Zeitschrift für Analysis und thre Anwendungen Bd. 8(4) 1989, S. 293-306

# **A. Symbol Calculus for the Algebra Generated by Shift Operators**

### S. ROCH and B. SILBERMANN

*Dedicated to S. G. Mikhlin on the occasion of his 801h birthday* 

Wir beschreiben die algebraische Struktur der durch den Verschiebungsoperator und einer seiner Linksinversen erzeugten Banaeh-Algebra und geben em Symbol für die Regularisierbarkeit eines Operators bezuglich des Quasikommutatorideals an. Die Beziehungen dieses Symbols zur Invertierbarkeit bzw. zu den Fredholm-Eigenschaften der Elemente dieser Algebra werden untersucht.

Описывается алгебраическая структура Банаховы алгебры, порождённой оператором сдвига и одного его левого обратного, и строится символ, с помощью которого дается. условие обратимости оператора относительно идеала полукоммутаторов. Исследуюутся соотношения между этим символом и обратимостью или нетеровостью элементов этой алгебры. Wir beschreiben die algebraische Strukeiner Linksinversen erzeugten Banach-<br>keit eines Operators bezüglich des Quasii<br>zur Invertierbarkeit bzw. zu den Fredho<br>untersucht.<br>Описывается алгебранческая структ<br>сдвига и одного е

We describe the algebraical structure of the algebra generated by the shift operator and by one of its left-inverses and construct a symbol for the invertibility of an operator modulo the quasicommutator ideal. The correspondence between this symbol and the invertibility and Fredholmness of elements, of this algebra are studied. COOTHOLIBUTER SCOTHOLIBUTER COOTHOLIBUTER SCOTHOLIBUTER AND SAFE OF THE COOTHOLIBUTER ON A SURVEY COOTHOLIBUTER OF THE COTTESPOND Fredholmness of elements of this algebra<br> **1.** Introduction<br> **I.** Introduction<br>
In 1936, S.

In 1936, S. G. MICHLIN [2, 3] was the first who created a symbol concept for twodimensional singular integral operators. Since that time the notion of symbols has gained an extraordinary significance in the theory of integral operators: It allows .to algebraize large classes of operators in such a manner that operations with operators can be transformed into operations with their symbols which leads to essential simplifications in their treatment. Important classes of operators possessing a natural symbol calculus are pseudodifferential operators of Fourier and Mellin type as well as convolution operators. -Meanwhile one has recognized that even the numerical, solution of certain convolution equations (as, e.g. singular integral equations) by projection, methods corresponds to the invertibility of special matrix- or operator-' 1. Introduction<br>
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In the present paper the authors raise a scheme due to I. Z. Goons examed I. A. FELDMAN [1] which refers to continuous functions of shift operators. By a shift we here mean an only one-sided' invertible operator *V* with the additional property that the spectrum of  $\check{V}$  and the spectrum of its one-sided inverse  $V_{-1}$  are both contained in the closed unit disk  $\{z: |z| \leq 1\}$ . To each operator *A* belonging to the closed algebra generated by  $V$  and  $V_{-1}$  we associate a complex-valued continuous function on the unit circle  $T -$  its symbol, and we examine the spectrum and the essential spectrum of the operator *A* in terms of the geometric behaviour of its symbol. Besides this we explain the algebraic structure of the algebra with generators  $V$  and  $V_{-1}$  and show  $that - in the best case - it decomposes into the direct sum of the lineal of all con$ tinuous functions of  $V$  and  $V_{-1}$  and of a certain ideal consisting of quasicommutators.

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### 2. Functions of shift operators

 

Throughout this paper let  $X$  be a Banach space with identity operator  $\overline{I}$  and let *V* denote a bounded linear operator on X which is only invertible from the left. We. fix one of its left inverses, say  $V_{-1}$ , and put for brevity tors<br>  $\mathbf{r} \times \mathbf{r}$  is a Banach<br>  $\mathbf{r}$  operator on  $X$  w<br>  $\mathbf{r}$ , say  $V_{-1}$ , and put<br>  $\mathbf{r} \times \mathbf{0}$ ,<br>  $\mathbf{r} \times \mathbf{0}$ .

$$
V_n = \begin{cases} V^n & \text{if } n \geq 0, \\ (V_{-1})^{-n} & \text{if } n < 0. \end{cases}
$$

For a given polynomial  $R(t) = \sum a_i t^i$  on the unit circle T we define an operator  $R(V)$  by  $R(V) = \sum_{i=1}^{n} a_i V_i$  and call  $R(V)$  a polynomial of *V.* Let  $L^0(V)$  stand for the set of all polynomials of V. Notice that there is a one-to-one correspondence between *• 1*  the operators in  $L^0(V)$  and the polynomials on T. Indeed, if  $0 = \sum a_i V_i$  and  $a_{-n} \neq 0$ , given  $\Gamma$ <br>by  $R(V)$ <br>all polyncerators is<br> $\Gamma \atop n \to \infty$ unded linear operator on X which is<br>eft inverses, say  $V_{-1}$ , and put for bre<br> $\begin{cases} V^n & \text{if } n \geq 0, \\ (V_{-1})^{-n} & \text{if } n < 0. \end{cases}$ <br>oolynomial  $R(t) = \sum_{j=-n}^n a_j t^j$  on the unit<br> $= \sum_{j=-n}^n a_j V_j$  and call  $R(V)$  a polynom<br>nomials  $\sum_{i=1}^{n} a_i V_i V_n = 0,$  and we find  $0 + a_{-n} I = V\left(-\sum_{i=1}^{n} a_i V_{i+n-1}\right)$ . This yields the right  $V_n = \begin{cases} V^n & \text{if } n \leq 0, \\ (V_{-1})^{-n} & \text{if } n < 0. \end{cases}$ <br> *For a given polynomial*  $R(t) = \sum_{i=0}^{n} a_i t^i$  on the unit circle **T** we<br>  $R(V)$  by  $R(V) = \sum_{j=-n}^{n} a_j V_j$  and call  $R(V)$  a polynomial of  $V$ . Let<br>
set of all polynomials invertibility of  $V$ , which contradicts our hypothesis. If  $R(V)$  is a polynomial with only positive powers of  $V$ , then the proof is similar.  $V_n = \begin{cases} V^n & \text{if } n \geq 0, \\ (V_{-1})^{-n} & \text{if } n < 0. \end{cases}$ <br>For a given polynomial  $R(t) = \sum_{i=-n}^{n} a_i t^i$  on the unit circ<br> $R(V)$  by  $R(V) = \sum_{j=-n}^{n} a_j V_j$  and call  $R(V)$  a polynomial of<br>set of all polynomials of *V*. Notice that the  $R(V)$  by  $R(V) = \sum_{i=1}^{n} a_i V_i$  and call  $R(V)$  a polynomial of  $V$ <br>set of all polynomials of  $V$ . Notice that there is a one-to-one<br>the operators in  $L^0(V)$  and the polynomials on T. Indeed, if<br>then  $\sum_{i=-n}^{n} a_i V_i V_n = 0$ , a *a*) e operators in  $L^0(V)$  and the polynomials on **T**. Indeed, if 0<br>  $\lim_{n \to \infty} a_i V, V_n = 0$ , and we find  $0 + a_{-n} I = V\left(-\sum_{j=-n+1}^{n} a_j V_{j+n-j}$ <br> *i*  $i = -n+1$ <br> *i*  $i = -n+1$ <br> *i a*) *i j eximility of <i>V*, which contradic

The following hypothesis (H) will figure prominently in deriving invertibility criinvertibility of  $V$ , which contradicts our hypothesis. If  $R(V)$  is a polynon<br>only positive powers of  $V$ , then the proof is similar.<br>The following hypothesis (H) will figure prominently in deriving invertit<br>teria for pol

1).

Here  $\sigma(\cdot)$  refers to the spectrum of a given operator in the Banach algebra  $\mathcal{L}(X)$  of

Theorem 1 [1: Chap. I, § 1.3]: *Let* (H) *be fulfilled. Then the following assertions hold.*

**b)** *An operator*  $R(V) \in L^{0}(V)$  *is at least one-sided invertible if the function*  $R(t)$  *has no zeros on*  $T \cdot If R(t) = 0$  *on*  $T$ *, then the invertibility of*  $R(V)$  *corresponds to the index of R(t), i.e. R is invertible, invertible only from the left or, only from the right if the winding number of*  $R(t)$ *,* bolynomials of  $V$ :<br>
pectra  $\sigma(V)$  and  $\sigma(V_{-1})$  of  $V$  and  $V_{-1}$  are contained in  $\{z \in \mathbb{C} : |z| \leq 1\}$ .<br>
refers to the spectrum of a given operator in the Banach algebra  $\mathcal{L}$ <br>
ed linear operators on  $X$ .<br>
em 1 Theorem 1 [1: Chap. I, § 1.3]: Let (H) be fulfilled. Then the following assertions<br>
hold.<br>
a)  $\sigma(V) = \sigma(V_{-1}) = \{z \in \mathbb{C} : |z| = 1\}$ .<br>
t b) An operator  $R(V) \in L^0(V)$  is at least one-sided invertible if the function  $R(t)$  has<br> b) An operator  $R(V) \in L^0$ <br> *xo zeros on* **T**. If  $R(t) \neq 0$  or<br> *ximpler of*  $R(t)$ , *i.e. R is invertible, inv*<br> *xumber of*  $R(t)$ ,<br>
wind  $R(t) := \frac{1}{2\pi}$  [an<br> *xs zero, positive, or negative, i*<br> *ximplerial pointing*

wind 
$$
R(t) := \frac{1}{2\pi} [\arg R(e^{i\alpha})]_{\alpha=0}^{2\pi}
$$
,

*is zero, positive, or negative, respectively.*<br>
c) If  $R(V)$  is one-sided invertible, then there exists a one-sided inverse of  $R(V)$  in the *algebra generated by V and*  $V_{-1}$ *. is zero, positive, or negative, respectively.*<br> *is zero, positive, or negative, respectively.*<br> *c) If*  $R(V)$  *is one-sided invertible, then there exists a one-sided inverse of*  $R(V)$  *idgebra generated by*  $V$  *and*

d). The spectral radius of  $R(V)$  equals  $\max |R(t)|$ .

*f*),  $If R(t_0) = 0$  for some  $t_0 \in \mathbf{T}$ , then R is neither a  $\Phi_+$ - nor a  $\Phi_-$ -operator.

Example 2.1 and the *Sate of Reprositive*, or negative, respectively.<br>  $\Gamma$ ) If  $R(V)$  is one-sided invertible, then there exists a one-sided inverse of  $R$ <br>  $\Gamma$  about *R(V)* is a  $\Phi$ -operator and if  $\kappa := \text{codim (im } V) < \infty$ Let  $L(V)$  stand for the closure of  $L^{o}(V)$  in  $\mathcal{L}(X)$ . The elements of  $L(V)$  will be called continuous functions of *V*. To each operator *R* in  $L(V)$  we can associate a continuous function  $R(t)$  on **T** – its symbol. In fact, by Theorem 1/d) we have f) If  $R(t_0) = 0$  for some  $t_0 \in \mathbb{T}$ , then R is neither a  $\Phi_+$ - nor a  $\Phi_-$ -operator.<br>This theorem justifies to speak about  $R(t)$  as the symbol of  $R(V)$ .<br>Let  $L(V)$  stand for the closure of  $L^0(V)$  in  $\mathcal{L}(X)$ . The e wind  $R(t) := \frac{1}{2\pi} \{ \arg R(e^x) \}_{x=0}^T$ ,<br>
sitive, or negative, respectively.<br>
(V) is one-sided invertible, then there exists a one-sided inverse of  $R(V)$  in the<br>
merated by V and  $V_{-1}$ ,<br>
spectral radius of  $R(V)$  equals  $\$ 

$$
\max \left\{ |R_n(t)| : t \in \mathbb{T} \right\} \leq ||R_n(V)||
$$

for each polynomial  $R_n(t)$ . If  $\{R_n\}$  denotes a sequence of polynomials converging to  $R \in L(V)$ , then, by (1), the sequence  $\{R_n(t)\}\$  converges uniformly to a certain con-

*1*

tinuous function  $R(t)$  which, moreover, does not depend on  $\{R_n\}$ . In Section 4 we shall extend this definition of symbols to a larger class of operators.

We emphasize once more that the symbol' concept for polynomials is distinguished by the following important aspects:

(i) An operator in  $L^0(V)$  is uniquely determined by its symbol.

(ii) The invertibility of an operator in  $L^0(V)$  depends only on its symbol. One might enquire whether these properties are passed on to  $L(V)$ . The following example which is due to A. Pomp (private communication) shows that at least the tinuous function  $R(t)$  wh<br>shall extend this definitie<br>We emphasize once mo<br>by the following importa<br>(i) An operator in  $L^0$ <br>(ii) The invertibility of<br>One might enquire whet<br>example which is due to<br>first of them does not. operator in  $L^0(V)$  depends only on its<br>these properties are passed on to  $L(\text{Pomp (private communication) show})$ <br> $L^0(\text{Pomp (private communication) show})$ <br> $L^1(\text{Rep}(X) \leq \text{Rep}(X))$ <br> $L^1(\text{Rep}(X)) = \lim_{k \to \infty} \frac{x_k}{k} + \sup_n |x_n - n \lim_{k \to \infty} \frac{x_k}{k}$ <br>makes E into a Banach space. Let<br>in

first of them does not. Example: Let *m* denote the Banach space of all bounded sequences of complex

and define  
\n
$$
E = \{(x_k) = \{ak + y_k\}_{k=1}^{\infty} \text{ with } a \in \mathbb{C}, \{y_k\}_{k=1}^{\infty} \in m\}.
$$
\n|v.

 

numbers and define

$$
E = \{(x_k) = (ak + y_k)_{k=1}^{\infty} \text{ with } a \in \mathbb{C}, \{y_k\}_{k=1}^{\infty} \in \mathbb{m}\}.
$$
  
\n
$$
E \ni x = \{x_k\}_{k=1}^{\infty} \mapsto ||x||_E := \left|\lim_{k \to \infty} \frac{x_k}{k}\right| + \sup_{n} \left|x_n - n \lim_{k \to \infty} \frac{x_k}{k}\right|
$$
  
\n*h* norm on *E* which makes *E* into a Banach space. Let *V* so *E* which are defined by  
\n
$$
V\{x_k\}_{k=1}^{\infty} = \{x_{k-1}\}_{k=1}^{\infty}, \quad x_0 = 0, \text{ and } V_{-1}\{x_k\}_{k=1}^{\infty} = V_{-1}V = I \text{ and } VV_{-1} \neq I. \text{ Moreover, it is easy to see that}
$$
\n
$$
V = \lim_{k \to \infty} \frac{1}{k} \sum_{k=1}^{n} \frac{1}{k} \sum
$$

defines a norm on *E* which makes *E* into a Banach space. Let *V* and *V\_ <sup>1</sup>* be the operators on  $E$  which are defined by

$$
V\{x_k\}_{k=1}^{\infty} = \{x_{k-1}\}_{k=1}^{\infty}, \quad x_0 = 0, \quad \text{and} \quad V_{-1}\{x_k\}_{k=1}^{\infty} = \{x_{k+1}\}_{k=1}^{\infty}.
$$

Clearly,  $V_{-1}V = I$  and  $VV_{-1} + I$ . Moreover, it is easy to see that the operators *V* and  $V_{-1}$  are bounded on *E* and that  $||V_n|| = |n| + 1$  for all  $n \in \mathbb{Z}$ . Consequently, the operators  $V, V_{-1}$  are subject of our hypothesis (H). Clearly,  $V_{-1}V = I$  and  $VV_{-1} + I$ . Moreover, it is easy to see that the operators  $V$  and  $V_{-1}$  are bounded on  $E$  and that  $||V_n|| = |n| + 1$  for all  $n \in \mathbb{Z}$ . Consequently, the operators  $V$ ,  $V_{-1}$  are subject of our  $V_{-1}$  and  $V_{-1}$  are boother and  $V_{-1}$  are boother and  $V_{-1}$  are both  $A + 0$ .

*•* 

Proof: Given  $x = \{x_k\} \in E$  with  $x_k/k \to a$  we define the operator *A* by  $Ax$ Proposition 1: There exists an operator  $A \in L(V)$  the symbol of which<br>  $a_0$  but  $A + 0$ .<br>
Proof: Given  $x = \{x_k\} \in E$  with  $x_k/k \to a$  we define the operat<br>  $a\{1, 1, 1, \ldots\} \in E$ . Evidently, *A* is bounded on *E* and  $||A|| =$ <br> 1. We claim **Example 1. I** *linting call is a <i>n b*-*p a n i c*<sub>*k***</sub>** *k*  $\rightarrow$  *a* **w** define the operator  $\therefore$   $= a\{1, 1, 1, \ldots\} \in E$ . Evidently, *A* is bounded on *E* and  $||A|| = 1$ . **V** that  $||1/n$   $V_{-n} - A|| \rightarrow 0$  as  $n$ that  $||1/n V_{-n} - A|| \to 0$  as  $n \to \infty$ : If  $x = \{ak + y_k\} \in E$ , then berators on *E* which are defined by<br>  $V\{x_k\}_{k=1}^{\infty} = \{x_{k-1}\}_{k=1}^{\infty}$ ,  $x_0 = 0$ <br>
Clearly,  $V_{-1}V = I$  and  $VV_{-1} + I$ . Morec<br>
and  $V_{-1}$  are bounded on *E* and that  $||V_n||$ <br>
operators *V*,  $V_{-1}$  are subject of our hy *Your, it is eas*<br>  $y = |n| + 1$  for thesis (H).<br>  $y \cdot A \in L(V)$  to  $k \to a$  we define the set of  $k \to a$  we define the set of  $k + y_k$ <br>  $\frac{y_{k+n}}{n} - a \Big|_{k=1}^{\infty}$ ce. Let V and  $V_{-1}$  be<br>  $x_k|_{k=1}^{\infty} = \{x_{k+1}\}_{k=1}^{\infty}$ .<br>
see that the operato<br>  $n \in \mathbb{Z}$ . Consequently<br>
mbol of which is identically<br>  $\therefore$  the operator A by<br>
and  $||A|| = 1$ . We c<br>
then<br>  $\frac{1}{n} \{ka + y_{k+n}\}_{k=1}^{\infty}$ *i V*,  $V_{-1}$  are subject of our<br>
sition 1: There exists an op<br>  $A = 0$ .<br> *i* Given  $x = \{x_k\} \in E$  with<br> *i*, ...}  $\in E$ . Evidently, *A*<br> *v*  $V_{-n} - A \mid$   $\to 0$  as  $n \to \infty$ :<br>  $\left(\frac{1}{n} V_{-n} - A\right) x = \left\{\frac{k+n}{n} \right\}$ by<br>  $\begin{align*}\n\dot{v}_0 &= 0, & \text{if } v_0 = 0, \\
\text{Moreover, if } ||V_n|| &= |n|, \\
\text{bypothesis} \\
\text{operator } A &\infty \\
\text{operator } B &\infty \\
\text{Hence, if } x = \{ak\} \\
\text{Hence, if } x = 0, \\
\text{Hence, if } x$ *J* as easy to see<br> *J* for all  $n \in V$ <br> *V )* the symbol<br> *V* be symbol<br> *V* define the<br>
on *E* and<br>  $y_k$   $\in E$ , then<br>  $\begin{cases} \infty \\ k=1 \end{cases} = \frac{1}{n}$  {ke *y* to see that the operators  $\vec{v}$ <br> *r* all  $n \in \mathbb{Z}$ . Consequently, the<br> *ie symbol of which is identically*<br>
efine the operator  $\vec{A}$  by  $A\vec{x}$ <br> *E* and  $||A|| = 1$ . We claim<br>  $\in E$ , then<br>  $= \frac{1}{n} \{ka + y_{k+n}\}_{k=1}^{\$ Given  $x = \{x_k\} \in E$  with  $x_k/k \to a$ <br>  $\left|\ldots\right| \in E$ . Evidently, *A* is bounde<br>  $\left|\frac{V_{-n}}{h} - A\right| \to 0$  as  $n \to \infty$ : If  $x = \{ak : \frac{1}{n} \quad V_{-n} - A\}$ <br>  $\frac{1}{n} \left|\frac{V_{-n}}{h} - A\right| = \frac{1}{n} \left|\frac{1}{n} \left|\frac{1}{h} \times \frac{1}{h} - A\right| \to 0\}$ <br>

$$
\left(\frac{1}{n} V_{-n} - A\right) x = \left\{\frac{k+n}{n} a + \frac{y_{k+n}}{n} - a\right\}_{k=1}^{\infty} = \frac{1}{n} \left\{ka + y_{k+n}\right\}_{k=1}^{\infty}.
$$

Hence,

$$
\left|\frac{1}{n}V_{-n} - A\right| x = \left|\frac{1}{n}a + \frac{x - a}{n} - a\right|_{k=1} = \frac{1}{n} \{k\}
$$

$$
\left\|\left(\frac{1}{n}V_{-n} - A\right)x\right\| = \frac{1}{n} \{ |a| + \sup_{k} |y_{k+n}|\} \le \frac{1}{n} \|x\|_{k}.
$$

Consequently, *A* is in  $L(V)$ ,  $A = 0$ , but the symbol of *A* is identically zero since the symbols of  $1/n$   $V_{-n}$  converge uniformly to zero  $\blacksquare$ .

In what follows, we shall only deal, with the problem of the invertibility of functions of shift operators. Concerning the unique determinationof an operator by its symbol we refer to [5], where, among other things, the following is proved. symbol of  $A$  is identically zero since the<br>
problem of the invertibility of functions<br>
ermination of an operator by its symbol<br>
the following is proved.<br> *If one of the conditions*<br>  $V_n V_{-n} || < \infty$ <br>
quely determined by it

*Theorem 2: A8süme that fl* im *V,, = {0}. I/ one of the conditions*  no ' a)closU <sup>k</sup>e<sup>r</sup> *V\_, =X,''* b) **SUP** *lI* VV\_ *]I <sup>&</sup>lt;*c nO , S

*is fulfilled, then every operator in*  $L(V)$  *is uniquely determined by its symbol.* 

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*\ 0*

### 3.  $V<sub>z</sub>$  dominating algebras

 

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Our next concern is to extend the assertions of Theorem 1 to a subset of  $L(V)$  as large as possible. Again, let *V* be an operator on X which is only invertible from the left and let  $V_{-1}$  be one of its left inverses. We do not assume the hypothesis (H) to be fulfilled. A commutative Banach algebra *D* with unit element *e* is called  $(V, V_{-1})$ -296 S. Rocu and B. SILBERMANN<br> **3.** *V*<sub>-</sub>dominating algebras<br>
Our next concern is to extend the assertions of Theor<br>
large as possible. Again, let *V* be an operator on *X* which<br>
left and let  $V_{-1}$  be one of its left 3. *V*<sub>-</sub>dominating algebras<br>
Our next concern is to extend the assertions<br>
large as possible. Again, let *V* be an operator<br>
left and let  $V_{-1}$  be one of its left inverses. We<br>
fulfilled. A commutative Banach algebra

a) there is an invertible element *d* in *D* which spans together with its inverse  $d^{-1}$  a dense subalgebra of *D,* 

b) the spectra  $\sigma_D(d)$  and  $\sigma_D(d^{-1})$  of *d* and  $d^{-1}$  in *D* are contained in  $\{z \in \mathbb{C} : |z| \leq 1\}$ , c) there exists a constant  $M > 0$  such that

$$
||P(V)||_{\mathcal{L}(X)} \leq M ||P(d)||_D
$$

 $t|^{'} = 1.$ 

Proposition 2: *If a*  $(V, V_{-1})$ -dominating algebra *D* exists, then *V* and  $V_{-1}$  are *subject of the hypothesis (H). More general, if P is a polynomial on* T, *then we have for the spectral radii*  $\varrho_{\mathcal{X}(X)}(P(V))$  *and*  $\varrho_{D}(P(d))$  *that ihe hypothesis* (H). Mo<br>*il radii*  $\rho_{X(X)}(P(V))$  and<br> $\rho_{X(X)}(P(V)) \leq \rho_P(P(d))$ 

$$
\varrho_{\mathcal{I}(X)}(P(V)) \leq \varrho_{\mathcal{D}}(P(d)). \tag{3}
$$

the spectral radii  $\rho_{X(X)}(P(V))$  and  $\rho_D(P(d))$  that<br>  $\rho_{X(X)}(P(V)) \leq \rho_D(P(d))$ . (3)<br>
Proof: It is sufficient to verify the estimation (3):  $B_Y'(2)$ ,  $\|P(V)^n\|_{X(X)} \leq M \|P(d)^n\|_D$ <br>  $(n \in \mathbb{Z}^+)$ . Thus,  $\|P(V)^n\|_1^{1/n} \leq M^{1/n} \|P(d)^n\$  $M^{1/n} ||P(d)^n||^{1/n}$ , and passing throught the limit yields the assertion  $\blacksquare$ 

Proposition 3: The maximal ideal space  $M(D)$  of D is homeomorphic to the unit *circle*  $\mathbf{T}$ *, and the Gelfand transform maps d into the function*  $t \mapsto t$  *(* $t \in \mathbf{T}$ *):* " 

• Proof: We shall show that the spectrum of *d* equals T. The remaining assertions follow immediately from the general theory of commutative Banach, algebras. By b) the spectrum  $\sigma_p(d)$  of *d* in *D* belongs to T. Assume that  $\sigma_p(d) = T$ . Then there is an inner point *z* of  $T \setminus \sigma_D(d)$ , and we can find a continuous *f* on **T** such that  $0 \leq f(t) \leq 1$  $(t \in \mathbf{T})$ ,  $f(z) = 1$  and  $f(t) = 0$  for  $t \in \sigma_D(d)$ . Given  $\varepsilon > 0$  we choose a polynomial b) the spectrum  $\sigma_D(d)$  of d in *D*<br>inner point z of  $T \setminus \sigma_D(d)$ , and<br> $(t \in T)$ ,  $f(z) = 1$  and  $f(t) = 0$ <br> $p \in C(T)$  so that max  $|f(t) - t|$  $p \in C(T)$  so that max  $|f(t) - p(t)| < \varepsilon$ . Since  $|p(t)| < \varepsilon$  for all  $t \in \sigma_p(d)$ , we obtain **leT**   $\rho_D(p(d)) \leq \varepsilon$ . On the other hand, by Theorem 1/d) and by Proposition 2 we have From the issue of the contract of the contract of  $(n \in \mathbb{Z}^+ )$ . Thus,  $||P(V)^n||^{1/n} \leq 1$ ,<br>the assertion  $\blacksquare$ <br>Proposition 3: The maximative circle T, and the Gelfand transf<br>follow immediately from the<br>b) the spectrum  $\$ 

 $\varrho_D(p(a)) \geq \varepsilon$ . On the other hand, by Theorem 1/a) and by Troposhion 2 we have  $\varrho_{T(X)}(p(V)) > 1 - \varepsilon$ . These two inequalities contradict the hypothesis (2) for  $\varepsilon$  suf-<br>ficiently small **i**<br>Since  $p(d)$  is uniquely deter Since  $p(d)$  is uniquely determined by its Gelfand transform' $p(t)$  (even in the case that D has a non-trivial radical), the mapping  $p(d) \mapsto p(V)$  is well-defined, linear, and by (2) bounded. Hence, we can extend this mapping continuously to the whole algebra  $D$  and its image, abbreviated to  $L_p(V)$ , is contained in  $L(V)$ . For  $a \in D$  let  $\varphi \in C(\mathbf{T})$  so that  $\max |f(t) - p(t)| < \varepsilon$ . Since  $|p(t)| < \varepsilon$  for all  $t \in \sigma_D(d)$ , we obtain  $\varrho_D(p(d)) \leq \varepsilon$ . On the other hand, by Theorem 1/d) and by Proposition 2 we have  $\varrho_{\mathbf{T}(X)}(p(V)) > 1 - \varepsilon$ . These two inequalities  $a(V)$  denote the image element of  $a$  under this mapping. Notice that If  $\sum_{t \in \mathbf{T}} |f(t) - p(t)| < \varepsilon$ .<br>  $\leq \varepsilon$ . On the other hand, by T<br>  $\sum_{t \in \mathbf{T}} |f(t) - p(t)| < \varepsilon$ .<br>  $\leq \varepsilon$ . On the other hand, by T<br>  $\sum_{t \in \mathbf{T}} |f(t) - f(t)|$ <br>  $\leq \varepsilon$ . These two inequal<br>
small  $\blacksquare$ <br>  $\blacksquare$ <br>  $\blacksquare$ <br>  $|p(t)| < \varepsilon$  for all  $t \in$ <br>em 1/d) and by Prope<br>contradict the hypotl<br> $\therefore$ <br> $\therefore$  Selfand transform' $p(t)$ <br> $\therefore$ <br> $p(d) \mapsto p(V)$  is we<br>this mapping continuation.<br> $(V)$ , is contained in  $L$ <br>is mapping. Notice the<br>coincides with t *d*  $f(x)(p(v)) > 1 - e$ . These two *n* ficiently small **g**<br>
Since  $p(d)$  is uniquely determin<br>
that *D* has a non-trivial radical),<br>
and by (2) bounded. Hence, we calgebra *D* and its image, abbrevia<br>  $a(V)$  denote the image elem

$$
||a(V)||_{\mathcal{I}(X)} \leq M ||a||_D \tag{4}
$$

for all  $a \in D$  and that the symbol of  $a(V)$  coincides with the Gelfand transform of  $a \in D$ . If the radical of *D* is trivial, then each element of  $L_p(V)$  is uniquely determined by its symbol even if this is unknown for arbitrary elements in  $L(V)$ . and by (2) bounded. Hence, we can extend on  $L_p(V)$ , is contained in  $L(V)$ . For  $a \in D$  let  $a(V)$  denote the image element of a under this mapping. Notice that  $||a(V)||_{\mathcal{I}(X)} \leq M ||a||_D$  (4)<br>for all  $a \in D$  and that the symbol

(2)

**0**

a) *An operator*  $R \in L_p(V)$  *is at least one-sided invertible if*  $R(t) \neq 0$  *for all*  $t \in T$ . *If the symbol*  $R(t)$  *of R does not vanish on*  $T$ , *then the invertibility of R corresponds to the*. *winding number of R(t).* 

*b) If*  $R(t_0) = 0$  *for some*  $t_0 \in \mathbb{T}$ , *then R is neither*  $a \Phi_{+}$  *nor*  $a \Phi_{-}$ *operator.* 

Proof: a) Let  $R \in L_p(V)$  and  $R(t) \neq 0$  ( $t \in T$ ). Then there is an element  $a \in T$ with  $a(V) = R$ , and a must be invertible since  $R(t)$  is the Gelfand transform of a. Thus, we can find a polynomial  $p(t) = \sum a_j t^j$   $(t \in T)$  such that  $r = p(d)$  is invertible with  $a(V) = R$ , and *a* must be invertible since  $R(t)$  is the Gelfand transform of *a*.<br>Thus, we can find a polynomial  $p(t) = \sum_{i=1}^{n} a_i t^i$   $(t \in T)$  such that  $r = p(d)$  is invertible<br>and  $r^{-1}a = e + c$  with  $||c||_D < 1/M$ . As in th [1: Chap. I, § 1.3]) there is a representation of *r* in the form  $r = r_d * r_t$ , where  $r_t$  and  $r_-$  are polynomials in *d* and  $d^{-1}$  with only non-negative and non-positive exponents, and  $r^{-1}a = e + c$  with  $||c||_D < 1/M$ . As in the proof of Theorem 1 (see Theorem 1.1 in [1: Chap. I, § 1.3]) there is a representation of r in the form  $r = r_d * r_+$  where  $r_+$  and  $r_-$  are polynomials in d and  $d^{-1}$  with only  $e + c$  with  $||c||_D < 1/M$ . As in<br>
§ 1.3]) there is a representation<br>
comials in d and  $d^{-1}$  with only<br>  $f$ , and  $r_{\pm}(V) \in L^0(V)$  are inver<br>  $= \begin{cases} r_-d^*(e + c) \, r_+ & \text{if } \varkappa \leq 0, \\ r_-(e + c) \, d^*r_+ & \text{if } \varkappa > 0, \end{cases}$ *a* - *a* -

Then

ely, and 
$$
r_{\pm}(V) \in L^0(V)
$$
 are invertible. Now write\n
$$
a = \begin{cases} r_- d^*(e + c) \ r_+ & \text{if } x \leq 0, \\ r_- (e + c) \, d^* r_+ & \text{if } x > 0, \end{cases}
$$
\n
$$
a(V) = \begin{cases} r_-(V) \ V_x(I + c(V)) \ r_+(V) & \text{if } x \leq 0, \\ r_-(V) \ (I + c(V)) \ V_{x} r_+(V) & \text{if } x > 0. \end{cases}
$$
\n
$$
V \parallel \lambda \leq 1, \text{ the element } I + c(V) \text{ is invertible, and}
$$

Since  $||c(V)|| < 1$ , the element  $I + c(V)$  is invertible, and we are done.

b) Assume  $R(t_0) = 0$ , R is  $\Phi_+$ . Then there is a  $\delta > 0$  such that  $||R - r|| < \delta$  implies that *r* is  $\Phi_+$ , too. Now take  $r \in L^0(V)$  so that  $||R - r|| < \delta/2$ . Because  $|r(t_0)| < \delta/2$ and  $||R - (r - r(t_0) I)|| < \delta$  the operator  $r - r(t_0) I \in L^p(V)$  is  $\Phi_+$ . But this contradicts Theorem 1/f), since  $r(t) - r(t_0)$  vanishes at  $t = t_0$ . The case that R is  $\Phi$  can be treated analogously  $\blacksquare$  $f(V) = \begin{cases} r_{-}(V) V_{*}(I + c(V)) r_{+}(V) & \text{if } x \leq 0, \\ r_{-}(V) (I + c(V)) V_{*}r_{+}(V) & \text{if } x > 0. \end{cases}$ <br>  $|||V_{*}(I)| = 0$ ,  $R$  is  $\Phi_{+}$ . Then there is a  $\delta > 0$  such that  $||R - r|| < \delta$  implies  $P_{+}$ , too. Now take  $r \in L^{q}(V)$  so that  $||R - r|| < \delta$ *PUP*  $\{V_{-1}, V_{-1} \mid V_{-1}, V_{+1} \mid V_{-1}, V_{+1} \mid V_{-1}, V_{+1} \mid V_{-1}, V_{-1} \mid V_{-1} \}$ <br> *PUP*  $\{U_{0}\} = 0$ *,*  $R$  *is*  $\Phi_{+}$ *. Then there is a*  $\delta > 0$  *such that*  $||R - r|| < \delta$  *implies<br>*  $\Phi_{+}$ *, too. Now take*  $r \in L^{0}(V)$  *so that ||R - r|| < \delta/* **Drawing**  $P(|V|| \le 1,$  **the effective**  $I + c(V)$  **is more than**  $V + c(V)$  **is the case.**<br>  $|N \log U| \le 1$ , the eigenvector  $\mathcal{L}(V)$  and  $||R - r|| \le \delta$  is  $\mathcal{L}(V_0)| \le \delta/2$ <br>
and  $||R - (r - r(t_0) I)|| \le \delta$  the operator  $r - r(t_0) I \in L^0(V)$  is  $\Phi_$ 

Now we are going to mention two examples of V-dominating algebras.

Example 1 (cp. [1: Chap. I, § 3.2]): Let *Y* be a Banach space with identity operator *I.* Assume that there are given a bounded projection operator *P* on *Y* and an invertible operator  $U \in \mathcal{L}(Y)$  such that



Let D stand for the smallest closed subalgebra of  $\mathcal{L}(Y)$  containing U and  $U^{-1}$ .

Theorem 4: a) *D* is a commutative Banach algebra with a maximal ideal space *homeomorphic .to* T.

*b) The operator PUP*  $\vert_{\text{im } P}$  *is invertible only from the left and PU<sup>-1</sup>P*  $\vert_{\text{im } P}$  *is one of its left inverses* 

Proof: First we verify b). Obviously,  $PU^{-1}PUP = P$ . Assume that  $PUPAP = P$  with some  $A \in \mathcal{L}(Y)$ . By (6),  $UPAP = P$  and  $PAP = U^{-1}P$ . Thus we get  $PUPU^{-1}P$ *b The operator PUP*  $\vert_{\text{Im } P}$  *is invertible only from the left and PU<sup>-1P</sup>*  $\vert_{\text{im } P}$  *<i>is one of its left inverses.*<br>
c) *D is (PUP*  $\vert_{\text{im } P}$ , *PU*<sup>-1</sup>*P*  $\vert_{\text{im } P}$ )-dominating.<br>
Proof: First we verify b For a proof of c) note that if  $p(t)$  is a polynomial on **T**, then, by (6),  $p(PUP) = Pp(U)P$  whence  $||p(PUP)|| \leq ||P||^2 ||p(U)||$ . Now part a) follows immediately from Propowith some  $A \in \mathcal{L}(Y)$ . By (6),  $UPAP = P$  and  $PAP = U^{-1}P$ . Thus we get  $PUPU^{-1}P$ <br>  $= P$ . Again by (6) this leads to  $UPU^{-1} = P$  and  $UP = PU$  which contradicts (7).<br>
For a proof of c) note that if  $p(t)$  is a polynomial on T, then, b **Proof:** First we verify b). Obviously,  $PU^{-1}PUP = P$ . Assume that  $PUPAP = P$  with some  $A \in \mathcal{L}(Y)$ . By (6),  $UPAP = P$  and  $PAP = U^{-1}P$ . Thus we get  $PUPU^{-1}P = P$ . Again by (6) this leads to  $UPU^{-1} = P$  and  $UP = PU$  which contradicts (7). Froot: First we verify 0). Obviously,  $PO^{-T}OP = P$ . Assume that  $PORAP$ <br>with some  $A \in \mathcal{L}(Y)$ . By (6),  $UPAP = P$  and  $PAP = U^{-1}P$ . Thus we get  $PUPU^{-1}$ <br>= *P*. Again by (6) this leads to  $UPU^{-1} = P$  and  $UP = PU$  which contradicts (<br>For

Example 2: A sequence  $f = {f_k}_{k=-\infty}^{\infty}$  of positive real numbers is called a *weight* if

S. RocH and B. SILBERMANN  
\nExample 2: A sequence 
$$
f = \{f_k\}_{k=-\infty}^{\infty}
$$
 of positive real numbers is called a *weight* if  
\n
$$
\lim_{n\to\infty} \sqrt[n]{f_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{f-n}} = 1
$$
\n(8)

/

*(* 

and if  $f^* = \sup_{k,n} f_{n+k}/f_n f_k < \infty$ . By  $W(f)$  we denote the collection of all complexvalued functions *a* on **T** the Fourier coefficients  $a_k$  of which satisfy  $\sum_{k=-\infty}^{\infty} |a_k| f_k < \infty$ , and put 298 S. Roch and<br>
Example 2: A i<br>  $\lim_{n\to\infty} \sqrt[n]{f_n} =$ <br>
and if  $f^* = \sup_{k,n} f_k$ <br>
valued functions a<br>
and put<br>  $||a||_{W(f)} :=$ <br>
Theorem 5: a) *•*  3. Roch and B. SILBERMANN<br>
plo 2: A sequence  $f = |f_k|_{k=-\infty}^{\infty}$  of positive real numbers is called<br>  $\lim_{n\to\infty} \sqrt[n]{f_n} = \lim_{n\to\infty} \sqrt[n]{f_{-n}} = 1$ <br>  $=\sup_{k,n} \int_{n+k} |f_n f_k < \infty$ . By  $W(f)$  we denote the collection of all<br>
inctions *• •*

$$
||a||_{W(f)} := f^* \sum_{k=-\infty} |a_k| f_k.
$$

Theorem 5: a) *The set W(f) forms a commutative Banach algebra under the norm*   $(9)$  whose maximal ideal space is homeomorphic to **.** 

b) The operator  $d: a(t) \mapsto ta(t)$  spans together with its inverse a dense subalgebra. *Of. W(f).*

*c) If V* is a shift operator, then the algebra  $W(||V_n||)_{n=-\infty}^{\infty}$  is  $(V, V_{-1})$ -dominating.

The proof follows from Proposition 3 if one takes into account (3) and the simple of  $W(f)$ .<br>
c) If V is a shift operator, then the algebra  $W(\{\}\)$ <br>
The proof follows from Proposition 3 if one to<br>
estimation  $||d^n a|| = ||t^n a(t)|| = f^* \sum |a_k| f_{k+n} \leq ||d^n|| \leq f^* f_n$  $(f^*)^2 f_n \sum |a_k| f_k$  which leads to  $\begin{align} \text{c) } \widetilde{I}_f \ \text{c} \ \text{d} \|\widetilde{d}^n\| \leq \end{align}$ 10n<br>*f\*f*。∎

# **4. The** algebra generated by *V* and *V\_1*

Our next objective is to study the smallest closed subalgebra of  $\mathcal{L}(X)$  containing  $\bar{V}$ our next objective is to study the smallest closed subalgebra of  $\mathcal{L}(X)$  containing  $V$ <br>and  $V_{-1}$ . Denote by algo  $(V, V_{-1})$  the (non-closed) subalgebra of  $\mathcal{L}(X)$  generated by<br>V and  $V_{-1}$  and by alg  $(V, V_{-1})$  it two-sided ideal of alg<sup>o</sup>  $(V, V_{-1})$  which contains all quasicommutator operators of the form.  $(R_1R_2)$  (V)  $\tilde{R}_1$ (V)  $R_2$ (V) where  $R_1$ ,  $R_2$  are arbitrary polynomials on T. The closure  $QC(V)$  of  $QC^{o}(V)$  in alg  $(V, V_{-1})$  is called the *quasicommutator ideal* of alg  $(V, V_{-1})$  $V_{-1}$ ). Henceforth, the quasicommutator  $I - V_n V_{-n}$  of the operators  $V_n$  and  $V_{-n}$ . will be denoted by  $P_n$   $(n \in \mathbb{Z}^+)$ , and we put  $Q_n = I - P_n$ . Obviously,  $P_n$  and  $Q_n$ are projection operators on  $\mathcal{L}(X)$ . In what follows we are mainly interested in the algebraic structure of alg  $(V, V_{-1})$  and in whether an invertible operator is invertible in this algebra. **4.** The algebra generated by  $V$  and  $V_{-1}$ . Denote by algo  $(V, V_{-1})$ <br>  $V$  and  $V_{-1}$  and by algo  $(V, V_{-1})$  its<br> *two*-sided ideal of algo  $(V, V_{-1})$  w<br>
form.  $(R_1R_2)(V) - R_1(V) R_2(V)$ <br>
closure  $QC(V)$  of  $QC^0(V)$  in algo  $(V_{$ *•* In *V*<sub>-1</sub> *behove by ang*  $(V, V_{-1})$  *ine* (non-closed)<br> *V* and *V*<sub>-1</sub> and by alg  $(V, V_{-1})$  is closure. Further<br> *two*-sidedideal of alg<sup>o</sup>  $(V, V_{-1})$  is closure. Further<br> *two*-sidedideal of alg<sup>o</sup>  $(V, V_{-1})$  which c *V* and  $V$ ,  $Y_{-1}$  *V* and  $V$  and  $V_{-n}$  of the operators  $V_n$ ,  $(n \in \mathbb{Z}^+)$ , and we put  $Q_n = I - P_n$ . Obviously,  $P_n$ <br>so on  $\mathcal{L}(X)$ . In what follows we are mainly interested  $g(V, V_{-1})$  and in whether an invertible o

Proposition 4: *The following conditions are equivalent for*  $K \in \text{alg } (V, V_{-1})$ :

this algebra.<br>
Proposition 4: The following conditions are equivalent for  $K \in \text{alg } (V, V_{-1})$ :<br>
a)  $K \in QC(V)$ .<br>
b)  $K$  belongs to the smallest closed ideal of alg  $(V, V_{-1})$  containing  $P_1$ .<br>
c)  $If \sup_n ||Q_n|| =: M < \infty$ , then a) is

Proof: a)  $\Rightarrow$  b): The quasicommutator ideal is generated by all operators  $(R_1R_2)(V)$  $- R_1(V) R_2(V)$  where  $R_1(V)$  and  $R_2(V)$  run through  $L^0(V)$ . Since  $V_{i+j} - V_i V_j$  $= P_i V_{i+j}$  and  $P_i$  $i \rightarrow$ <br>The<br>ere<br> $i-1$ <br> $\sum_{j=0}$ <br>a cu *D*) is denoted by  $P_n$  ( $n \in \mathbb{Z}^n$ ), and we put  $Q_n = I - P_n$ . Obviously,  $P_n$  and  $Q_n$ <br>gebraic structure of alg  $(V, V_{-1})$  and in whether an invertible operator is invertible<br>this algebra.<br>Proposition 4: *The following co* Proposition 4: The following conditions a:<br>
a)  $K \in QC(V)$ .<br>
b) *K* belongs to the smallest closed ideal of all<br>
c) If sup<sub>n</sub>,  $||Q_n|| = : M < \infty$ , then a) is equiva<br>
and  $||KQ_n|| \to 0$  as  $n \to \infty$ .<br>
Proof: a)  $\Rightarrow$  b): The quasicommu **•** (b) *K e QC*(*V*).<br> **b**) *K belongs to the smallest closed ideal of alg* (*V*,  $V_{-1}$ ) *contains* (*o If* supp.  $\|\varphi_n\| = : M < \infty$ , then a) is equivalent to each of the f  $\int$  and  $\|\mathcal{K}Q_n\| = 0$  as  $n \to \infty$ 

b)  $\Rightarrow$  c): We only prove that b) implies  $\|\mathcal{Q}_n K\| \rightarrow 0$ . First we show that  $\|\mathcal{Q}_n A P_1\| \rightarrow 0$  as  $n \rightarrow \infty$  if *A* is in alg  $(V, V_{-1})$ . Given  $\varepsilon > 0$  write  $A = A_{\epsilon} + (A - A_{\epsilon})$  with  $\|\tilde{A} - A_{\epsilon}\| < \varepsilon$  and  $A_{\epsilon}$  a  $P_i = P_i V_{i+j}$  and  $P_i = \sum_{j=0}^{i-1} V_j P_1 V_{-j}$ , the inclusion follows.<br>
b)  $\Rightarrow$  a):  $P_1$  is the quasicommutator of  $V$  and  $V_{-1}$ .<br>
b)  $\Rightarrow$  c): We only prove that b) implies  $||Q_n K|| \rightarrow 0$ . First we show that  $||Q_n A P_1|| \rightarrow 0$ <br>

$$
[Q_n V_{i_1} V_{i_2} \dots V_{i_k} P_1 = Q_n P_{i_1 + i_2 + \dots + i_k} V_{i_1} V_{i_1} \dots V_{i_k}
$$

Symbol Calculus for an Algebra 299<br>  $(i_1, i_2, \ldots, i_k \in \mathbb{Z})$  which is zero for  $i_1 + i_2 + \cdots + i_k + 1 < n$ . This yields that,<br>
given  $\varepsilon > 0$ , we can find an  $n_0$  such that  $||Q_nAP_1|| < \varepsilon$  if only  $n > n_0$ . Now let  $K \in QC(V)$ . The  $=\sum_{i=1}^{n} B_i P_1 C_j$  (where  $B_j$ ,  $C_j \in \text{alg } (V, V_{-1})$ ). By what has already been proved each  $C(V)$ . Then we write  $K = K_{\epsilon} + (K - K_{\epsilon})$  with  $||K - K_{\epsilon}|| < \epsilon$  and with  $K_{\epsilon}$ .<br>  $\sum_{j=1}^{r} B_j P_1 C_j$  (where  $B_j$ ,  $C_j \in \text{alg } (V, V_{-1})$ ). By what has already been proved each  $m$  of  $Q_n K_{\epsilon}$  converges to zero, and we are don *c*)  $\Rightarrow$  a): If  $||Q_nK|| \rightarrow 0$  as  $n \rightarrow \infty$ , then K is the uniform limit of the operators  $P_nK$  which are in  $QC(V)$ Symbol Calculus for<br> *(i<sub>1</sub>, i<sub>2</sub>, ..., i<sub>k</sub>*  $\in \mathbb{Z}$ ) which is zero for  $i_1 + i_2 + \cdots + i_k + 1 < n$ .<br>
given  $\varepsilon > 0$ , we can find an  $n_0$  such that  $||Q_nAP_1|| < \varepsilon$  if only  $n \in QC(V)$ . Then we write  $K = K_{\epsilon} + (K - K_{\epsilon})$  with  $||K$ 

*Corollary 1: If codim (im*  $V$ *)*  $<$  $\infty$ *, then,*  $QC(V)$  *consists only of compact operators. If, moreover, codim (im V) = 1 and if {P<sub>n</sub>} and {P<sub>n</sub><sup>\*</sup>} converge strongly to the identity operator on*  $X$  and  $X^*$ , respectively, then  $QC(V)$  equals the ideal of all compact operators

Proof: The first assertion is obvious from Proposition 4/b) since codim' (im  $V$ ) <  $\infty$ implies dim (im  $P_1$ ) <  $\infty$ . Now let codim (im  $V$ ) = 1. Then each operator  $P_n - P_{n-1}$ has rank 1. Consequently, for each linear bounded operator *A* on  $\overline{X}$  we can find con-<br>
stants  $a_{ij} \in C$  so that<br>  $P_n A P_n = \sum_{j=1}^n (P_j - P_{j-1}) A(P_i - P_{i-1}) = \sum_{i,j=1}^n a_{ij} V_{j-1} P_1 V_{-i+1}$ . (10)<br>
Hence  $P_A P$  belongs to  $O C$ stants  $a_{ij} \in \mathbb{C}$  so that *PAP*  $P_1$   $\rightarrow$  *Ps*  $\rightarrow$  *Ps I*(*I*)  $I: I \in \text{codim } W$  = 1 and if  $\{P_n\}$  and  $\{P_n^*\}$  converge strongly to the identity operator on  $X$  and  $X^*$ , respectively, then  $QC(V)$  equals the ideal of all compact operators on  $X$ .<br>
Proof: The first assertio

$$
P_n A P_n = \sum_{i,j=1}^n (P_{ij} - P_{j-1}) A (P_i - P_{i-1}) = \sum_{i,j=1}^n a_{ij} V_{j-1} P_1 V_{-i+1}.
$$
 (10)

Hence,  $P_nAP_n$  belongs to  $QC(V)$  for each  $A \in \mathcal{L}(X)$ . In particular, if  $A = K$  is a compact operator, then  $||P_nKP_n - K|| \to 0$  which implies that  $K \in QC(V)$   $\blacksquare$ 

The following construction will allow us to define a symbol calculus for the whole algebra alg  $(V, V_1)$ : Let alg<sup>\*</sup>  $(V, V_{-1})$  stand for the quotient algebra alg  $(V, V_{-1})/$  $QC(V)$  and denote by  $\pi$  the corresponding canonical homomorphism. Obviously, alg<sup>*i*</sup> (*V*,  $V_{-1}$ ) is a commutative Banach algebra generated by  $\pi(V)$  and by its inverse  $\pi(V_{-1}).$ partially the probability of  $\mathbb{R}P_n$   $\leq f(X)$ . In particular, in the operator, then  $||P_nKP_n - K|| \to 0$  which implies that  $K \in QC(V|$ <br>llowing construction will allow us to define a symbol calculus for<br>lead  $(V, V_{-1})$ : Let  $\text{$ 

Proposition 5: *Assume that* (H) is fulfilled. Then the spectrum  $\sigma(\pi(V))$  of  $\pi(V)$ *coincides with the unit circle* T, and for each polynomial  $p(t)$ ,  $|t| = 1$ , we have

$$
\max_{t \in \mathbf{T}} |p(t)| \leq ||p(\pi(V))|| = ||\pi(p(V))|| \leq ||p(V)||. \tag{11}
$$

*Hence, the maximal ideal space of alg<sup>\*</sup>*  $(V, V_{-1})$  *is homeomorphic to* T, and by (11) the symbol of an operator  $A \in L(V)$  coincides with the Gelfand transform of  $\pi(A)$ .

Proof: We have only to verify that  $\sigma(\pi(V)) = T$ . The other assertions follow immediately from the general theory of Banch algebras. By (H) the spectra of  $\pi(V)$  and  $(\pi(V))^{-1} = \pi(V_{-1})$  are contained in **T**. Assume that  $\sigma(\pi(V))$  + **T** and choose for the quotient algebra algorithm.<br>
anonical homomorphism. (<br>
generated by  $\pi(V)$  and by i<br>
d. Then the spectrum  $\sigma(\pi(V))$ <br>
d. Then the spectrum  $\sigma(\pi(V))$ <br>  $d$ . Then the spectrum  $\sigma(\pi(V))$ <br>
is homeomorphic to  $T$ , and the G  $z_0 \in \mathbf{T} \setminus \sigma(\pi(V))$ . Then there are operators  $B \in \text{alg}(V, V_{-1})$  and  $K \in QC(V)$  such that <sup>3.</sup>  $\frac{n}{m}$ Proof: We have only to verify that  $\sigma(\pi(V)) = \mathbf{T}$ . The other assertions follow<br>immediately from the general theory of Banach algebras. By (H) the spectra of<br> $\pi(V)$  and  $(\pi(V))^{-1} = \pi(V_{-1})$  are contained in **T**. Assume that Further, approximate *K* by  $K_0 = \sum_{i=1}^{n} \prod_{j=1}^{m} K_{ij}$ ,  $K_{ij} \in L^0(V)$ , so that  $||K - K_0|| < 1/2$ . What results is that  $B_0(V - z_0 I) = I + K_0 + C_0$  with  $||C_0|| < 1$ . Now we represent  $B_0(V)$  and  $(\pi(V))^{-1} = \pi(V_{-1})$  are contained in **T**. Assume that  $\sigma(\pi(V)) + T$  and choose  $z_0 \in T \setminus \sigma(\pi(V))$ . Then there are operators  $B \in \text{alg}(V, V_{-1})$  and  $K \in QC(V)$  such that  $B(V - z_0 I) = I + K$ . Approximate  $B$  by  $B_0 = \sum_{i=1}^{n}$  $B_0 \in \text{alg}^0(V, V_{-1})$  in the form  $B_0 = R_0 + S_0$  with  $R_0 \in L^0(V)$  and  $S_0 \in QC^0(V)$ . (This is always possible; moreover, the representation is unique.) Thus,  $R_0(V - z_0 I)$ is always possible; moreover, the representation is unique.) Thus,  $R_0(V-z_0I)$ <br>=  $I + K_0 + C_0 - S_0(V-z_0I)$ . Since  $K_0 - S_0(V-z_0I) \in QC^0(V)$ , we can find an  $n \in \mathbb{Z}^+$  such that  $(K_0 - S_0(V - z_0 I))$   $V_n = 0$  (cp. the proof of Proposition 4). Finally,  $R_0(V - z_0 I) V_n = (I + C_0) V_n$ , and the right-hand side of this equation is invertible from the left. On the other hand,  $R_0(V - z_0 I) V_n \in L^0(V)$ , and  $R_0(t)$  $\times (t - z_0)$  *t*<sup>n</sup> vanishes at  $z_0 \in$  **T** which is a contradiction to Theorem 1 **I** 

.5.

• 

 

**If (H) is fulfilled,** then the previous theorem enables us to assign to every operator  $A \in \text{alg } (V, V_{-1})$  a continuous function smb *A* on **T** called its symbol, namely the Gelfand transform of  $\pi(A)$ . Obviously, the mapping  $A \mapsto \text{smb } A$  is a continuous homomorphism and its kernel includes the quasicommutator ideal. The kernel equals  $QC(V)$  if and only if alg<sup>*n*</sup>  $(V, V_{-1})$  has a trivial radical. Proposition 5 implies that an operator  $A \in \text{alg } (V, V_{-1})$  is regularizable with respect to  $QC(V)$  (that is, there exists an operator  $B \in \text{alg } (V, V_{-1})$  such that both  $\overline{AB} - I$  and  $\overline{BA} - I$  belong to  $QC(V)$ ) if and only if (smb *A*)  $(t) \neq 0$  for all  $t \in T$ . So it is natural to ask whether an operator  $A \in \text{alg}(V, V_{-1})$  being invertible in  $\mathcal{L}(X)$  must be invertible If (H) is fulfilled, then the previous theorem enables us to assign to every operator  $A \in \text{alg}(V, V_{-1})$  a continuous function smb  $A$  on  $T$  called its symbol, namely the Gelfand transform of  $\pi(A)$ . Obviously, the mappin 300 S. Roch and B. Sultanuary<br>
If (H) is fulfilled, then the<br>  $A \in \text{alg } (V, V_{-1})$  a continuo<br>
Gelfand transform of  $\pi(A)$ .<br>
homomorphism and its kee<br>
equals  $QC(V)$  if and only if<br>
that an operator  $A \in \text{alg } (I)$ <br>
there exist in alg  $(V, V_{-1})$ . This question will be answered positively by Theorem 6 below.

Proposition 6: If  $K \in QC(V)$  and if  $I + K$  is at least one-sided invertible (in  $\mathcal{L}(X)$ ), then  $I' + K$  is two-sided invertible in alg  $(V, V_{-1})$ , and  $(I + K)^{-1} - I' \in QC(V)$ ;

Proof: Approximate *K* by operators  $K_n \in QC^0(V)$ . As in the proof of Proposition Proposition 6: If  $K \in QC(V)$  and if  $I + K$  is at least one-sided invertible (in  $\mathcal{L}(X)$ ), then  $I + K$  is two-sided invertible in alg  $(V, V_{-1})$ , and  $(I + K)^{-1} - I' \in QC(V)$ .<br>Proof: Approximate  $K$  by operators  $K_n \in QC(V)$ . As in th 4 there exists a sequence  $\{a_n\} \subseteq \mathbb{Z}^+$  such that  $Q_{a_n}K_n = K_n\bar{Q}_{a_n} = 0$ . Hence,  $K_n = P_{a_n}K_nP_{a_n}$ , and the one-sided invertibility of  $I + K$  yields that  $I + K_n = I$  $+ P_{an} K_n P_{a_n} = Q_{a_n} + P_{a_n} (K_n + I) P_{a_n}$  is one-sided invertible in  $\mathcal{I}(X)$  for *n* large enough. This shows that  $P_{a_n} + P_{a_n} K_n P_{a_n}|_{\text{im } P_{a_n}}$  must be one-sided invertible for *n* large enough. Now a little thought shows that the algebra of all operators  $P_{a_n}RP_{a_n}|_{\text{im}P_{a_n}}$  with. arbitrary  $R \in \text{alg } (V, V_{-1})$  is isomorphic to the algebra  $C^{a_n \times a_n}$  (compare with the representation (10)). Thus, the one-sided invertibility of the operators  $P_a$ , exists an operation of exists an operation of  $QC(V)$  if  $i$ <br>increase an operation of  $(V, V_{-1})$ . This<br>opposition 6, then  $I' + K$ <br>oof: Approximation of  $K_n P_{a_n}$ , and<br> ${}_{n}K_n P_{a_n} = Q_{a_n}$ <br>a little thou ary  $R \in \text{alg}$  (and send  $+ P_{a_n} K_n P_{a_n}|_{\text{im } P_{a_n}}$  gives their two-sided invertibility, and their inverses are in alg *(V<sub>i</sub>*)  $V_{-1}$ ). So we find that the operators  $I + K_n = Q_{a_n} + P_{a_n}(K_n + I)P_{a_n}$  must be two-sided<br>invertible in alg  $(V, V_{-1})$ . Now assume for definiteness that  $A = I + K$  is left-invertible<br>and that A' is a left inverse for A. Then  $A'($ invertible in alg  $(V, V_{-1})$ . Now assume for definiteness that  $A = I + K$  is left-invertible and that  $A'$  is a left inverse for  $A$ . Then  $A'(I + K_n)$  tends in the norm to the identity operator. Thus, for *n* large enough, we get the invertibility of  $A'(I + K_n)$  and that of  $(I + K_n)$ . Therefore, the operator  $A'$  must be invertible and, consequently,  $A$  is invertible. Finally we note that  $(I + K_n)^{-1} = I + R_n$  with  $R_n \in QC(V)$ , and the closedness of  $QC(V)$  gives  $A' - I = (I + K)^{-1} - I \in QC(V)$ mvertible in alg  $(V, V_{-1})$ . Now assume for definiteness that  $A = I + K$  is left-invertible<br>and that  $A'$  is a left inverse for  $A$ . Then  $A'(I + K_n)$  tends in the norm to the identity<br>of  $o(f + K_n)$ . Therefore, the operator  $A'$  m

Theorem 6: Let (H) be fulfilled and assume that  $A \in \text{alg } (V, V_{-1})$  and let  $A \cdot$  be *invertible in*,  $\mathcal{L}(X)$ . Then  $A^{-1} \in \text{alg } (V, V_{-1})$  and, moreover, smb  $A^{-1} = (\text{smb } A)^{-1}$ *and* wind smb  $A = 0$ .

$$
\|A^{-1}-A_n^{-1}\|\leq \|A_n^{-1}\|\|A^{-1}\|\|A_n-A\|<2\|A^{-1}\|^2\|A_n-A\|,
$$

and this estimation shows that it suffices to verify the assertion for operators in alg<sup>o</sup>  $(V, V_{-1})$ . Given  $A \in \text{alg}^0(V, V_{-1})$  we find operators  $B \in L^0(V)$  and  $K \in QC^0(V)$ and a number  $l \in \mathbb{Z}^+$  such that  $A = B + K$  and  $KV_l = 0$ . Thus,  $AV_l = BV_l$ , and the invertibility of  $A$  implies the invertibility of  $BV_t$  from the left. Theorem 1 shows that then *B* must be one-sided invertible and that its one-sided inverse belongs to alg  $(V, V_{-1})$ . Assume, e.g., *B* to be invertible from the left and take  $C \in \text{alg } (V, V_{-1})$  so that  $CB = I$ . Then solonest of  $QC(V)$  gives  $A' - I = (I + K)^{-1}$ <br>
Theorem 6: Let (H) be fulfilled and assumentible in  $\mathcal{L}(X)$ . Then  $A^{-1} \in$  alg  $(V, V_{-1})$ <br>
and wind smb  $A = 0$ .<br>
Proof: Approximate A by operators  $A_n \in \mathfrak{g}$ <br>
then  $A_n$  is inv em 6: Let (H) be fulfilled and assume that  $A \in \text{alg } (V, V_{-1})$  and let  $A \cdot b$ <br>  $\in$  in,  $\mathcal{I}(X)$ . Then  $A^{-1} \in \text{alg } (V, V_{-1})$  and, moreover, smb  $A^{-1} = (\text{smb } A)^{-1}$ <br>
smb  $A = 0$ .<br>
Approximate A by operators  $A_n \in \text{alg}^{\mathfrak{$ 

$$
A = B + K = B + KCB = (I + KC) B
$$
 (12)

with  $KC \in QC(V)$ . Since A is invertible, we conclude from (12) that  $I + KC$  must be invertible from the right, which leads by Proposition 6 to the two-sided invertibility *of*  $I + KC$  in alg  $(V, V_{-1})$ . Put  $T := (I + KC)^{-1} - I \in QC(V)$ . Then  $C(I + T)$  A so that  $\overrightarrow{CB} = I$ . Then<br>  $A = B + K = B + KCB = (I + KC) B$  (12)'<br>
with  $KC \in QC(V)$ . Since *A* is invertible, we conclude from (12) that  $I + KC$  must be<br>
invertible from the right, which leads by Proposition 6 to the two-sided invertibility

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and we arrive at  $A^{-1} = C(I + T) \in \text{alg } (V, V_{-1})$ . Finally, the<br>
(cuth  $A$ )-1 is obvious and the index formula fallows firm (19) is an inverse for *A* and we arrive at  $A^{-1} = C(I + T) \in \text{alg } (V, V_{-1})$ . Finally, the equation smb  $A^{-1} = (\text{smb } A)^{-1}$  is obvious and the index formula follows from (12) by invoking Theorem 1  $\blacksquare$ 

If the operator  $A \in \text{alg } (V, V_{-1})$  is of a special structure, we can specify the preceding theorem as follows.

Theorem *<sup>7</sup> 1 Let* (H) *be fulfilled and assume there is a V-dominating algebra D. 1/ A* =  $B + K$  (with  $B \in L_p(V)$  and  $K \in QC(V)$ ). is an invertible operator, then A and *B* are both invertible in alg  $(V, V_{-1})$ . Moreover, there exist operators  $B' \in L_p(V)$  and  $K' \in QC(V)$ -such that  $A^{-1} = (B + K)^{-1} = B' + K'$ , and for the symbols we have  $\begin{array}{c} \begin{array}{c} \mathbf{1} \ \mathbf{1} \ \mathbf{B} \ \mathbf{1} \ \mathbf{1} \end{array} \end{array}$ 

 $\sinh A^{-1} = \sinh B^{-1} = (\sinh B)^{-1} = (\sinh A)^{-1} = \sinh B'$ .

Proof: If *A* is invertible, then, by Theorem 6,  $A^{-1} \in \text{alg } (V, V_{-1})$  and wind smb *A*  $= 0$ . Because smb  $A = \text{smb } B$  and  $B \in L_p(V)$ , we conclude via Theorem 3 that *B* is invertible and via Theorem 6 that  $B^{-1} \in \text{alg}(V, V_{-1})$ . Let  $B = a(V)$ ,  $a \in D$ . The invertibility of *B* implies the invertibility of *a* in *D*. Put  $B' = (a^{-1})(V) \in L_p(V)$ . Then  $B'(B + K) = (a^{-1})(V)(a(V) + K) = I + K_1$  with  $K_1 \in QC(V)$ . Again by Theorem 3, the operator *B'* is invertible. So the operator  $I + K_1$  must be invertible, Theorem 3, the operator *B'* is invertible. So the operator  $I + K_1$  must be invertible, too, and Proposition 6 states that  $(I + K_1)^{-1} = I + K_2$  with  $K_2 \in QC(V)$ . Finally, the operator  $(I + K_2) B' =: B' + K'$  is an inverse for *A*. too, and Proposition 6 states that  $(I + K_1)^{-1} = I + K_2$  with  $K_2 \in QC(V)$ . Finally,<br>the operator  $(I + K_2) B' =: B' + K'$  is an inverse for *A*. The symbol identity is<br>obvious  $\blacksquare$ Theorem 7: Let (<br>
If  $A = B + K$  (with  $B$  are both invertible<br>  $K' \in QC(V)$  such that<br>  $\text{smb } A^{-1} = \text{s}$ <br>  $\text{Proof: If } A \text{ is inv} = 0.$  Because smb  $A$ <br>
invertible and via T<br>
invertiblity of  $B$  im<br>
Then  $B'(B + K) =$ <br>
Theorem 3, the operat

The following theorem reifies Theorems 6 and 7 for the case when the operator *<sup>V</sup>* has a finite cokernel.

Theorem 8: Let  $(H)$  be fulfilled and  $A \in \text{alg } (V, V_{-1})$ .

*a)* If A is  $\Phi_+$  or  $\Phi_-,$  then  $(\operatorname{smb} A)(t) \neq 0$  for all  $t \in \mathbb{T}$ .

*b)*  $I f x = \text{codim } (\text{im } V) < \infty$ , *then A is a*  $\Phi$ *-operator if and only if*  $(\text{smb } A)$   $(t) \neq 0$ *for all t*  $\in$  **T***. If this condition is fulfilled, then ind*  $A =$  *—wind (smb <i>A*).

We omit the proof since it follows from Theorem 1 by similar arguments as those in the proof of Theorem 6 and by the stability of ind and wind under small perturbations I

# 5. Decomposing algebras

Generally, all what we know about the relations between  $L(V)$ ,  $QC(V)$  and alg  $(V, V_{-1})$ is that the algebraical sum  $L(V) + QC(V)$  is dense in alg  $(V, V_{-1})$ . Under additional conditions one can get essentially more information about the structure of alg  $(V, V_{-1})$ . *II* the proof since it follows from Theorem 1 by si<br> *II* coof of Theorem 6 and by the stability of ind and<br> *P* **II**. **II** what we know about the relations between  $L(V$ <br> *I* all what we know about the relations between quments as those<br>
ler small pertur-<br>  $\blacksquare$ <br>
and alg  $(V, V_{-1})$ <br>
Jnder additional<br>
re of alg  $(V, V_{-1})$ <br>  $\blacksquare$ 

Theorem 9: *Assume that* (H) is fulfilled and that for all polynomials  $R_{ij}(t)$  on  $\mathbf{T}$ 

$$
\|(\sum_{i} \prod_{j} R_{ij})(V)\| \leq M \|\sum_{i} \prod_{j} R_{ij}(V)\|
$$
\n(13)

with some  $M > 0$ . Then the algebra alg  $(V, V_{-1})$  decomposes into the direct sum  $\text{alg}(V, V_{-1}) = L(V) + QC(V),$  *i.e.* there is a projection  $S \in \mathcal{L}(X)$  mapping alg  $(V, V_{-1})$ *onto L(V) parallel to QC(V). Conversely, if alg*  $(V, V_{-1})$  *decomposes, then (13) holds viih*  $M = ||S||$ . **5. Decomposing algebras**<br> **Generally, all what we know about the relations between**  $L(V)$ **,**  $QC$ **<br>
is that the algebraical sum**  $L(V) + QC(V)$  **is dense in alg**  $(V, V_{-1})$ **<br>
conditions one can get essentially more information about th** *F*, all what we know about the relations between  $L(V)$ ,  $QC$  (i.e algebraical sum  $L(V) + QC(V)$  is dense in alg  $(V, V_{-1})$  is one can get essentially more information about the struct em 9: Assume that (H) is fulfilled and t

Proof: Define a linear mapping S on alg<sup>o</sup>  $(V, V_{-1})$  by

$$
S: \sum_{i} \prod_{j} R_{ij}(V) \mapsto (\sum_{i} \prod_{j} R_{ij}) (V) \in L^{0}(V).
$$

 

S is well-defined and bounded by (13). Since  $S(R) = R$  for  $R \in L^{0}(V)$ , the mapping S is a continuous projection operator mapping alg<sup>o</sup>  $(V, V_1)$  onto  $L^0(V)$ . Its continuous extension onto the whole algebra alg  $(V, V_{-1})$  will again be denoted by *S*. Thus, alg  $(V, V_{-1})$  $V_{-1}$  = im S  $+$  ker S. Obviously, im  $S = L(V)$ , and it only remains to show that ker S  $= QC(V)$ . By definition,  $QC(V) \subseteq$  ker *S*. Now let  $A \in$  ker *S*. Approximate *A* by  $A_n$ ,  $A_n \in \text{alg}^0 (V, V_{-1})$ . Then  $||S(A_n)|| = ||S(A_n) - S(A)|| \to 0$ . Since  $S(A_n) - A_n \in QC^0 (V)$ , we get that *A* is in  $QC(V)$   $\Box$  $QC(V)$ . By definition,  $QC(V) \subseteq$  ker *S*. Now let *A*<br>  $\in$  alg<sup>o</sup>  $(V, V_{-1})$ . Then  $||S(A_n)|| = ||S(A_n) - S(A)|| -$ <br>  $\in$  get that *A* is in  $QC(V) \blacksquare$ <br>
Remarks: (i) It is easy to see that the following is supplying that  $\frac{1}{n!n-0}$  of l *= QC(V).* By do <br>*A<sub>n</sub>*  $\in$  alg<sup>o</sup> (*V, V<sub>-</sub>* we get that *A* is<br>Remarks: (i)<br> $\{W_n\}_{n=0}^{\infty}$  of left-in<br> $1. VW_n = W_n$ ]<br>2. sup<sub>n</sub>: $\|W_n\| <$ <br>3.  $\|W_n^{(-1)}KW_n$ <br>In particular, if s<br>(ii) Obviously,<br>see that (13) is eq<br>alg<sup>n</sup> (

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Remarks: (i) It is easy to see that the following is sufficient for (13): There is a family *FICHTERS*: (1) It is easy to see that the following is sufficient for  $\{W_n\}_{n=0}^{\infty}$  of left-invertible operators with left inverses  $W_n^{(-1)}$  such that

 $\mathcal{O}(\mathcal{O}_\mathcal{O})$  and  $\mathcal{O}(\mathcal{O}_\mathcal{O})$  . The contribution of  $\mathcal{O}(\mathcal{O}_\mathcal{O})$ 

 $2. \sup_{n} \|W_n\| < \infty, \sup_{n} \|W_n^{(-1)}\| < \infty,$ 

3.  $\|\hat{W}_n^{(-1)}\hat{K}\hat{W}_n\| \to 0$  as  $n \to \infty$  for any  $K \in QC(V)$ .

In particular, if  $\sup_{n\in\mathbb{Z}}||V_n|| < \infty$ , we may take  $\{V_n\}_{n\in\mathbb{Z}^+}$  as the family  $\{W_n\}_{n\in\mathbb{Z}^+}$ .

(ii) Obviously,  $(\sum_i \prod_j R_{ij})$   $(V) - \sum_i \prod_j R_{ij}(V) \in QC^0(V)$ . Since  $QC^0(V)$  is dense in  $QC(V)$ , we see that (13) is equivalent to the assertion  $||R(V)|| \le M ||\pi(R(V))||$  for each polynomial *R*, i.e.  $\text{alg}^{\pi}(V, V_{-1})$  is V-dominating. So we arrived at the following:

*The algebra alg*  $(V, V_{-1})$  *decomposes into*  $L(V) + QC(V)$  *if and only if alg<sup>-1</sup>*  $(V, V_{-1})$  *is dominating.* Moreover, if  $D = \mathrm{alg}^{\pi}(V, V_{-1})$  is  $V$  dominating, then  $L(V) = L_D(V)$ .

(iii) By (ii) we can endow  $L(V)$  with an equivalent norm and by a multiplication  $\circ$  which makes  $L(V)$  into a commutative Banach algebra,  $(L(V), \circ)$  which is isomorphic (and the isomorphism is continuous) to the quotient algebra alg<sup>*n*</sup> (*V*, *V*<sub>-1</sub>). Obviously,  $(L(V), \circ)$  is *V*dominating and  $L(L(V), \bullet)(V) = L(V)$ . 3.  $||W_n^{(-1)}KW_n|| \to 0$  as  $n \to \infty$  for any  $K \in QC(V)$ .<br>In particular, if  $\sup_{n \in \mathbb{Z}} ||V_n|| < \infty$ , we may take  $\{V_n\}_{n \in \mathbb{Z}}$  as the family  $\{V_n\}$  (ii) Obviously,  $(\sum_i \prod_j R_{ij}) (V) - \sum_i \prod_j R_{ij}(V) \in QC(V)$ . Since  $QC^0(V)$  is see tha

We conclude these remarks by the following concrete realization, which is thought as an illustration of the general statements from the preceding sections. Let  $X = l^{p,r}$ 6. Toeplitz operators on  $P^{y}$ <br>We conclude these remarks by the following concrete realization, which is thought as<br>an illustration of the general statements from the preceding sections. Let  $X = l^{p}$ <br> $(p \ge 1, \gamma \in \mathbb{R})$  b numbers with the norm r, if  $D = \text{alg}^{\pi} (V, V_{-1})$  is<br>
n endow  $L(V)$  with an eq<br>
commutative Banach alge<br>
uous) to the quotient all<br>  $V_{1,\bullet}(V) = L(V)$ .<br> **IF** on  $\mathbf{P}^{r}$ <br> **IF** on  $\mathbf{P}^{r}$ <br> **IF** on  $\mathbf{P}^{r}$ <br> **IFF on**  $\mathbf{P}^{r}$ <br> **IFF on** dominating and  $L(L(V), \rho)(V) =$ <br>
6. Toeplitz operators on  $\mathbf{P}^{ij}$ .<br>
We conclude these remarks<br>
an illustration of the gener<br>  $(p \ge 1, \gamma \in \mathbf{R})$  be the Ban<br>
numbers with the norm<br>  $||x||_{p,y} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)$ <br>
and cons *1*llowing<br> *nents* fr<br> *e* of al<br> *1*<br> *1*<br> *1*<br> *1*<br> *1*<br> *1* 6. Toeplitz operators on  $\mathbb{P}^{\nu}$ <br>
We conclude these remarks by the following concrete<br>
an illustration of the general statements from the p<br>  $(p \ge 1, \gamma \in \mathbb{R})$  be the Banach space of all sequeno<br>
mumbers with the nor *y* the following concrete realization, which is thou<br>statements from the preceding sections. Let X<br>h space of all sequences  $x = (\dot{x}_0, x_1, ...)$  of co<br> $+ .1)^{p\gamma}$ <br>and  $\ddot{V}_{-1}$  on X given by<br> $V_{-1}x = (x_1, x_2, ...)$ <br>if  $\gamma \ge 0$ , (x operators on<br>
ude these rema<br>
ation of the ge<br>  $\in \mathbb{R}$ ) be the 1<br>
with the norm<br>  $||x||_{p,\nu} = \left(\sum_{k=0}^{\infty} |x_k|\right)$ <br>
ider the operator<br>  $Vx = (0, x_0, x_1)$ <br>  $t^+,$ <br>  $||V_n|| = \begin{cases} (1 + 1) & \text{if } n \neq 0 \\ 1 & \text{if } n \neq 1 \end{cases}$ <br>
(V) a

which the normal  
\n
$$
||x||_{p,\nu} = \left(\sum_{k=0}^{\infty} |x_k|^p (k+1)^{p\nu}\right)^{1/p}
$$
\n
$$
V = (0, x_0, x_1, \ldots), \qquad V_{-1}x
$$
\n
$$
V_{-1}x
$$

and consider the operators  $V$ , and  $\overline{V}_{-1}$  on  $\overline{X}$  given by.

$$
Vx = (0, x_0, x_1, \ldots), \qquad V_{-1}x = (x_1, x_2, \ldots).
$$

$$
||x||_{p,y} = \left(\sum_{k=0} |x_k|^p (k+1)^{p_y}\right)
$$
  
\nhsider the operators  $V$  and  $V_{-1}$  on  $X$  given by  
\n
$$
Vx = (0, x_0, x_1, \ldots), \qquad V_{-1}x = (x_1, x_2, \ldots).
$$
  
\n
$$
Z^+, \qquad |V_n|| = \begin{cases} (1+n)^y & \text{if } y \ge 0, \\ 1 & \text{if } y < 0, \end{cases} \qquad ||V_{-n}|| = \begin{cases} (1+n)^{-y} & \text{if } y \le 0, \\ 1 & \text{if } y > 0. \end{cases}
$$
  
\n
$$
|V_{-n}|| = \begin{cases} (1+n)^{-y} & \text{if } y \le 0, \\ 1 & \text{if } y > 0. \end{cases}
$$
  
\n
$$
|V_{-n}|| = \begin{cases} (1+n)^{-y} & \text{if } y \le 0, \\ 1 & \text{if } y > 0. \end{cases}
$$

In particular, the operators *V* and  $V_{-1}$  are subject of the hypothesis (H). The operators in  $L(V)$  are called Toeplitz operators. By Theorem 2, each Toeplitz operator is uniquely determined by its symbol. As it is usual, if  $A \in L(V)$  and if a is the symbol of *A*, we shall write  $T(a)$  instead of *A*. The Banach spaces  $l^{p,y}$  and  $l^{p,0}$  are isometrically isomorphic, and the isometry is given by and consider the operators V and  $V_{-1}$  on X given by<br>  $Vx = (0, x_0, x_1, \ldots), \qquad V_{-1}x = (x_1, x_2, \ldots).$ <br>
For  $n \in \mathbb{Z}^+,$ <br>  $||V_n|| = \begin{cases} (1 + n)^{\gamma} & \text{if } \gamma \geq 0, \\ 1 & \text{if } \gamma < 0, \end{cases}$   $||V_{-n}|| = \begin{cases} (1 + n)^{-\gamma} & \text{if } \gamma \leq 1, \\ 1 & \text{if }$  $Vx = (0, x_0, x_1, ...)$ ,  $V_{-1}x = (x_1, x_2, ...)$ .<br>  $Z^+,$ <br>  $||V_n|| = \begin{cases} (1 + n)^r & \text{if } \gamma \ge 0, \\ 1 & \text{if } \gamma < 0, \end{cases} ||V_{-n}|| = \begin{cases} ||V_{-n}|| = \begin{cases} (1 + n)^r & \text{if } \gamma \ge 0, \\ 1 & \text{if } \gamma < 0, \end{cases} ||V_{-n}|| = \begin{cases} ||V_{-n}|| = \begin{cases} (1 + n)^r & \text{if } \gamma \ge 0, \\ (V) \text{ are called Toeplitz operators$  $\log_{10}$ .<br>  $\log_{10}$  operator is symbol<br>  $\log_{10}$ <br>  $\log_{10}$ <br>  $\log_{10}$ *i*  $|V_n|| = \begin{cases} 1 & \text{if } \gamma < 0, \end{cases}$   $||V_{-n}|| = \begin{cases} 1 & \text{if } \gamma > 0. \end{cases}$ <br> *In* particular, the operators *V* and  $V_{-1}$  are subject of the hypothesis (H). The operators in  $L(V)$  are called Toeplitz operators. By Theorem 2,  $||\nabla_n|| = \begin{cases} (1+n)^{\gamma} & \text{if } \gamma \geq 0, \\ 1 & \text{if } \gamma < 0, \end{cases}$   $||V_{-n}|| = \begin{cases} (1+n)^{-\gamma} & \text{if } \gamma \leq 0 \\ 1 & \text{if } \gamma > 0 \end{cases}$ <br>
In particular, the operators *V* and *V<sub>-1</sub>* are subject of the hypothesis (H). The<br>
tors in  $L(V)$  are call

$$
\Lambda\colon l^{p,0}\to l^{p,\gamma},\qquad (x_k)_{k=0}^{\infty}\mapsto (x_k(k+1)^{-\gamma})_{k=0}^{\infty}.
$$

In the sequel we have to distinguish between Toeplitz operators on  $l^{p,\gamma}$  and on  $l^{p,0}$ , both generated by the same symbol. Let us agree to designate the operators on the

Symbol Calculus for an Algebra  $303$ 

••.

Proposition 7.: *We have the inclusions,*

 

$$
A^{-1}V_{\pm 1}A - V'_{\pm 1} \in QC(V'), \quad A V'_{\pm 1}A^{-1} - V_{\pm 1} \in QC(V).
$$

Symbol Calculus for an Algebra<br>
A-i V<sub>±1</sub> $A - V'_{\pm 1} \in QC(V')$ ,  $AV'_{\pm 1}A^{-1} - V_{\pm 1} \in QC(V)$ .<br>  $\therefore$  We only demonstrate the first assertion. The matrix representation  $V'$  in the canonical basis of  $l^{p,0}$  is Proof: We only demonstrate the *first* assertion. The matrix representation of  $A^{-1}VA - V'$  in the canonical basis of  $l^{p,0}$  is Symbol Calculus for an Algebra<br>
on 7: We have the inclusions,<br>  $\pm_1A - V_{\pm 1} \in QC(V'), \quad A V'_{\pm 1}A^{-1} - V_{\pm 1} \in QC(V).$ <br>
only demonstrate the first assertion. The matrix representation<br>
in the canonical basis of  $l^{p,0}$  is<br>  $0 \q$ 

2 ion 7: We<br>  $V_{\pm 1}A - V$ <br>  $V_{\pm 1}A - V$ Symbox<br>
have the inclusions<br>  $\downarrow$ ,  $\in QC(V'), \qquad AV'_{\pm 1}A^{-1}$ <br>
nonstrate the first assertion<br>
onical basis of  $l^{p,0}$  is<br>
0  $\cdot$  0  $\cdot$ .<br>
0 0  $\cdot$ <br>
0  $\cdot$ <br>
-1 0  $\cdot$ on 7: We have the<br>  $\frac{1}{\pm 1}A - V'_{\pm 1} \in QC($ <br>
only demonstrate<br>
in the canonical be<br>  $0 \t 0 \t 0$ <br>  $-1 \t 0 \t 0$ <br>  $0 \t \frac{3^y}{2^y} - 1 \t 0$ <br>  $\vdots \t \vdots \t \vdots$ <br>
ollows immediately

So the proof follows immediately from the characterizationi of the quasicommutator ideal given in Proposition  $4/c$ ) by invoking a simple norm estimation  $\blacksquare$ 

Notice that for  $p \neq 1$  the operators  $A^{-1}V_{\pm 1}A - V'_{\pm 1}$  are compact (this is due to Corollary 1). It is easy to see that this holds even for  $p = 1$ .

 $\begin{array}{|l|l|} \hline 2^{\gamma}-1 & 0 & 0 & \cdots \\ \hline 1^{\gamma}-1 & 0 & 0 & \frac{3\gamma}{2^{\gamma}}-1 & 0 & \cdots \\ \hline \end{array}$ <br>
the proof follows immediately from the characterization of the quasicommutator<br>
al given in Proposition 4/c) by invoking a simple norm est  $T' \in QC(V')$  for each polynomial P. As in the proof of Proposition 4, there is an  $n \in \mathbb{Z}^+$  such that, for each  $T' \in QC^0(V')$ ,  $V'_{-n}(P(V') + T')$   $V_n = P(V')$ . Since the norms  $||V'_{\pm n}||$  are equal to 1 for  $n \in \mathbb{Z}^+$ , we conclude that *III* P(*Y'*) for each polynomial  $P$ .<br> *II* P(*Y'*) for each polynomial  $P$ .<br> *III* III are equal to 1 for  $n \in \mathbb{Z}^+$ <br>  $||P(V')|| \leq ||P(V') + T'||$ <br> **IATY** Let  $T(a)$  be a Toenlitz **for all**  $\frac{3r}{2r} - 1$  of  $\frac{3r}{2r} - 1$  of  $\frac{3r}{2r} - 1$  of  $\frac{1}{2r} - 1$  is the proof follows immediately from the characterization of the quasicommutator deal given in Proposition 4/c) by invoking a simple norm esti

$$
||P(V')|| \leq ||P(V') + T'|| \qquad \text{for all } T' \in QC(V'). \tag{14}
$$

Forms  $||V_{\pm n}||$  are equal to *I* for  $n \in \mathbb{Z}$ , we conclude that<br>  $||P(V')|| \leq ||P(V') + T'||$  for all  $T' \in QC(V')$ .<br>
Corollary: Let  $T(a)$  be a Toeplitz operator on  $l^{p,y}$ . Then a is also a symbol of a<br> *Toeplitz operator on*  $l^{p,$ The proof follows immediately from (14) *•*  Corollary: Let  $T(a)$  be a Toeplitz operator on  $l^{p,p}$ . Then a is also a symbol of a<br> *Toeplitz operator on*  $l^{p,0}$ . Moreover,  $\Lambda^{-1}T(a) \Lambda - T'(a) \in QC(V')$  and  $||T'(a)|| \leq ||T(a)||$ .<br>
The proof follows immediately from (14)<br> **Theore** 

*invertible if and only if its synthot does not degenerate on T. If this condition is fulfilled, then the one-sided invertibility of*  $T(a)$  corresponds to the winding number of a.

Proof outline: If  $T(a)$  is invertible, then, by Theorem 6, its symbol  $a$  must be invertible. Now let  $a_i$ , be invertible. Consider the operator  $T'(a)$  defined on  $l^{p,0}$  by the same symbol a. By Remark (ii) we conclude that  $T'(a)$  is at least one-sided invertible. On the other hand, Theorem 8 yields that  $T(a)$  is a  $\Phi$ -operator with ind  $T(a)$  = —wind *a*. To finish the proof it remains to show that the kernels of  $T(a)$  and of  $||P(V')|| \leq ||P(V') + T'||$  for all  $T' \in QC(V')$ .<br>Corollary: Let  $T(a)$  be a Toeplitz operator on  $l^{p,q}$ . Then a is all<br>Toeplitz operator on  $l^{p,q}$ . Moreover,  $A^{-1}T(a)$   $A - T'(a) \in QC(V')$  and  $||$ <br>The proof follows immediately from (14)  $||$ Toeplitz operator on  $l^{p,0}$ . Moreover,  $\Lambda^{-1}T(a)$   $\Lambda$ <br>The proof follows immediately from (14)<br>Theorem 10: Let  $T(a)$  be a Toeplitz operal<br>invertible if and only if its symbol does not deget<br>then the one-sided invertibil of outline: If  $T(a)$  is invertible, then, by Theorem 6, its sy<br>ble. Now let  $a_i$  be invertible. Consider the operator  $T'(a)$  de<br>e symbol  $a$ . By Remark (ii) we conclude that  $T'(a)$  is at least<br>n the other hand, Theorem 8

Now we turn our attention to the structure of the algebra alg  $(V, V_{-1})$  generated

Proposition 8: The algebra alg  $(V, V_1)$  decomposes into  $L(V) + QC(V)$  if and *only if*  $\gamma = 0$ .

Proof: It is easy to see that  $||\pi(V_n)|| = 1$  for all *n*, *p* and *y* (cp. [5]). Comparing this with the norms of  $V_n$  quoted above we find that the projection S defined in Section 5 is bounded if and only if  $\gamma = 0$ 

Now fix  $p = 1$  and let  $\gamma \neq 0$ . Since  $\|\pi(V_n)\| = 1$ , the symbol algebra alg<sup>\*</sup> (V,'V<sub>-1</sub>) contains a copy of the Wiener algebra consisting of all functions  $a(t) = \sum_{i \in \mathbb{Z}} a_i t^i$   $(t \in T)$  with  $\sum_{i \in \mathbb{Z}} |a_i| < \infty$ . On the other hand, the matrix representation

S. RoCH and B. SILEERMANN  
\nwith 
$$
\sum_{i\in\mathbb{Z}}|a_i| < \infty
$$
. On the other hand, the matrix  
\n
$$
T(a) \sim \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}
$$

other hand, the matrix represents<br>  $\begin{cases}\n\frac{1}{2} \cos \theta \, dt\n\end{cases}$   $\begin{cases}\n\frac{1}{2} \cos \theta \, dt\n\end{cases}$   $\begin{cases}\n\frac{1}{2} \cos \theta \, dt\n\end{cases}$   $\begin{cases}\n\frac{1}{2} \cos \theta \, dt\n\end{cases}$ of the operator  $T(a)$  in the canonical basis of  $l^{1,\gamma}$  shows that the sequence  $(a_0, a_1, \ldots)$ must necessarily belong to  $l^{1,y}$ . Hence, there are symbols the corresponding operator of which is not the sum of a Toeplitz operator and an operator in  $QC(V)$  (which has the zero symbol by definition). Such operators can be found among the operators of the form  $AT'(a)A^{-1}$  where  $T'(a)$  runs through the set  $L(V')$  of all Toeplitz operators on *lP,0* as the following proposition indicates.  $r(t \in T)$  with  $\sum_{i \in \mathbb{Z}} |a_i| < \infty$ . On the other hand,<br>  $T(a) \sim \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$ <br>
of the operator  $T(a)$  in the canonical basis of  $l^{1}$ .<br>
must necessarily

*Proposition 9: Let V and V' be the shift operators on*  $l^{p,y}$  *and*  $l^{p,0}$  *(* $p \geq 1, \gamma \in \mathbb{R}$ *),* 

a) alg  $(V, V_{-1}) =$  alg  $(AV'A^{-1}, AV'_{-1}A^{-1}),$ 

*b) the algebras*  $\text{alg}^{\pi}(V, V_{-1}) = \text{alg}(V, V_{-1})/QC(V)$  and  $(L(V'), \circ)$  (on  $l^{p,0}$ ) are iso*morphic.*

 $\Pr{\text{coof: a)}} \text{ By Proposition 7, } AV'_{\pm 1}A^{-1} \in \text{alg } (V, V_{-1}), \text{ and so all } (AV'A^{-1}, AV'_{-1}A^{-1})$  $\mathbf{a}$ lg(*V, V<sub>-1</sub>*). Analogously,  $\mathbf{a}$ lg( $A^{-1}V$  $A$ ,  $A^{-1}V_{-1}A$ )  $\subseteq$   $\mathbf{a}$ lg(*V', V'<sub>-1</sub>*), and since the  $\equiv$  alg  $(V, V_{-1})$ . Analogously, alg  $\langle A \cdot V \cdot A, A \cdot V_{-1} \rangle$  and the algebras alg  $\langle A \cdot V \cdot A \cdot A \cdot V' \cdot A$  $A^{-1}_{-1}V_{-1}A$ ) and alg  $(V, V_{-1})$  are obviously isomorphic we get a). respectively. Then<br> **a)** alg(V', V<sub>+1</sub>) = alg(N'/-1<sup>-1</sup>,  $AV'_{-1}A'V_{-1}A^{-1}$ ),<br>
b) *Be algebras* alg<sup>n</sup> (V',  $V_{-1}$ ) = alg(V',  $V_{-1}$ )/ $QC(V)$  and  $(L(V'), o)$  (on  $l^{p,0}$ ) are so-<br> *morphic.*<br>
Proof: a) By Proposition 7,  $AV'$ 

b) Since  $I - A V' A^{-1} A V'_{-1} A^{-1} = P_1$ , the quasicommutator ideals  $QC(V, V_{-1})$  of alg  $(V, V_{-1})$  and  $QC(AV'A^{-1}, A'_{-1}A^{-1})$  of alg  $(AV'A^{-1}, AV'_{-1}A^{-1})$  coincide. Thus, the following algebras are isomorphic to each other:

, alg  $(V, V_{-1}^{'})/QC(V) \cong$  alg  $(AV'A^{-1}, AV_{-1}^{'}A^{-1})/QC(AV'A^{-1})$ 

where the last isomorphy follows from the fact that alg  $(V', V'_{-1})$  decomposes  $\blacksquare$ 

**-** The somewhat unexpected result of Proposition 9 is that the symbol algebras do not depend on  $\gamma$  (only on  $p$ ). Hint: Proposition 9 does not mean that each symbol of a Toeplitz operator on  $l^{p,0}$  is again a symbol of a Toeplitz operator on  $l^{p,\gamma}$ ! In case  $p = 2$  we can complete this picture as follows. both the last isomorphic to each other.<br>
alg  $(\tilde{V}, V_{-1}^{'})/QC(V) \cong$  alg  $(\Lambda V^{'}\Lambda^{-1}, \Lambda V_{-1}^{'}\Lambda^{-1})$ <br>  $\cong$  alg  $(V', V_{-1}^{'})/QC(V') \cong$ <br>
where the last isomorphy follows from the fact that all<br>
The somewhat unexpected result where the contract the solution of the solutio

Proposition 10: Let  $p = 2$ ,  $\gamma \in \mathbb{R}$ . Then the algebra  $\text{alg}(V, V_{-1})/QC(V)$  is a  $C^*$  *ebra which is isomorphic to the algebra*  $C(\mathbb{T})$  *of all continuous functions on the unit cle T. algebra which is isomorphic to the algebra C(T) of all'continuous functions on the unit* 

**Proof:** Denote by  $(\cdot, \cdot)_0$  the usual inner product on the Hilbert space  $l^{2.0}$ . The Banach space  $l^{2.9}$  ( $\gamma$   $\neq$  0) can be made into a Hilbert space on defining an inner product by  $(x, y)_y = (A^{-1}x, A^{-1}y)_0$ . Since

where the last isomorphy follows from the fact that alg 
$$
(V', V_{-1}) \in (V', N')
$$
,  
\nwhere the last isomorphy follows from the fact that alg  $(V', V'_{-1})$  decomposes. The somewhat unexpected result of Proposition 9 is that the symbol algebras do  
\nnot depend on  $\gamma$  (only on  $p$ ). Hint: Proposition 9 does not mean that each symbol of  
\na Toeplitz operator on  $l^{p,0}$  is again a symbol of a Toeplitz operator on  $l^{p,\gamma}$ ! In case,  
\n $p = 2$  we can complete this picture as follows.  
\nProposition 10: Let  $p = 2$ ,  $\gamma \in \mathbb{R}$ . Then the algebra alg  $(V, V_{-1})/QC(V)$  is a C\*-  
\nalgebra which is isomorphic to the algebra  $C(T)$  of all continuous functions on the unit  
\ncircle T.  
\nProof: Denote by  $(\cdot, \cdot)_0$  the usual inner product on the Hilbert space  $l^{2,0}$ . The  
\nBanach space  $l^{2,\gamma}$  ( $\gamma \neq 0$ ) can be made into a Hilbert space on defining an inner  
\nproduct by  $(x, y)_\gamma = (A^{-1}x, A^{-1}y)_0$ . Since  
\n $(x, Vy)_\gamma = (A^{-1}x, A^{-1}Y)_0 = (A^{-1}x, A^{-1}VAA^{-1}y)_0$   
\n $= (A^{-1}x, (V' + T'') A^{-1}y)_0$  (with  $T' \in QC(V')$ )  
\n $= (A^{-1}A(V'_{-1} + T'')A^{-1}x, A^{-1}y)_0$  (with  $T' \in QC(V')$ )  
\n $= (A^{-1}U_{-1} + T')x, A^{-1}y)_0$  (with  $T \in QC(V)$ )  
\n $= (I - 1 + (V_{-1} + T')x, A^{-1}y)_0$  (with  $T \in QC(V)$ )  
\n $= ((V_{-1} + T) x, y)_\gamma$ 

V

•

we see that  $V^* = V_{-1} \in QC(V)$  and, analogously, we get  $V_{-1}^* = V \in QC(V)$ . Hence, alg  $(V, V_{-1})$  is a  $C^*$ -algebra, and it remains to verify that the mapping  $\mathcal{L}(l^{2,0}) \ni A$  $\mapsto A A A^{-1} \in \mathcal{L}(l^{2,\gamma})$  is a  $\bullet$ -isomorphism. This statement follows by similar conclusions as we have used above. In fact, we get  $B^* = A^{-1}(A^{-1}BA)^*A$  for  $B \in \text{alg } (V, V_{-1})$ where the star on the left denotes the adjoint with respect to  $(., .)_r$  and the star on the right refers to the usual adjoint on  $l^{2,0}$ . Thus, the  $C^*$ -algebras alg<sup>*n*</sup> (*V*,  $V_{-1}$ ) and  $a\vert g^{\pi}(\overline{A}V'A^{-1}, A V'_{-1}A^{-1})$  are star isomorphic, and the proof is complete since the latter is isomorphic to  $C(T)$  by standard arguments (cp. Section 5)  $\blacksquare$ 

Our final goal is the finite section method for operators in alg  $(V, V_{-1})$ . Put  $P_n$  $I = I - V_n V_{-n}$  and assume that  $P_n \rightarrow I$  strongly. We say that the finite section method applies to  $A \in \text{alg } (V, V_{-1})$  if there is an  $n_0$  such that the equation  $P_n A P_n x_n$  $= P_n y$  has a unique solution  $x_n \in \text{im } P_n$  for each  $y \in X$  and for each  $n \geq n_0$  and if these solutions  $x_n$  converge in the norm of X to a solution x of the equation  $Ax = y$ . It is well known (Theorem 2.1 in [1: Chap. II, §2]) that the finite section method applies to A if and only if the operator A is invertible and if the sequence  $\{P_nAP_n\}$  is stable, i.e. there must exist an  $n_0$  such that the operators  $P_nAP_n|_{\text{im }P_n}$  are invertible for  $n \geq n_0$  and sup  $||(P_nAP_n|_{\text{im }P_n})^{-1}|| < \infty$ , or, equivalently, if *A* is invertible and the sequence  ${Q_n A^{-1} Q_n}$  is stable. method applies to  $A \in \text{aug}(V, V_{-1})$  true is an  $w_0$  such that the equation  $T_n A^n x_{n+1}$ <br>  $= P_n y$  has a unique solution  $x_n \in \text{im } P_n$  for each  $y \in X$  and for each  $n \ge n_0$  and if<br>
these solutions  $x_n$  converge in the norm ations  $x_n$  converge in the norm of X<br> *A* if and only if the operator A is is<br> *A* if and only if the operator A is is<br> *A*<sub>0</sub> and sup  $||(P_nAP_n|_{\text{im }P_n})^{-1}|| < \infty$ ,<br>  $\sum_{n \ge n_i}$ <br>
Paper  $|Q_nA^{-1}Q_n|$  is stable.<br>  $\text{em 11:}$  *e* finite sect<br>the sequence<br> $AP_n|_{\text{im }P_n}$  are<br>*i* f *A* is inv<br>*i i s* inv<br>*i i section met*<br>*i i y y l j* and *a* =<br>*i y y* and *a* =<br>*i b i sides by*  $\zeta$ 

Theorem 11: Let  $A \in \text{alg } (V, V_{-1}) \subseteq \mathcal{L}(l^{p, \gamma})$ . Then the finite section method applies *to A if and only if the operator A is invertible on ip.:* 

Proof: Let *A* be invertible. Then, by Theorem 6,  $A^{-1} \in \text{alg}(V, V_{-1}) = \text{alg}(AV'A^{-1}, V_{-1})$  $AV'_{-1}A^{-1}$ . Hence,  $A^{-1}A^{-1}A \in \text{alg}(V', V'_{-1})$ . Since the latter algebra decomposes we find that  $A^{-1}A^{-1}A = T'(a) + K'$  with  $T'(a) \in L(V')$ ,  $K' \in QC(V')$  and  $a = (\text{smb } A)^{-1}$ .

$$
A^{-1} = A T'(a) A^{-1} + K
$$

with some  $K \in QC(AV'A^{-1}) = QC(V)$ . Multiply (15) from both sides by  $Q_n = V_n V_{-n}$  $= A V_n^T \overline{V}_{n}^T A^{-1}$  to find

$$
Q_n A^{-1} Q_n = A V_n' V_{-n}' T'(a) V_n' V_{-n}' A^{-1} + Q_n K Q_n
$$
  
=  $A V_n' T'(a) V_{-n}' A^{-1} + Q_n K Q_n$ .

The invertibility of *A* implies (Theorem 6) that wind  $a = 0$ . Since the algebra alg  $(W, V'_{-1})$  decomposes, the operator  $T'(a)$  must be invertible (in alg  $(V', V'_{-1})$ ). Consequently, the operators  $AV_nT^i(a)V_{-n}^1A^{-1}|_{\text{im}\mathcal{Q}_n}$  are invertible, and the norms of their inverses  $AV_n'(T'(a))^{-1} V'_{-n}A^{-1} \mid_{\text{im}\mathcal{Q}_n}$  are uniformly bounded. Moreover, by Proposition  $4, ||Q_n KQ_n|| \rightarrow 0$  as  $n \rightarrow \infty$ , and these two facts lead to the stability of  $\{Q_n A^{-1}Q_n\}$  as desired **B** 

Remark: We emphasize that our approach to the theory of Toeplitz operators with continuous symbols on  $l^{p,\gamma}$  also applies (with minor modifications) to Wiener-Hopf operators as well as to operators in finite differences on weighted  $L^p$ -spaces. Fired  $\blacksquare$ <br>
Lemark: We emphasize that our approach to the lous symbols on  $l^{p,z}$  also applies (with minor mas to operators in finite differences on weighted<br>
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COCHBERG, I. Z., und I. A. FELDMAN: Faltungs,<br>
the

## REFERENCES

'S

- [1] GoeHBERG, I. Z., und I. A. FELDMAN: Faltungsgleichungen und Projektionsverfahren zu ihrer Lösung. Berlin: Akademie-Verlag 1974.
- [2] Михлип, С. Г.: Композиция двойных сингулярных, интегралов. Докл.: Акад. Наук.<br>СССР 2 (11), 1 (87) (1936), 3-6.

(15)

C

# 306 S. Roch and B. SILBEBMANN

- [3] Михлин, С. Г.: Сингулярные интегральные уравнения с двумя независимыми переменными. Мат. сб. 1 (43) (1936), 535-550. nepeMeHHbMM, C. Г.: Сингулярные интегральные уравнения с двумя независимыми<br>
Mat. C. Г.: Сингулярные интегральные уравнения с двумя независимыми<br>
(4) PROSSDORF, S.: Some classes of singular equations. Amsterdam—London: No
- Fubl. Comp. 1978.<br>[5] Roch, S., and B. SILBERMANN: Functions of shifts on Banach spaces Invertibility, dila-
- S. ROCH and B. SILBERMANN<br>
MUXJIMH, C. F.: CMHTYJIAPHE MITEPPAJEME YPABER<br>
(HOPSSDORF, S.: Some classes of singular equations. Amst<br>
PROSSDORF, S.: Some classes of singular equations. Amst<br>
Publ. Comp. 1978.<br>
ROCH, S., and tions, and numerical analysis. Preprint. Berlin: Akad. Wiss. DDR 1987, Preprint P-Math.-11j87. S. Roch and B. SLEEEMANN<br>
H, C. Г.: СИНГУЛЯРНЫЕ ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ С ДВУМЯ НЕЗАВИ<br>
ННЫМИ. Mat. c6. 1 (43) (1936), 535—550.<br>
SORE, S.: Some classes of singular equations. Amsterdam — London: North<br>
Comp. 1978.<br>
S., and 5] ROCH, S., and B. SILBERMANN: Functions of shifts on Banach spaces -<br>
ions, and numerical analysis. Preprint. Berlin: Akad. Wiss. DDR 1987<br>
11/87.<br>
Manuskripteingang: 05. 05. 1988<br>
VERFASSER:<br>
Dipl. Math. STEFFEN ROCH un [3] Михлин, С. Г.: Сингулярные интегральные уравнения (переменными. Мат. сб. 1 (43) (1936), 535 – 550.<br>
[4] Рабsбловг, S.: Some classes of singular equations. Amsterdam<br>
Publ. Comp. 1978.<br>
[5] ROCR, S., and R<sub>b</sub> SLLBERMA

0 S <sup>S</sup>

Manuskripteingang: 05. 05.1988

Sektion Mathematik der Technischen Universitht

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