Zeitschrift für Analysis
und ihre Anwendungen
Bd. 8 (3) 1989, S. 261 – 288

Local Approximation Spaces

H. Triebel

Dedicated to Professor S. G. Mikhlin on the occasion of his 80th birthday

Die Arbeit beschäftigt sich mit Charakterisierungen der Räume F^s_{pq} und B^s_{pq} auf $\mathbf{R}^n$, $\mathbf{R}_+{}^n$ und beschränkten Gebieten durch lokale Oszillationen und Differenzen von Funktionen. Außerdem werden Morrey-Campanato-Räume behandelt.

В работе характеризуются пространства F^s_{pq} и B^s_{pq} на $\mathbf{R}^n$, $\mathbf{R}_+{}^n$ и ограниченных областях локальными осцилляциями и разностями функций. Кроме этого рассматриваются пространства типа Моррей-Кампанато.

The paper deals with characterizations of spaces F^s_{pq} and B^s_{pq} on $\mathbf{R}^n$, $\mathbf{R}_+{}^n$ and bounded domains via local oscillations and differences of functions. Furthermore Morrey-Campanato spaces are treated.

1. Introduction and historical comments

Let Ω be a smooth bounded domain in $\mathbf{R}^n$ and let $B(x, t) = \{y \mid y \in \Omega, |x - y| < t\}$, where $x \in \Omega$ and $t > 0$. Then

$$\operatorname{osc}_p{}^M f(x, t) = \inf \left(⨍_{B(x,t)} |f(y) - P(y)|^p \, dy \right)^{1/p}, \qquad x \in \Omega, \tag{1.1}$$

denotes local oscillations of $f \in L_p(\Omega)$ where the infimum is taken over all polynomials P of degree less than or equal to M with $M \in \mathbf{N}_0$. Furthermore $0 < p < \infty$ and $⨍_V g(y) \, dy = |V|^{-1} \int_V g(y) \, dy$ denotes the mean value. We complement (1.1) by $\operatorname{osc}_p{}^{-1} f(x, t) = \left(⨍_{B(x,t)} |f(y)|^p \, dy \right)^{1/p}$. Since the early sixties oscillations of this type have been systematically used in order to describe smoothness properties of functions. Let $L_p{}^s(\Omega)$ with

$$1 \leqq p < \infty, \qquad M = -1, 0, 1, 2, \ldots \quad \text{and} \quad -n/p \leqq s < M + 1 \tag{1.2}$$

be the collection of all $f \in L_p(\Omega)$ such that $f_p{}^{M,s} \in L_\infty(\Omega)$, where

$$f_p{}^{M,s}(x) = \sup_{0<t<1} t^{-s} \operatorname{osc}_p{}^M f(x, t) \tag{1.3}$$

is a so-called sharp maximal function. Naturally normed $L_p{}^s(\Omega)$ coincides with the well-known Morrey-Campanato spaces. The case $M = -1$ and hence $-n/p \leqq s < 0$ goes back to C. B. Morrey [22]. Recall $L_p{}^{-n/p}(\Omega) = L_p(\Omega)$. Furthermore $L_p{}^0(\Omega) = BMO(\Omega)$ is essentially the John-Nirenberg space of all functions with bounded mean oscillation, see [16]; and $L_p{}^s(\Omega) = \mathcal{C}^s(\Omega)$ if $s > 0$, where the latter stands for the Hölder-Zygmund spaces. The theory of these spaces has been systematically developed in the middle of the sixties by several authors, we mention especially S. Campanato [6–9], G. N. Meyers [21], S. Spanne [25] and G. Stampacchia [26]. We refer also to the surveys [23] and [20: Chapter 4, in particular 4.10]. In [27] G. Stampac-

CHIA replaced the sup-norm in (1.3) by an L_q-norm. This idea was extended by V. P. IL'IN who described on that way the classical Besov spaces $B^s_{pq}(\Omega)$ via oscillations and who introduced large classes of new spaces in a Besov space setting, we refer to the survey of his results given in [1: § 28], see also the paper by JU. A. BRUDNYJ [3]. In this connection it should be mentioned that the idea of approximation of functions by best polynomials can also be used to study spaces of Besov type in rather general non-smooth domains, see the surveys [17, 33] and the papers mentioned there. In the last few years oscillations and sharp maximal functions of type (1.1) and (1.3) attracted new attention. Apparently this new development began with the work by A. P. CALDERÓN and R. SCOTT, see [4, 5], who connected sharp maximal functions of type (1.3) with the theory of Sobolev spaces. In particular A. P. CALDERÓN proved in [4]

$$W_p^k(\Omega) = \{f \mid f \in L_p(\Omega),\ f_u^{k-1,k} \in L_p(\Omega)\} \tag{1.4}$$

with $k \in \mathbb{N}_0$, $1 \leqq u < p$, where $W_p^k(\Omega)$ stands for the classical Sobolev spaces, see also [5, 10, 11]. Formula (1.4) with $k = 0$ is essentially the Hardy-Littlewood maximal inequality. These ideas were modified by R. A. DEVORE and R. C. SHARPLEY who introduced spaces of the type

$$C_p^s(\Omega) = \{f \mid f \in L_{\bar{p}}(\Omega),\ f_p^{[s],s} \in L_p(\Omega)\}, \tag{1.5}$$

where $s > 0$, $0 < p < \infty$ and $\bar{p} = \max(1, p)$, see [11: §§ 6 and 12] (these authors deal mostly with the case $p \geqq 1$). Independently B. BOJARSKI defined in [2] spaces which cover more or less both the Morrey-Campanato spaces L_p^s and the above spaces C_p^s (with $p \geqq 1$), see also [10]. Furthermore it was noted by J. R. DORRONSORO in [12] that oscillations of type (1.1) can be used in order to characterize Bessel-potential spaces in the sense of Paley-Littlewood characterizations. The extension of this observation to some spaces of type $F^s_{pq}(\Omega)$ is due to A. SEEGER [24], see also [13]. (Recall that the Besov space counterpart of this part of the theory is more or less covered by Il'in's work, $p \geqq 1$, see also [3, 14, 17, 33].) The theory of the spaces B^s_{pq} and F^s_{pq} in its full extent; i.e. $-\infty < s < \infty$, $0 < p \leqq \infty$ ($p < \infty$ in the case of the F-spaces), $0 < q \leqq \infty$; has been developed in [28]. These two scales cover many well-known spaces: classical Besov spaces, Hölder-Zygmund spaces, (fractional) Sobolev spaces and inhomogeneous (Sobolev-) Hardy spaces. One of the main results of the present paper reads as follows: Let

$$\begin{gathered} 0 < p < \infty;\quad 0 < q \leqq \infty,\quad 1 \leqq r \leqq \infty,\quad s > n\left(\frac{1}{p} - \frac{1}{r}\right)_+, \\ s > n\left(\frac{1}{q} - \frac{1}{r}\right)_+, \end{gathered} \tag{1.6}$$

let $0 < u \leqq r$ and $M \geqq [s]$, then $F^s_{pq}(\Omega)$ is the collection of all $f \in L_{\max(p,r)}(\Omega)$ such that

$$\|f \mid L_p(\Omega)\| + \left\| \left(\int_0^1 t^{-sq} \operatorname{osc}_u^M f(\cdot, t)^q \frac{dt}{t} \right)^{1/q} \middle| L_p(\Omega) \right\| < \infty \tag{1.7}$$

in the sense of equivalent quasi-norms (modification if $q = \infty$). With $u = r$ this assertion is essentially covered by A. SEEGER [24: Corollary 1]. The extension to $u < r$ is useful for several reasons, in particular it follows immediately

$$C_p^s(\Omega) = F^s_{p\infty}(\Omega), \qquad 0 < p < \infty,\ s > n\left(\frac{1}{p} - 1\right)_+, \tag{1.8}$$

where $C_p^s(\Omega)$ are the spaces from (1.5). Oscillations are closely connected with differences of functions

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^{M-j} \binom{M}{j} f(x+jh), \qquad x \in \mathbf{R}^n, \quad h \in \mathbf{R}^n. \tag{1.9}$$

Characterizations of function spaces via differences are much better known than characterizations via oscillations. Descriptions of function spaces with the help of differences have a long history. In the theory of the Hölder-Zygmund spaces $\mathscr{C}^s$ with $s > 0$ and the classical Besov spaces B_{pq}^s; $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$, they played a decisive role from the very beginning. The extension of this part of the theory to more general spaces B_{pq}^s, including those ones with $p < 1$, and corresponding spaces F_{pq}^s may be found in [28], where we gave also detailed references, see also [29] for a more recent and systematic approach. In [28] we mentioned the problem to find intrinsic descriptions of the spaces $F_{pq}^s(\Omega)$ via differences under reasonable restrictions for the involved parameters. This problem was solved by G. A. Kaljabin [18] in a satisfactory way for the spaces $F_{pq}^s(\Omega)$ with $s > 0$, $1 < p < \infty$, $1 < q < \infty$. It is the second main aim of the present paper to extend Kaljabin's characterization to spaces $F_{pq}^s(\Omega)$ with (1.6) ($r = 1$). We add a technical but important remark: If one takes one of the two characterizations of function spaces in question, i.e. via oscillations or via differences, as granted, then such an assumption is of great help to derive the other one. Since characterizations via differences are known, it was quite natural to use them in order to treat characterizations via oscillations, see [24]. Our intention in the present paper is different. We start from scratch, what means in our context that we begin with characterizations of $F_{pq}^s(\mathbf{R}^n)$ via local means in the sense of [29] and [31]. Then we derive inequalities for oscillations for their own sake which can be used both for F_{pq}^s spaces and Morrey-Campanato spaces. Then we give a new and almost trivial proof of the extension property for the spaces $F_{pq}^s(\mathbf{R}_+^n)$ and arrive finally at characterizations of type (1.7). On this basis and the distinguished local means from [31] we deal afterwards with characterizations via differences.

The plan of the paper is the following. Definitions and main results are collected in Section 2. Proofs and more technical assertions are presented in Section 3. As usual unimportant positive constants are denoted by c, occasionally with additional marks. They may differ from formula to formula (but not within the same formula or inequality).

2. Definitions and main results

2.1. Definitions

2.1.1. Let $\mathbf{N}$ be the collection of all natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ be the collection of all non-negative integers and $\mathbf{N}_{-1} = \mathbf{N}_0 \cup \{-1\}$. Let $\mathbf{R}^n$ with $n \in \mathbf{N}$ be the Euclidean n-space. Then S and S' stand for the Schwartz space of all infinitely differentiable rapidly decreasing complex-valued functions on $\mathbf{R}^n$ and the collection of all complex-valued tempered distributions on $\mathbf{R}^n$, respectively. For sake of brevity we adopt here the following convention: for spaces and quasi-norms which are defined on $\mathbf{R}^n$ we omit "$\mathbf{R}^n$" in the respective notations, otherwise the underlying domain will be mentioned explicitly (mostly $\mathbf{R}_+^n$ or bounded smooth domains in $\mathbf{R}^n$). Similarly $\int = \int\limits_{\mathbf{R}^n}$, otherwise the region of integration will be specified explicitly. Let $0 < p \leq \infty$, then

$$\|f \mid L_p\| = \left(\int |f(x)|^p \, dx\right)^{1/p} \tag{2.1}$$

(usual modification if $p = \infty$). Let $k \in S$, then we introduce the means

$$k(t, f)(x) = \int k(y) f(x + ty)\, dy, \qquad x \in \mathbf{R}^n, \quad t > 0, \tag{2.2}$$

which make sense for any $f \in S'$ (appropriate interpretation). Let $k_0 \in S$ with

$$\operatorname{supp} k_0 \subset \{y \mid |y| < 1\}, \qquad \hat{k}_0(0) \neq 0, \tag{2.3}$$

where $\hat{k}_0$ denotes the Fourier transform of k_0. Let $a_+ = \max(a, 0)$ where a is a real number. Let $\Delta^l = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right)^l$ with $l \in \mathbf{N}$.

2.1.2. Definition. Let $-\infty < s < \infty$ and $0 < q \leqq \infty$.

(i) Let $0 < p < \infty$. Let $\varepsilon > 0$ be small and $l \in \mathbf{N}$ with $2l > \max\left(s, \bar{n}(1/p - 1)_+\right)$. Let $k = \Delta^l k_0$, where k_0 is the above function. Then

$$F^s_{pq} = \Big\{ f \mid f \in S', \|f \mid F^s_{pq}\| = \|k_0(\varepsilon, f) \mid L_p\| + \Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \Big)^{1/q} \mid L_p \Big\| < \infty \Big\} \tag{2.4}$$

(modification if $q = \infty$).

(ii) Let $0 < p \leqq \infty$. Let l be the same number and k be the same function as in (i). Then

$$B^s_{pq} = \Big\{ f \mid f \in S', \|f \mid B^s_{pq}\| = \|k_0(\varepsilon, f) \mid L_p\| + \Big(\sum_{j=1}^{\infty} 2^{jsq} \|k(2^{-j}, f) \mid L_p\|^q \Big)^{1/q} < \infty \Big\} \tag{2.5}$$

(modification if $q = \infty$).

2.1.3. Remark. The original Fourier analytical definition of F^s_{pq} and B^s_{pq} looks somewhat different, see [28: 2.3.1]. The above version is covered by [29], see also [30: Remark 1] for more detailed references. Of course $k(2^{-j}, f)$ is given by (2.2). We shall not distinguish between equivalent quasi-norms in a given space. This justifies the above abuse of notations to write simply $\|f \mid F^s_{pq}\|$ although it is quite clear that the above $\|f \mid F^s_{pq}\|$ depends on ε, l and the chosen function k_0. We recall that these two scales B^s_{pq} and F^s_{pq} of quasi-Banach spaces cover many well-known spaces: B^s_{pq} with $s > 0$, $1 < p < \infty$, $1 \leqq q \leqq \infty$, are the classical Besov spaces; $B^s_{\infty\infty} = \mathcal{C}^s$ with $s > 0$ are the Hölder-Zygmund spaces; $H_p^s = F^s_{p2}$ with $-\infty < s < \infty$, $1 < p < \infty$ are the Bessel-potential spaces with the Sobolev spaces as a subcase; $H_p = F^0_{p2}$ with $0 < p < \infty$ are inhomogeneous Hardy spaces. The theory of these spaces has been systematically developed in [28].

2.1.4. Distinguished kernels. We need some results proved in [31] which we describe in this subsection and the next one. There exist two C^∞ functions φ and ψ on the real line with $\operatorname{supp} \varphi \subset (0, 1)$, $\operatorname{supp} \psi \subset (0, 1)$ and

$$\int_{-\infty}^{\infty} \varphi(t)\, dt = 1, \qquad \varphi(t) - \frac{1}{2} \varphi\left(\frac{t}{2}\right) = \psi^{(M)}(t), \tag{2.6}$$

where $M \in \mathbf{N}$ is a given number. Let

$$\Phi(x) = \prod_{l=1}^{n} \varphi(x_l), \qquad x = (x_1, \ldots, x_n) \in \mathbf{R}^n, \tag{2.7}$$

$$k_0(x) = \frac{(-1)^{M+1}}{M!} \sum_{r=1}^{M} \sum_{m=1}^{M} (-1)^{r+m} \binom{M}{r} \binom{M}{m} r^{-n} m^{M-n} \Phi\left(\frac{x}{rm}\right), \tag{2.8}$$

and

$$k(x) = k_0(x) - 2^{-n} k_0(x/2). \tag{2.9}$$

These are the distinguished kernels introduced in [31]. Constructions of type (2.8) play a crucial role in the Russian school of the theory of function spaces. The particular structure of (2.8) with a double sum goes back to G. A. Kaljabin [18], see also [19] and Lizorkin's appendix D.2.3 in the Russian edition of [28: p. 411]. Let $f \in S'$, then

$$f_j(x) = \begin{cases} k(2^{-j}, f)(x) & \text{if } j \in \mathbf{N}, \\ k_0(2^{-j}, f)(x) & \text{if } -j \in \mathbf{N}_0. \end{cases} \tag{2.10}$$

Furthermore, any $f \in S'$ can be represented as

$$f = \sum_{j=0}^{\infty} f_j \qquad \text{(convergence in } S'\text{)}. \tag{2.11}$$

For proofs and details we refer to [31].

2.1.5. Equivalent quasi-norms. Let k_0 and k be the functions from (2.8) and (2.9), respectively, and let $k(t, f)$ and $k_0(t, f)$ be the corresponding means, see (2.2). Let $-\infty < s < \infty$, $0 < p \leqq \infty$ (with $p < \infty$ in the case of F-spaces) and $0 < q \leqq \infty$. Let $M \in \mathbf{N}$ be the same number as in 2.1.4 with $M > \max\left(s, n(1/p - 1)_+\right)$. Then there exists a number $K \in \mathbf{N}_0$ such that

$$F^s_{pq} = \left\{f \mid f \in S', \left\| \left(\sum_{j=-k}^{\infty} 2^{jsq} |f_j(\cdot)|^q\right)^{1/q} \mid L_p \right\| < \infty\right\} \tag{2.12}$$

and

$$B^s_{pq} = \left\{f \mid f \in S', \left(\sum_{j=-k}^{\infty} 2^{jsq} \|f_j \mid L_p\|^q\right)^{1/q} < \infty\right\} \tag{2.13}$$

in the sense of equivalent quasi-norms (modification if $q = \infty$). The functions f_j have the same meaning as in (2.10). A proof of this assertion may be found in [31: Theorem 2.2.4].

2.1.6. Spaces on domains. Let Ω be either $\mathbf{R}_+^n = \{x \mid x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ with $x_n > 0\}$ or a bounded C^∞ domain in $\mathbf{R}^n$. Then $F^s_{pq}(\Omega)$ is the restriction of F^s_{pq} on Ω, quasi-normed by

$$\|f \mid F^s_{pq}(\Omega)\| = \inf \|g \mid F^s_{pq}\|, \tag{2.14}$$

where the infimum is taken over all $g \in F^s_{pq}$ with $g \mid \Omega = f$ (in the sense of $D'(\Omega)$). Similarly one defines $B^s_{pq}(\Omega)$. The parameters s, p, q have the same meaning as above.

2.2. Characterizations via oscillations

2.2.1. Oscillations and sharp maximal functions. Let Ω be an arbitrary domain in $\mathbf{R}^n$. Let

$$B(x, t) = \{y \mid |x - y| < t\} \cap \Omega, \qquad x \in \Omega, t > 0. \tag{2.15}$$

For sake of brevity we write

$$\mathop{⨍}_{B(x,t)} g(y)\,dy = |B(x,t)|^{-1} \int_{B(x,t)} g(y)\,dy.$$

Let $M \in \mathbf{N}_{-1}$ and $0 < u \leqq \infty$, let $\mathbf{P}_M$ be the collection of all polynomials (with complex coefficients) of degree less than or equal to M, where we put $\mathbf{P}_{-1} = \{0\}$. Then

$$\operatorname{osc}_u^M f(x, t) = \inf \Big(\mathop{⨍}_{B(x,t)} |f(y) - P(y)|^u\,dy \Big)^{1/u}, \qquad x \in \Omega,\ t > 0, \tag{2.16}$$

denotes the local oscillation where the infimum is taken over all $P \in \mathbf{P}_M$ (usual modification if $u = \infty$). Of course the notation $\operatorname{osc}_u^M f$ depends on Ω, but in general we shall not indicate this dependence. Furthermore it is tacitly assumed that $|f|^u$ is integrable in $B(x, t)$. We introduce the sharp maximal functions

$$f_u^{M,s}(x) = \sup_{0<t<1} t^{-s} \operatorname{osc}_u^M f(x, t), \qquad x \in \Omega, \tag{2.17}$$

where s is a real number, $M \in \mathbf{N}_{-1}$ and $0 < u \leqq \infty$. Let $a_+ = \max(a, 0)$ where a is a real number.

2.2.2. Theorem. *Let Ω be either $\mathbf{R}^n$, $\mathbf{R}_+^n$ or a bounded C^∞ domain in $\mathbf{R}^n$.*

(i) *Let $0 < p < \infty$, $0 < q \leqq \infty$, $1 \leqq r \leqq \infty$ and*

$$s > n \left(\frac{1}{p} - \frac{1}{r}\right)_+ \quad \text{and} \quad s > n \left(\frac{1}{q} - \frac{1}{r}\right)_+. \tag{2.18}$$

Let $0 < u \leqq r$ and $M \in \mathbf{N}_0$ with $M \geqq [s]$, then

$$F^s_{pq}(\Omega) = \Big\{ f \mid f \in L_{\max(p,r)}(\Omega),\ \|f \mid L_p(\Omega)\| + \Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{osc}_u^M f(\cdot, 2^{-j})^q \Big)^{1/q} \Big| L_p(\Omega) \Big\| < \infty \Big\} \tag{2.19}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

(ii) *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$, $1 \leqq r \leqq \infty$ and*

$$s > n \left(\frac{1}{p} - \frac{1}{r}\right)_+. \tag{2.20}$$

Let $0 < u \leqq r$ and $M \in \mathbf{N}_0$ with $M \geqq [s]$. Then

$$B^s_{pq}(\Omega) = \Big\{ f \mid f \in L_{\max(p,r)}(\Omega),\ \|f \mid L_p(\Omega)\| + \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\operatorname{osc}_u^M f(\cdot, 2^{-j}) \mid L_p(\Omega)\|^q \Big)^{1/q} < \infty \Big\} \tag{2.21}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

2.2.3. Remark. As we said in the introduction there exists some characterizations of type (2.19) in the literature. As far as fractional Sobolev spaces $H_p^s = F^s_{p2}$ with $s > 0$ and $1 < p < \infty$ are concerned we refer to J. R. DORRONSORO [12]. Extensions to more general spaces F^s_{pq} are due to J. R. DORRONSORO [13] and A. SEEGER [24], where the latter paper covers more or less (2.19) with $u = r$. Characterizations of type (2.21) may be found in a somewhat hidden form in the work of V. P. IL'IN, see

[1: §28], we refer also to Ju. A. Brudnyj [3], H. Wallin [33: Theorem 7] and the references given in the latter paper to further papers by Ju. A. Brudnyj, J. R. Dorronsoro, F. Ricci, M. Taibleson, A. Jonsson, H. Wallin, R. A. DeVore and V. Popov in this connection. We concentrate ourselves on the more complicated F-spaces, and formulate the corresponding results for the B-spaces only for sake of completeness. We prefered in (2.19) and (2.21) a discrete version. But it follows immediately from (2.16) that one can replace the sum over j by an integral. For example,

$$\|f \mid L_p(\Omega)\| + \left\| \left(\int\limits_0^1 t^{-sq} \operatorname{osc}_u^M f(\cdot, t)^q \frac{dt}{t} \right)^{\frac{1}{q}} L_p(\Omega) \right\| \tag{2.22}$$

is an equivalent quasi-norm in $F^s_{pq}(\Omega)$ under the same conditions for s, p, q as in the theorem.

2.2.4. Definition. Let Ω be a bounded C^∞ domain.

(i) Let $1 \leqq p < \infty$, $s \geqq -n/p$, and $M = \max(-1, [s])$. Then

$$\begin{aligned} L_p{}^s(\Omega) = \{f \mid f \in L_p(\Omega), \|f \mid L_p{}^s(\Omega)\| \\ = \|f \mid L_p(\Omega)\| + \sup_{x \in \Omega} f_p{}^{M,s}(x) < \infty\}. \end{aligned} \tag{2.23}$$

(ii) Let $s > 0$, $0 < p \leqq \infty$ and $\bar{p} = \max(1, p)$. Then

$$\begin{aligned} C_p{}^s(\Omega) = \{f \mid f \in L_{\bar{p}}(\Omega), \|f \mid C_p{}^s(\Omega)\| \\ = \|f \mid L_p(\Omega)\| + \|f_p{}^{[s],s} \mid L_p(\Omega)\| < \infty\}. \end{aligned} \tag{2.24}$$

2.2.5. Remark. The spaces $L_p{}^s(\Omega)$ are the Morrey-Campanato spaces. The case $-n/p \leqq s < 0$, and hence $M = -1$, goes back to C. B. Morrey [22]. Furthermore $L_p{}^0(\Omega) = BMO(\Omega)$ is essentially the John-Nirenberg space of all functions with bounded mean oscillation, see [16]. (If one extends the case $M = -1$ to $s = 0$, in contrast to our definition, then one obtains $L_\infty(\Omega)$, and it is well known that BMO is strictly larger than L_∞.) The extension of the spaces $L_p{}^s(\Omega)$ to $s > 0$ has been done in the sixties by S. Campanato [6–9], G. N. Meyers [21], S. Spanne [25] and G. Stampacchia [26, 27]. The original notations are different. The above notations are adapted to the main subject of this paper, the spaces F^s_{pq}. Further details and references may be found in [23] and [20: Chapter 4, in particular 4.10]. The spaces $C_p{}^s(\Omega)$ have been introduced by R. A. DeVore and R. C. Sharpley in [11] and M. Christ [10]. The difference between (2.23) and (2.24) is obvious: One replaces the sup-norm in (2.23) by the L_p-quasi-norm. The idea to replace the sup-norm by some L_q-norms is not new, see G. Stampacchia [27] and V. P. Il'in [1: § 28]. A discussion of this point and further references may also be found in [2]. In order to define $L_p{}^s(\Omega)$ and $C_p{}^s(\Omega)$ it is not necessary that the underlying domain Ω is bounded or smooth. It is simply convenient for us. However if Ω is unbounded, then it seems to be desirable to replace $\|f \mid L_p(\Omega)\|$ in (2.23) by $\sup_{x \in \Omega} \|f \mid L_p(B(x, 1))\|$, see (2.15). As we shall see one can prove well-known classical results for the spaces $L_p{}^s(\Omega)$ more or less as a by-product of the technique which we develop in order to handle the spaces F^s_{pq}. Maybe this justifies to incorporate these results in Theorem 2.2.7 below.

2.2.6. **Hölder-Zygmund and Sobolev spaces.** Let Ω be a bounded C^∞ domain in $\mathbf{R}^n$, let $1 < p < \infty$ and $k \in \mathbf{N}_0$. Then

$$W_p^k(\Omega) = \left\{f \mid f \in L_p(\Omega), \|f \mid W_p^k(\Omega)\| = \sum_{|\alpha| \leq k} \|D^\alpha f \mid L_p(\Omega)\| < \infty\right\} \tag{2.25}$$

are the well-known Sobolev spaces. Let $M \in \mathbf{N}$, $t > 0$ and $x \in \Omega$, then

$$V^M(x, t) = \{h \mid h \in \mathbf{R}^n, x + \tau h \in \Omega, |h| < t, 0 \leq \tau \leq M\} \tag{2.26}$$

is a maximal subset of a ball of radius t and with its centre at the origin such that $x + MV^M(x, t) \subset \Omega$. Again we omit to indicate the dependence of $V^M(x, t)$ on Ω. Let $s > 0$ and let $M \in \mathbf{N}$ be the smallest number with $M > s$, then

$$\begin{aligned}\mathscr{C}^s(\Omega) = \{f \mid f \in L_\infty(\Omega), \|f \mid \mathscr{C}^s(\Omega)\| \\ = \|f \mid L_\infty(\Omega)\| + \sup |h|^{-s} |\Delta_h^M f(x)| < \infty\}\end{aligned} \tag{2.27}$$

are the Hölder-Zygmund spaces, where the supremum is taken over all $x \in \Omega$, $0 < t < 1$ and $h \in V^M(x, t)$, and

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^{M-j} \binom{M}{j} f(x + jh) \tag{2.28}$$

are the usual differences. It is known that both spaces $W_p^k = F_{p2}^k$ and $\mathscr{C}^s = B_{\infty\infty}^s$ are covered by the spaces introduced above, see [28: 2.5.6 and 2.5.7], but this is unimportant at the moment. Furthermore one can replace the smallest $M \in \mathbf{N}$ with $M > s$ in (2.27) by an arbitrary $M \in \mathbf{N}$ with $M > s$ (equivalent norms).

2.2.7. **Theorem.** *Let Ω be a bounded C^∞ domain in $\mathbf{R}^n$.*

(i) *Let $1 \leq p < \infty$ and $s > 0$. Then*

$$L_p^{-n/p}(\Omega) = L_p(\Omega) \quad \text{and} \quad L_p^s(\Omega) = \mathscr{C}^s(\Omega). \tag{2.29}$$

(ii) *Let $1 \leq p < \infty$, $M \in \mathbf{N}_{-1}$ and $-n/p \leq s < M + 1$. Then*

$$\|f \mid L_p^s(\Omega)\|^M = \|f \mid L_p(\Omega)\| + \sup_{x \in \Omega} f_p^{M,s}(x) \tag{2.30}$$

is an equivalent norm on $L_p^s(\Omega)$.

(iii) *Let $0 < p \leq \infty$ and $s > n(1/p - 1)_+$. Then*

$$C_p^s(\Omega) = F_{p\infty}^s(\Omega) \tag{2.31}$$

and for any $M \in \mathbf{N}_0$ with $M \geq [s]$

$$\|f \mid C_p^s(\Omega)\|^M = \|f \mid L_p(\Omega)\| + \|f_p^{M,s} \mid L_p(\Omega)\| \tag{2.32}$$

is an equivalent quasi-norm in $C_p^s(\Omega)$.

(iv) *Let $k \in \mathbf{N}_0$ and $1 \leq u < p < \infty$. Then*

$$W_p^k(\Omega) = \{f \mid f \in L_p(\Omega), \|f \mid L_p(\Omega)\| + \|f_u^{k-1,k} \mid L_p(\Omega)\| < \infty\} \tag{2.33}$$

(equivalent norms).

2.2.8. **Remark.** As we said we consider the above theorem as a complement of Theorem 2.2.2. For this reason we did not try to give most general formulations. One has corresponding assertions if one replaces Ω by $\mathbf{R}^n$ or $\mathbf{R}_+^n$ (with the modifications indicated in Remark 2.2.5). Furthermore Ω need not be C^∞. Almost all assertions of the above theorem are known: (i) and (ii) are known since the sixties in the framework of the theory of the Morrey-Campanato spaces, see the references given in 2.2.5. Furthermore (iv) and also the assertion about the equivalent quasi-norms in (iii) are covered by [11], including some generalizations, whereas (2.33) is essentially due to A. P. CALDERÓN [4], however see also [5, 10]. Formula (2.31) seems to be new, but it is more or less a consequence of [24: Corollary 1].

2.3. Extensions and characterizations via differences

2.3.1. The extension problem. Let Ω be either $\mathbf{R}_+^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. Let $-\infty < s < \infty$, $0 < p < \infty$, $0 < q \leqq \infty$ and let re be the restriction operator from $F^s_{pq}(\mathbf{R}^n)$ onto $F^s_{pq}(\Omega)$ in the sense of 2.1.6. Of course re is linear and bounded. We are looking for a linear and bounded operator ext from $F^s_{pq}(\Omega)$ into $F^s_{pq}(\mathbf{R}^n)$ with

$$\text{re} \circ \text{ext} = \text{id} \qquad (\text{identity in } F^s_{pq}(\Omega)). \tag{2.34}$$

This extension property is well known for the classical spaces (Sobolev, fractional Sobolev, Besov, Hölder-Zygmund). For general spaces F^s_{pq} (and also B^s_{pq}) including in particular smoothness parameters $s \leqq 0$ and values $p < 1$ this property has been studied in [32: 2.9.3] and [28: 2.9]. However there remained some gaps. The first full proof is due to J. FRANKE [15]. In [31] we gave a new proof of this assertion which was based on the kernels and the quasi-norms in 2.1.4 and 2.1.5, respectively. In the present paper we give a new and almost trivial proof of the extension property for the spaces F^s_{pq} under the same restrictions for p, q and s as in Theorem 2.2.2 which is not only of interest for its own sake but which is crucial for our method: First we prove all assertions on $\mathbf{R}^n$, for example those ones from Theorem 2.2.2, then we prove the extension property and obtain on this way corresponding assertions on $\mathbf{R}_+^n$ and transfer finally these assertions to bounded C^∞ domains with the help of diffeomorphic maps.

2.3.2. Theorem. *Let*

$$0 < p < \infty, \qquad 0 < q \leqq \infty, \qquad s > n\left(\frac{1}{p} - 1\right)_+, \qquad s > n\left(\frac{1}{q} - 1\right)_+. \tag{2.35}$$

Let $L \in \mathbf{N}_0$ with $L \geqq [s]$. Let $0 < \lambda_0 < \lambda_1 < \dots < \lambda_L$ and let $a_0, \dots, a_L$ be real numbers with

$$\sum_{k=0}^{L} a_k(-\lambda_k)^l = 1 \quad \textit{where} \quad l = 0, \dots, L. \tag{2.36}$$

Then ext_L *with*

$$\text{ext}_L f(x) = \begin{cases} f(x) & \textit{if } x \in \mathbf{R}_+^n \\ \sum_{k=0}^{L} a_k f(x', -\lambda_k x_n) & \textit{if } x = (x', x_n) \text{ with } x_n \leqq 0 \end{cases} \tag{2.37}$$

is an extension operator from $F^s_{pq}(\mathbf{R}_+^n)$ into $F^s_{pq}(\mathbf{R}^n)$.

2.3.3. Remark. Extension operators of type (2.37) are not new, they go back to Fichtengol'z and we used them several times, see [32: 2.9.1 and 2.9.3], [28: 2.9] and [31], in an extended form which covers also values $s \leqq 0$. The theorem in the above form can be extended immediately to the spaces B^s_{pq} with $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s > n(1/p - 1)_+$.

2.3.4. Corollary. *For any $\varepsilon > 0$ and all p, q, s with*

$$\varepsilon < p < \infty, \quad \varepsilon < q \leqq \infty, \quad |s| < \varepsilon^{-1} \tag{2.38}$$

there exists a common extension operator ext^ε from $F^s_{pq}(\mathbf{R}_+{}^n)$ into $F^s_{pq}(\mathbf{R}^n)$.

What is meant by this a little sloppy formulation is that there is a linear operator ext^ε defined on the union of all spaces in question such that its restriction to a particular space $F^s_{pq}(\mathbf{R}_+{}^n)$ has the desired property. We incorporated this corollary because it follows rather simply from Theorem 2.3.2 and some techniques used in [28: 2.9]: duality and complex interpolation. Hence one has a new proof of the full theorem (without any restrictions of the parameters). The same operator ext^ε is also an extension operator from $B^s_{pq}(\mathbf{R}_+{}^n)$ into $B^s_{pq}(\mathbf{R}^n)$ if

$$\varepsilon < p \leqq \infty, \quad 0 < q \leqq \infty, \quad |s| < \varepsilon^{-1}. \tag{2.39}$$

2.3.5. Means of differences. Let Ω be either $\mathbf{R}^n$ or $\mathbf{R}_+{}^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. Let $V^M(x, t)$ and $\Delta_h{}^M f(x)$ be given by (2.26) and (2.28), respectively. Let $f \in F^s_{pq}(\Omega)$ with the same restrictions for s, p, q as in Theorem 2.2.2(i) with $r = 1$. Then $f \in L_{\bar{p}}(\Omega)$ where $\bar{p} = \max(p, 1)$. In particular,

$$d_t{}^M f(x) = \fint_{V^M(x,t)} |\Delta_h{}^M f(x)| \, dh, \quad x \in \Omega, t > 0, \tag{2.40}$$

makes sense where again we omit to indicate the dependence of $d_t{}^M f$ on the given domain Ω. Furthermore $\fint$ stands for the mean value, see 2.2.1. This is the counterpart of $\mathrm{osc}_u{}^M f(x, t)$ from (2.16) with $u = 1$. There is no problem to replace the L_1-means in (2.40) by L_u-means and to generalize the theorem below to this case. But for sake of simplicity we restrict ourselves to the above means.

2.3.6. Theorem. *Let Ω be either $\mathbf{R}^n$, or $\mathbf{R}_+{}^n$ or a bounded C^∞ domain in $\mathbf{R}^n$.*

(i) *Let $0 < p < \infty$, $0 < q \leqq \infty$ and*

$$s > n\left(\frac{1}{p} - 1\right)_+, \qquad s > n\left(\frac{1}{q} - 1\right)_+. \tag{2.41}$$

Let $M \in \mathbf{N}$ with $M > s$. Then

$$F^s_{pq}(\Omega) = \left\{ f \mid f \in L_{\bar{p}}(\Omega), \|f \mid L_p(\Omega)\| + \left\| \left(\sum_{j=0}^\infty 2^{jsq} d^M_{2^{-j}} f(\cdot)^q \right)^{1/q} \Big| L_p(\Omega) \right\| < \infty \right\} \tag{2.42}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

(ii) *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s > n(1/p - 1)_+$. Let $M \in \mathbf{N}$ with $M > s$. Then*

$$B^s_{pq}(\Omega) = \left\{ f \mid f \in L_{\bar{p}}(\Omega), \|f \mid L_p(\Omega)\| + \left(\sum_{j=0}^\infty 2^{jsq} \|d^M_{2^{-j}} f \mid L_p(\Omega)\|^q \right)^{1/q} < \infty \right\} \tag{2.43}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

2.3.7. Remark. We prefered again the discrete version. But there is no problem to replace the sum over j in (2.42) and (2.43) by corresponding integrals. For example

$$\|f \mid L_p(\Omega)\| + \left\| \left(\int_0^1 t^{-sq} d_t{}^M f(\cdot)^q \frac{dt}{t} \right)^{1/q} \middle| L_p(\Omega) \right\| \tag{2.44}$$

is an equivalent quasi-norm in $F^s_{pq}(\Omega)$ under the same conditions for s, p, q as in the theorem.

2.3.8. Remark. Characterizations of function spaces via differences have been studied extensively. This is well known for the Hölder-Zygmund spaces $\mathscr{C}^s$ with $s > 0$ and the classical Besov spaces B^s_{pq} with $s > 0$, $1 < p < \infty$, $1 \leqq q \leqq \infty$. Corresponding results for fractional Sobolev spaces have been proved in the late sixties and early seventies by R. S. Strichartz and P. I. Lizorkin, see [28: 2.5.10, Remark 3] for further references. Extensions to F^s_{pq} are due to G. A. Kaljabin and the author. An extensive treatment has been given in [28: 2.5.9—2.5.12], where we gave many references, see also [29] for a more recent systematic study. The characterization (2.42) with $\Omega = \mathbf{R}^n$ coincides with the corollary in [28: 2.5.11]. However the extension of characterizations of this type to $\mathbf{R}_+{}^n$ or to bounded domains Ω caused some trouble. In [18] G. A. Kaljabin proved the characterization (2.42) for bounded domains Ω (even more general than C^∞ domains) for the spaces $F^s_{pq}(\Omega)$ with $s > 0$, $1 < p < \infty$, $1 < q < \infty$. In other words: (2.42) extends Kaljabin's result to $0 < p < \infty$, $0 < q \leqq \infty$ and s with (2.41).

2.3.9. Remark. Our study of the spaces $F^s_{pq}(\Omega)$ where Ω is a bounded C^∞ domain in $\mathbf{R}^n$ and p, q, s obey (2.35) is based on the results for the spaces F^s_{pq} (on $\mathbf{R}^n$) and the extension property from Theorem 2.3.2. In particular we obtain as a by-product that all the spaces $F^s_{pq}(\Omega)$, and also $B^s_{pq}(\Omega)$, have the extension property (from Ω to $\mathbf{R}^n$). If Ω is a non-smooth bounded domain, the situation is different and the extension problem from Ω to $\mathbf{R}^n$ cannot be reduced to Theorem 2.3.2. This question attracted some attention in recent times. We refer to [18, 24] as far as F^s_{pq}-spaces are concerned and to [10, 11] in connection with $C_p{}^s$-spaces.

3. Proofs and further inequalities

3.1. Inequalities for oscillations

3.1.1. The aim of Section 3 it twofold. First of all we prove the main assertions of this paper formulated in the Theorems 2.2.2, 2.2.7, 2.3.2, 2.3.6 and in Corollary 2.3.4. Secondly we derive several inequalities for oscillations which are (as we hope) of some interest for their own sake.

3.1.2. Preliminaries. Let Ω be either $\mathbf{R}^n$, or $\mathbf{R}_+{}^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. In the latter case we assume without restriction of generality $0 \in \partial\Omega$ and that $\partial\Omega$ can be represented near the origin as $x_n = \psi(x')$, $x' \in \mathbf{R}^{n-1}$, where ψ is a C^∞ function near the origin in $\mathbf{R}^{n-1}$. Let k_0 and k be the kernels from (2.8) and (2.9), then

$$\operatorname{supp} k_0 \subset \mathbf{R}_+{}^n \quad \text{and} \quad \operatorname{supp} k \subset \mathbf{R}_+{}^n. \tag{3.1}$$

Let $f \in L_p(\Omega)$ with $1 \leqq p \leqq \infty$. Then $k(t, f)(x)$ from (2.2) and similarly $k_0(t, f)(x)$ make sense at least for $x \in \Omega$ near the origin and $t > 0$ small. We may assume that these means make sense for all $x \in \Omega$ near the origin and $0 < t \leqq 1$. Let f_j with $j \in \mathbf{N}_0$ be given by (2.10). Let

$$f^j(x) = \sum_{k=0}^{j} f_k(x), \qquad \text{where } j \in \mathbf{N}_0. \tag{3.2}$$

3.1.3. Proposition (Optimal polynomials). *Let Ω be either $\mathbf{R}^n$, or $\mathbf{R}_+^n$ or the bounded C^∞ domain from 3.1.2. Let $1 \leqq p \leqq \infty$ and $M \in \mathbf{N}_0$ then there exist positive numbers c and c' such that*

$$\begin{aligned} \operatorname{osc}_p^{M-1} f(x, 2^{-j}) &\leqq \left(\fint_{B(x,2^{-j})} \left| f(y) - \sum_{|\alpha| \leqq M-1} \frac{1}{\alpha!} (D^\alpha f^j)(x) (y-x)^\alpha \right|^p dy \right)^{1/p} \\ &\leqq c' \operatorname{osc}_p^{M-1} f(x, c2^{-j}) \end{aligned} \tag{3.3}$$

for all $f \in L_p(\Omega)$, $j \in \mathbf{N}_0$ and all $x \in \Omega$ (near the origin).

3.1.4. Proof. The left-hand side of (3.3) is obvious. We prove the right-hand side. By (2.2) and (2.9) we have

$$\begin{aligned} f^j(x) &= \int k_0(y) f(x+y)\, dy + \sum_{l=1}^{j} 2^{ln} \int k(2^l y) f(x+y)\, dy \\ &= \int k_0(y) f(x + 2^{-j} y)\, dy. \end{aligned} \tag{3.4}$$

By (2.8), see also [31: (3.12) and (3.8)], it follows

$$\begin{aligned} f^j(x) - f(x) &= \frac{(-1)^{M+1}}{M!} \sum_{m=1}^{M} (-1)^{M-m} \binom{M}{m} m^{M-n} \\ &\quad \times \int \Phi\left(\frac{y}{m}\right) \sum_{r=0}^{M} (-1)^{M-r} \binom{M}{r} f(x + r2^{-j}y)\, dy \\ &= \int \tilde{k}(y)\, \Delta^M_{2^{-j}y} f(x)\, dy, \end{aligned} \tag{3.5}$$

where $\tilde{k}$ is a compactly supported C^∞ function with $\operatorname{supp} \tilde{k} \subset \mathbf{R}_+^n$. We can replace f on the right-hand side of (3.5) by $f - P$ with $P \in \mathbf{P}_{M-1}$. Then it follows

$$|f(x) - f^j(x)| \leqq c' |f(x) - P(x)| + c' \fint_{B(x,c2^{-j})} |f(y) - P(y)|\, dy \tag{3.6}$$

for all $P \in \mathbf{P}_{M-1}$ and some $c > 0$ and $c' > 0$. Hence we have

$$\left(\fint_{B(x,2^{-j})} |f(y) - f^j(y)|^p\, dy \right)^{1/p} \leqq c' \operatorname{osc}_p^{M-1} f(x, c2^{-j}) \tag{3.7}$$

for some $c > 0$ and $c' > 0$. On the other hand the polynomial in (3.3) is the Taylor expansion of f^j. Let $y \in B(x, 2^{-j})$ then we have

$$\left| f^j(y) - \sum_{|\alpha| \leqq M-1} \frac{1}{\alpha!} (D^\alpha f^j)(x) (y-x)^\alpha \right| \leqq c 2^{-jM} \sup_{z \in B(x,2^{-j})} \sum_{|\alpha| = M} |D^\alpha f^j(z)|. \tag{3.8}$$

By (3.4) it follows

$$D^\alpha f^j(x) = 2^{j|\alpha|} \int D^\alpha k_0(y) f(x + 2^{-j} y)\, dy, \qquad |\alpha| = M. \tag{3.9}$$

Hence one can again replace f on the right-hand side of (3.9) by $f - P$ with $P \in \mathbf{P}_{M-1}$ and we obtain by the same arguments as above

$$|D^\alpha f^j(y)| \leqq c' 2^{jM} \operatorname{osc}_p^{M-1} f(x, c2^{-j}), \qquad y \in B(x, 2^{-j}), \tag{3.10}$$

for some $c > 0$ and $c' > 0$. Now (3.7) and (3.8), (3.10) prove the right-hand side of (3.3).

3.1.5. Proposition. *Let Ω be either $\mathbf{R}^n$, or $\mathbf{R}_+^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. Let $1 \leqq p \leqq \infty$ and $M \in \mathbf{N}_0$, then there exist positive numbers c and c' such that*

$$\operatorname{osc}_p^{M-1} f(x, 2^{-j}) \leqq c' \sum_{l=0}^{j} 2^{-(j-l)M} \operatorname{osc}_p^M f(x, c2^{-l}) + c'2^{-jM} \int_{B(x,c)} |f(y)| \, dy \quad (3.11)$$

for all $f \in L_p(\Omega)$, $j \in \mathbf{N}_0$ and all $x \in \Omega$.

3.1.6. Proof. We may assume that Ω has the properties described in 3.1.2 and that $x \in \Omega$ is a point near the origin. Then we can apply (3.3) and obtain

$$\operatorname{osc}_p^{M-1} f(x, 2^{-j}) \leqq c' \operatorname{osc}_p^M f(x, c2^{-j}) + c'2^{-jM} \sum_{|\alpha|=M} |D^\alpha f^j(x)|. \quad (3.12)$$

By (3.4) and [31: (3.14)] we have

$$D^\alpha f^l(x) = 2^{lM} \int k'(y) \, \Delta^M_{2^{-l}y} f(x) \, dy, \quad |\alpha| = M, \quad (3.13)$$

and $l \in \mathbf{N}_0$, where k' is a compactly supported C^∞ function with $\operatorname{supp} k' \subset \mathbf{R}_+^n$, a linear combination of $D^\alpha \Phi(y/m)$ similar as in (3.5) (obvious modification if $M = 0$). Let $l \in \mathbf{N}_0$, then we have

$$D^\alpha f^{l+1}(x) - D^\alpha f^l(x) = 2^{(l+1)M} \int k'(y) \left[\Delta^M_{2^{-l-1}y} f(x) - 2^{-M} \Delta^M_{2^{-l}y} f(x)\right] dy. \quad (3.14)$$

Next we use the formula

$$\Delta^M_{2^{-l-1}y} f(x) - 2^{-M} \Delta^M_{2^{-l}y} f(x) = \sum_{r=0}^{M-1} a_r \Delta^{M+1}_{2^{-l-1}y} f(x + r2^{-l-1}y), \quad (3.15)$$

where a_r are some constants, see [28: 2.5.9, formula (45)]. This formula makes clear that one can replace f on the right-hand side of (3.14) by $f - P$ with $P_M \in \mathbf{P}$. We obtain

$$|D^\alpha f^{l+1}(x) - D^\alpha f^l(x)| \leqq c'2^{lM} \int_{B(x,c2^{-l})} |f(y) - P(y)| \, dy \quad (3.16)$$

for some $c > 0$ and $c' > 0$, where we used the structure of $k'(y)$, in particular $\int k'(y) \, dy = 0$ if $M \in \mathbf{N}$. Hence we can replace the right-hand side of (3.16) by $c'2^{lM} \operatorname{osc}_p^M f(x, c2^{-l})$. Finally we arrive at

$$\begin{aligned} 2^{-Mj} |D^\alpha f^j(x)| &\leqq 2^{-jM} \sum_{l=0}^{j-1} |D^\alpha f^{l+1}(x) - D^\alpha f^l(x)| + 2^{-jM} |D^\alpha f_0(x)| \\ &\leqq c' \sum_{l=0}^{j-1} 2^{-(j-l)M} \operatorname{osc}_p^M f(x, c2^{-l}) + c'2^{-jM} \int_{B(x,1)} |f(y)| \, dy. \quad (3.17) \end{aligned}$$

Now (3.12) and (3.17) prove (3.11).

3.1.7. Proposition. *Let Ω be either $\mathbf{R}^n$ or $\mathbf{R}_+^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. Let $M \in \mathbf{N}_{-1}$,*

$$0 < u < 1 < r \leqq \infty \quad \text{and} \quad 1 = \frac{1-\theta}{u} + \frac{\theta}{r}. \quad (3.18)$$

There exist positive numbers c and c' such that

$$\begin{aligned} \operatorname{osc}_1^M f(x, 2^{-j}) \leqq c' \left[\sum_{l=0}^{\infty} \operatorname{osc}_r^M f(x, c2^{-j-l})\right]^\theta \Bigg[\sup_{l \in \mathbf{N}} \operatorname{osc}_u^M f(x, c2^{-j-l}) \\ + \sup_{l \in \mathbf{N}} \left(\quad \fint_{\{w \mid x+w \in B(x,c2^{-j})\}} \operatorname{osc}_u^M f(x+w, c2^{-j-l})^u \, dw\right)^{1/u}\Bigg]^{1-\theta} \end{aligned} \quad (3.19)$$

for all $f \in L_r(\Omega)$, $j \in \mathbf{N}_0$ and all $x \in \Omega$.

3.1.8. Proof. We assume that Ω satisfies the hypotheses of 3.1.2 and that x is near the origin, in particular (3.3) (now with M instead of $M-1$) is at our disposal. Then all the calculations below are justified. We may assume $M \in \mathbf{N}_0$ because the case $M = -1$ is obvious by Hölder's inequality. We use the same technique as above, i.e.

$$\operatorname{osc}_1^M f(x, 2^{-j}) \leqq \fint_{B(x,2^{-j})} |f(y) - f^j(y)|\, dy + \fint_{B(x,2^{-j})} \Big| f^j(y) - \sum_{|\alpha| \leqq M} \frac{1}{\alpha!} D^\alpha f^j(x)\,(y-x)^\alpha \Big|\, dy. \tag{3.20}$$

In order to estimate the second term we need the integral version of Taylor's expression,

$$f^j(y) - \sum_{|\alpha| \leqq M} \frac{1}{\alpha!} D^\alpha f^j(x)\,(y-x)^\alpha = \sum_{|\beta| = M+1} \frac{M+1}{\beta!}\,(y-x)^\beta \int_0^1 (1-\tau)^M D^\beta f^j\big(x + \tau(y-x)\big)\, d\tau. \tag{3.21}$$

We may assume that $x + \tau(y-x) \in B(x, 2^{-j})$ for all $0 \leqq \tau \leqq 1$. Then we have

$$\fint_{B(x,2^{-j})} |D^\beta f^j(x + \tau(y-x))|\, dy \leqq \fint_{B(x,\tau 2^{-j})} |D^\beta f^j(y)|\, dy. \tag{3.22}$$

On the other hand (3.13) with D^β, $|\beta| = M+1$, instead of D^α, $|\alpha| = M$, yields

$$\begin{aligned} &2^{-j(M+1)}\, |D^\beta f^j(y)| \\ &\leqq \Big(\int |k'(z)|^u \big|\Delta^{M+1}_{2^{-j}z} f(y)\big|^u dz\Big)^{(1-\theta)/u} \Big(\int |k'(z)|^r\, |\Delta^{M+1}_{2^{-j}z} f(y)|^r\, dz\Big)^{\theta/r} \\ &\leqq c' \Big(|f(y) - P_1(y)|^u + \fint_{B(y,c2^{-j})} |f(z) - P_1(z)|^u\, dz\Big)^{(1-\theta)/u} \\ &\quad \times \Big(|f(y) - P_2(y)|^r + \fint_{B(y,c2^{-j})} |f(z) - P_2(z)|^r\, dz\Big)^{\theta/r}, \end{aligned} \tag{3.23}$$

where the polynomials $P_1 \in \mathbf{P}_M$ and $P_2 \in \mathbf{P}_M$ are at our disposal. By (3.18) and Hölder's inequality it follows

$$\begin{aligned} &2^{-j(M+1)} \fint_{B(x,\tau 2^{-j})} |D^\beta f^j(y)|\, dy \\ &\leqq \Big[\fint_{B(x,\tau 2^{-j})} |f(y) - P_1(y)|^u\, dy + \fint_{\{w \mid x+w \in B(x,c2^{-j})\}} \fint_{B(x,\tau 2^{-j})} |f(y+w) - P_1(y+w)|^u\, dy\, dw\Big]^{(1-\theta)/u} \\ &\quad \times \Big[\fint_{B(x,\tau 2^{-j})} |f(y) - P_2(y)|^r\, dy + \fint_{B(x,\tau 2^{-j})} \fint_{B(y,c2^{-j})} |f(z) - P_2(z)|^r\, dz\, dy\Big]^{\theta/r}. \end{aligned} \tag{3.24}$$

The two factors will be treated differently. As for the second factor we choose P_2 as the optimal polynomial in the sense of (3.3) (with M instead of $M-1$). Then the second summand in the second factor can be estimated from above by $c' \operatorname{osc}_r^M f(x, c2^{-j})^r$ with a new constant $c > 0$. We may assume $\tau = 2^{-l}$ with $l \in \mathbf{N}_0$ in the first summand of the second factor. By our choice of P_2 and by (3.3) with $j + l$ and M instead of j and $M - 1$, respectively, we have

$$\begin{aligned} &\Big(\fint_{B(x,\tau 2^{-j})} |f(y) - P_2(y)|^r\, dy\Big)^{1/r} \\ &\leqq c \operatorname{osc}_r^M f(x, 2^{-l-j}c) + c \sum_{k=0}^{l-1} \sum_{|\alpha| \leqq M} 2^{-(j+l)|\alpha|}\, |D^\alpha f_{j+k}(x)|, \end{aligned} \tag{3.25}$$

where we used (3.2). We may assume that

$$D^\alpha f_{j+k}(x) = 2^{(j+k)|\alpha|} \int \sum_{|\beta|=M+1} D^\beta k_{\alpha\beta}(y)\, \Delta^{M+1}_{2^{-j-k}y} f(x)\, dy \tag{3.26}$$

holds for $|\alpha| = 0$ and $|\alpha| = M + 1$, where the kernels $k_{\alpha\beta}$ are compactly supported C^∞ functions in $\mathbf{R}^n$ with supports in $\mathbf{R}_+{}^n$. This claim is covered by [31: (2.23), (2.24)]. We can replace f in (3.26) by $f - P$ with $P \in \mathbf{P}_M$. Because the integral over the kernels vanishes we obtain

$$|D^\alpha f_{j+k}(x)| \leqq c 2^{(j+k)|\alpha|} \operatorname{osc}_\tau{}^M f(x, c2^{-j-k}) \tag{3.27}$$

for $|\alpha| = 0$ and $|\alpha| = M + 1$. It is not hard to see that this estimate can be extended afterwards to all α with $0 \leqq |\alpha| \leqq M + 1$. Hence, (3.25) with (3.27) has the desired form, the second factor in (3.24) can be estimated from above by the factor in (3.19). If we choose P_1 in the first factor in (3.24) in an optimal way, then the first factor in (3.24) has also the desired form. By (3.21), (3.22) and (3.24) it follows now that the second summand in (3.20) can be estimated from above by the right-hand side of (3.19). In order to estimate the first summand in (3.20) we use (3.5) with $M + 1$ instead of M. Then we are in the same position as in the right-hand sides of (3.23) and (3.24), now with $\tau = 1$. Hence we have again the desired estimate.

3.2. Proof of Theorem 2.2.2, the case $\Omega = \mathbf{R}^n$

3.2.1. We prove the theorem for the F^s_{pq} spaces. The proof for the B^s_{pq} spaces is similar, but technically simpler. Let $k(2^{-j}, f)(\cdot)$ be the same means as in Definition 2.1.2 and let s be restricted by (2.18). Then we can replace (2.4) by

$$F^s_{pq} = \Big\{ f \mid f \in L_{\max(p,r)},\ \|f \mid L_p\| + \Big\| \Big(\sum_{j=1}^\infty 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \Big)^{1/q} \Big| L_p \Big\| < \infty \Big\} \tag{3.28}$$

(equivalent quasi-norms) where we again omit to indicate $\mathbf{R}^n$ as the underlying domain. We refer to [29: Theorem 1] for the replacement of $\|k_0(\varepsilon, f) \mid L_p\|$ in (2.4) by $\|f \mid L_p\|$ and to the embedding theorems in [28: 2.7.1] as far as $f \in L_{\max(p,r)}$ is concerned.

3.2.2. Let the numbers p, q, r, s and M be the same as in Theorem 2.2.2(i) and let $1 \leqq u \leqq r$. Let $f \in L_{\max(p,r)}$ such that the quasi-norm in (2.19) is finite. Recall $k = \Delta^l k_0$ in the sense of Definition 2.1.2 where l is as large as we want. Then we have

$$|k(2^{-j}, f)(x)| = \Big| \int k(y)\, (f - P)\, (x + 2^{-j}y)\, dy \Big| \leqq c \operatorname{osc}_u{}^M f(x, 2^{-j}), \tag{3.29}$$

$j \in \mathbf{N}$, where we used that the first equality holds for any $P \in \mathbf{P}_M$. Then it follows by (3.28), $f \in F^s_{pq}$, and

$$\|f \mid F^s_{pq}\| \leqq \|f \mid L_p\| + \Big\| \Big(\sum_{j=1}^\infty 2^{jsq} \operatorname{osc}_u{}^M f(\cdot, 2^{-j})^q \Big)^{1/q} \Big| L_p \Big\|. \tag{3.30}$$

3.2.3. We wish to prove the converse inequality under the same restrictions for the parameters as in 3.2.2, in particular $1 \leqq u \leqq r$. Let $f \in F^s_{pq}$. We use the same technique as in 3.1.3 and 3.1.4 and obtain

$$\operatorname{osc}_u^M f(x, 2^{-j}) \leq \Big(\fint_{B(x,2^{-j})} |f(y) - f^j(y)|^u \, dy \Big)^{1/u} + \left(\fint_{B(x,2^{-j})} \Big| f^j(y) - \sum_{|\alpha| \leq M} \frac{1}{\alpha!} (D^\alpha f^j)(x)(y-x)^\alpha \Big|^u dy \right)^{1/u}$$
$$\leq \sum_{l=1}^{\infty} \Big(\fint_{B(x,2^{-j})} |f_{j+l}(y)|^u \, dy \Big)^{1/u} + c' \sup \Big| \int D^\alpha k_0(y) f(z + 2^{-j}y) \, dy \Big|, \tag{3.31}$$

where the supremum is taken over $z \in B(x, c2^{-j})$ and $|\alpha| = M + 1$. We used (2.11), (3.2) for the first term and (3.8), (3.9) (with $M + 1$ instead of M) for the second term. However the last term in (3.31) is a maximal function which fits in the scheme of [29: Corollary 1] because of $M + 1 > s$, which we denote temporarily by $\tilde{f}_j$. In other words

$$\Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} \tilde{f}_j(\cdot)^q \Big)^{1/q} \Big| L_p \Big\| \leq c \, \|f \mid F^s_{pq}\|. \tag{3.32}$$

In order to handle the first terms in (3.31) we need more maximal functions. Let g^* be the usual Hardy-Littlewood maximal function of g and let

$$f_j^a(x) = \sup_{y \in \mathbf{R}^n} \frac{|f_j(x+y)|}{1 + |2^j y|^a} \quad \text{with} \quad a > \frac{n}{\min(p, q)}. \tag{3.33}$$

Because of $\dfrac{s}{n} > \dfrac{1}{\min(p,q)} - \dfrac{1}{r}$ we find a real number $\varkappa$ with $\varkappa < 1$ and

$$\frac{1}{u} \min(p, q) > \varkappa > 1 - \frac{s}{n} \min(p, q) \quad \text{if } 1 \leq u \leq r. \tag{3.34}$$

Then we have

$$\Big(\fint_{B(x,2^{-j})} |f_{j+l}(y)|^u \, dy \Big)^{1/u} \leq c' 2^{la(1-\varkappa)} f^a_{j+l}(x)^{1-\varkappa} (|f_{j+l}|^{\varkappa u})^{*1/u}(x). \tag{3.35}$$

Now (3.31), (3.32) and (3.35) yield

$$\Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{osc}_u^M f(\cdot, 2^{-j})^q \Big)^{1/q} \Big| L_p \Big\| \leq c' \, \|f \mid F^s_{pq}\| + c' \sum_{l=1}^{\infty} 2^{-(s-\varepsilon)l} 2^{la(1-\varkappa)}$$
$$\times \Big\| \Big(\sum_{j=1}^{\infty} 2^{(j+l)sq} f^a_{j+l}(\cdot)^{q(1-\varkappa)} (|f_{j+l}|^{\varkappa u})^{*q/u}(\cdot) \Big)^{1/q} \Big| L_p \Big\|, \tag{3.36}$$

where ε is an arbitrary positive number. We may choose a in (3.33), $\varkappa$ in (3.34) and $\varepsilon > 0$ in such a way that $s - \varepsilon > a(1 - \varkappa)$. Then the left-hand side of (3.36) can be estimated from above by

$$c \, \|f \mid F^s_{pq}\| + c \, \Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} f_j^a(\cdot)^q \Big)^{1/q} \Big| L_p \Big\|^{1-\varkappa}$$
$$\times \Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} (|f_j|^{\varkappa u})^{*q/\varkappa u} \Big)^{1/q} \Big| L_p \Big\|^{\varkappa}. \tag{3.37}$$

The first factor of the second summand can be estimated from above by $c \, \|f \mid F^s_{pq}\|^{1-\varkappa}$, see again [29: Corollary 1]. Because both $q > \varkappa u$ and $p > \varkappa u$, see (3.34), we can apply the vector-valued maximal inequality due to C. Fefferman and E. M. Stein, see [28: 1.2.3] for references (which works also if $q = \infty$). Then it follows that the second factor of the second summand in (3.37) can be estimated from above by $c \|f \mid F^s_{pq}\|^*$. We arrive at

$$\|f \mid L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{osc}_u{}^M f(\cdot, 2^{-j})^q \right)^{1/q} \middle| L_p \right\| \leqq c \|f \mid F^s_{pq}\|. \tag{3.38}$$

Now (3.38) and 3.2.2 prove Theorem 2.2.2(i) with $\Omega = \mathbf{R}^n$ under the additional restriction $1 \leqq u \leqq r$.

3.2.4. The extension of (3.38) to $0 < u < 1$ is almost obvious because we have by Hölder's inequality

$$\operatorname{osc}_u{}^M f(x, t) \leqq \operatorname{osc}_v{}^M f(x, t), \quad 0 < u < v \leqq \infty, t > 0. \tag{3.39}$$

It remains to prove (3.30) under the hypotheses that the right-hand side of (3.30) is finite and $f \in L_r$. We begin with some preparations. If (2.18) is satisfied for $r = 1$, then it is also satisfied for some $r > 1$. Furthermore again by (3.39) it is sufficient to prove (3.30) for small values of $u > 0$. Hence we may assume without restriction of generality

$$0 < u < 1 < r \leqq \infty \qquad \text{and} \qquad 0 < u < \min(p, q). \tag{3.40}$$

Temporarily we take it for granted that under these hypotheses f belongs to F^s_{pq}, we return to this question in 3.2.5. We use (3.19) and obtain

$$\begin{aligned} \operatorname{osc}_1{}^M f(x, 2^{-j}) \leqq{}& \varepsilon \left(\sum_{l=0}^{\infty} 2^{l\eta q} \operatorname{osc}_r{}^M f(x, c2^{-j-l})^q \right)^{1/q} \\ &+ c_\varepsilon \left(\sum_{l=0}^{\infty} |\operatorname{osc}_u{}^M f(x, c2^{-j-l})^u|^{*q/u} \right)^{1/q}, \end{aligned} \tag{3.41}$$

where again the star indicates the Hardy-Littlewood maximal function. The positive numbers ε and η may be chosen arbitrarily small, in particular $0 < \eta < s$. We multiply (3.41) with 2^{js} and take the l_q-quasi-norm with respect to j. Then we obtain

$$\begin{aligned} &\left(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{osc}_1{}^M f(x, 2^{-j})^q \right)^{1/q} \\ &\leqq \varepsilon \left(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{osc}_r{}^M f(x, c2^{-j})^q \right)^{1/q} + c_\varepsilon' \left(\sum_{j=1}^{\infty} 2^{jsq} |\operatorname{osc}_u{}^M f(x, c2^{-j})^u|^{*q/u} \right)^{1/q}. \end{aligned} \tag{3.42}$$

Next we apply the L_p-quasi-norm to (3.42) and add on both sides $\|f \mid L_p\|$. Then we obtain by the above results

$$\begin{aligned} \|f \mid F^s_{pq}\| \leqq{}& \varepsilon \|f \mid F^s_{pq}\| + c \|f \mid L_p\| \\ &+ \tilde{c}_\varepsilon \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} |\operatorname{osc}_u{}^M f(\cdot, c2^{-j})^u|^{*q/u} \right)^{1/q} L_p \right\|, \end{aligned} \tag{3.43}$$

where $\varepsilon > 0$ is at our disposal. Because $u < \min(p, q)$ we can again apply the vector-valued Hardy-Littlewood maximal inequality to the last summand in (3.43). We obtain (3.30).

3.2.5. It remains to prove the following assertion: Let $f \in L_r$ and let the right-hand side of (3.30) be finite. Then $f \in F^s_{pq}$. We begin with two preliminaries. Let $K \in \mathbf{N}$ with $K \geqq [s]$ and let φ be a C^∞ function in $\mathbf{R}^n$ with

$$|D^\gamma \varphi(x)| \leqq 1 \quad \text{if} \quad |\gamma| \leqq K + 1 \quad \text{and} \quad x \in \mathbf{R}^n. \tag{3.44}$$

Then we have

$$\operatorname{osc}_u^{K+M} (\varphi f)(x, 2^{-j}) \leqq \left(\fint_{B(x,2^{-j})} \left| \varphi(y) f(y) - \sum_{|\alpha| \leqq k} \frac{1}{\alpha!} D^\alpha \varphi(x) (y-x)^\alpha P(y) \right|^u dy \right)^{1/u},$$

where $P \in \mathbf{P}_M$ is an arbitrary polynomial. It follows

$$\operatorname{osc}_u^{K+M} (\varphi f)(x, 2^{-j}) \leqq c \operatorname{osc}_u^M f(x, 2^{-j}) + c 2^{-j(K+1)} \left(\fint_{B(x,2^{-j})} |f(y)|^u dy \right)^{1/u}. \tag{3.45}$$

The last integral can be estimated from above by $(|f|^u)^{*1/u}$, where again the star indicates the Hardy-Littlewood maximal function. We have

$$\left\| \left(\sum_{j=1}^\infty 2^{jsq} \operatorname{osc}_u^{K+M} (\varphi f)(\cdot, 2^{-j})^q \right)^{1/q} \Big| L_p \right\| \leqq c' \|(|f|^u)^{*1/u} \mid L_p\| + \left\| \left(\sum_{j=1}^\infty 2^{jsq} \operatorname{osc}_u^M f(\cdot, 2^{-j})^q \right)^{1/q} \Big| L_p \right\|. \tag{3.46}$$

We estimate the first summand on the right-hand side of (3.46) by $c \|f \mid L_p\|$, this follows from the Hardy-Littlewood maximal inequality and $p > u$, see (3.40). Hence the left-hand side of (3.46) can be estimated from above by a constant which is independent of φ with (3.43). We need a second preparation. Let ψ be a compactly supported C^∞ function in $\mathbf{R}^n$ with $\int \psi(y)\, dy = 1$ and let f be the above function, i.e. $f \in L_r$ such that the right-hand side of (3.30) is finite. Then $\psi(t, f)$ in the sense of (2.2) is a mollification. We have

$$\psi(t_l, f)(x) \to f(x) \quad \text{almost everywhere in } \mathbf{R}^n \tag{3.47}$$

at least for some sequence $t_l \to 0$. Furthermore,

$$\operatorname{osc}_u^L \psi(t, f)(x, 2^{-j}) \leqq \left(\fint_{B(x,2^{-j})} |\psi(t, f - P)(y)|^u dy \right)^{1/u} \tag{3.48}$$

for any $P \in \mathbf{P}_L$ because $\psi(t, P) \in \mathbf{P}_L$. Now we combine these two preparations. Let $f \in L_r$ such that the right-hand side of (3.30) is finite. Let φ be a cut-off function with (3.44) and $\varphi(y) = 1$ if $|y| \leqq R$. Then $\psi(t, \varphi f)$ is a compactly supported C^∞ function and belongs to F^s_{pq}. We can apply the arguments from 3.2.4 to $\psi(t, \varphi f)$ instead of f and obtain

$$\|\psi(t, \varphi f) \mid F^s_{pq}\| \leqq \|\psi(t, \varphi f) \mid L_p\| + \left\| \left(\sum_{j=1}^\infty 2^{jsq} \operatorname{osc}_u^{K+M} \psi(t, \varphi f)(\cdot, 2^{-j})^q \right)^{1/p} \Big| L_p \right\|. \tag{3.49}$$

We use (3.48) with $L = K + M$ and φf instead of f, choose $P = 0$ and estimate $\operatorname{osc}_u^{K+M} \psi(t, \varphi f)$ by $\left(\fint_{B(x,2^{-j})} |f(y)|^r dy \right)^{1/r}$. Hence by Lebesgue's bounded convergence theorem and (3.47) we have for any bounded domain ω and some sequence $t_l \to 0$

$$\left\|\left(\sum_{j=1}^{J} 2^{jsq} \operatorname{osc}_u^{K+M} \psi(t_l, \varphi f)(\cdot, 2^{-j})^q\right)^{1/q} \middle| L_p(\omega)\right\|$$

$$\to \left\|\left(\sum_{j=1}^{J} 2^{jsq} \operatorname{osc}_u^{K+M} (\varphi f)(\cdot, 2^{-j})^q\right)^{1/q} \middle| L_p(\omega)\right\|$$

$$\leq \|f \mid L_p\| + \left\|\left(\sum_{k=1}^{\infty} 2^{ksq} \operatorname{osc}_u^{M} f(\cdot, 2^{-k})^q\right)^{1/q} \middle| L_p\right\|, \tag{3.50}$$

where we used (3.46) and the subsequent arguments, $J \in \mathbf{N}$. Now by Fatou's lemma we can replace J on the left-hand side of (3.50) by ∞ and ω by $\mathbf{R}^n$. By the definition of $\|\cdot \mid F^s_{pq}\|$ in (2.4), (3.49), (3.50) and Fatou's lemma follows now $f \in F^s_{pq}$.

3.3. Spaces on $\mathbf{R}_+^n$

3.3.1. In this section we prove Theorem 2.3.2, Theorem 2.2.2 with $\Omega = \mathbf{R}_+^n$ and we add remarks about Corollary 2.3.4.

3.3.2. Let p, q and s be given by (2.35). Then we have (2.19) with $\Omega = \mathbf{R}^n$. In particular $F^s_{pq}(\mathbf{R}_+^n) \subset L_{\max(p,r)}(\mathbf{R}_+^n)$ for some $r \geq 1$ and $\operatorname{ext}_L f(x)$ from (2.37) makes sense for any $f \in F^s_{pq}(\mathbf{R}_+^n)$. In order to prove the extension property from Theorem 2.3.2 we have to distinguish between oscillations based on $\mathbf{R}^n$, denoted by $\operatorname{osc}_u^M f$, and oscillations based on $\mathbf{R}_+^n$, which we denote temporarily by $\operatorname{Osc}_u^M f$. Let $P \in \mathbf{P}_L$, then we have by (2.36)

$$\operatorname{ext}_L P(x) = P(x) \quad \text{if} \quad x \in \mathbf{R}^n. \tag{3.51}$$

In particular $\operatorname{ext}_L f(x) - P(x) = \operatorname{ext}_L (f - P)(x)$ for any $f \in F^s_{pq}(\mathbf{R}_+^n)$ and $P \in \mathbf{P}_L$. Let $x = (x', x_n)$ with $x' \in \mathbf{R}^{n-1}$. We have

$$\operatorname{osc}_u^L (\operatorname{ext}_L f)(x, 2^{-j}) = \operatorname{Osc}_u^L f(x, 2^{-j}) \tag{3.52}$$

if $x \in \mathbf{R}^n$, $x_n > 2^{-j}$, and

$$\operatorname{osc}_u^L (\operatorname{ext}_L f)(x, 2^{-j}) \leq c \sum_{k=0}^{L} \operatorname{Osc}_u^L f\big((x', -\lambda_k x_n), c2^{-j}\big) \tag{3.53}$$

if $x \in \mathbf{R}^n$, $x_n < -2^{-j}$, for some $c > 0$. Finally if $x = (x', x_n)$ with $|x_n| \leq 2^{-j}$, then it follows

$$\operatorname{osc}_u^L (\operatorname{ext}_L f)(x, 2^{-j}) \leq c \operatorname{Osc}_u^L f\big((x', |x_n|), c2^{-j}\big) \tag{3.54}$$

for some $c > 0$. Let now $L \geq [s]$ and $f \in F^s_{pq}(\mathbf{R}_+^n)$, then we claim

$$\begin{aligned}\|f \mid F^s_{pq}(\mathbf{R}_+^n)\| &\leq \|\operatorname{ext}_L f \mid F^s_{pq}\| \\ &\leq c' \|f \mid L_p(\mathbf{R}_+^n)\| + c' \left\|\left(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{Osc}_u^L f(\cdot, c2^{-j})^q\right)^{1/q} \middle| L_p(\mathbf{R}_+^n)\right\| \\ &\leq c'' \|f \mid F^s_{pq}(\mathbf{R}_+^n)\|\end{aligned} \tag{3.55}$$

for some positive numbers c, c' and c'' which are independent of f. Furthermore we assume that p, q, s and also r and u are the same numbers as in Theorem 2.2.2(i). By (3.52)–(3.54) and Theorem 2.2.2(i) with $\Omega = \mathbf{R}^n$ it follows both $\operatorname{ext}_L f \in F^s_{pq}$ and the second inequality in (3.55). The first inequality is obvious and the last inequality in (3.55) follows again from Theorem 2.2.2(i) with $\Omega = \mathbf{R}^n$. In particular ext_L is an extension operator and Theorem 2.3.2 is proved.

3.3.3. We prove Theorem 2.2.2(i) with $\Omega = \mathbf{R}_+^n$. If $f \in F^s_{pq}(\mathbf{R}_+^n)$, then we have $f \in L_{\max(p,r)}(\mathbf{R}_+^n)$ and the right-hand side of (3.55). If $f \in L_{\max(p,r)}(\mathbf{R}_+^n)$ such that the third term in (3.55) is finite, then it follows by the above arguments and by Theorem 2.2.2(i) with $\Omega = \mathbf{R}^n$ that $\mathrm{ext}_L f \in F^s_{pq}$ and hence $f \in F^s_{pq}(\mathbf{R}_+^n)$. Now (2.19) with $\Omega = \mathbf{R}_+^n$ is a consequence of (3.55).

3.3.4. We add some remarks about the proof of Corollary 2.3.4. By duality arguments and on the basis of Theorem 2.3.2 one can prove the extension property for the spaces F^s_{pq} with $-\infty < s < \infty$, $1 < p < \infty$, $1 < q < \infty$, see [28: 2.9.2] for details. Afterwards one can use complex interpolation which proves the extension property for all spaces F^s_{pq} with $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q < \infty$, see [28: 2.9.4] for details. Similarly one can prove the extension property for the spaces B^s_{pq} for all $-\infty < s < \infty$, $0 < p \leqq \infty$, $0 < q \leqq \infty$. Now complex interpolation between the spaces $F^{s_0}_{p\infty}$, $s_0 > n(1/p - 1)_+$, covered by Theorem 2.3.2 and $B^{s_1}_{\infty\infty} = F^{s_1}_{\infty\infty}$ covers also the case $F^s_{p\infty}$.

3.4. Proof of Theorem 2.2.2

3.4.1. The aim of this section is to prove Theorem 2.2.2 for bounded C^∞ domains Ω. Recall that we proved Theorem 2.2.2(i) if the underlying domain is either $\mathbf{R}^n$ or $\mathbf{R}_+^n$, see 3.2 and 3.3.3. In the same way one proves corresponding assertions with B^s_{pq} instead of F^s_{pq}. Now we concentrate ourselves again to the spaces $F^s_{pq}(\Omega)$, the proof for the spaces $B^s_{pq}(\Omega)$ is similar but easier.

3.4.2. Let Ω be a bounded C^∞ domain. Let p, q, s be the same numbers as in Theorem 2.2.2(i). We have to distinguish between oscillations based on $\mathbf{R}^n$, which we denote by $\mathrm{osc}_u^M f$, and oscillations based on Ω, which we denote temporarily by $\mathrm{Osc}_u^M f$ (not to be mixed with the corresponding notation used temporarily in connection with $\mathbf{R}_+^n$). Let ext be an extension operator from $F^s_{pq}(\Omega)$ into F^s_{pq} obtained in the usual way via a resolution of unity, local diffeomorphisms and the operator ext_L from (2.37) (we refer for details about this procedure to [28: 3.3.4], see also the considerations below). We begin with a local consideration and assume that the hypotheses of 3.1.2 are satisfied and that $f \in F^s_{pq}(\Omega)$ has a support near the origin. Let χ be a diffeomorphic map of, say, the unit ball B in $\mathbf{R}^n$ into itself such that $\chi(B \cap \Omega) = B \cap \mathbf{R}_+^n$ and $\chi(0) = 0$. Then ext can be described locally as

$$\mathrm{ext}\, f(x) = \mathrm{ext}_L\, f \circ \chi^{-1}\big(\chi(x)\big), \tag{3.56}$$

where ext_L has the meaning of Theorem 2.3.2. Let $1 \leqq u \leqq r$ in the sense of Theorem 2.2.2. Then we can use Proposition 3.1.3 both for $\mathrm{osc}_u^L\, \mathrm{ext}\, f$ and $\mathrm{Osc}_u^L f$. We deal only with the most critical case which corresponds to (3.54) in the case of $\mathbf{R}_+^n$. The transform $\lambda(x) = (x', |x_n|)$ in this case must be replaced now by $\mu(x) = \chi^{-1} \circ \lambda \circ \chi(x)$. We have

$$\begin{aligned} &\mathrm{osc}_u^L\, \mathrm{ext}\, f\left(\chi^{-1}(x), 2^{-j}\right) \\ &\leqq \Big(\fint_{\{y \mid |\chi^{-1}(x) - y| < 2^{-j}\}} \Big|\mathrm{ext}_L\, f \circ \chi^{-1}\big(\chi(y)\big) - \sum_{|\alpha| \leqq L} a_\alpha(x)\, (y - \chi^{-1}(x))^\alpha\Big|^u\, dy\Big)^{1/u}. \end{aligned} \tag{3.57}$$

We wish to apply (3.3) with Osc instead of osc at the point $\mu \circ \chi^{-1}(x)$ instead of x. We substitute $y = \chi^{-1}(z)$ in (3.57) and may assume

$$\big(\chi^{-1}(z) - \chi^{-1}(x)\big)^\alpha = c_\alpha(x)\, (z - x)^\alpha + \dots + O(|z - x|^{L+1}) \tag{3.58}$$

with $c_\alpha(x) \neq 0$ where $+ \ldots$ indicates explicitly calculated terms of order between $|\alpha| + 1$ and L or of type $(z - x)^\beta$ with $|\beta| = |\alpha|$ and $\beta_n < \alpha_n$. Now we determine the coefficients $a_\alpha(x)$ by induction with respect to $|\alpha|$ and α_n such that

$$\sum_{|\alpha| \leq L} a_\alpha(x) \left(\chi^{-1}(z) - \chi^{-1}(x)\right)^\alpha = \sum_{|\alpha| \leq L} \frac{1}{\alpha!} D^\alpha f^j\left(\mu \circ \chi^{-1}(x)\right) (z - x)^\alpha + O(|z - x|^{L+1}). \tag{3.59}$$

The coefficients $a_\alpha(x)$ are uniquely determined, they are linear combinations of $D^\beta f^j\left(\mu \circ \chi^{-1}(x)\right)$ with $|\beta| \leqq L$, multiplied with harmless C^∞-functions. Now we choose L large, for example $L = 2M$ with $M \in \mathbf{N}$ and $M \geqq [s]$. Then we have $L + 1 - M > M$ and by (3.9) (with $|\beta| \leqq M$ instead of $|\alpha| = M$)

$$\begin{aligned} 2^{-j(L+1)} \left|D^\beta f^j\left(\mu \circ \chi^{-1}(x)\right)\right| &= 2^{-j(M+1)} 2^{-Mj} \left|D^\beta f^j\left(\mu \circ \chi^{-1}(x)\right)\right| \\ &\leqq c 2^{-j(M+1)} f^*\left(\mu \circ \chi^{-1}(x)\right), \qquad |\beta| \leqq M, \end{aligned} \tag{3.60}$$

where the star indicates again the Hardy-Littlewood maximal function restricted to $\mathbf{R}_+^n$. If $M < |\beta| \leqq L$, then we estimate $D^\beta f^j\left(\mu \circ \chi^{-1}(x)\right)$ by (3.10) (now with Osc instead of osc). Recall that we are only interested in the counterpart of (3.54). Now we use (3.51) in the same way as in the case of $\mathbf{R}_+^n$, the optimal polynomial in (3.59), see (3.3), and obtain the counterpart of (3.54)

$$\begin{aligned} &\mathrm{osc}_u^L \operatorname{ext} f\left(\chi^{-1}(x), 2^{-j}\right) \\ &\leqq c' \sum_{k=M}^{L} \mathrm{Osc}_u^k f\left(\mu \circ \chi^{-1}(x), c2^{-j}\right) + c' 2^{-j(M+1)} f^*\left(\mu \circ \chi^{-1}(x)\right). \end{aligned} \tag{3.61}$$

Similarly but simpler one obtains the counterparts of (3.52) and (3.53). We replace $\chi^{-1}(x)$ in (3.61) by x, multiply (3.61) and the just mentioned counterparts of (3.52) and (3.53) with 2^{js}, take the l_q-quasi-norm and then the L_p-quasi-norm and obtain

$$\begin{aligned} &\left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \mathrm{osc}_u^L \operatorname{ext} f(\cdot, 2^{-j})^q \right)^{1/q} \middle| L_p \right\| \\ &\leqq c' \sum_{k=M}^{L} \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \mathrm{Osc}_u^k f(\cdot, c2^{-j})^q \right)^{1/q} \middle| L_p(\Omega) \right\| + c' \|f^* \mid L_p(\Omega)\|, \end{aligned} \tag{3.62}$$

where now f^* stands for the Hardy-Littlewood maximal function with respect to Ω. We may assume $r > 1$ in the sense of Theorem 2.2.2. Let $t = \max(p, r)$. By our assumptions we can replace $L_p(\Omega)$ in the last term in (3.62) by $L_t(\Omega)$ (Hölder's inequality). Hence by the maximal inequality the last term on the right-hand side of (3.62) can be estimated from above by $c \|f \mid L_t(\Omega)\|$. Next we use (3.11) with k instead of $M - 1$. Because $k + 1 > s$ we obtain

$$\begin{aligned} &\left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \mathrm{osc}_u^L \operatorname{ext} f(\cdot, 2^{-j})^q \right)^{1/q} \middle| L_p \right\| \\ &\leqq c' \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \mathrm{Osc}_u^L f(\cdot, c2^{-j})^q \right)^{1/q} \middle| L_p(\Omega) \right\| + c' \|f \mid L_t(\Omega)\|. \end{aligned} \tag{3.63}$$

Again by Hölder's inequality and known embedding theorems, see [28: 2.7.1], we have

$$\|\operatorname{ext} f \mid L_t\| \leqq \varepsilon \|\operatorname{ext} f \mid F^s_{pq}\| + c_\varepsilon \|f \mid L_p(\Omega)\|, \tag{3.64}$$

where $\varepsilon > 0$ is at our disposal. Now by Theorem 2.2.2 with $\mathbf{R}^n$ it follows

$$\begin{aligned}\|f \mid F^s_{pq}(\Omega)\| &\leqq \|\text{ext}\, f \mid F^s_{pq}\| \\ &\leqq \|f \mid L_p(\Omega)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \operatorname{Osc}_u{}^L f(\cdot, c2^{-j})^q \right)^{1/q} \middle| L_p(\Omega) \right\| \\ &\leqq c' \|f \mid F^s_{pq}(\Omega)\|. \end{aligned} \tag{3.65}$$

In other words, the last but one term is an equivalent quasi-norm on $F^s_{pq}(\Omega)$ provided that $L = 2M$ with $M \geqq [s]$ and $1 \leqq u \leqq r$ in the sense of Theorem 2.2.2. However we can extend this assertion immediately to all $L \in \mathbf{N}_0$ with $L \geqq [s]$. This follows again from (3.11). Finally we assume $f \in L_t(\Omega)$ (with a support near the origin) such that the quasi-norm in (2.19) is finite. Then we have (3.63) by the above arguments. Hence ext $f \in F^s_{pq}$, again by Theorem 2.2.2, and consequently $f \in F^s_{pq}(\Omega)$. Hence Theorem 2.2.2 is proved provided that the hypotheses of 3.1.2 are satisfied, f has a support near the origin and $1 \leqq u \leqq r$.

3.4.3. Now let $f \in F^s_{pq}(\Omega)$ and let $1 = \sum_{k=1}^{k} \varphi^k(x)$ if $x \in \Omega$ be an appropriate resolution of unity. If $\operatorname{supp} \varphi^k \cap \partial\Omega \neq \emptyset$, then we may assume that we can apply the middle part of (3.65) to $\varphi^k f$ instead of f. We choose L large and apply the counterpart of (3.45). By the same technique as above we estimate

$$\|(|f|^u)^{*1/u} \mid L_p(\Omega)\| \leqq c\, \|(|f|^u)^{*1/u} \mid L_p(\Omega)\| \leqq c' \|f \mid L_r(\Omega)\| \tag{3.66}$$

by the Hardy-Littlewood maximal inequality, where we assumed $1 \leqq u < r \leqq \infty$ without restriction of generality (the case $u = r = \infty$ fits also in this scheme). We use (3.64) with L_r instead of L_t and obtain the desired estimate. Let $\operatorname{supp} \varphi^k \cap \partial\Omega = \emptyset$, then we have

$$\operatorname{osc}_u{}^L \varphi^k f(x, c2^{-j}) = \operatorname{Osc}_u{}^L \varphi^k f(x, c2^{-j})$$

at least for $j \geqq J$. The terms with $1 \leqq j < J$ can be treated in the above way. This proves the middle part of (3.65) provided that $f \in F^s_{pq}(\Omega)$, L is large and $1 \leqq u < r$. This inequality can be extended to all $L \in \mathbf{N}_0$ with $L \geqq [s]$ by the same arguments as in 3.4.2. Furthermore the first inequality in (3.65) is obvious and the last inequality follows from Theorem 2.2.2 with $\mathbf{R}^n$. Hence the last but one quasi-norm in (3.65) is an equivalent quasi-norm on $F^s_{pq}(\Omega)$ for all $L \in \mathbf{N}_0$ with $L \geqq [s]$ and $1 \leqq u \leqq r$. If $f \in L_t(\Omega)$ with $t = \max(p, r)$, then we argue in the same way as at the end of 3.4.2. This complete the proof of Theorem 2.2.2 provided that $u \geqq 1$.

3.4.4. Let $0 < u < 1$. By the same arguments as in 3.2.4 it follows that the last but one term in (3.65) is an equivalent quasi-norm in $F^s_{pq}(\Omega)$, now for all $0 < u \leqq r$. Now let $f \in L_r(\Omega)$ such that the corresponding quasi-norm in (3.65) is finite. By the same arguments as at the beginning of 3.2.5 we may assume that the hypotheses of 3.1.2 are satisfied and that f has a support near the origin. Now we combine the mollification 3.2.5 with an additional translation $f(x) \to f(x', x_n + \varepsilon_l)$ for some $\varepsilon_l > 0$ and $\varepsilon_l \to 0$. The rest is the same as in 3.2.5. The proof is complete.

3.5. Proof of Theorem 2.2.7

3.5.1. Part (iii) of the theorem follows from (2.24) and (2.19), see also (2.22) and (2.17).

3.5.2. We prove part (ii). We compare (2.23) and (2.30). Then it is clear that the proof is reduced to

$$\|f \mid L_p{}^s(\Omega)\|^M \leqq c\, \|f \mid L_p{}^s(\Omega)\|^{M+1} \tag{3.67}$$

for some $c > 0$ and all $f \in L_p^s(\Omega)$, provided that $M \in \mathbf{N}_{-1}$ and $-n/p \leqq s < M + 1$. However (3.67) follows from (3.11).

3.5.3. We prove part (i). Let $s = -n/p$, then we have

$$t^{-s} \operatorname{osc}_p^{-1} f(x,t) = t^{n/p} \Big(\fint_{B(x,t)} |f(y)|^p \, dy \Big)^{1/p} = c \cdot \Big(\fint_{B(x,t)} |f(y)|^p \, dy \Big)^{1/p}.$$

Now by (2.17) and (2.23) it follows $L_p(\Omega) = L_p^{-n/p}(\Omega)$. Let $s > 0$ and $f \in \mathscr{C}^s(\Omega)$, see (2.27). In order to prove the second equality in (2.29) we have to calculate $\operatorname{osc}_p^M f(x,t)$ where $1 \leqq p < \infty$ and $M \geqq [s]$. We assume without restriction of generality that the hypotheses of 3.1.2 are satisfied and that all calculations take place near the origin. We use the same technique as in 3.1.4, in particular (3.5), (3.8) and (3.13) (with $M + 1$ instead of M). We obtain

$$\operatorname{osc}_p^M f(x, 2^{-j}) \leqq c \sup_{z \in B(x, 2^{-j})} \fint_{V^{M+1}(z, c2^{-j})} |\Delta_y^{M+1} f(z)| \, dy. \tag{3.68}$$

Now it follows

$$\begin{aligned} \|f \mid L_p^s(\Omega)\| &\leqq \|f \mid L_p(\Omega)\| + \sup 2^{js} \operatorname{osc}_p^M f(x, 2^{-j}) \\ &\leqq c \, \|f \mid L_\infty(\Omega)\| + \sup |h|^{-s} \, |\Delta_h^{M+1} f(x)| = \|f \mid \mathscr{C}^s(\Omega)\|, \end{aligned} \tag{3.69}$$

where the first supremum is taken over all $x \in \Omega$ and $j \in \mathbf{N}$, whereas the second supremum is taken over $x \in \Omega$, $0 < t < 1$ and $h \in V^{M+1}(x, t)$, see (2.27). In particular $f \in L_p^s(\Omega)$. We prove the converse inequality to (3.69), temporarily under the assumption $f \in \mathscr{C}^s(\Omega)$. By (2.11), (2.10) and the properties of the kernel $k(x)$ it follows

$$\begin{aligned} |f(x)| &= \Big| \sum_{j=0}^{\infty} f_j(x) \Big| \\ &\leqq \Big| \int k_0(y) \, f(x + y) \, dy \Big| + \sum_{j=1}^{\infty} \Big| \int k(y) \, (f - P_j) \, (x + 2^{-j} y) \, dy \Big| \end{aligned} \tag{3.70}$$

with $P_j \in \mathbf{P}_M$. In particular

$$\begin{aligned} \|f \mid L_\infty(\Omega)\| &\leqq c \, \|f \mid L_p(\Omega)\| + \sup_{x \in \Omega, j \in \mathbf{N}} 2^{j\varepsilon} \operatorname{osc}_p^M f(x, c2^{-j}) \\ &\leqq c' \, \|f \mid L_p^s(\Omega)\|, \end{aligned} \tag{3.71}$$

where $0 < \varepsilon < s$ is at our disposal. Next we estimate $\Delta_h^{M+1} f(x)$ with $M \geqq [s]$ and $|h| \sim 2^{-j}$. We use (2.11) and (3.2) and obtain

$$\Delta_h^{M+1} f(x) = \Delta_h^{M+1} f^{j+R}(x) + \sum_{r=1}^{\infty} \Delta_h^{M+1} f_{j+R+r}(x), \tag{3.72}$$

where $R \in \mathbf{N}_0$ will be determined later on. By (3.4) we have

$$\begin{aligned} \Delta_h^{M+1} f^{j+R}(x) &= \Delta_h^{M+1} \int k_0(2^{j+R} y - 2^{j+R} x) \, f(y) \, dy \\ &= \int \Delta_{-2^{j+R} h}^{M+1} k_0(y) \, f(x + 2^{-j-R} y) \, dy. \end{aligned} \tag{3.73}$$

We can replace f on the right-hand side of (3.73) by $f - P$ with $P \in \mathbf{P}_M$. It follows

$$|\Delta_h^{M+1} f^{j+R}(x)| \leqq c \operatorname{osc}_p^M f(x, c' 2^{-j}), \tag{3.74}$$

where c' depends on R. Similar as in (3.33) we introduce the maximal function

$$f_k^a(x) = \sup |f_k(x + y)| / (1 + |2^k y|^a) \quad \text{with } a > 0, \tag{3.75}$$

where $k = j + R + r$ with $r \in \mathbf{N}$ and the supremum is taken over y with $x + y \in B(x, c2^{-j})$ for some $c > 0$. Then we have

$$|\Delta_h^{M+1} f_{j+R+r}(x)| \leqq c 2^{(R+r)a} f^a_{j+R+r}(x) \tag{3.76}$$

and

$$|h|^{-s} |\Delta_h^{M+1} f_{j+R+r}(x)| \leqq c 2^{-(R+r)(s-a)} 2^{(j+R+r)s} f^a_{j+R+r}(x) \tag{3.77}$$

and finally

$$\sum_{r=1}^{\infty} |h|^{-s} |\Delta_h^{M+1} f_{j+R+r}(x)| \leqq \varepsilon \sup_{k \in \mathbf{N}} 2^{ks} f_k^a(x), \tag{3.78}$$

where $\varepsilon > 0$ is at our disposal (if $0 < a < s$ and if R is chosen large enough). We may assume that $f \in \mathscr{C}^s(\Omega)$ is the restriction of $g \in \mathscr{C}^s = \mathscr{C}^s(\mathbf{R}^n)$. However $\mathscr{C}^s = B^s_{\infty\infty}$, see [28: 2.5.7] and we can use the theory of these spaces developed in [28] and [29]. In particular by [29: Corollary 7] and the extension property for $\mathscr{C}^s$-spaces it follows that the right-hand side of (3.78) can be estimated from above by $\varepsilon \, \|f \mid \mathscr{C}^s(\Omega)\|$. Now (3.71), (3.72), (3.74) and (3.78) yield $\|f \mid \mathscr{C}^s(\Omega)\| \leqq c \, \|f \mid L_p^s(\Omega)\| + \varepsilon \, \|f \mid \mathscr{C}^s(\Omega)\|$ and hence

$$\|f \mid \mathscr{C}^s(\Omega)\| \leqq c \, \|f \mid L_p^s(\Omega)\|. \tag{3.79}$$

Let $f \in L_p^s(\Omega)$. Then we use the approximation procedure from 3.2.5 and 3.4.4: translations and mollifications. For the approximating functions we have (3.79). Now by the same limiting arguments as in 3.2.5 and 3.4.4 we can extend (3.79) to the given function $f \in L_p^s(\Omega)$. The proof of (2.29) is complete.

3.5.4. We prove part (iv). The case $k = 0$ follows from the Hardy-Littlewood maximal inequality. Let $f \in W_p^k(\Omega)$ with $k \in \mathbf{N}$ and $1 < p < \infty$. Let $1 \leqq u < p$. We use again the above technique, in particular

$$\operatorname{osc}_u^{k-1} f(x, 2^{-j}) \leqq \sum_{r=1}^{\infty} \left(\fint_{B(x,2^{-j})} |f_{j+r}(y)|^u \, dy \right)^{1/u} + c 2^{-jk} \sup_{z \in B(x,2^{-j})} \sum_{|\alpha|=k} |D^\alpha f^j(z)|, \tag{3.80}$$

see (3.8), (2.11) and (3.2). By (3.4) we have

$$|D^\alpha f^j(z)| = \left|\int k_0(y) \, D^\alpha f(z + 2^{-j} y) \, dy\right| \leqq c (D^\alpha f)^* (x), \tag{3.81}$$

where again the star indicates the Hardy-Littlewood maximal function, $z \in B(x, 2^{-j})$. Then (3.80) yields

$$2^{jk} \operatorname{osc}_u^{k-1} f(x, 2^{-j}) \leqq c \sum_{|\alpha|=k} (D^\alpha f)^* (x) + c \left(\sup_r 2^{rku} |f_r|^u \right)^{*1/u} (x). \tag{3.82}$$

We take the supremum with respect to j on the left-hand side of (3.82) and obtain

$$\|f_u^{k-1,k} \mid L_p(\Omega)\|$$
$$\leqq c \sum_{|\alpha|=k} \|(D^\alpha f)^* \mid L_p(\Omega)\| + c \left\| \left(\sup_r 2^{rku} |f_r|^u \right)^{*1/u} \Big| L_p(\Omega) \right\|. \tag{3.83}$$

Recall $1 \leqq u < p < \infty$. By the Hardy-Littlewood maximal inequality we have

$$\|f_u^{k-1,k} \mid L_p(\Omega)\| \leqq c \sum_{|\alpha|=k} \|D^\alpha f \mid L_p(\Omega)\| + c \left\| \sup_r 2^{rk} |f_r| \, \Big| \, L_p(\Omega) \right\|. \tag{3.84}$$

As we mentioned in 3.5.3 we may apply the Fourier-analytical characterization of the spaces F^s_{pq}. In particular the last term on the right-hand side of (3.84) equals

$\|f \mid F^k_{p\infty}\|$, which is less than $\|f \mid W_p{}^k(\Omega)\|$. Consequently,

$$\|f \mid L_p(\Omega)\| + \|f_u{}^{k-1,k} \mid L_p(\Omega)\| \leqq c\, \|f \mid W_p{}^k(\Omega)\|. \tag{3.85}$$

Next we prove the converse inequality. Let $f \in L_p(\Omega)$ such that the left-hand side of (3.85) is finite. We assume temporarily that (2.11) with $D^\alpha f$ instead of f holds pointwise almost everywhere, $|\alpha| = k$. Then we have by (3.2) and (3.4)

$$\begin{aligned} D^\alpha f(x) &= \lim_{j\to\infty} D^\alpha f^j(x) = \lim_{j\to\infty} \int k_0(y)\, D^\alpha f(x + 2^{-j}y)\, dy \\ &= \lim_{j\to\infty} 2^{jk} \int D^\alpha k_0(y)\, (f - P_j)\, (x + 2^{-j}y)\, dy \end{aligned} \tag{3.86}$$

with $|\alpha| = k$ and $P_j \in \mathbf{P}_{k-1}$. We obtain

$$|D^\alpha f(x)| \leqq c f_u{}^{k-1,k}(x), \qquad |\alpha| = k, \tag{3.87}$$

and consequently $f \in W_p{}^k(\Omega)$, including the converse inequality to (3.85). If $f \in L_p(\Omega)$ such that the left-hand side of (3.85) is finite, then we use the same limiting arguments as in 3.2.5 and 3.4.4, see also the end of 3.5.3. We obtain (3.87) and hence $f \in W_p{}^k(\Omega)$. The proof is complete.

3.6. Proof of Theorem 2.3.6

3.6.1. There is no difference in the proof whether Ω is $\mathbf{R}^n$, $\mathbf{R}_+{}^n$ or a bounded C^∞ domain in $\mathbf{R}^n$. So we may assume that Ω is a bounded C^∞ domain. All considerations are local and we may assume that the hypotheses of 3.1.2 are satisfied and that the points $x \in \Omega$ of interest are near the origin. Recall that the supports of the kernels of all involved means are located in $\mathbf{R}_+{}^n$. This justifies all the considerations below. Furthermore we restrict ourselves to the proof of (2.42). The proof of (2.43) is the same but simpler.

3.6.2. Let $0 < p < \infty$, $0 < q \leqq \infty$, let s be restricted by (2.41) and let $M \in \mathbf{N}_0$ with $M \geqq [s]$. Then we have to prove (2.42) with $M + 1$ instead of M. Let $f \in F^s_{pq}(\Omega)$. We use the same splitting technique as in (3.31) with $u = 1$. Then (2.40) yields

$$d^{M+1}_{2^{-j}} f(x) \leqq \sum_{l=1}^{\infty} \;\; ⨍_{V^{M+1}(x,2^{-j})} |\Delta_h{}^{M+1} f_{j+l}(x)|\, dh + ⨍_{V^{M+1}(x,2^{-j})} |\Delta_h{}^{M+1} (f^j - P)\, (x)|\, dh, \tag{3.88}$$

where we choose the same polynomial $P \in \mathbf{P}_M$ as in (3.31). We obtain

$$d^{M+1}_{2^{-j}} f(x) \leqq c \sum_{l=1}^{\infty} \Big[|f_{j+l}(x)| + ⨍_{B(x,c2^{-j})} |f_{j+l}(y)|\, dy \Big] + c \sup \ldots, \tag{3.89}$$

where sup ... is the same supremum as in (3.31). By the same arguments as in 3.2.3 we arrive at (3.38) with $d^{M+1}_{2^{-j}} f(x)$ instead of $\operatorname{osc}_1{}^M f(\cdot, 2^{-j})$.

3.6.3. Next we prove the converse inequality again under the assumption $f \in F^s_{pq}(\Omega)$. Let $0 < u < \min(p, q, 1)$, then we wish to estimate $\operatorname{osc}_u{}^M f(x, 2^{-j})$, where we assume that we can apply (3.3). We have by (3.5) and (3.21)

$$\begin{aligned} &\Big| f(y) - \sum_{|\alpha| \leqq M} \frac{1}{\alpha!} (D^\alpha f^j)\, (x)\, (y - x)^\alpha \Big|^u \\ &\leqq |f(y) - f^j(y)|^u + \Big| f^j(y) - \sum_{|\alpha| \leqq M} \frac{1}{\alpha!} (D^\alpha f^j)\, (x)\, (y - x)^\alpha \Big|^u \leqq c' d^{M+1}_{c2^{-j}} f(y)^u \\ &\quad + c' 2^{-j(M+1)u} \sum_{|\beta| = M+1} \left(\int_0^1 |D^\beta f^j(x + \tau(y - x))|^u\, d\tau \right)^u \sup |D^\gamma f^j(z)|^{(1-u)u} \end{aligned} \tag{3.90}$$

where the last supremum is taken over $|\gamma| = M + 1$ and $z \in B(x, c2^{-j})$. However multiplied with $2^{-j(M+1)u(1-u)}$ it is the same supremum as in (3.31) which fits in our scheme, see also (3.9) with $M + 1$ instead of M. We obtain

$$\operatorname{osc}_u^M f(x, 2^{-j}) \leqq c' \left(\fint_{B(x,2^{-j})} d_{c2^{-j}}^{M+1} f(y)^u \, dy \right)^{1/u} + \varepsilon 2^{-j(M+1)} \sup |D^\gamma f^j(z)|$$
$$+ c_\varepsilon \sum_{|\beta|=M+1} 2^{-j(M+1)} \left(\int_0^1 \fint_{B(x,2^{-j})} |D^\beta f^j(x + \tau(y - x))|^u \, dy \, d\tau \right)^{1/u}, \tag{3.91}$$

where the supremum is the same as in (3.90) and $\varepsilon > 0$ is at our disposal. As for the inner integral we have (3.22) with $|\cdot|^u$ instead of $|\cdot|$. We use (3.13) with $|\beta| = M + 1$ instead of $|\alpha| = M$ and arrive at

$$\operatorname{osc}_u^M f(x, 2^{-j}) \leqq c_\varepsilon \big(d_{c2^{-j}}^{M+1} f^u\big)^{*1/u}(x) + \varepsilon 2^{-j(M+1)} \sup |D^\gamma f^j(z)|, \tag{3.92}$$

where the supremum has the same meaning as in (3.91) and the star indicates the Hardy-Littlewood maximal function restricted to Ω. We multiply (3.92) with 2^{js}, take the l_q-quasi-norm with respect to j and the $L_p(\Omega)$-quasi-norm. By Theorem 2.2.2 and the remarks after (3.31) we obtain

$$\|f \mid F_{pq}^s(\Omega)\| \leqq c' \|f \mid L_p(\Omega)\| + \varepsilon \|f \mid F_{pq}^s(\Omega)\|$$
$$+ c_\varepsilon \left\| \left(\sum_{j=1}^\infty 2^{jsq} \big(d_{c2^{-j}}^{M+1} f^u\big)^{*1/u}(\cdot)^q \right)^{1/q} \middle| L_p(\Omega) \right\|. \tag{3.93}$$

Because $u < \min(p, q)$ we can apply the vector-valued maximal inequality and arrive at

$$\|f \mid F_{pq}^s(\Omega)\| \leqq c' \|f \mid L_p(\Omega)\| + c' \left\| \left(\sum_{j=1}^\infty 2^{jsq} d_{c2^{-j}}^{M+1} f(\cdot)^q \right)^{1/q} \middle| L_p(\Omega) \right\|. \tag{3.94}$$

The terms $d_{c2^{-j}}^{M+1}$ in (3.94) with $c2^{-j} \geqq 1/2$ can be estimated from above either by $c' \|f \mid L_p(\Omega)\|$ if $p \geqq 1$ or by $c' \|f \mid L_1(\Omega)\|$ if $p < 1$, which in turn can be estimated by $c_\varepsilon \|f \mid L_p(\Omega)\| + \varepsilon \|f \mid F_{pq}^s(\Omega)\|$ where $\varepsilon > 0$ is at our disposal. Hence we may assume $c = 1$ in (3.94) which yields the desired estimate.

3.6.4. Let $f \in L_{\bar{p}}(\Omega)$ such that the quasi-norm in (2.42) with $M + 1$ instead of M is finite. We use the same approximation scheme as in 3.4.4 and 3.2.5, translation and mollification, and apply (3.94) to the approximating functions. Then by the above limiting arguments (3.94) with $c = 1$ can be extended to all functions $f \in L_{\bar{p}}(\Omega)$ with a finite quasi-norm. The proof is complete.

REFERENCES

[1] Бесов, О. В., Ильин, В. П., и С. М. Никольский: Интегральные представления функций и теоремы вложения. Москва: Изд-во Наука 1975. Engl. transl.: Besov, O. V., Il'in, V. P., and S. M. Nikol'skij: Integral representations of functions and imbedding theorems. Scripta Series in Math. Washington, D. C.: V. H. Winston & Sons 1978/79.

[2] Bojarski, B.: Sharp maximal operator of fractional order and Sobolev imbedding inequalities. Bull. Polish Acad. Sci., Ser. Math., 33 (1985), 7–16.

[3] Брудный, Ю. А.: Пространства, определяемые с помощью локальных приближений. Труды Моск. мат. общества **24** (1971), 69–132.
[4] Calderón, A. P.: Estimates for singular integral operators in terms of maximal functions. Studia Math. **44** (1972), 563–582.
[5] Calderón, A. P., and R. Scott: Sobolev type inequalities for $p > 0$. Studia Math. **62** (1978), 75–92.
[6] Campanato, S.: Propriétà di inclusione per spazi di Morrey. Ricerche Mat. **12** (1963), 67–86.
[7] Campanato, S.: Propriétà di hölderianità di alcune classi di funzioni. Ann. Scuola Norm. Sup. Pisa **17** (1963), 175–188.
[8] Campanato, S.: Propriétà di una famiglia di spazi funzionali. Ann. Scuola Norm. Sup. Pisa **18** (1964), 137–160.
[9] Campanato, S.: Teoremi di interpolazione per transformazioni che applicano L^p in $C^{k,\alpha}$. Ann. Scuola Norm. Sup. Pisa **18** (1964), 345–360.
[10] Christ, M.: The extension problem for certain function spaces involving fractional orders of differentiability. Ark. Mat. **22** (1984), 63–81.
[11] DeVore, R. A., and R. C. Sharpley: Maximal functions measuring smoothness. Mem. Amer. Math. Soc. **293** (1984), 1–115.
[12] Dorronsoro, J. R.: A characterization of potential spaces. Proc. Amer. Math. Soc. **95** (1985), 21–31.
[13] Dorronsoro, J. R.: Poisson integrals of regular functions. Trans. Amer. Math. Soc. **297** (1986), 669–685.
[14] Dorronsoro, J. R.: On the differentiability of Lipschitz-Besov functions. Trans. Amer. Math. Soc. **303** (1987), 229–240.
[15] Franke, J.: On the spaces F^s_{pq} of Triebel-Lizorkin type: pointwise multipliers and spaces on domains. Math. Nachr. **125** (1986), 29–68.
[16] John, F., and L. Nirenberg: On functions of bounded mean oscillation. Comm. Pure Appl. Math. **14** (1961), 415–426.
[17] Jonsson, A., and H. Wallin: Function spaces on subsets of $\mathbf{R}^n$. Math. Reports **2** (1984), 1–221.
[18] Калябин, Г. А.: Теоремы о продолжении, мультипликаторах и диффеоморфизмах для обобщенных классов Соболева-Лиувилля в областях с липшицевой границей. Труды мат. и-та Стеклова **172** (1985), 173–185.
[19] Kaljabin, G. A., and P. I. Lizorkin: Spaces of functions of generalized smoothness. Math. Nachr. **133** (1987), 7–32.
[20] Kufner, A., John, O., and S. Fučik: Function spaces. Prague: Academia 1977.
[21] Meyers, G. N.: Mean oscillation over cubes and Hölder continuity. Proc. Amer. Math. Soc. **15** (1964), 717–721.
[22] Morrey, C. B.: On the solutions of quasi linear elliptic partial differential equations. Trans. Amer. Math. Soc. **43** (1938), 126–166.
[23] Peetre, J.: On the theory of $L_{p,\lambda}$ spaces. J. Funct. Anal. **4** (1969), 71–87.
[24] Seeger, A.: A note on Triebel-Lizorkin spaces. In: Approximation Theory and Function Spaces, Proc. Banach Center Semester, Spring 1986, Warsaw (to appear).
[25] Spanne, S.: Sur l'interpolation entre les espaces $L_k^{p\Phi}$. Ann. Scuola Norm. Sup. Pisa **20** (1966), 625–648.
[26] Stampacchia, G.: $L^{(p,\lambda)}$-spaces and interpolation. Comm. Pure Appl. Math. **17** (1964), 293–306.
[27] Stampacchia, G.: The spaces $L^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation. Ann. Scuola Norm. Sup. Pisa **19** (1965), 443–462.
[28] Triebel, H.: Theory of function spaces. Leipzig: Akad. Verlagsges. Geest & Portig 1983, and Boston: Birkhäuser 1983. (Russian translation, with an appendix by P. I. Lizorkin, Moskva: Mir 1986.)
[29] Triebel, H.: Characterizations of Besov-Hardy-Sobolev spaces: a unified approach. J. Approx. Theory **52** (1988), 162–203.
[30] Triebel, H.: Diffeomorphism properties and pointwise multipliers for function spaces. In: Function Spaces. Proc. Int. Conf. Poznań 1986 (Teubner-Texte zur Mathematik: Vol. 103). Leipzig: B. G. Teubner Verlagsges. 1988, p. 75–84.

[31] Triebel, H.: Characterizations of F^s_{pq} spaces via local means, the extension problem. Trudy mat. inst. Streklov (to appear).
[32] Triebel, H.: Interpolation theory, function spaces, differential operators. Berlin: Dt. Verlag Wiss. 1978, and Amsterdam: North-Holland 1978.
[33] Wallin, H.: New and old function spaces. In: Function Spaces and Applications. Lect. Notes Math. **1302** (1988), 97–114.

Manuskripteingang: 26. 05. 1988

VERFASSER:

Prof. Dr. Hans Triebel
Sektion Mathematik der Friedrich-Schiller-Universität
Universitätshochhaus
DDR-6900 Jena