lambda}{(\lambda + \mu)^{q+1}} = \frac{N(\gamma\mu)}{q\mu^q}. \tag{1.10}$$

On the other hand, by (1.9) we have for any $\varepsilon > 0$, if $\gamma\mu$ is sufficiently large,

$$\int_{\gamma\mu}^\infty (\lambda + \mu)^{-q-1} N(\lambda)\, d\lambda \leqq (\varrho_2 + \varepsilon)\, 2^q \int_{\gamma\mu}^\infty (\lambda + \mu)^{-q-1} \lambda^\delta d\lambda$$

$$\leqq 2^q(\varrho_2 + \varepsilon) \int_{\gamma\mu}^\infty \lambda^{\delta-q-1} d\lambda = 2^q(\varrho_2 + \varepsilon)(q - \delta)^{-1} (\gamma\mu)^{\delta-q}. \tag{1.11}$$

From (1.6), (1.10), (1.11) it follows that

$$\mu^{q-\delta} S_q(\mu) - q 2^q(\varrho_2 + \varepsilon)(q - \delta)^{-1} \gamma^{\delta-q} \leqq \mu^{-\delta} N(\gamma\mu). \tag{1.12}$$

Choose μ_0 so large that $\mu^{q-\delta} S_q(\mu) > \varrho_1 - \varepsilon$ for $\mu > \mu_0$ (see (1.7)) and γ so large that the second term in the left-hand side of (1.12) is less than ε. Then $N(\gamma\mu) > (\varrho_1 - 2\varepsilon)\mu^\delta$ $(\mu > \mu_0)$, and therefore

$$\underline{\lim}_{\lambda\to\infty} \lambda^{-\delta} N(\lambda) \geqq \gamma^{-\delta}(\varrho_1 - 2\varepsilon). \tag{1.13}$$

From (1.9) and (1.13) we obtain the conclusion of the theorem ∎

Remark 1.2: In the proof of (1.9) only (1.8) has been used, whereas in the proof of (1.13) we have used both (1.7) and (1.8).

Remark 1.3: We do not try to obtain the best estimates for $\underline{\lim}\ \lambda^{-\delta}N(\lambda)$ and $\overline{\lim}\ \lambda^{-\delta}N(\lambda)$ in terms of ϱ_1 and ϱ_2.

Remark 1.4: In the proof of Theorem 1.1 we have shown that (1.7) and (1.9) imply $\underline{\lim}\ \lambda^{-\delta}N(\lambda) > 0$.

We shall need also the following statement which is inverse to Theorem 1.1 (Abelian theorem). It is valid in a sharper form, and the requirement that $N(\lambda)$ should have a finite variation on each finite segment, instead of monotonicity, is sufficient.

Let (1.1) *hold. Then*

$$\underline{\lim}_{\mu\to\infty} \mu^{q-\delta}S_q(\mu) \geqq b_{\delta,q} \underline{\lim}_{\lambda\to\infty} \lambda^{-\delta}N(\lambda), \tag{1.14}$$

$$\overline{\lim}_{\mu\to\infty} \mu^{q-\delta}S_q(\mu) \leqq b_{\delta,q} \overline{\lim}_{\lambda\to\infty} \lambda^{-\delta}N(\lambda), \tag{1.15}$$

where $b_{\delta,q}$ *is defined in* (1.3).

The proof is elementary; see e.g. [29: Chapter V].

1.2. Asymptotic estimates for eigenvalues of a compact operator by its singular values. In this subsection K is a compact operator in a Hilbert space. Let $\{\lambda_n(K)\}_1^\infty$ be the sequence of its eigenvalues, counted according to their multiplicities (i.e. the dimensions of the corresponding root subspaces) and arranged so that $|\lambda_1(K)| \geqq |\lambda_2(K)| \geqq \ldots$. If K has only a finite number of non-zero eigenvalues, we complete the sequence by zeros. The numbers $s_n(K) = \lambda_n\big((K^*K)^{1/2}\big)$ are called the *singular values* of K. The eigenvalues and the singular values are connected by the well-known *Weyl inequalities* (see e.g. [8: Chapter II, § 3]):

$$\prod_{j=1}^{n} |\lambda_j(K)| \leqq \prod_{j=1}^{n} s_j(K) \quad (n = 1, 2, \ldots). \tag{1.16}$$

They have many consequences; in particular,

$$\sum_{j=1}^{n} |\lambda_j(K)|^p \leqq \sum_{j=1}^{n} s_j^p(K) \quad (n = 1, 2, \ldots; p > 0). \tag{1.17}$$

Denote by $n(t)$ (respectively by $\nu(t)$) the counting function for $|\lambda_j^{-1}(K)|$ (respectively for $s_n^{-1}(K)$). As it is pointed out in [19], (1.16) can be rewritten in the form

$$\int_0^\lambda t^{-1}n(t)\,dt \leqq \int_0^\lambda t^{-1}\nu(t)\,dt \quad (\lambda > 0). \tag{1.18}$$

Here we want to establish some connections between the asymptotic behaviour of $n(t)$ and that of $\nu(t)$ following from (1.17) and (1.18).

Theorem 1.5: *For any* $\delta > 0$,

$$\underline{\lim}_{\lambda\to\infty} \lambda^{-\delta}n(\lambda) \leqq \overline{\lim}_{\lambda\to\infty} \lambda^{-\delta}\nu(\lambda). \tag{1.19}$$

Proof: If K has only a finite number of non-zero eigenvalues, then the left-hand side of (1.19) is equal to 0. Therefore we can assume that all the $\lambda_n(K)$ (and by (1.16) all the $s_n(K)$) are distinct from 0. Set $\mu_n = |\lambda_n^{-1}(K)|$ and $\sigma_n = s_n^{-1}(K)$. Obviously,

$$\overline{\lim}_{\lambda\to\infty} \lambda^{-\delta}\nu(\lambda) \geqq \overline{\lim}_{n\to\infty} \sigma_n^{-\delta}n \tag{1.20}$$

and for each $\varepsilon > 0$ we have

$$\varliminf_{\lambda\to\infty} \lambda^{-\delta} n(\lambda) \leqq \varliminf_{n\to\infty} (\mu_n - \varepsilon)^{-\delta} n = \varliminf_{n\to\infty} \mu_n{}^{-\delta} n. \tag{1.21}$$

Suppose that (1.19) is false. Then (1.20) and (1.21) imply $\varliminf \mu_n{}^{-\delta} n > \varlimsup \sigma_n{}^{-\delta} n$, and therefore, with some $a > 0$, $\mu_n{}^{-\delta} - \sigma_n{}^{-\delta} > an^{-1}$ $(n \geqq n_0)$. Hence $\sum\limits_{n=n_0}^{m} \mu_n{}^{-\delta} - \sum\limits_{n=n_0}^{m} \sigma_n{}^{-\delta} > a \sum\limits_{n=n_0}^{m} n^{-1} \to \infty$ as $m \to \infty$, which contradicts (1.17) with $p = \delta$ ∎

Theorem 1.6: *For any $\delta > 0$,*

$$\varlimsup_{\lambda\to\infty} \lambda^{-\delta} n(\lambda) \leqq e \varlimsup_{\lambda\to\infty} \lambda^{-\delta} \nu(\lambda). \tag{1.22}$$

Proof: It follows from (1.18) that for any $\gamma > 0$

$$\int_0^{\lambda\gamma} t^{-1}\nu(t)\, dt \geqq \int_0^{\lambda\gamma} t^{-1} n(t)\, dt \geqq \int_\lambda^{\lambda\gamma} t^{-1} n(t)\, dt \geqq n(\lambda) \ln \gamma. \tag{1.23}$$

Let d_2 be the upper limit in the right-hand side of (1.22) (we assume it to be finite) and ε be an arbitrary positive number. Then $\nu(\lambda) \leqq (d_2 + \varepsilon)\lambda^\delta$ for $\lambda \geqq \lambda_0$, and therefore we obtain from (1.23)

$$n(\lambda) \leqq (\ln \gamma)^{-1} \left(\int_0^{\lambda_0} t^{-1}\nu(t)\, dt + (d_2 + \varepsilon) \int_{\lambda_0}^{\lambda\gamma} t^{\delta-1}\, dt \right)$$

$$= (\ln \gamma)^{-1} (\text{Const} + (d_2 + \varepsilon)\, \delta^{-1} \lambda^\delta \gamma^\delta),$$

or

$$\lambda^{-\delta} n(\lambda) \leqq (\ln \gamma)^{-1} \left(\lambda^{-\delta} \text{Const} + (d_2 + \varepsilon)\, \delta^{-1} \gamma^\delta\right).$$

Setting here $\gamma = e^{1/\delta}$, we obtain the inequality

$$\lambda^{-\delta} n(\lambda) \leqq \delta\left(\lambda^{-\delta} \text{Const} + (d_2 + \varepsilon)\, \delta^{-1} e\right),$$

from which (1.22) follows ∎

Remark 1.7: The constant e in (1.22) is the best possible. Indeed, by Horn's theorem [11], for any integers l and m $(0 \leqq l < m)$ there exists an operator $A_{l,m}$, acting in a Hilbert space of finite dimension $m - l$, such that $\lambda_j(A_{l,m}) = (l!/m!)^{1/(m-l)}$ and $s_j(A_{l,m}) = (l + j)^{-1}$ $(j = 1, \dots, m - l)$. Choose an increasing sequence $\{m_q\}_1^\infty$ of positive integers such that $m_q/m_{q-1} \to \infty$ and denote the operator A_{m_{q-1},m_q} by K_q $(q = 1, 2, \dots; m_0 = 0)$. Let K be the orthogonal sum of K_q. Evidently, $\lambda_j(K) = (m_{q-1}!/m_q!)^{1/(m_q - m_{q-1})}$ $(m_{q-1} < j \leqq m_q,\ q = 1, 2, \dots)$ and $s_j(K) = j^{-1}$ $(j = 1, 2, \dots)$. Hence $\nu(t) = [t]$ and $n\left((m_q!/m_{q-1}!)^{1/(m_q - m_{q-1})}\right) = m_q$ $(q = 1, 2, \dots)$. Using Stirling's formula, we obtain

$$\varlimsup_{t\to\infty} \frac{n(t)}{t} \geqq \lim_{q\to\infty} \left(\frac{m_{q-1}!}{m_q!}\right)^{\frac{1}{m_q - m_{q-1}}} = e \lim_{q\to\infty} \left(\frac{m_{q-1}}{m_q}\right)^{\frac{m_{q-1}+1/2}{m_q - m_{q-1}}} = e.$$

So $\varlimsup\limits_{t\to\infty} t^{-1} n(t) \geqq e = e \lim\limits_{t\to\infty} t^{-1}\nu(t)$ ∎

1.3. Theorems on rough asymptotics for modules of eigenvalues of elliptic operators. Let M be an n-dimensional C^∞-manifold provided with a positive C^∞-density dx. We consider a classical (i.e. polyhomogeneous) scalar pseudo-differential operator A of order $t > 0$. Let $a_0(x, \xi)$ be its principal symbol. It is a C^∞-function on $T^*M \setminus 0$, positively homogeneous of order t in ξ (see e.g. [27]). The values of $a_0(x, \xi)$ $\big((x, \xi) \in T^*M\big)$ cover a sector with vertex at the origin. We assume that the sector does not coincide with the whole plane $\mathbf{C}$. We may also suppose that its bisectrix is $\mathbf{R}_+$, and then our assumption is as follows:

$$|\arg a_0(x, \xi)| \leqq \theta \quad (\theta < \pi). \tag{1.24}$$

This condition means that A is elliptic with a parameter in any sector

$$\{\lambda: |\arg \lambda - \pi| \leqq \pi - \theta - \varepsilon\} \qquad (0 < \varepsilon < \pi - \theta). \tag{1.25}$$

It follows (see [23]) that A (as an operator in $L^2(M) = H_0(M)$ with the domain $H_t(M)$) has the compact resolvent $R_A(\lambda) = (A - \lambda I)^{-1}$ and that in any sector (1.25) A may have only a finite number of eigenvalues. Moreover each half-line $\{\lambda: \arg \lambda = \varphi\}$ lying outside the sector $\{\lambda: |\arg \lambda| \leqq \theta\}$ is a ray of *maximal decrease* (by the terminology in [1], a ray of minimal growth) of the resolvent, i.e. $\|R_A(\lambda)\| = O(|\lambda|^{-1})$ as $\lambda \to \infty$ along such a half-line. Replacing if necessary λ by $\lambda - c$ with an appropriate c, we assume (without loss of generality) that all the points in some sector (1.25), including O, are regular for A.

Let $lt > n$ ($l \in \mathbf{N}$); then $[R_A(\lambda)]^l$ belongs to the trace class, so we may consider its trace $\operatorname{tr} [R_A(\lambda)]^l$. It is well known (see e.g. [14]) that

$$\operatorname{tr} [R_A(-\mu)]^l \sim c_l \mu^{n/t-l} \qquad (\mu \to +\infty), \tag{1.26}$$

where

$$c_l = (2\pi)^{-n} \int\limits_{T^*M} \big(a_0(x, \xi) + 1\big)^{-l} \, dx \, d\xi. \tag{1.27}$$

The coefficient c_l may be expressed also, using a sufficiently small partition of unity $\{\varphi_k(x)\}_1{}^m$ on M and values $a_0{}^{(k)}(x, \xi)$ of the principal symbol in corresponding local coordinates, in the form

$$c_l = \frac{b_{n/t,l}}{(2\pi)^n \, n} \int\limits_M \sum_{k=1}^m \varphi_k(x) \int\limits_{|\xi|=1} \big(a_0{}^{(k)}(x, \xi)\big)^{-n/t} \, dS_\xi \, dx. \tag{1.28}$$

Here $d\xi = \varrho^{n-1} \, d\varrho \, dS_\xi$; $\varrho = |\xi|$ in local coordinates, and by $\big(a_0{}^{(k)}(x, \xi)\big)^{-n/t}$ we mean the main value of the function $z^{-n/t}$ for $z = a_0{}^{(k)}(x, \xi)$ (if $z = r\,\mathrm{e}^{i\psi}$, $-\pi < \psi \leqq \pi$, then $z^{-n/t} = r^{-n/t} \mathrm{e}^{-i\psi n/t}$). From (1.27) to (1.28) one may pass applying the well-known formula

$$\int\limits_0^\infty \frac{\varrho^{n-1} \, d\varrho}{(a\varrho^\nu + 1)^\mu} = \nu^{-1} a^{-n/\nu} B(n/\nu, \mu - n/\nu) \quad (1 \leqq n < \nu\mu, a \notin (-\infty, 0]) \tag{1.29}$$

(see e.g. [9: p. 299]). Introduce two quantities

$$d = b_{n/t,l}^{-1} (2\pi)^{-n} \int\limits_{T^*M} \big(a_0(x, \xi) + 1\big)^{-l} \, dx \, d\xi, \tag{1.30}$$

$$\varDelta = b_{n/t,l}^{-1} (2\pi)^{-n} \int\limits_{T^*M} \big(|a_0(x, \xi)| + 1\big)^{-l} \, dx \, d\xi. \tag{1.31}$$

Using (1.29), we can rewrite (1.30), (1.31) in the form

$$d = \frac{1}{(2\pi)^n \, n} \int\limits_M \sum_{k=1}^m \varphi_k(x) \int\limits_{|\xi|=1} \big(a_0{}^{(k)}(x, \xi)\big)^{-n/t} \, dS_\xi \, dx, \tag{1.32}$$

$$\varDelta = \frac{1}{(2\pi)^n \, n} \int\limits_M \sum_{k=1}^m \varphi_k(x) \int\limits_{|\xi|=1} |a_0{}^{(k)}(x, \xi)|^{-n/t} \, dS_\xi \, dx. \tag{1.33}$$

From this we see that $|d| \leqq \Delta$ and that d, Δ do not depend on l. Since (1.24) implies

$$\operatorname{Re}\left[\left(a_0^{(k)}(x, \xi)\right)^{-n/t}\right] \geqq |a_0^{(k)}(x, \xi)|^{-n/t} \cos(\theta n/t),$$

we obtain from (1.32), (1.33) also

$$|d| \geqq \Delta \cos(\theta n/t). \tag{1.34}$$

Denote by $N(\lambda)$ the counting function for modules of eigenvalues of A, i.e. the number of them in the circle $\{z: |z| \leqq \lambda\}$. Recall that each eigenvalue is counted according to its multiplicity.

Theorem 1.8: *Under the above assumptions,*

$$|d| \leqq \varlimsup_{\lambda\to\infty} \lambda^{-n/t} N(\lambda) \leqq e\Delta. \tag{1.35}$$

Proof: Let us verify that in the proof of the inequality

$$|d| \leqq \varlimsup_{\lambda\to\infty} \lambda^{-n/t} N(\lambda) \tag{1.36}$$

t may be replaced by t/p, where p is an arbitrary positive integer. By this replacement $a_0(x, \xi)$ turns into $\left(a_0(x, \xi)\right)^{1/p}$ while d, as it is seen from (1.32), remains the same. Further, if $N^{(p)}(\lambda)$ is the counting function for modules of new eigenvalues $\lambda_\nu^{1/p}$, then $\mu^{n/t}N(\mu) = \lambda^{np/t}N^{(p)}(\lambda)$ for $\mu = \lambda^p$. Therefore the right-hand side of (1.36) also does not change. Hence we may assume that all the eigenvalues λ_ν of A lie in $\{\lambda: |\arg \lambda| \leqq \varphi\}$ where $\varphi < \pi/2$. We shall first establish the inequality of the type (1.36) for the counting function $N_R(\lambda)$ of the real parts of λ_ν:

$$|d| \leqq \varlimsup_{\lambda\to\infty} \lambda^{-n/t} N_R(\lambda). \tag{1.37}$$

It is obvious that, for $\mu > 0$, $\sum_\nu (\operatorname{Re} \lambda_\nu + \mu)^{-l} \geqq \sum_\nu |\lambda_\nu + \mu|^{-l} \geqq \left|\sum_\nu (\lambda_\nu + \mu)^{-l}\right|$, and consequently

$$\varlimsup_{\mu\to\infty} \mu^{l-n/t} \sum_\nu (\operatorname{Re} \lambda_\nu + \mu)^{-l} \geqq \lim_{\mu\to\infty} \left|\mu^{l-n/t} \sum_\nu (\lambda_\nu + \mu)^{-l}\right|. \tag{1.38}$$

Since $\operatorname{tr}[R_A(-\mu)]^{-l} = \sum_\nu (\lambda_\nu + \mu)^{-l}$, by (1.26) the right-hand side of (1.38) is equal to $|c_l|$, and we may rewrite (1.38) in the form

$$\varlimsup_{\mu\to\infty} \mu^{l-n/t} \int_0^\infty (\lambda + \mu)^{-l} \, dN_R(\lambda) \geqq |c_l|.$$

Now (1.15) and the equality $c_l = b_{n/t,l}d$ imply (1.37). Since $|\arg \lambda_\nu| \leqq \varphi (< \pi/2)$, we have $N(\lambda) \geqq N_R(\lambda \cos\varphi)$, so from (1.37) it follows that $|d| (\cos\varphi)^{n/t} \leqq \varlimsup_{\lambda\to\infty} \lambda^{-n/t}N(\lambda)$. Replace here t by t/p: $|d| \left(\cos(\varphi/p)\right)^{np/t} \leqq \varlimsup_{\lambda\to\infty} \lambda^{-n/t}N(\lambda)$. Evidently $\left(\cos(\varphi/p)\right)^{np/t} \to 1$ as $p \to \infty$, so in the limit we obtain (1.36).

Now pass to the proof of the right inequality in (1.35). Without loss of generality, assume that $0 \notin \sigma(A)$. If $K = A^{-1}$, then obviously $N(\lambda) = n(\lambda)$, where $n(\lambda)$ is the counting function for modules of the characteristic values of the compact operator K. By Theorem 1.6

$$\varlimsup_{\lambda\to\infty} \lambda^{-n/t} n(\lambda) \leqq e \varlimsup_{\lambda\to\infty} \lambda^{-n/t} \nu(\lambda). \tag{1.39}$$

It is easily seen that $\nu(t)$ coincides with the counting function for eigenvalues of $B = (A^*A)^{1/2}$. It is a positive elliptic operator of order t, and its principal symbol is equal $|a_0(x, \xi)|$. Therefore

$$\nu(\lambda) \sim \Delta \lambda^{n/t}, \quad \lambda \to \infty \tag{1.40}$$

(see e.g. [27: § 15]). Thus (1.39) yields the right inequality in (1.35) ∎

Using (1.40) and Theorem 1.5, we obtain the following assertion.

Theorem 1.9: *Under the above assumptions,*

$$\varliminf_{\lambda \to \infty} \lambda^{-n/t} N(\lambda) \leqq \Delta . \tag{1.41}$$

Remark 1.10: As it is seen from the proofs, the right inequality in (1.34) and (1.41) are both valid for each elliptic pseudo-differential operator with a discrete spectrum (the condition (1.24) is not necessary).

Theorem 1.11: *Let $d \neq 0$. Then*

$$\varliminf_{\lambda \to \infty} \lambda^{-n/t} N(\lambda) > 0. \tag{1.42}$$

Proof: Since, according to our assumption, all the eigenvalues λ_ν lie in some sector $\{\lambda: |\arg \lambda| \leqq \theta_1\}$ $(\theta_1 < \pi)$, we have for $\mu > 0$

$$|\lambda_\nu + \mu|^2 = |\lambda_\nu|^2 + \mu^2 + 2\,|\lambda_\nu|\,\mu \cos \arg \lambda_\nu \geqq (|\lambda_\nu| + \mu)^2 \cos^2 \frac{\theta_1}{2}, \tag{1.43}$$

and hence

$$\sum_{\nu=1}^{\infty} (|\lambda_\nu| + \mu)^{-l} \geqq C \sum_{\nu=1}^{\infty} |\lambda_\nu + \mu|^{-l} \geqq C \left| \sum_{\nu=1}^{\infty} (\lambda_\nu + \mu)^{-l} \right|,$$

where $C = \left(\cos (\theta_1/2)\right)^{-l}$. From this we obtain

$$\varliminf_{\mu \to \infty} \mu^{l-n/t} \sum_{\nu=1}^{\infty} (|\lambda_\nu| + \mu)^{-l} \geqq C \lim_{\mu \to \infty} \mu^{l-n/t} \left| \sum_{\nu=1}^{\infty} (\lambda_\nu^\circ + \mu)^{-l} \right|.$$

By (1.26) the latter limit is equal to $|c_l| = |b_{n/t,l} d|$. So

$$\varliminf_{\mu \to \infty} \mu^{l-n/t} \int_0^\infty (\lambda + \mu)^{-l} \, dN(\lambda) > 0. \tag{1.44}$$

On the other hand, by Theorem 1.8

$$\varlimsup_{\lambda \to \infty} \lambda^{-n/t} N(\lambda) < \infty . \tag{1.45}$$

According to Remark 1.4, (1.44) and (1.45) imply (1.42) ∎

From Theorems 1.8 and 1.11 follows

Corollary 1.12: *If $d \neq 0$, then $N(\lambda) \asymp \lambda^{n/t}$.*

Remark 1.13: If

$$n\theta < \pi t/2, \tag{1.46}$$

where θ is the same as in (1.24), then $d \neq 0$. This follows from (1.34). If M is connected and $n > 1$, the sign $<$ in (1.46) may be replaced by $\leqq$. Indeed, in this case the real part of the integrand in (1.32) is non-negative, and if $d = 0$, we have $\operatorname{Re} \left(a_0^{(k)}(x, \xi)\right)^{-n/t} = 0$ on $U_k = \{(x, \xi):$

$\varphi_k(x) > 0$, $|\xi| = 1\}$ $(k = 1, \ldots, m)$. But then, since M and the unit sphere in $\mathbf{R}^n$ $(n > 1)$ is connected, there exist a number k and a point $(x_0, \xi_0) \in U_k$ such that $\operatorname{Im}\,(a_0^{(k)}(x_0, \xi_0))^{-n/t} = 0$, so that $a_0^{(k)}(x_0, \xi_0) = 0$, which contradicts the ellipticity of A.

Remark 1.14: Corollary 1.12 shows that A has "many" eigenvalues if $d \neq 0$. As it has been mentioned in the Introduction, the condition (1.46) assures the completeness of root functions of the operator.

Theorems 1.8 and 1.9 imply

Corollary 1.15: *If the limit of $\lambda^{-n/t}N(\lambda)$ as $\lambda \to \infty$ exists, it belongs to the segment* $[|d|, \Delta]$.

Remark 1.16: If $|d| = \Delta$, then the limit of $\lambda^{-n/t}N(\lambda)$ as $\lambda \to \infty$ exists (and obviously coincides with $|d| = \Delta$). Indeed, from (1.32) and (1.33) it follows that in this case $\arg a_0(x, \xi) =$ const. But then $A = \alpha(A_0 + B)$, where $\alpha \in \mathbf{C}$, A_0 is a selfadjoint elliptic pseudo-differential operator (with the principal symbol $\alpha^{-1}a_0(x, \xi) > 0$), and B is a pseudo-differential operator of order $\leqq t - 1$. Therefore for A the formula of the type (0.1) is valid with $\alpha^{-1}a_0$ instead of a_0 (see e.g. [18]).

1.4. Generalizations to matrix elliptic operators. Let A be a $(r \times r)$-matrix elliptic pseudo-differential operator of order $t > 0$ with the principal symbol $a_0(x, \xi)$. Denote by $\lambda_j(x, \xi)$ $(j = 1, \ldots, r)$ the eigenvalues of the matrix $a_0(x, \xi)$. Under the condition $|\arg \lambda_j(x, \xi)| \leqq \theta < \pi$ $(j = 1, \ldots, r)$, Theorems 1.8, 1.9, 1.11 and Corollaries 1.12, 1.15 can be easily extended to the matrix case, with the replacement of $(a_0 + 1)^{-l}$ and $(|a|_0 + 1)^{-l}$ in (1.30) and (1.31) by $\operatorname{tr}(a_0 + E)^{-l}$ and $\operatorname{tr}\big((a_0^* a_0)^{1/2} + E\big)^{-l}$, respectively. Assume now that the eigenvalues of $a_0(x, \xi)$ lie in two closed sectors Λ_1 and Λ_2 with vertex at the origin and without any other common points. For definiteness assume that

$$\Lambda_1 = \{\zeta : |\arg \zeta| \leqq \theta_1\}, \quad \Lambda_2 = \{\zeta : |\arg \zeta| \geqq \theta_2\},$$

where $0 \leqq \theta_1 < \theta_2 \leqq \pi$. For arbitrary small $\varepsilon > 0$, A is elliptic with a parameter in $\Lambda_\varepsilon^\pm = \{\zeta : \theta_1 + \varepsilon \leqq \pm\arg \zeta \leqq \theta_2 - \varepsilon\}$ $\big(\varepsilon < (\theta_2 - \theta_1)/2\big)$, and these sectors can contain only a finite number of eigenvalues $\lambda_\nu(A)$. Fixing $\varepsilon > 0$, denote by $N_1(\lambda)$ the counting function for modules of those $\lambda_\nu(A)$ which lie in $\Lambda_1(\varepsilon) = \{\zeta : |\arg \zeta| \leqq \theta_1 + \varepsilon\}$. Generalizations we are going to obtain concern $N_1(\lambda)$ (instead of $N(\lambda)$). Since $N_1(\lambda) \leqq N(\lambda)$, some upper bounds for $N_1(\lambda)$ come from appropriate bounds for $N(\lambda)$, so we shall deal only with lower bounds. Set

$$c_l^{(1)} = (2\pi)^{-n} \int\limits_{T^*M} \sum_{\lambda_j \in \Lambda_1} \big(\lambda_j(x, \xi) + 1\big)^{-l}\, dx\, d\xi, \qquad d^{(1)} = b_{n/t,l}^{-1} c_l^{(1)}, \tag{1.47}$$

$$S_{1,l}(\zeta) = \sum_{\lambda_\nu \in \Lambda_1(\varepsilon)} \big(\lambda_\nu(A) - \zeta\big)^{-l}, \tag{1.48}$$

where l is an arbitrary positive integer greater than n/t (one can verify easily that $d^{(1)}$ does not depend on l). The desired generalizations will be derived from the following

Theorem 1.17: *If $l - 1 < n/t < l$, then*

$$S_{1,l}(\zeta) = c_l^{(1)}(-\zeta)^{n/t - l} + o(|\zeta|^{n/t - l}) \qquad (\zeta \to \infty) \tag{1.49}$$

uniformly in $\{\zeta : |\arg \zeta| \geqq \theta_1 + \varepsilon\}$.

Proof: We begin with the known formula (see e.g. [14])

$$\operatorname{tr}\big(R_A(\zeta)\big)^l = (2\pi)^{-n} \int\limits_{T^*M} \operatorname{tr}\,[a_0(x, \xi) - \zeta E]^{-l}\, dx\, d\xi + O\Big(|\zeta|^{\frac{n-1}{t} - l}\Big),$$

which is valid in $\Lambda_\varepsilon^\pm$. Setting $S_{2,l}(\zeta) = \operatorname{tr}(R_A(\zeta))^l - S_{1,l}(\zeta)$, we put down this formula in the form

$$S_{1,l}(\zeta) + S_{2,l}(\zeta) = (2\pi)^{-n} \int_{T^*M} \sum_{\lambda_j \in \Lambda_1}' (\lambda_j(x, \xi) - \zeta)^{-l}\, dx d\xi$$
$$+ (2\pi)^{-n} \int_{T^*M} \sum_{\lambda_j \in \Lambda_2} (\lambda_j(x, \xi) - \zeta)^{-l}\, dx\, d\xi + O\left(|\zeta|^{\frac{n-1}{t}-l}\right).$$

Here the first integral makes sense if $|\arg \zeta| > \theta_1$ and we can transform it by setting $\xi = \mu^{1/t}\eta$ for $\zeta = -\mu < 0$ and using the holomorphic extension in ζ. The second integral makes sense if $|\arg \zeta| < \theta_2$ and admits a similar transformation: we set $\xi = \mu^{1/t}\eta$ for $\zeta = \mu > 0$ and then use the holomorphic extension in ζ. So we obtain for $\zeta \in \Lambda_\varepsilon^\pm$

$$S_{1,l}(\zeta) + S_{2,l}(\zeta) = c_l^{(1)}(-\zeta)^{n/t-l} + c_l^{(2)}\zeta^{n/t-l} + O\left(|\zeta|^{\frac{n-1}{t}-l}\right), \tag{1.50}$$

where $c_l^{(1)}$ is defined by the first equality in (1.47) and $c_l^{(2)}$ by the analogous equality with $\lambda_j(x, \xi) \in \Lambda_2$.

Let us estimate the growth of $S_{1,l}(\zeta)$ when $|\arg \zeta| \geqq \theta_1 + \varepsilon$. Since $|\arg \lambda_\nu| \leqq \theta_1 + \varepsilon/2$ in each term in (1.48) with sufficiently large ν, we have, by the inequality analogous to (1.43),

$$|S_{1,l}(\zeta)| \leqq \sum_{\lambda_\nu \in \Lambda_1(\varepsilon)} |\lambda_\nu(A) - \zeta|^{-l} \leqq C_1 \sum_{\lambda_\nu \in \Lambda_1(\varepsilon)} (|\lambda_\nu(A)| + |\zeta|)^{-l}$$

for sufficiently large $|\zeta|$. Since $N(\lambda) = O(\lambda^{n/t})$ (see (1.35)), we have $|\lambda_\nu^{-1}(A)| = O(\nu^{-t/n})$, and therefore

$$|S_{1,l}(\zeta)| \leqq C_1 \sum_{\nu=1}^{\infty} (\nu^{t/n} + |\zeta|)^{-l} \leqq C_1 \int_0^\infty \frac{dx}{(x^{t/n} + |\zeta|)^l} = C_1 |\zeta|^{n/t-l} \int_0^\infty \frac{dy}{(y^{t/n} + 1)^l}$$

(we use the substitution $x = |\zeta|^{n/t} y$). Thus

$$S_{1,l}(\zeta) = O(|\zeta|^{n/t-l}) \qquad (\zeta \to \infty, |\arg \zeta| \geqq \theta_1 + \varepsilon). \tag{1.51}$$

Similarly we can verify that

$$S_{2,l}(\zeta) = O(|\zeta|^{n/t-l}) \qquad (\zeta \to \infty, |\arg \zeta| \leqq \theta_2 - \varepsilon). \tag{1.52}$$

The formula (1.49) we shall derive from (1.50) by "separating" the asymptotics of $S_{1,l}(\zeta)$. To do this, take the contour Γ consisting of two half-lines $\{\zeta : \arg \zeta = \pm(\theta_1 + \varepsilon)\}$, passing from ∞ to 0 on the lower and from 0 to ∞ on the upper half-line. If some $\lambda_\nu(A)$ are found on Γ (there is at most a finite number of such $\lambda_\nu(A)$), we slightly deform Γ near such points so as to avoid them, but make this to that all the eigenvalues of A, contained in the sector $\{\zeta : |\arg \zeta| > \theta_1 + \varepsilon\}$, remain at the left of Γ and so that the origin remains the unique point common to Γ and R. Divide (1.50) by $2\pi i(\zeta - z)$ and integrate along Γ, assuming z lies at the left of Γ:

$$\frac{1}{2\pi i} \int_\Gamma \frac{S_{1,l}(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_\Gamma \frac{S_{2,l}(\zeta)}{\zeta - z}\, d\zeta$$
$$= \frac{c_l^{(1)}}{2\pi i} \int_\Gamma \frac{(-\zeta)^{n/t-l}}{\zeta - z}\, d\zeta + \frac{c_l^{(2)}}{2\pi i} \int_\Gamma \frac{\zeta^{n/t-l}}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_\Gamma \frac{R(\zeta)}{\zeta - z}\, d\zeta, \tag{1.53}$$

where $R(\zeta)$ is the remainder in (1.50) (all the integrals converge absolutely in virtue of the condition $n/t < l$ and relations (1.51) and (1.52)). In the first term of the right-hand side we can replace Γ by a closed contour surrounding z and lying at the left of Γ. From the Cauchy integral formula it follows immediately that this term is equal to $c_l^{(1)}(-z)^{n/t-l}$. Analogously, in the second term of the right-hand side we can replace Γ by a closed contour lying at the right of Γ; so this term is equal to 0. Similar arguments permit to calculate easily the terms in the left-hand side of (1.53). The first of them is equal to $S_{1,l}(z)$, the second is equal to 0. It remains to estimate the third term in the right-hand side of (1.53). Let r be so large that the part Γ'' of Γ lying outside the disk $\{\zeta : |\zeta| \leqq r\}$ consists of half-lines, and let Γ' be the part of Γ lying inside the disk. Then

$$\left| \int_\Gamma \frac{R(\zeta)}{\zeta - z}\, d\zeta \right| \leqq \int_{\Gamma'} \frac{|R(\zeta)|}{|\zeta - z|}\, |d\zeta| + C_3 \int_{\Gamma''} \frac{|\zeta|^{\frac{n-1}{t}-l}}{|\zeta| + |z|}\, |d\zeta|$$

for $|\arg z| \geqq \theta_1 + \varepsilon$ (we again apply the inequality analogous to (1.43)). The first term on the right-hand side obviously has the order $O(|z|^{-1})$. The second term is not greater than

$$C_3\, |z|^{\frac{n-1}{t}-l} \left(\int_{r|z|^{-1}}^{1} \tau^{\frac{n-1}{t}-l}\, d\tau + \int_1^\infty \tau^{\frac{n-1}{t}-l} (1+\tau)^{-1}\, d\tau \right)$$

(here we use the substitution $|\zeta| = \tau\, |z|$ and the inequality $(1 + \tau)^{-1} < 1$ for $\tau \in (0, 1)$). The order of this quantity is $O\left(|z|^{\frac{n-1}{t}-l}\right)$, when $(n-1)/t - l > -1$, $O(|z|^{-1} \ln z)$, when $(n-1)/t - l = -1$, and $O(|z|^{-1})$, when $(n-1)/t - l < -1$. Since $l - 1 < n/t < l$, in all the cases we obtain the estimate $o(|z|^{n/t-l})$ for the third integral in the right-hand side of (1.53) ∎

The main result of the present subsection is

Theorem 1.18: *If* $d^{(1)} \neq 0$, *then*

$$\varliminf_{\lambda\to\infty} \lambda^{-n/t} N_1(\lambda) > 0, \qquad \varlimsup_{\lambda\to\infty} \lambda^{-n/t} N_1(\lambda) \geqq |d^{(1)}|. \tag{1.54}$$

Proof: The proof of the first inequality is quite similar to that of Theorem 1.11. To prove the second inequality, it is convenient to change the notations and assume that the bisectrix of one of the sectors separating Λ_1 and Λ_2 coincides with $\mathbf{R}_-$. For this, A is to be replaced by $e^{i\psi}A$ with an appropriate ψ. Theorem 1.17 gives the asymptotics of $S_{1,l}(\zeta)$ outside the angular neighbourhood of Λ_1 in its new position and, in particular, along $\mathbf{R}_-$ if n/t is not an integer. Now we note that it is sufficient to obtain the desired result for $A_\alpha = A^\alpha$ with an arbitrary $\alpha \in (0, 1)$ (taking into account that the first formula in (1.47) can be rewritten in a form analogous to (1.32) with $\sum \lambda_j^{-n/t}$ instead of $a_0^{-n/t}$). Therefore we may assume that t is irrational and that we have a formula of type (1.49) for $A^{1/p}$ along $\mathbf{R}_-$. Now we can prove the second inequality in (1.54) in the same way as (1.36) ∎

Note that if $\theta_1 = 0$, i.e. if Λ_1 is reduced to $\mathbf{R}_+$, one of the authors [3] has obtained the regular asymptotics $N_1(\lambda) \sim d^{(1)}\lambda^{n/t}$ $(\lambda \to \infty)$. Here we have used the way of reasoning employed in [3].

1.5. Results for elliptic boundary value problems. Let G be a bounded domain in $\mathbf{R}^n$ with a C^∞-boundary ∂G. Consider an elliptic boundary value problem

$$Au = f \text{ in } G, \qquad B_j u|_{\partial G} = 0 \qquad (j = 1, \ldots, m) \tag{1.55}$$

(see e.g. [15]) with homogeneous boundary conditions. Here u, f are scalar (for simplicity) functions, A is a differential operator of order $t = 2m$, elliptic in $\overline{G}$, B_j are differential operators of order $t_j < t$ and all the operators have C^∞-coefficients. Denote by A_B the corresponding closed operator in $L^2(G)$; its domain is the subspace in $H_{2m}(G)$ defined by the boundary conditions $B_j u|_{\partial G} = 0$ $(j = 1, \ldots, m)$. Suppose the problem, obtained from (1.55) by replacing A with $A - \lambda I$, is elliptic with a parameter in a (closed) sector $\mathcal{L}$ with the bisectrix $\mathbf{R}_-$. Then the boundary operators form a normal system (see e.g. [24]), the resolvent $R_{A_B}(\lambda)$ exists for $\lambda \in \mathcal{L}$ with sufficiently large $|\lambda|$ and satisfies the estimate $\|R_{A_B}(\lambda)\| = O(|\lambda|^{-1})$ (see [1]). As in Subsection 1.3, we may assume that $R_{A_B}(\lambda)$ exists for all $\lambda \in \mathcal{L}$.

Define d and Δ by (0.2) (where $a_0(x, \xi)$ is the principal symbol of A and $t = 2m$) with G instead of M. Let $N(\lambda)$ be the counting function for modules of eigenvalues $\lambda_\nu(A_B)$.

Theorem 1.19: *Under the above assumptions,*

$$|d| \leqq \overline{\lim_{\lambda\to\infty}}\, \lambda^{-n/t} N(\lambda) \leqq e\Delta, \qquad \underline{\lim}_{\lambda\to\infty}\, \lambda^{-n/t} N(\lambda) \leqq \Delta;$$

furthermore, if $d \neq 0$, *then* $\underline{\lim}_{\lambda\to\infty} \lambda^{-n/t} N(\lambda) > 0$.

The proof is similar in the main to the proofs of Theorems 1.8, 1.9, 1.11, and we restrict ourselves to the following explanations. First of all, a formula for A_B of the form (1.40) is valid. It comes from the fact that the composition $A_B{}^* A_B$ corresponds to the self-adjoint elliptic boundary value problem in G for the differential operator A^*A with the principal symbol $|a_0(x, \xi)|^2$ (see e.g. [10]). Secondly, one can define the powers $A_B{}^\alpha$ of A_B, $0 < \alpha < 1$ [25]. Set $R_{\alpha,q}(\lambda) = (A_B{}^\alpha - \lambda I)^{-q}$ $(q \in \mathrm{N})$. If $2m\alpha q > n$, this operator belongs to the trace class and the following lemma is valid.

Lemma 1.20: *Under the above assumptions,*

$$\operatorname{tr} R_{\alpha,q}(\lambda) = b_{n/2m\alpha,q}\, d(-\lambda)^{\frac{n}{2m\alpha} - q} + O\left(|\lambda|^{\frac{n-1}{2m\alpha} - q}\right) \quad (\lambda \to -\infty).$$

To prove this formula one must apply the equality

$$R_{\alpha,q}(\lambda) = \frac{1}{2\pi i} \int_\Gamma (\mu^\alpha - \lambda)^{-q} R_{A_B}(\mu)\, d\mu$$

and Seeley's formulas [24] for the parametrix, which approximates R_{A_B} in $\mathcal{L}$. Here Γ is the contour consisting of two half-lines $\{\mu: \arg \mu = \pm\psi, |\mu| > \delta\}$ and the arc $\{\mu: |\mu| = \delta, |\arg \mu| \leqq \psi\}$; δ is a small positive number; $0 < \psi < \pi$ and ψ is sufficiently close to π; the direction of passing is counter-clockwise on the arc.

One can take a compact manifold with boundary instead of G and consider elliptic boundary value problems for vector functions, including the case of two sectors of ellipticity with a parameter. But we shall not dwell on that.

2. Examples and counterexamples

2.1. Elliptic operators with empty spectrum and with spectrum filling the whole plane. Consider an elliptic differential operator of the form

$$A = e^{i\beta\cdot x} P(D)$$

on the n-dimensional torus $\mathbf{T}^n$. Here $P(\xi)$ is a polynomial and β is a non-zero multi-index. (We identify functions on $\mathbf{T}^n$ with appropriate functions of $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

2π-periodic in every x_j.) Assume first that $P(\alpha) \neq 0$ for each $\alpha \in \mathbf{Z}^n$. Suppose $Au = \lambda u$ for some $\lambda \in \mathbf{C}$ and some function $u \in L^2(\mathbf{T}^n)$. Substituting here the Fourier expansion

$$u(x) = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \, e^{i\alpha \cdot x} \tag{2.1}$$

we obtain

$$P(\alpha) \, c_\alpha = \lambda c_{\alpha+\beta}. \tag{2.2}$$

If $\lambda = 0$, from (2.2) it follows that $c_\alpha = 0$ for all α, so that $u(x) \equiv 0$. Now let $\lambda \neq 0$. If $c_{\alpha_0} \neq 0$ for some α_0, then by (2.2)

$$c_{\alpha_0+k\beta} = \lambda^{-k} P(\alpha_0) \, P(\alpha_0 + \beta) \dots P\big(\alpha_0 + (k-1)\,\beta\big) \, c_{\alpha_0} \qquad (k \in \mathbf{N}),$$

so that $c_{\alpha_0+k\beta} \to \infty$ as $k \to \infty$. This contradicts the condition $u \in L^2(\mathbf{T}^n)$. Hence $c_\alpha = 0$ for all α, i.e. $u(x) \equiv 0$. Thus A has no eigenvalues. The same fact can be similarly established for A^*. So the spectrum of A is empty.

Now consider the case when $P(\alpha_0) = 0$ for some $\alpha_0 \in \mathbf{Z}^n$. Since A is elliptic, $P(\xi) \neq 0$ for sufficiently large $|\xi|$. Hence we may assume that $P(\alpha_0 - l\beta) \neq 0$ for $l \in \mathbf{N}$. For an arbitrary $\lambda \in \mathbf{C}$, set $c_{\alpha_0} = 1, c_{\alpha_0 - l\beta} = \lambda^l [P(\alpha_0 - \beta) \cdots P(a_0 - l\beta)]^{-1}$ $(l \in \mathbf{N})$ and $c_\alpha = 0$ for all other α. Evidently $Au = \lambda u$ where $u \ \big(\in L^2(\mathbf{T}^n)\big)$ is given by (2.1). Thus in this case the eigenvalues of A cover the whole plane.

2.2. Elliptic operator with an incomplete system of eigenfunctions. Consider the differential operator

$$A = a_0(x) \, D + a_1(x) \qquad (D = -i \, d/dx) \tag{2.3}$$

on the circle $\mathbf{T}$ with complex functions a_k $(k = 1, 2)$. For simplicity assume $a_k \in C^\infty$; we identify functions on $\mathbf{T}$ with appropriate 2π-periodic functions on $\mathbf{R}$. Assume also that A is elliptic: $a_0(x) \neq 0$ for all x. Each solution of $Au = \lambda u$ has the form

$$u(x) = C \exp \left[i \left(\lambda \int_0^x a_0^{-1}(t) \, dt - \int_0^x a_1(t) \, a_0^{-1}(t) \, dt \right) \right]. \tag{2.4}$$

For $C \neq 0$ this function is 2π-periodic if and only if

$$\lambda \int_0^{2\pi} a_0^{-1}(t) \, dt - \int_0^{2\pi} a_1(t) \, a_0^{-1}(t) \, dt \in 2\pi \mathbf{Z}.$$

From this it follows that in the case when

$$\int_0^{2\pi} a_0^{-1}(t) \, dt = 0, \tag{2.5}$$

the spectrum of A either covers the whole plane (if $\int_0^{2\pi} a_1(t) \, a_0^{-1}(t) \, dt \in 2\pi\mathbf{Z}$) or is empty (if the condition is not fulfilled). In the case when

$$\int_0^{2\pi} a_0^{-1}(t) \, dt \neq 0, \tag{2.6}$$

the spectrum of A consists of the eigenvalues $\lambda_k = a(k + c)$ $(k \in \mathbf{Z})$, where $a = \left(\frac{1}{2\pi} \int_0^{2\pi} a_0^{-1}(t) \, dt \right)^{-1}$, $c = \frac{1}{2\pi} \int_0^{2\pi} a_1(t) \, a_0^{-1}(t) \, dt$. The λ_k approach the line $z = at$ $(t \in \mathbf{R})$ as $k \to \pm\infty$, and the counting function $N(\lambda)$ for their modules has the regular asymptotics $N(\lambda) = 2\,|a|^{-1}\,\lambda + O(1)$ $(\lambda \to \infty)$. By (2.4) the eigenfunction corresponding to the eigenvalue λ_k has the form

$$\varphi_k(x) = g^k(x)\,h(x) \quad (k \in \mathbf{Z}), \tag{2.7}$$

where

$$g(x) = \exp\left(ia \int_0^x a_0^{-1}(t)\,dt\right),$$
$$h(x) = \exp\left[i\left(ac \int_0^x a_0^{-1}(t)\,dt - \int_0^x a_1(t)\,a_0^{-1}(t)\,dt\right)\right]. \tag{2.8}$$

These assertions are contained in [26].

Let us now discuss the properties of the system of eigenfunctions (2.7) of A (we assume that (2.6) holds unless otherwise is specified). One can easily deduce from (2.6) that the equation $Au - \lambda_k u = \varphi_k$ has no 2π-periodic solutions. This means that all the root functions of A are eigenfunctions and that the multiplicity of each eigenvalue is equal to 1. Denote by Γ the closed curve given by the equation $z = g(x)$ $(0 \leqq x \leqq 2\pi)$. Since $g'(x) \neq 0$, Γ is smooth. It does not pass through the origin. Since $\operatorname{Re}\left(a \int_0^x a_0^{-1}(t)\,dt\right)$ is a continuously depending on x value of $\arg g(x)$, Γ goes around the origin once in the positive direction while x goes from 0 to 2π (the index of $g(x)$ is equal to 1).

Proposition 2.1: *The system $\{\varphi_k\}_{-\infty}^{\infty}$ of eigenfunctions of (2.3) is complete in $L^2(\mathbf{T})$ if and only if Γ has no points of self-intersection. If this condition is not satisfied, then the system has an infinite defect.*

Proof: If $g(x_1) \neq g(x_2)$ for $0 \leqq x_1 < x_2 < 2\pi$, then the function $z = g(x)$ defines a mapping of the segment $[0, 2\pi]$ with identified endpoints onto Γ which is one-to-one and continuous and has continuous inverse. It generates the mapping $f(z) \to f[g(x)]$ of $L^2(\Gamma)$ onto $L^2(\mathbf{T})$ which is a continuous (in both directions) isomorphism. Hence the study of geometric properties of $\{g^k(x)\}_{-\infty}^{\infty}$ (and by the inequality $h(x) \neq 0$ also of $\{\varphi_k(x)\}_{-\infty}^{\infty}$) in $L^2(\mathbf{T})$ is reduced to the study of appropriate properties of $\{z^k\}_{-\infty}^{\infty}$ in $L^2(\Gamma)$. In the case under consideration the system $\{z^k\}$ is complete in $C(\Gamma)$ (see e.g. [28: Chapter II, Theorem 7]) and hence in $L^2(\Gamma)$. Therefore $\{\varphi_k(x)\}$ is complete in $L^2(\mathbf{T})$.

Now assume Γ to have at least one point of self-intersection. Since Γ has no cusps and goes around the origin exactly once, the set $\mathbf{C} \setminus \Gamma$ has at least one bounded connected component G not containing the origin. The functions z^k $(k \in \mathbf{Z})$ are holomorphic in G; and if some sequence of their linear combinations converges to a function $f(z)$ in $L^2(\Gamma)$, then evidently $f(z)$ must belong to Smirnov's class $E^2(G)$ (see e.g. [20: Chapter III, Section 17.2]). Therefore $\{z^k\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\Gamma)$; thus $\{\varphi_k(x)\}_{-\infty}^{\infty}$ has an infinite defect in $L^2(\mathbf{T})$ ∎

Let us consider a particular example.

Example 2.2: Let $a_0(x) = (1 + ib\,e^{ix})$ $(b \in \mathbf{R},\ b \neq \pm 1)$. Then the function (2.8) has the form $g(x) = \exp[i(x + b\,e^{ix} - b)]$. By Proposition 2.1, $\{\varphi_k(x)\}_{-\infty}^{\infty}$ is complete in $L^2(\mathbf{T})$ if and only if for some $k \in \mathbf{Z}$ the system of equations $x_1 - x_2 + b(\cos x_1 - \cos x_2) = 2k\pi$, $\sin x_1 = \sin x_2$ has a solution (x_1, x_2) with $0 \leqq x_1 < x_2 < 2\pi$. It is easily seen that, if $|b| > \pi/2$, such a solution exists for $k = -1$ and, if $|b| \in (1, \pi/2]$, for $k = 0$. If $b \in (-1, 1)$, then the function $\arg g(x) = x + b(\cos x - 1)$ is increasing, so that $g(x_1) \neq g(x_2)$ for $0 \leqq x_1 < x_2 < 2\pi$. We see that the system $\{\varphi_k(x)\}_{-\infty}^{\infty}$ is complete if and only if $b \in (-1; 1)$. In particular, in SEELEY's [26] example $(b = 2)$ the system is incomplete.

Remark 2.3: The completeness of $\{\varphi_k(x)\}_{-\infty}^{\infty}$ in $L^2(\mathbf{T})$ yields its completeness in Sobolev's space $H_t(\mathbf{T})$ for each $t \in \mathbf{R}$. Indeed, if λ_0 is a regular point of A, then $B_n = (A - \lambda_0 I)^{-n}$ maps $L^2(\mathbf{T})$ onto $H_n(\mathbf{T})$ isomorphically and continuously (in both direction). Therefore $\{B_n\varphi_k\}_{k=-\infty}^{\infty}$ is complete in $H_n(\mathbf{T})$, and it remains to note that $B_n\varphi_k = (\lambda_k - \lambda_0)^{-n}\varphi_k$.

Remark 2.4: By means of Levy's theorem [16: § 34] one can easily show that Γ has no points of self-intersection if and only if the equality $\int_{x_1}^{x_2} a_0^{-1}(t)\,dt = \int_0^{2\pi} a_0^{-1}(t)\,dt$ $(x_1, x_2 \in \mathbf{R})$ implies that $x_2 - x_1 = 2\pi$.

Remark 2.5: If (2.5) holds and the spectrum of (2.3) covers the whole plane, to any $\lambda \in \mathbf{C}$ there corresponds the infinite chain of root functions $u_{\lambda;k}(x) = d^k u_\lambda(x)/d\lambda^k$ $(k = 0, 1, \ldots)$, where $u_{\lambda;0}(x) = u_\lambda(x)$ is an eigenfunction. In accordance with (2.4), $u_\lambda(x)$ can be put down in the form

$$u_\lambda(x) = v(x) \exp\left(i\lambda\delta^{-1}w(x)\right), \quad w(x) = \delta \int_0^x a_0^{-1}(t)\,dt,$$

where $\delta > 0$ has been chosen so small that $|\mathrm{Re}\, w(x)| < \pi/2$ $(0 \leq x \leq 2\pi)$. Then the curve $\gamma = \{z = \exp(iw(x)) : 0 \leq x \leq 2\pi\}$ lies in the open right half-plane and the functions $d^k z^{\lambda/\delta}/d\lambda^k$ $(\lambda \in \mathbf{C};\ k = 0, 1, \ldots)$ are holomorphic (in z) in each bounded component of the complement of γ. It follows immediately that the closed linear span of the root functions of A has an infinite defect in $L^2(\mathbf{T})$ (cf. with the second part of the proof of Proposition 2.1).

Now we shall give some examples of operators on a two-dimensional manifold with incomplete systems of eigenfunctions.

Example 2.6: Consider the elliptic differential operator $A = a_0(x)(D_x + iD_y)$ on $\mathbf{T}^2$ (we write (x, y) instead of (x_1, x_2)). The C^∞-function $a_0(x)$ is normalized so that $\int_0^{2\pi} a_0^{-1}(t)\,dt = 2\pi$. Let $u(x, y)$ be an eigenfunction of A. Expand it in Fourier series in y: $u(x, y) = \sum_{-\infty}^{\infty} v_l(x)\, e^{ily}$. Substituting this into the equation $Au = \lambda u$, we obtain $a_0(x)(D_x + il)\, v_l(x) = \lambda v_l(x)$ for every $l \in \mathbf{Z}$, from which $v_l(x) = \exp\left(i\lambda \int_0^x a_0^{-1}(t)\,dt + lx\right)$ (up to a numerical multiplier). The condition of 2π-periodicity of this function gives $2\pi i\lambda + 2\pi l = 2\pi ik$, so we obtain the set of eigenvalues

$$\lambda_{k,l} = k - il \qquad (k, l \in \mathbf{Z}) \tag{2.9}$$

and the set of eigenfunctions

$$u_{k,l}(x, y) = \exp\left[i(k - il) \int_0^x a_0^{-1}(t)\,dt + l(x + iy)\right]. \tag{2.10}$$

One can easily verify that $\{\bar{\lambda}_{k,l}\}$ is the set of all eigenvalues of A^* and that there are no root functions of A except eigenfunctions. So the spectrum of A coincides with the set (2.9) of its eigenvalues and all of them are simple. It is not difficult to verify also that the system (2.10) of eigenfunctions of A is complete in $L^2(\mathbf{T}^2)$ if and only if the system of eigenfunctions of $a_0(x)\, D_x$ is complete in $L^2(\mathbf{T})$. Using Proposition 2.1, we obtain examples of two-dimensional elliptic differential operators with incomplete systems of eigenfunctions. Note that the modules of the eigenvalues (2.9) are equal to $(k^2 + l^2)^{1/2}$ and coincide with the eigenvalues of the self-adjoint pseudo-differential operator $(D_x^2 + D_y^2)^{1/2}$. They clearly have the regular asymptotics.

2.3. Elliptic operators with complete systems of eigenfunctions which are not bases.

Proposition 2.7: *Let the system of eigenfunctions of the operator* (2.3) *be complete in* $L^2(\mathbf{T})$. *This system is a basis in* $L^2(\mathbf{T})$ *if and only if* $\arg a_0(x) \equiv \text{const}$.

Proof: For $z \in \Gamma$, $|z| = \exp\left[-\operatorname{Im}\left(a \int_0^x a_0^{-1}(t)\,dt\right)\right]$, hence Γ is a circle with the center at the origin if and only if $\operatorname{Im}\left(a \int_0^x a_0^{-1}(t)\,dt\right) \equiv \text{const}$, i.e. if $\operatorname{Im}\left(aa_0^{-1}(x)\right) \equiv 0$, which is equivalent to $\arg a_0(x) \equiv \text{const}$. So it remains to show that $\{z^k\}_{-\infty}^{\infty}$ is a basis in $L^2(\Gamma)$ if and only if Γ is the circle with the center in the origin. The sufficiency is obvious (and the basis in this case is orthogonal); we must verify the necessity. Let $x = x(z)$ be the function inverse to $z = g(x)$. Then obviously the system

$$u_k(z) = \mathrm{i}\bar{z}^{-k-1}\, \overline{g'(x(z))}\, \left(2\pi\, |g'(x(z))|\right)^{-1} \qquad (k \in \mathbf{Z})$$

is biorthogonal to $\{z^k\}_{-\infty}^{\infty}$. Set $r = \min\{|z| : z \in \Gamma\}$, $R = \max\{|z| : z \in \Gamma\}$ and suppose $r < R$. Fix numbers r_0, R_0 with $r < r_0 < R_0 < R$ and set $E_1 = \{z \in \Gamma : |z| > R_0\}$, $E_2 = \{z \in \Gamma : |z| < r_0\}$. It is easily seen that, for $k \in \mathbf{N}$, $\|z^k\|_{L^2(\Gamma)} \geqq R_0^k \delta_1^{1/2}$, $\|u_k\|_{L^2(\Gamma)} \geqq (2\pi)^{-1} r_0^{-k-1} \delta_2^{1/2}$, where $\delta_k = \operatorname{mes} E_k$. Hence

$$\|z^k\|_{L^2(\Gamma)}\, \|u_k\|_{L^2(\Gamma)} \to \infty \qquad (k \to +\infty). \tag{2.11}$$

It follows (see e.g. [17: Chapter III, § 6, p. 170]) that $\{z^k\}_{-\infty}^{\infty}$ is not a basis in $L^2(\Gamma)$ (and no permutation can make it a basis) ∎

Remark 2.8: It is easily seen that in case $\arg a_0(x) \equiv \text{const}$ the system $\{\varphi_k\}_{-\infty}^{\infty}$ is an unconditional basis in $H_t(\mathbf{T})$ for each $t \in \mathbf{R}$.

Remark 2.9: In case $\arg a_0(x) \not\equiv \text{const}$ the system $\{\varphi_k\}_{-\infty}^{\infty}$ is also not a *basis with parentheses*. For, suppose the contrary. Then $\{z^k\}_{-\infty}^{\infty}$ is a basis with parentheses in $L^2(\Gamma)$, i.e. there exist increasing sequences $\{m_k\}_1^{\infty}$, $\{n_k\}_1^{\infty}$ of positive integers such that

$$\left\| \sum_{j=-m_k}^{n_k-1} c_j z^j - f(z) \right\|_{L^2(\Gamma)} \to 0 \qquad (k \to \infty) \text{ for any } f \in L^2(\Gamma),$$

where c_j are the Fourier coefficients of $f(z)$ with respect to $\{z^j\}$. Let P be the natural projector to the corresponding Smirnov's space $E^2(G)$. Since it is bounded, we obtain

$$\left\| \sum_{i=k}^{\infty} \sum_{j=n_i}^{n_{i+1}-1} c_j z^j \right\|_{L^2(\Gamma)} \to 0, \text{ i.e. } \|z^{n_k} P(z^{-n_k} f)\|_{L^2(\Gamma)} \to 0 \qquad (k \to \infty).$$

Therefore the norms of the operators $z^{n_k}P(z^{-n_k}\,\cdot)$ in $L^2(\Gamma)$ are uniformly bounded. It follows that $\sup_k \|P\|_{L^2(\Gamma, |z|^{2n_k})} < \infty$, where $L^2(\Gamma, |z|^{2n_k})$ is the L^2-space with appropriate weight. By the Stein-Weiss theorem (see e.g. [5: Section 5.4]), we obtain $\sup\{\|P\|_{L^2(\Gamma, |z|^{2n})} : n \geqq n_1\} < \infty$. So we may conclude that the norms of $z^n P(z^{-n}\,\cdot)$ $(n \geqq 1)$ in $L^2(\Gamma)$ are uniformly bounded. Since $c_n z^n = z^n P(z^{-n} f) - z^{n-1} P(z^{1-n} f)$, we have $|c_n|\, \|z^n\|_{L^2(\Gamma)} \leqq c\, \|f\|_{L^2(\Gamma)}$ $(n \geqq 1)$. This contradicts (2.11) because of $c_n = \int_\Gamma f(z)\, \overline{u_n(z)}\, |dz|$.

Remark 2.10: Assume that $|\arg a_0(x)| \leqq \theta < \pi/2$. Then, for any $\alpha > 0$, the Fourier series of $u \in H_\tau(\mathbf{T})$ with respect to the eigenfunctions of (2.3) is summable by Abel's method of order α if α is greater than 1 and close enough to 1 (see [2: § 35]). In the present case there is no need of parenthesis, which generally one puts into the series with Abelian factors $\exp(-\lambda_k^{\alpha} t)$ for convergence, since the series $\sum \exp(-\lambda_k^{\alpha} t)\, c_k \varphi_k(x)$ converges for all $t > 0$ (here c_k are the Fourier coefficients of $u(x)$ with respect to $\{\varphi_k\}$).

Remark 2.11: Returning to Example 2.6, we can easily verify that the system of eigenfunctions of $a_0(x)(D_x + iD_y)$ on $\mathbf{T}^2$ is complete but is not a basis in $H_t(\mathbf{T}^2)$ $(t \geqq 0)$ if and only if the same is true for the system of eigenfunctions of $a_0(x) D_x$ on $\mathbf{T}$.

2.4. Examples for theorems on rough asymptotics from Subsection 1.3. Let A be an elliptic pseudo-differential operator of order $t > 0$ on the n-dimensional torus $\mathbf{T}^n$. If A is a *normal* operator (i.e. $A^*A = AA^*$; for example if A is a differential operator with constant coefficients), then the modules of its eigenvalues coincide with eigenvalues of the elliptic pseudo-differential operator $(A^*A)^{1/2}$, and in virtue of (1.40)

$$N(\lambda) \sim \Delta \lambda^{n/t} \qquad (\lambda \to \infty), \tag{2.12}$$

where Δ is defined by (1.33).

Assume for simplicity that $n = 2$ and consider the following differential operator on $\mathbf{T}^2$ which admits the separation of variables:

$$A = [a_0(x) D_x]^2 + [b_0(y) D_y]^2; \tag{2.13}$$

here the functions $a_0(x)$, $b_0(y)$ are C^∞, 2π-periodic and non-zero everywhere. Assume that

$$|\arg a_0(x)| \leqq \theta_1, \quad |\arg b_0(y)| \leqq \theta_2, \quad \theta_1 + \theta_2 < \pi/2. \tag{2.14}$$

This provides the ellipticity of A and even its ellipticity with a parameter in some angular neighbourhood of $\mathbf{R}_-$. Hence the spectrum of A does not cover the whole plane.

Recall that the spectra of $a_0(x) D_x$ and $b_0(y) D_y$ consist of eigenvalues ak $(k \in \mathbf{Z})$ and bl $(l \in \mathbf{Z})$, respectively, where $a = \left(\frac{1}{2\pi}\int_0^{2\pi} a_0^{-1}(x)\,dx\right)^{-1}$, $b = \left(\frac{1}{2\pi}\int_0^{2\pi} b_0^{-1}(y)\,dy\right)^{-1}$, and that the systems $\{\varphi_k(x)\}_{-\infty}^{\infty}$ and $\{\psi_l(y)\}_{-\infty}^{\infty}$ of corresponding eigenfunctions are complete in $L^2(\mathbf{T})$. This follows, for instance, from asserted in Subsection 2.2. Indeed, by (2.14) $\operatorname{Re} a_0^{-1}(x) > 0$ and $\operatorname{Re} b_0^{-1}(y) > 0$, therefore $a_0(x)$ and $b_0(y)$ satisfy (2.6) and the arguments of the corresponding functions (2.8) are monotonic.

Evidently A has the eigenvalues

$$a^2k^2 + b^2l^2 \qquad (k, l \in \mathbf{Z}), \tag{2.15}$$

corresponding to the eigenfunctions $\varphi_k(x)\,\psi_l(y)$. Since the system $\{\varphi_k(x)\,\psi_l(y)\}$ is complete in $L^2(\mathbf{T}^2)$, the set of all eigenvalues of A (repeated according to their multiplicities) coincides with (2.15). The normal differential operator $a^2D_x^2 + b^2D_y^2$ has the same eigenvalues, and by (2.12) the counting function $N(\lambda)$ for modules of eigenvalues of A has the asymptotics $N(\lambda) \sim d_0\lambda^{n/t}$, where

$$d_0 = \frac{1}{2}\int_0^{2\pi} |a^2 \cos^2\theta + b^2 \sin^2\theta|^{-1}\,d\theta. \tag{2.16}$$

Write down the quantities d and Δ for (2.13). By (1.32)

$$d = \frac{1}{2\pi^2}\int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\pi/2} \left(a_0^2(x)\cos^2\theta + b_0^2(y)\sin^2\theta\right)^{-1} d\theta$$

$$= \frac{1}{2\pi^2}\int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\infty} \left(a_0^2(x) + b_0^2(y)\,\tau^2\right)^{-1} d\tau = \frac{1}{4\pi}\int_0^{2\pi}\frac{dx}{a_0(x)}\int_0^{2\pi}\frac{dy}{b_0(y)}. \tag{2.17}$$

Thus

$$d = \frac{\pi}{ab} = \frac{1}{2}\int_0^{2\pi} (a^2 \cos^2\theta + b^2 \sin^2\theta)^{-1}\, d\theta. \tag{2.18}$$

Comparing (2.16) with (2.18), we see that $|d| = d_0$ if and only if $\arg a = \arg b$. Further, comparing the initial expression for d in (2.17) with the equality

$$\Delta = \frac{1}{2\pi^2}\int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{\pi/2} |a_0{}^2(x)\cos^2\theta + b_0{}^2(y)\sin^2\theta|^{-1}\, d\theta,$$

which follows from (1.33), we conclude that $|d| < \Delta$ if at least one of the functions $a_0(x), b_0(y)$ has a non-constant argument. Take $a_0(x)$ with a non-constant argument. Setting $b_0(x) = a_0(x)$ we have $|d| = d_0 < \Delta$, while if $b_0(x) = a_0(x)\, e^{i\varepsilon}$ we have (for sufficiently small $\varepsilon > 0$) $|d| < d_0 < \Delta$. Now take constant functions $a_0(x) \equiv a$ and $b_0(y) \equiv b$ with $\arg a \neq \arg b$. Then $|d| < d_0 = \Delta$. The last equality holds because in this case A is normal. So we have shown that for (2.13) all the cases (0.7) are possible.

2.5. Another class of elliptic operators with a regular behaviour of eigenvalues. Consider the differential operator

$$A = \sum_{k=0}^{\infty} a_k(e^{ix})\, D_x{}^{n-k} \tag{2.19}$$

on the circle $\mathbf{T}$. Assume that $a_k(\zeta)$ belongs to $C^\infty(\mathbf{T})$ and admit holomorphic extensions into the disk $\{\zeta : |\zeta| < 1\}$, continuous up to the boundary, and that $a_0(\zeta) \neq 0$ for $|\zeta| \leq 1$.

Proposition 2.12: *The set of eigenvalues of* (2.19) *coincides with the set*

$$\sum_{k=0}^{n} a_k(0)\, m^{n-k} \qquad (m \in \mathbf{Z}); \tag{2.20}$$

moreover, the multiplicity of each eigenvalue λ_0 is equal to the number of such $m \in \mathbf{Z}$ that λ_0 can be written down in the form (2.20).

To prove this, we shall need two lemmas.

Lemma 2.13: *Let the numbers $a_{n,k}$ ($k = 1, \ldots, n$; $n = 1, 2, \ldots$) satisfy the recurrence relations*

$$a_{n,k} = a_{n-1,k} + (n - k + 1)\, a_{n-1,k-1} \qquad (1 < k < n) \tag{2.21}$$

and $a_{n,1} = a_{n,n} = 1$. Then, for all ϱ,

$$\sum_{k=1}^{n} a_{n,k}\varrho(\varrho - 1)\ldots(\varrho - n + k) = \varrho^n. \tag{2.22}$$

Proof: We shall verify (2.22) by induction with respect to n. As a preliminary, we note that (2.22) is obviously valid for $\varrho = 0$, and that dividing both the sides by ϱ and setting $\varrho = k + 1$ ($k = 0, \ldots, n - 1$) we obtain, for these k,

$$a_{n,n} + k a_{n,n-1} + k(k-1)\, a_{n,n-2} + \ldots + k!\, a_{n,n-k} = (k+1)^{n-1}. \tag{2.23}$$

On the other hand, since both the sides of (2.22) are polynomials of degree n in ϱ, (2.22) follows from (2.23) with $k = 0, \ldots, n - 2$. For $n = 1$ the equality (2.22) obviously holds. Suppose now that it holds for some positive integer n, and verify that it remains true after replacing n by $n + 1$. From (2.21) it follows that, for $k = 0, \ldots, n - 1$,

$$\begin{aligned} &a_{n+1,n+1} + ka_{n+1,n} + k(k-1)\, a_{n+1,n-1} + \ldots + k!\, a_{n+1,n+1-k} \\ &= a_{n,n} + k(a_{n,n} + 2a_{n,n-1}) + k(k-1)(a_{n,n-1} + 3a_{n,n-2}) + \ldots \\ &\quad + k!\,(a_{n,n+1-k} + (k+1)\, a_{n,n-k}) \\ &= (k+1)\, a_{n,n} + (k+1)\, ka_{n,n-1} + (k+1)\, k(k-1)\, a_{n,n-2} + \ldots \\ &\quad + (k+1)!\, a_{n,n-k} \\ &= (k+1)\,(a_{n,n} + ka_{n,n-1} + k(k-1)\, a_{n,n-2} + \ldots + k!\, a_{n,n-k}) \\ &= (k+1)\,(k+1)^{n-1} = (k+1)^n \end{aligned}$$

(in the next to the last equality we have used (2.23) with $k = 0, \ldots, n - 1$). Thus we have proved (2.23) with the replacement of n by $n + 1$, for $k = 0, \ldots, n - 1$. This yields (2.22) with $n + 1$ instead of n ∎

Set $\partial = \partial_\zeta = d/d\zeta$. Consider the equation

$$\zeta^n \partial^n y + \sum_{k=0}^{n-1} \zeta^k r_k(\zeta)\, \partial^k y = 0. \tag{2.24}$$

Lemma 2.14: *Let $r_k(\zeta)$ be holomorphic in $\{\zeta: |\zeta| < 1\}$ and continuous in $\{\zeta: |\zeta| \leqq 1\}$. If the equation (2.14) has a non-trivial solution on $\{\zeta: |\zeta| = 1\}$, then at least one root of the equation*

$$\varrho(\varrho - 1) \ldots (\varrho - n + 1) + \sum_{k=1}^{n-1} r_k(0)\, \varrho(\varrho - 1) \ldots (\varrho - k + 1) + r_0(0) = 0 \tag{2.25}$$

is an integer.

Proof: Let $y(\zeta)$ ($\not\equiv 0$) be a solution of (2.24) on $\{\zeta: |\zeta| = 1\}$. Since the coefficients of the equation are holomorphic for $0 < |\zeta| < 1$ and continuous for $0 < |\zeta| \leqq 1$, the function $y(\zeta)$ can be extended on $\{\zeta: 0 < |\zeta| < 1\}$ as an analytic solution there, continuous in $\{\zeta: 0 < |\zeta| \leqq 1\}$ (see e.g. [7: Chapter III]). This solution is single-valued. Indeed, if $y_0(\zeta)$ and $y_1(\zeta)$ are two branches of $y(\zeta)$ in the domain $\{\zeta: 0 < |\zeta| \leqq 1\}$ with the cut along $(0, 1]$, then $y_0^{(k)}(\zeta_0) = y_1^{(k)}(\zeta_0)$ ($k = 0, 1, \ldots, n - 1$) for each ζ_0 with $|\zeta_0| = 1$, and by the uniqueness theorem [7] $y_0(\zeta) = y_1(\zeta)$. So (2.24) has the holomorphic solution in $\{\zeta: 0 < |\zeta| < 1\}$.

The equation (2.24) has a regular singular point at $\zeta = 0$, and (2.25) is called the *indicial equation* of (2.24). If ϱ_k ($k = 1, \ldots, n$) are all the roots of (2.25), then (2.24) has the following fundamental system of solutions in $\{\zeta: 0 < |\zeta| < 1\}$ (see e.g. [12: Chapter 1, Section 18.2]). If ϱ_k is such a root that no difference $\varrho_k - \varrho_j$ ($j \neq k$) is an integer, then to ϱ_k there corresponds the solution $y_k(\zeta) = \zeta^{\varrho_k} \varphi_k(\zeta)$ of (2.24) where $\varphi_k(\zeta)$ is holomorphic for $|\zeta| < 1$. Further, if $\varrho_l, \ldots, \varrho_{l+m}$ is a set of such roots that all its differences are integers and, moreover, $\varrho_k - \varrho_{k+1} \geqq 0$ ($k = l, \ldots, l + m - 1$), then to this set there corresponds the set of solutions of the form

$$y_k(\zeta) = \sum_{j=l}^{k} \zeta^{\varrho_j} \varphi_{kj}(\zeta) \ln^{k-j} \zeta \qquad (k = l, \ldots, l + m),$$

where $\varphi_{kj}(\zeta)$ are holomorphic for $|\zeta| < 1$. It is easily seen that if no root ϱ_k is an integer, then the fundamental system of solutions of (2.24) just indicated contains no function holomorphic for $0 < |\zeta| < 1$; moreover, no non-trivial combination of these solutions is holomorphic for $0 < |\zeta| < 1$ ∎

Proof of Proposition 2.12: Divide the equation $Ay = \lambda y$ by $a_0(e^{ix})$ and set $a_0^{-1}(e^{ix}) = q(e^{ix})$, $a_k(e^{ix})/a_0(e^{ix}) = p_k(e^{ix})$ $(k = 1, \ldots, n)$:

$$D^n y + \sum_{k=0}^{n-1} p_{n-k}(e^{ix})\, D^k y = \lambda q(e^{ix})\, y. \tag{2.26}$$

Make the substitution $e^{ix} = \zeta$. A simple induction shows that

$$D_x{}^k = \sum_{j=1}^{k} a_{k,j} \zeta^{k-j+1} \partial_\zeta{}^{k-j+1},$$

where $a_{k,j}$ satisfy the conditions of Lemma 2.13. After the substitution the equation (2.26) looks as follows (for convenience we set $p_0(\zeta) \equiv 1$):

$$\sum_{k=1}^{n} p_{n-k}(\zeta) \sum_{j=1}^{k} a_{k,j} \zeta^{k-j+1} \partial_\zeta{}^{k-j+1} y + \big(p_n(\zeta) - \lambda q(\zeta)\big)\, y = 0.$$

If λ is an eigenvalue of A, the latter equation has a non-trivial solution in the unit circle. By Lemma 2.14 at least one root of the equation

$$\sum_{k=1}^{n} p_{n-k}(0) \sum_{j=1}^{k} a_{k,j} \varrho(\varrho - 1) \ldots (\varrho - k + 1) + p_n(0) - \lambda q(0) = 0$$

is an integer. By Lemma 2.13 this equation may be rewritten in the form

$$\sum_{k=1}^{n} p_{n-k}(0)\, \varrho^k + p_n(0) - \lambda q(0) = 0.$$

So every eigenvalue λ of (2.19) is given by the formula

$$\lambda = q^{-1}(0) \sum_{k=0}^{n} p_{n-k}(0)\, m^k = \sum_{k=0}^{n} a_{n-k}(0)\, m^k$$

for some $m \in \mathbf{Z}$.

Since $a_0(\zeta) \neq 0$ for $|\zeta| \leq 1$, we can select a one-valued branch of its logarithm. In other words, there exists a function $u(\zeta)$ holomorphic for $|\zeta| < 1$, continuous for $|\zeta| \leq 1$, and such that $a_0 = \exp u$. Set

$$a_0{}^{(\tau)}(\zeta) = \exp\big(\tau u(\zeta) + (1 - \tau)\, u(0)\big),$$

$$a_k{}^{(\tau)}(\zeta) = \tau a_k(\zeta) + (1 - \tau)\, a_k(0) \qquad (k > 0),$$

$$A^{(\tau)} = \sum_{k=0}^{n} a_k{}^{(\tau)}(e^{ix})\, D^{n-k}.$$

By the result above the eigenvalues of $A^{(\tau)}$, for each $\tau \in [0, 1]$, are contained in (2.20). On the other hand, the set of eigenvalues of the operator $A^{(0)} = \sum_{k=0}^{n} a_k(0)\, D^{n-k}$ coincides with the set (2.20), the multiplicity of each eigenvalue λ_0 being equal to the number of $m \in \mathbf{Z}$ such that λ_0 admits the representation (2.20). Applying the theorem about the stability of root multiplicity (see e.g. [13: Chapter IV, Theorem 3.18]), we conclude that these assertions are valid for $A^{(1)} = A$ as well ∎

Corollary 2.15: *The counting function for modules of eigenvalues of the operator* (2.19) *has the regular asymptotics*

$$N(\lambda) = 2\,|a_0(0)|^{-1/n}\,\lambda^{1/n} + O(1) \qquad (\lambda \to \infty).$$

REFERENCES

[1] Agmon, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math. **15** (1962), 119—147.

[2] Агранович, М. С.: Спектральные свойства задач дифракции. Дополнение к книге: Войтович, Н. Н., Каценеленбаум, Б. З., и А. Н. Сивов: Обобщенный метод собственных колебаний в теории дифракции. Москва: Изд-во Наука 1977, 289—416.

[3] Агранович, М. С.: Некоторые асимтотические формулы для эллиптических псевдодифференциальных операторов. Функц. анализ и его прил. **21** (1987), 63—65.

[4] Агранович, М. С., и А. С. Маркус: Замечания о спектре несамосопряженных эллиптических операторов. Успехи мат. наук **42** (1987) 4, 142.

[5] Bergh, J., and J. Löfström: Interpolation spaces. Berlin—Heidelberg—New York: Springer-Verlag 1976.

[6] Бойматов, К. Х.: Асимптотическое поведение собственных значений несамосопряженных операторов. Функц. анализ и его прил. **11** (1977) 4, 74—75.

[7] Coddington, E. A., and N. Levinson: Theory of ordinary differential equations. New York—Toronto—London: McGraw-Hill Book Co. 1955.

[8] Гохберг, И. Ц., и М. Г. Крейн: Введение в теорию несамосопряженных операторов. Москва: Изд-во Наука 1965.

[9] Градштейн, И. С., и И. М. Рыжик: Таблицы интегралов, сумм, рядов и произведений. Москва: Физматгиз 1962.

[10] Grubb, G.: Functional calculus of pseudo-differential boundary problems. Boston—Basel—Stuttgart: Birkhäuser Verlag 1986.

[11] Horn, A.: On the eigenvalues of a matrix with prescribed singular values. Proc. Amer. Math. Soc. **5** (1954), 4—7.

[12] Kamke, E.: Differentialgleichungen. Lösungsmethoden und Lösungen. I. Leipzig: Akad. Verlagsges. Geest & Portig 1946.

[13] Kato, T.: Perturbation theory for linear operators. Berlin—Heidelberg—New York: Springer-Verlag 1966.

[14] Кожевников, А. Н.: Об асимптотике собственных значений эллиптических систем. Функц. анализ и его прил. **11** (1977) 4, 84—85.

[15] Lions, J.-L., et E. Magenes: Problèmes aux limites non homogènes et applications. Vol. I. Paris: Dunod 1968.

[16] Люстерник, Л. А.: Выпуклые фигуры и многогранники. Москва: Гостехиздат 1956.

[17] Люстерник, Л. А., и В. И. Соболев: Элементы функционального анализа. Москва: Изд-во Наука 1965.

[18] Маркус, А. С.: Введение в спектральную теорию полиномиальных операторных пучков. Кишинев: Изд-во Штиница 1986.

[19] Маркус, А. С., и В. И. Параска: Об оценке числа собственных значений линейного оператора. Изв. Акад. наук МолдССР (1965) 7, 101—104.

[20] Привалов, И. И.: Граничные свойства аналитических функций. Москва—Ленинград: Гостехиздат 1950.

[21] Розенблюм, Г. В.: Спректральная асимптотика нормальных операторов. Функц. анализ и его прил. **16** (1982) 2, 82—83.

[22] Розенблюм, Г. В.: Угловая асимптотика спектра операторов, близких к нормальным. В кн.: Линейные и нелинейные краевые задачи. Спектральная теория. Ленинград: Изд-во Ленингр. ун-та 1986, 180—195.

[23] Seeley, R. T.: Complex powers of an elliptic operator. In: Singular Integrals (Proc. Symp. Pure Math. 10). Providence, R. I.: Amer. Math. Soc. 1967, 288—307.

[24] Seeley, R. T.: The resolvent of an elliptic boundary problem. Amer. J. Math. **91** (1969), 889—920.

[25] Seeley, R. T.: Analytic extension of the trace associated with elliptic boundary problems. Amer. J. Math. 91 (1969), 963–983.
[26] Seeley, R. T.: A simple example of spectral pathology for differential operators. Comm. Part. Diff. Eq. 11 (1986), 595–598.
[27] Шубин, М. А.: Псевдодифференциальные операторы и спектральная теория. Москва: Изд-во Наука 1978.
[28] Walsh, J. L.: Interpolation and approximation by rational functions in the complex domain. Providence, R. I.: Amer. Math. Soc. 1960.
[29] Widder, D. V.: The Laplace transform. Princeton: Princeton Univ. Press 1946.

Manuskripteingang: 25. 03. 1988

VERFASSER:

Проф. М. С. Агранович
Московский институт электронного машиностроения
Б. Вузовский пер. 3/12
СССР-109028 Москва

Ст. н. с. А. С. Маркус
Институт математики с ВЦ АН Молдавской ССР
ул. Гросула 5
СССР-277028 Кишинев