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A Symbol Calculus for the Algebra Generated by Shift Operators

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Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Wir beschreiben die algebraische Struktur der durch den Verschiebungsoperator und einer seiner Linksinversen erzeugten Banach-Algebra und geben ein Symbol für die Regularisierbarkeit eines Operators bezüglich des Quasikommutatorideals an. Die Beziehungen dieses Symbols zur Invertierbarkeit bzw. zu den Fredholm-Eigenschaften der Elemente dieser Algebra werden untersucht.

Описывается алгебраическая структура Банаховы алгебры, порождённой оператором сдвига и одного его левого обратного, и строится символ, с помощью которого дается условие обратимости оператора относительно идеала полукоммутаторов. Исследуются соотношения между этим символом и обратимостью или нетеровостью элементов этой алгебры.

We describe the algebraical structure of the algebra generated by the shift operator and by one of its left-inverses and construct a symbol for the invertibility of an operator modulo the quasicommutator ideal. The correspondence between this symbol and the invertibility and Fredholmness of elements of this algebra are studied.

1. Introduction

In 1936, S. G. Michlin [2, 3] was the first who created a symbol concept for two-dimensional singular integral operators. Since that time the notion of symbols has gained an extraordinary significance in the theory of integral operators: It allows to algebraize large classes of operators in such a manner that operations with operators can be transformed into operations with their symbols which leads to essential simplifications in their treatment. Important classes of operators possessing a natural symbol calculus are pseudodifferential operators of Fourier and Mellin type as well as convolution operators. Meanwhile one has recognized that even the numerical solution of certain convolution equations (as, e.g. singular integral equations) by projection methods corresponds to the invertibility of special matrix- or operator-valued symbol functions on some compact.

In the present paper the authors raise a scheme due to I. Z. Gochberg and I. A. Feldman [1] which refers to continuous functions of shift operators. By a shift we here mean an only one-sided invertible operator V with the additional property that the spectrum of V and the spectrum of its one-sided inverse V_{-1} are both contained in the closed unit disk $\{z: |z| \leq 1\}$. To each operator A belonging to the closed algebra generated by V and V_{-1} we associate a complex-valued continuous function on the unit circle $\mathbf{T}$ — its symbol, and we examine the spectrum and the essential spectrum of the operator A in terms of the geometric behaviour of its symbol. Besides this we explain the algebraic structure of the algebra with generators V and V_{-1} and show that — in the best case — it decomposes into the direct sum of the lineal of all continuous functions of V and V_{-1} and of a certain ideal consisting of quasicommutators.

2. Functions of shift operators

Throughout this paper let X be a Banach space with identity operator I and let V denote a bounded linear operator on X which is only invertible from the left. We fix one of its left inverses, say V_{-1}, and put for brevity

$$V_n = \begin{cases} V^n & \text{if } n \geqq 0, \\ (V_{-1})^{-n} & \text{if } n < 0. \end{cases}$$

For a given polynomial $R(t) = \sum_{j=-n}^{n} a_j t^j$ on the unit circle $\mathbf{T}$ we define an operator $R(V)$ by $R(V) = \sum_{j=-n}^{n} a_j V_j$ and call $R(V)$ a polynomial of V. Let $L^0(V)$ stand for the set of all polynomials of V. Notice that there is a one-to-one correspondence between the operators in $L^0(V)$ and the polynomials on $\mathbf{T}$. Indeed, if $0 = \sum_{j=-n}^{n} a_j V_j$ and $a_{-n} \neq 0$, then $\sum_{j=-n}^{n} a_j V_j V_n = 0$, and we find $0 \neq a_{-n} I = V\left(-\sum_{j=-n+1}^{n} a_j V_{j+n-1}\right)$. This yields the right invertibility of V, which contradicts our hypothesis. If $R(V)$ is a polynomial with only positive powers of V, then the proof is similar.

The following hypothesis (H) will figure prominently in deriving invertibility criteria for polynomials of V:

(H) *The spectra $\sigma(V)$ and $\sigma(V_{-1})$ of V and V_{-1} are contained in* $\{z \in \mathbf{C}: |z| \leqq 1\}$.

Here $\sigma(\cdot)$ refers to the spectrum of a given operator in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on X.

Theorem 1 [1: Chap. I, § 1.3]: *Let* (H) *be fulfilled. Then the following assertions hold.*

a) $\sigma(V) = \sigma(V_{-1}) = \{z \in \mathbf{C}: |z| = 1\}$.

b) *An operator $R(V) \in L^0(V)$ is at least one-sided invertible if the function $R(t)$ has no zeros on $\mathbf{T}$. If $R(t) \neq 0$ on $\mathbf{T}$, then the invertibility of $R(V)$ corresponds to the index of $R(t)$, i.e. R is invertible, invertible only from the left or only from the right if the winding number of $R(t)$,*

$$\text{wind } R(t) := \frac{1}{2\pi} [\arg R(e^{iz})]_{z=0}^{2\pi},$$

is zero, positive, or negative, respectively.

c) *If $R(V)$ is one-sided invertible, then there exists a one-sided inverse of $R(V)$ in the algebra generated by V and V_{-1}.*

d) *The spectral radius of $R(V)$ equals* $\max_{t \in \mathbf{T}} |R(t)|$.

e) *If $R \in L^0(V)$ is a Φ-operator and if* $\varkappa := \operatorname{codim}(\operatorname{im} V) < \infty$, *then* $\operatorname{ind} R = -\varkappa \times \text{wind } R(t)$.

f) *If $R(t_0) = 0$ for some $t_0 \in \mathbf{T}$, then R is neither a Φ_+- nor a Φ_--operator.*

This theorem justifies to speak about $R(t)$ as the *symbol* of $R(V)$.

Let $L(V)$ stand for the closure of $L^0(V)$ in $\mathcal{L}(X)$. The elements of $L(V)$ will be called continuous functions of V. To each operator R in $L(V)$ we can associate a continuous function $R(t)$ on $\mathbf{T}$ — its symbol. In fact, by Theorem 1/d) we have

$$\max \{|R_n(t)| : t \in \mathbf{T}\} \leqq \|R_n(V)\| \tag{1}$$

for each polynomial $R_n(t)$. If $\{R_n\}$ denotes a sequence of polynomials converging to $R \in L(V)$, then, by (1), the sequence $\{R_n(t)\}$ converges uniformly to a certain continuous function $R(t)$ which, moreover, does not depend on $\{R_n\}$. In Section 4 we shall extend this definition of symbols to a larger class of operators.

We emphasize once more that the symbol concept for polynomials is distinguished by the following important aspects:

(i) An operator in $L^0(V)$ is uniquely determined by its symbol.

(ii) The invertibility of an operator in $L^0(V)$ depends only on its symbol.

One might enquire whether these properties are passed on to $L(V)$. The following example which is due to A. Pomp (private communication) shows that at least the first of them does not.

Example: Let m denote the Banach space of all bounded sequences of complex numbers and define

$$E = \left\{\{x_k\} = \{ak + y_k\}_{k=1}^{\infty} \text{ with } a \in \mathbf{C},\ \{y_k\}_{k=1}^{\infty} \in m\right\}.$$

Obviously,

$$E \ni x = \{x_k\}_{k=1}^{\infty} \mapsto \|x\|_E := \left|\lim_{k\to\infty} \frac{x_k}{k}\right| + \sup_n \left|x_n - n \lim_{k\to\infty} \frac{x_k}{k}\right|$$

defines a norm on E which makes E into a Banach space. Let V and V_{-1} be the operators on E which are defined by

$$V\{x_k\}_{k=1}^{\infty} = \{x_{k-1}\}_{k=1}^{\infty}, \quad x_0 = 0, \quad \text{and} \quad V_{-1}\{x_k\}_{k=1}^{\infty} = \{x_{k+1}\}_{k=1}^{\infty}.$$

Clearly, $V_{-1}V = I$ and $VV_{-1} \neq I$. Moreover, it is easy to see that the operators V and V_{-1} are bounded on E and that $\|V_n\| = |n| + 1$ for all $n \in \mathbf{Z}$. Consequently, the operators V, V_{-1} are subject of our hypothesis (H).

Proposition 1: *There exists an operator $A \in L(V)$ the symbol of which is identically zero but $A \neq 0$.*

Proof: Given $x = \{x_k\} \in E$ with $x_k/k \to a$ we define the operator A by $Ax = a\{1, 1, 1, \ldots\} \in E$. Evidently, A is bounded on E and $\|A\| = 1$. We claim that $\|1/n\ V_{-n} - A\| \to 0$ as $n \to \infty$: If $x = \{ak + y_k\} \in E$, then

$$\left(\frac{1}{n} V_{-n} - A\right) x = \left\{\frac{k+n}{n} a + \frac{y_{k+n}}{n} - a\right\}_{k=1}^{\infty} = \frac{1}{n} \{ka + y_{k+n}\}_{k=1}^{\infty}.$$

Hence,

$$\left\|\left(\frac{1}{n} V_{-n} - A\right) x\right\| = \frac{1}{n} \left(|a| + \sup_k |y_{k+n}|\right) \leqq \frac{1}{n} \|x\|_E.$$

Consequently, A is in $L(V)$, $A \neq 0$, but the symbol of A is identically zero since the symbols of $1/n\ V_{-n}$ converge uniformly to zero ∎

In what follows we shall only deal with the problem of the invertibility of functions of shift operators. Concerning the unique determination of an operator by its symbol we refer to [5], where, among other things, the following is proved.

Theorem 2: *Assume that $\bigcap_{n\geqq 0} \operatorname{im} V_n = \{0\}$. If one of the conditions*

$$\text{a) } \operatorname{clos} \bigcup_{n\geqq 0} \ker V_{-n} = X, \qquad \text{b) } \sup \|V_n V_{-n}\| < \infty$$

is fulfilled, then every operator in $L(V)$ is uniquely determined by its symbol.

3. V-dominating algebras

Our next concern is to extend the assertions of Theorem 1 to a subset of $L(V)$ as large as possible. Again, let V be an operator on X which is only invertible from the left and let V_{-1} be one of its left inverses. We do not assume the hypothesis (H) to be fulfilled. A commutative Banach algebra D with unit element e is called (V, V_{-1})-*dominating* (or, shortly, *V-dominating*) if

a) there is an invertible element d in D which spans together with its inverse d^{-1} a dense subalgebra of D,

b) the spectra $\sigma_D(d)$ and $\sigma_D(d^{-1})$ of d and d^{-1} in D are contained in $\{z \in \mathbf{C}: |z| \leqq 1\}$,

c) there exists a constant $M > 0$ such that

$$\|P(V)\|_{\mathcal{L}(X)} \leqq M \|P(d)\|_D \tag{2}$$

for any polynomial $P(t) = \sum_{j=-n}^{n} a_j t^j$, $|t| = 1$.

Proposition 2: *If a (V, V_{-1})-dominating algebra D exists, then V and V_{-1} are subject of the hypothesis* (H). *More general, if P is a polynomial on* $\mathbf{T}$, *then we have for the spectral radii $\varrho_{\mathcal{L}(X)}(P(V))$ and $\varrho_D(P(d))$ that*

$$\varrho_{\mathcal{L}(X)}(P(V)) \leqq \varrho_D(P(d)). \tag{3}$$

Proof: It is sufficient to verify the estimation (3): By (2), $\|P(V)^n\|_{\mathcal{L}(X)} \leqq M \|P(d)^n\|_D$ $(n \in \mathbf{Z}^+)$. Thus, $\|P(V)^n\|^{1/n} \leqq M^{1/n} \|P(d)^n\|^{1/n}$, and passing throught the limit yields the assertion ∎

Proposition 3: *The maximal ideal space $M(D)$ of D is homeomorphic to the unit circle* $\mathbf{T}$, *and the Gelfand transform maps d into the function $t \mapsto t$ $(t \in \mathbf{T})$.*

Proof: We shall show that the spectrum of d equals $\mathbf{T}$. The remaining assertions follow immediately from the general theory of commutative Banach algebras. By b) the spectrum $\sigma_D(d)$ of d in D belongs to $\mathbf{T}$. Assume that $\sigma_D(d) \neq \mathbf{T}$. Then there is an inner point z of $\mathbf{T} \setminus \sigma_D(d)$, and we can find a continuous f on $\mathbf{T}$ such that $0 \leqq f(t) \leqq 1$ $(t \in \mathbf{T})$, $f(z) = 1$ and $f(t) = 0$ for $t \in \sigma_D(d)$. Given $\varepsilon > 0$ we choose a polynomial $p \in C(\mathbf{T})$ so that $\max_{t \in \mathbf{T}} |f(t) - p(t)| < \varepsilon$. Since $|p(t)| < \varepsilon$ for all $t \in \sigma_D(d)$, we obtain $\varrho_D(p(d)) \leqq \varepsilon$. On the other hand, by Theorem 1/d) and by Proposition 2 we have $\varrho_{\mathcal{L}(X)}(p(V)) > 1 - \varepsilon$. These two inequalities contradict the hypothesis (2) for ε sufficiently small ∎

Since $p(d)$ is uniquely determined by its Gelfand transform $p(t)$ (even in the case that D has a non-trivial radical), the mapping $p(d) \mapsto p(V)$ is well-defined, linear, and by (2) bounded. Hence, we can extend this mapping continuously to the whole algebra D and its image, abbreviated to $L_D(V)$, is contained in $L(V)$. For $a \in D$ let $a(V)$ denote the image element of a under this mapping. Notice that

$$\|a(V)\|_{\mathcal{L}(X)} \leqq M \|a\|_D \tag{4}$$

for all $a \in D$ and that the symbol of $a(V)$ coincides with the Gelfand transform of $a \in D$. If the radical of D is trivial, then each element of $L_D(V)$ is uniquely determined by its symbol even if this is unknown for arbitrary elements in $L(V)$.

Theorem 3: *Let V be only invertible from the left and let the algebra D be (V, V_{-1})-dominating.*

a) *An operator* $R \in L_D(V)$ *is at least one-sided invertible if* $R(t) \neq 0$ *for all* $t \in \mathbf{T}$. *If the symbol* $R(t)$ *of* R *does not vanish on* $\mathbf{T}$, *then the invertibility of* R *corresponds to the winding number of* $R(t)$.

b) *If* $R(t_0) = 0$ *for some* $t_0 \in \mathbf{T}$, *then* R *is neither a* Φ_+- *nor a* Φ_--*operator*.

Proof: a) Let $R \in L_D(V)$ and $R(t) \neq 0$ $(t \in \mathbf{T})$. Then there is an element $a \in D$ with $a(V) = R$, and a must be invertible since $R(t)$ is the Gelfand transform of a. Thus, we can find a polynomial $p(t) = \sum_{j=-n}^{n} a_j t^j$ $(t \in \mathbf{T})$ such that $r = p(d)$ is invertible and $r^{-1}a = e + c$ with $\|c\|_D < 1/M$. As in the proof of Theorem 1 (see Theorem 1.1 in [1: Chap. I, § 1.3]) there is a representation of r in the form $r = r_- d^\varkappa r_+$ where r_+ and r_- are polynomials in d and d^{-1} with only non-negative and non-positive exponents, respectively, and $r_\pm(V) \in L^0(V)$ are invertible. Now write

$$a = \begin{cases} r_- d^\varkappa (e + c)\, r_+ & \text{if } \varkappa \leqq 0, \\ r_-(e + c)\, d^\varkappa r_+ & \text{if } \varkappa > 0, \end{cases}$$

Then

$$a(V) = \begin{cases} r_-(V)\, V_\varkappa \big(I + c(V)\big)\, r_+(V) & \text{if } \varkappa \leqq 0, \\ r_-(V) \big(I + c(V)\big)\, V_\varkappa r_+(V) & \text{if } \varkappa > 0. \end{cases}$$

Since $\|c(V)\| < 1$, the element $I + c(V)$ is invertible, and we are done.

b) Assume $R(t_0) = 0$, R is Φ_+. Then there is a $\delta > 0$ such that $\|R - r\| < \delta$ implies that r is Φ_+, too. Now take $r \in L^0(V)$ so that $\|R - r\| < \delta/2$. Because $|r(t_0)| < \delta/2$ and $\|R - (r - r(t_0)\, I)\| < \delta$ the operator $r - r(t_0)\, I \in L^0(V)$ is Φ_+. But this contradicts Theorem 1/f), since $r(t) - r(t_0)$ vanishes at $t = t_0$. The case that R is Φ_- can be treated analogously ▮

Now we are going to mention two examples of V-dominating algebras.

Example 1 (cp. [1: Chap. I, § 3.2]): Let Y be a Banach space with identity operator I. Assume that there are given a bounded projection operator P on Y and an invertible operator $U \in \mathcal{L}(Y)$ such that

$$\varrho_{\mathcal{L}(Y)}(U) \leqq 1, \quad \varrho_{\mathcal{L}(Y)}(U^{-1}) \leqq 1, \tag{5}$$

$$PUP = UP, \quad PU^{-1}P = PU^{-1}, \tag{6}$$

$$UP \neq PU. \tag{7}$$

Let D stand for the smallest closed subalgebra of $\mathcal{L}(Y)$ containing U and U^{-1}.

Theorem 4: a) *D is a commutative Banach algebra with a maximal ideal space homeomorphic to* $\mathbf{T}$.

b) *The operator* $PUP|_{\operatorname{im} P}$ *is invertible only from the left and* $PU^{-1}P|_{\operatorname{im} P}$ *is one of its left inverses.*

c) *D is* $(PUP|_{\operatorname{im} P}, PU^{-1}P|_{\operatorname{im} P})$-*dominating.*

Proof: First we verify b). Obviously, $PU^{-1}PUP = P$. Assume that $PUPAP = P$ with some $A \in \mathcal{L}(Y)$. By (6), $UPAP = P$ and $PAP = U^{-1}P$. Thus we get $PUPU^{-1}P = P$. Again by (6) this leads to $UPU^{-1} = P$ and $UP = PU$ which contradicts (7). For a proof of c) note that if $p(t)$ is a polynomial on $\mathbf{T}$, then, by (6), $p(PUP) = Pp(U)P$ whence $\|p(PUP)\| \leqq \|P\|^2\, \|p(U)\|$. Now part a) follows immediately from Proposition 3 ▮

Notice that, given a shift V, one can always find operators P and U given on a Banach space $Y \supset X$ such that (5)—(7) hold and $PU^{\pm 1}P = V_{\pm 1}$ [1: Chap. I, § 3.2].

Example 2: A sequence $f = \{f_k\}_{k=-\infty}^{\infty}$ of positive real numbers is called a *weight* if

$$\lim_{n\to\infty} \sqrt[n]{f_n} = \lim_{n\to\infty} \sqrt[n]{f_{-n}} = 1 \tag{8}$$

and if $f^* = \sup_{k,n} f_{n+k}/f_n f_k < \infty$. By $W(f)$ we denote the collection of all complex-valued functions a on $\mathbf{T}$ the Fourier coefficients a_k of which satisfy $\sum_{k=-\infty}^{\infty} |a_k| f_k < \infty$, and put

$$\|a\|_{W(f)} := f^* \sum_{k=-\infty}^{\infty} |a_k| f_k. \tag{9}$$

Theorem 5: a) *The set $W(f)$ forms a commutative Banach algebra under the norm* (9) *whose maximal ideal space is homeomorphic to* $\mathbf{T}$.

b) *The operator d: $a(t) \mapsto ta(t)$ spans together with its inverse a dense subalgebra of $W(f)$.*

c) *If V is a shift operator, then the algebra $W(\{\|V_n\|\}_{n=-\infty}^{\infty})$ is (V, V_{-1})-dominating.*

The proof follows from Proposition 3 if one takes into account (3) and the simple estimation $\|d^n a\| = \|t^n a(t)\| = f^* \sum |a_k| f_{k+n} \leqq (f^*)^2 f_n \sum |a_k| f_k$ which leads to $\|d^n\| \leqq f^* f_n$ ∎

4. The algebra generated by V and V_{-1}

Our next objective is to study the smallest closed subalgebra of $\mathcal{L}(X)$ containing V and V_{-1}. Denote by $\text{alg}^0 (V, V_{-1})$ the (non-closed) subalgebra of $\mathcal{L}(X)$ generated by V and V_{-1} and by $\text{alg}\,(V, V_{-1})$ its closure. Further we let refer $QC^0(V)$ to the smallest two-sided ideal of $\text{alg}^0 (V, V_{-1})$ which contains all quasicommutator operators of the form $(R_1R_2)(V) - R_1(V) R_2(V)$ where R_1, R_2 are arbitrary polynomials on $\mathbf{T}$. The closure $QC(V)$ of $QC^0(V)$ in $\text{alg}\,(V, V_{-1})$ is called the *quasicommutator ideal* of $\text{alg}\,(V, V_{-1})$. Henceforth, the quasicommutator $I - V_n V_{-n}$ of the operators V_n and V_{-n} will be denoted by P_n ($n \in \mathbf{Z}^+$), and we put $Q_n = I - P_n$. Obviously, P_n and Q_n are projection operators on $\mathcal{L}(X)$. In what follows we are mainly interested in the algebraic structure of $\text{alg}\,(V, V_{-1})$ and in whether an invertible operator is invertible in this algebra.

Proposition 4: *The following conditions are equivalent for $K \in \text{alg}\,(V, V_{-1})$:*

a) $K \in QC(V)$.

b) *K belongs to the smallest closed ideal of* $\text{alg}\,(V, V_{-1})$ *containing* P_1.

c) *If $\sup_n \|Q_n\| =: M < \infty$, then* a) *is equivalent to each of the following*: $\|Q_n K\| \to 0$ *and* $\|KQ_n\| \to 0$ *as* $n \to \infty$.

Proof: a) $\Rightarrow$ b): The quasicommutator ideal is generated by all operators $(R_1R_2)(V) - R_1(V) R_2(V)$ where $R_1(V)$ and $R_2(V)$ run through $L^0(V)$. Since $V_{i+j} - V_i V_j = P_i V_{i+j}$ and $P_i = \sum_{j=0}^{i-1} V_j P_1 V_{-j}$, the inclusion follows.

b) $\Rightarrow$ a): P_1 is the quasicommutator of V and V_{-1}.

b) $\Rightarrow$ c): We only prove that b) implies $\|Q_n K\| \to 0$. First we show that $\|Q_n A P_1\| \to 0$ as $n \to \infty$ if A is in $\text{alg}\,(V, V_{-1})$. Given $\varepsilon > 0$ write $A = A_\varepsilon + (A - A_\varepsilon)$ with $\|A - A_\varepsilon\| < \varepsilon$ and A_ε a finite sum of products of shifts. Because $V_s P_r = P_{r+s} V_s$ for all $r, s \in \mathbf{Z}$, we have

$$Q_n V_{i_1} V_{i_2} \dots V_{i_k} P_1 = Q_n P_{i_1+i_2+\dots+i_k} V_{i_1} V_{i_2} \dots V_{i_k}$$

$(i_1, i_2, \ldots, i_k \in \mathbf{Z})$ which is zero for $i_1 + i_2 + \cdots + i_k + 1 < n$. This yields that, given $\varepsilon > 0$, we can find an n_0 such that $\|Q_n A P_1\| < \varepsilon$ if only $n > n_0$. Now let $K \in QC(V)$. Then we write $K = K_\varepsilon + (K - K_\varepsilon)$ with $\|K - K_\varepsilon\| < \varepsilon$ and with $K_\varepsilon = \sum_{j=1}^{r} B_j P_1 C_j$ (where $B_j, C_j \in \text{alg}\,(V, V_{-1})$). By what has already been proved each item of $Q_n K_\varepsilon$ converges to zero, and we are done with the implication b) $\Rightarrow$ c).

c) $\Rightarrow$ a): If $\|Q_n K\| \to 0$ as $n \to \infty$, then K is the uniform limit of the operators $P_n K$ which are in $QC(V)$ ∎

Corollary 1: *If* codim (im V) $< \infty$, *then* $QC(V)$ *consists only of compact operators. If, moreover,* codim (im V) $= 1$ *and if* $\{P_n\}$ *and* $\{P_n^*\}$ *converge strongly to the identity operator on* X *and* X^*, *respectively, then* $QC(V)$ *equals the ideal of all compact operators on* X.

Proof: The first assertion is obvious from Proposition 4/b) since codim (im V) $< \infty$ implies dim (im P_1) $< \infty$. Now let codim (im V) $= 1$. Then each operator $P_n - P_{n-1}$ has rank 1. Consequently, for each linear bounded operator A on X we can find constants $a_{ij} \in \mathbf{C}$ so that

$$P_n A P_n = \sum_{i,j=1}^{n} (P_j - P_{j-1})\, A (P_i - P_{i-1}) = \sum_{i,j=1}^{n} a_{ij} V_{j-1} P_1 V_{-i+1}. \tag{10}$$

Hence, $P_n A P_n$ belongs to $QC(V)$ for each $A \in \mathcal{L}(X)$. In particular, if $A = K$ is a compact operator, then $\|P_n K P_n - K\| \to 0$ which implies that $K \in QC(V)$ ∎

The following construction will allow us to define a symbol calculus for the whole algebra alg (V, V_{-1}): Let alg$^\pi$ (V, V_{-1}) stand for the quotient algebra alg $(V, V_{-1})/QC(V)$ and denote by π the corresponding canonical homomorphism. Obviously, alg$^\pi$ (V, V_{-1}) is a commutative Banach algebra generated by $\pi(V)$ and by its inverse $\pi(V_{-1})$.

Proposition 5: *Assume that* (H) *is fulfilled. Then the spectrum* $\sigma(\pi(V))$ *of* $\pi(V)$ *coincides with the unit circle* $\mathbf{T}$, *and for each polynomial* $p(t)$, $|t| = 1$, *we have*

$$\max_{t \in \mathbf{T}} |p(t)| \leqq \|p(\pi(V))\| = \|\pi(p(V))\| \leqq \|p(V)\|. \tag{11}$$

Hence, the maximal ideal space of alg$^\pi$ (V, V_{-1}) *is homeomorphic to* $\mathbf{T}$, *and by* (11) *the symbol of an operator* $A \in L(V)$ *coincides with the Gelfand transform of* $\pi(A)$.

Proof: We have only to verify that $\sigma(\pi(V)) = \mathbf{T}$. The other assertions follow immediately from the general theory of Banach algebras. By (H) the spectra of $\pi(V)$ and $(\pi(V))^{-1} = \pi(V_{-1})$ are contained in $\mathbf{T}$. Assume that $\sigma(\pi(V)) \neq \mathbf{T}$ and choose $z_0 \in \mathbf{T} \setminus \sigma(\pi(V))$. Then there are operators $B \in$ alg (V, V_{-1}) and $K \in QC(V)$ such that $B(V - z_0 I) = I + K$. Approximate B by $B_0 = \sum_{i=1}^{n} \prod_{j=1}^{m} B_{ij}$, $B_{ij} \in L^0(V)$, so that $\|(B - B_0)(V - z_0 I)\| < 1/2$. Hence, $B_0(V - z_0 I) = I + K + C$ with $\|C\| < 1/2$. Further, approximate K by $K_0 = \sum_{i=1}^{n} \prod_{j=1}^{m} K_{ij}$, $K_{ij} \in L^0(V)$, so that $\|K - K_0\| < 1/2$. What results is that $B_0(V - z_0 I) = I + K_0 + C_0$ with $\|C_0\| < 1$. Now we represent $B_0 \in$ alg^0 (V, V_{-1}) in the form $B_0 = R_0 + S_0$ with $R_0 \in L^0(V)$ and $S_0 \in QC^0(V)$. (This is always possible; moreover, the representation is unique.) Thus, $R_0(V - z_0 I) = I + K_0 + C_0 - S_0(V - z_0 I)$. Since $K_0 - S_0(V - z_0 I) \in QC^0(V)$, we can find an $n \in \mathbf{Z}^+$ such that $(K_0 - S_0(V - z_0 I))\, V_n = 0$ (cp. the proof of Proposition 4). Finally, $R_0(V - z_0 I)\, V_n = (I + C_0)\, V_n$, and the right-hand side of this equation is invertible from the left. On the other hand, $R_0(V - z_0 I)\, V_n \in L^0(V)$, and $R_0(t) \times (t - z_0)\, t^n$ vanishes at $z_0 \in \mathbf{T}$ which is a contradiction to Theorem 1 ∎

If (H) is fulfilled, then the previous theorem enables us to assign to every operator $A \in \operatorname{alg}(V, V_{-1})$ a continuous function smb A on $\mathbf{T}$ called its symbol, namely the Gelfand transform of $\pi(A)$. Obviously, the mapping $A \mapsto \operatorname{smb} A$ is a continuous homomorphism and its kernel includes the quasicommutator ideal. The kernel equals $QC(V)$ if and only if $\operatorname{alg}^\pi(V, V_{-1})$ has a trivial radical. Proposition 5 implies that an operator $A \in \operatorname{alg}(V, V_{-1})$ is regularizable with respect to $QC(V)$ (that is, there exists an operator $B \in \operatorname{alg}(V, V_{-1})$ such that both $AB - I$ and $BA - I$ belong to $QC(V)$) if and only if $(\operatorname{smb} A)(t) \neq 0$ for all $t \in \mathbf{T}$. So it is natural to ask whether an operator $A \in \operatorname{alg}(V, V_{-1})$ being invertible in $\mathcal{L}(X)$ must be invertible in $\operatorname{alg}(V, V_{-1})$. This question will be answered positively by Theorem 6 below.

Proposition 6: *If $K \in QC(V)$ and if $I + K$ is at least one-sided invertible (in $\mathcal{L}(X)$), then $I + K$ is two-sided invertible in* $\operatorname{alg}(V, V_{-1})$, *and* $(I + K)^{-1} - I \in QC(V)$.

Proof: Approximate K by operators $K_n \in QC^0(V)$. As in the proof of Proposition 4 there exists a sequence $\{a_n\} \subseteq \mathbf{Z}^+$ such that $Q_{a_n}K_n = K_nQ_{a_n} = 0$. Hence, $K_n = P_{a_n}K_nP_{a_n}$, and the one-sided invertibility of $I + K$ yields that $I + K_n = I + P_{a_n}K_nP_{a_n} = Q_{a_n} + P_{a_n}(K_n + I)P_{a_n}$ is one-sided invertible in $\mathcal{L}(X)$ for n large enough. This shows that $P_{a_n} + P_{a_n}K_nP_{a_n}|_{\operatorname{im} P_{a_n}}$ must be one-sided invertible for n large enough. Now a little thought shows that the algebra of all operators $P_{a_n}RP_{a_n}|_{\operatorname{im} P_{a_n}}$ with arbitrary $R \in \operatorname{alg}(V, V_{-1})$ is isomorphic to the algebra $\mathbf{C}^{a_n \times a_n}$ (compare with the representation (10)). Thus, the one-sided invertibility of the operators $P_{a_n} + P_{a_n}K_nP_{a_n}|_{\operatorname{im} P_{a_n}}$ gives their two-sided invertibility, and their inverses are in $\operatorname{alg}(V, V_{-1})$. So we find that the operators $I + K_n = Q_{a_n} + P_{a_n}(K_n + I)P_{a_n}$ must be two-sided invertible in $\operatorname{alg}(V, V_{-1})$. Now assume for definiteness that $A = I + K$ is left-invertible and that A' is a left inverse for A. Then $A'(I + K_n)$ tends in the norm to the identity operator. Thus, for n large enough, we get the invertibility of $A'(I + K_n)$ and that of $(I + K_n)$. Therefore, the operator A' must be invertible and, consequently, A is invertible. Finally we note that $(I + K_n)^{-1} = I + R_n$ with $R_n \in QC(V)$, and the closedness of $QC(V)$ gives $A' - I = (I + K)^{-1} - I \in QC(V)$ ∎

Theorem 6: *Let* (H) *be fulfilled and assume that* $A \in \operatorname{alg}(V, V_{-1})$ *and let A be invertible in $\mathcal{L}(X)$. Then* $A^{-1} \in \operatorname{alg}(V, V_{-1})$ *and, moreover,* $\operatorname{smb} A^{-1} = (\operatorname{smb} A)^{-1}$ *and* $\operatorname{wind} \operatorname{smb} A = 0$.

Proof: Approximate A by operators $A_n \in \operatorname{alg}^0(V, V_{-1})$. If $\|A_n - A\| < 1/2\,\|A^{-1}\|$, then A_n is invertible and $\|A_n^{-1}\| < 2\,\|A^{-1}\|$. Thus, for large n,

$$\|A^{-1} - A_n^{-1}\| \leqq \|A_n^{-1}\|\,\|A^{-1}\|\,\|A_n - A\| < 2\,\|A^{-1}\|^2\,\|A_n - A\|,$$

and this estimation shows that it suffices to verify the assertion for operators in $\operatorname{alg}^0(V, V_{-1})$. Given $A \in \operatorname{alg}^0(V, V_{-1})$ we find operators $B \in L^0(V)$ and $K \in QC^0(V)$ and a number $l \in \mathbf{Z}^+$ such that $A = B + K$ and $KV_l = 0$. Thus, $AV_l = BV_l$, and the invertibility of A implies the invertibility of BV_l from the left. Theorem 1 shows that then B must be one-sided invertible and that its one-sided inverse belongs to $\operatorname{alg}(V, V_{-1})$. Assume, e.g., B to be invertible from the left and take $C \in \operatorname{alg}(V, V_{-1})$ so that $CB = I$. Then

$$A = B + K = B + KCB = (I + KC)\,B \tag{12}$$

with $KC \in QC(V)$. Since A is invertible, we conclude from (12) that $I + KC$ must be invertible from the right, which leads by Proposition 6 to the two-sided invertibility of $I + KC$ in $\operatorname{alg}(V, V_{-1})$. Put $T := (I + KC)^{-1} - I \in QC(V)$. Then $C(I + T)\,A = C(I + T)\,(I + KC)\,B = I$, i.e. $C(I + T)$ is a left inverse for A. Hence, $C(I + T)$

is an inverse for A and we arrive at $A^{-1} = C(I + T) \in \operatorname{alg}(V, V_{-1})$. Finally, the equation $\operatorname{smb} A^{-1} = (\operatorname{smb} A)^{-1}$ is obvious and the index formula follows from (12) by invoking Theorem 1 ∎

If the operator $A \in \operatorname{alg}(V, V_{-1})$ is of a special structure, we can specify the preceding theorem as follows.

Theorem 7: *Let* (H) *be fulfilled and assume there is a V-dominating algebra D. If $A = B + K$ (with $B \in L_D(V)$ and $K \in QC(V)$) is an invertible operator, then A and B are both invertible in* $\operatorname{alg}(V, V_{-1})$. *Moreover, there exist operators $B' \in L_D(V)$ and $K' \in QC(V)$ such that $A^{-1} = (B + K)^{-1} = B' + K'$, and for the symbols we have*

$$\operatorname{smb} A^{-1} = \operatorname{smb} B^{-1} = (\operatorname{smb} B)^{-1} = (\operatorname{smb} A)^{-1} = \operatorname{smb} B'.$$

Proof: If A is invertible, then, by Theorem 6, $A^{-1} \in \operatorname{alg}(V, V_{-1})$ and wind smb $A = 0$. Because smb A = smb B and $B \in L_D(V)$, we conclude via Theorem 3 that B is invertible and via Theorem 6 that $B^{-1} \in \operatorname{alg}(V, V_{-1})$. Let $B = a(V)$, $a \in D$. The invertibility of B implies the invertibility of a in D. Put $B' = (a^{-1})(V) \in L_D(V)$. Then $B'(B + K) = (a^{-1})(V)\big(a(V) + K\big) = I + K_1$ with $K_1 \in QC(V)$. Again by Theorem 3, the operator B' is invertible. So the operator $I + K_1$ must be invertible, too, and Proposition 6 states that $(I + K_1)^{-1} = I + K_2$ with $K_2 \in QC(V)$. Finally, the operator $(I + K_2) B' =: B' + K'$ is an inverse for A. The symbol identity is obvious ∎

The following theorem reifies Theorems 6 and 7 for the case when the operator V has a finite cokernel.

Theorem 8: *Let* (H) *be fulfilled and* $A \in \operatorname{alg}(V, V_{-1})$.

a) *If A is Φ_+ or Φ_-, then* $(\operatorname{smb} A)(t) \neq 0$ *for all* $t \in \mathbf{T}$.

b) *If* $\varkappa = \operatorname{codim}(\operatorname{im} V) < \infty$, *then A is a Φ-operator if and only if* $(\operatorname{smb} A)(t) \neq 0$ *for all $t \in \mathbf{T}$. If this condition is fulfilled, then* $\operatorname{ind} A = -\operatorname{wind}(\operatorname{smb} A)$.

We omit the proof since it follows from Theorem 1 by similar arguments as those in the proof of Theorem 6 and by the stability of ind and wind under small perturbations ∎

5. Decomposing algebras

Generally, all what we know about the relations between $L(V)$, $QC(V)$ and $\operatorname{alg}(V, V_{-1})$ is that the algebraical sum $L(V) + QC(V)$ is dense in $\operatorname{alg}(V, V_{-1})$. Under additional conditions one can get essentially more information about the structure of $\operatorname{alg}(V, V_{-1})$.

Theorem 9: *Assume that* (H) *is fulfilled and that for all polynomials $R_{ij}(t)$ on* $\mathbf{T}$

$$\|(\textstyle\sum_i \prod_j R_{ij})(V)\| \leq M \, \|\textstyle\sum_i \prod_j R_{ij}(V)\| \tag{13}$$

with some $M > 0$. Then the algebra $\operatorname{alg}(V, V_{-1})$ *decomposes into the direct sum* $\operatorname{alg}(V, V_{-1}) = L(V) \dotplus QC(V)$, *i.e. there is a projection $S \in \mathcal{L}(X)$ mapping* $\operatorname{alg}(V, V_{-1})$ *onto $L(V)$ parallel to $QC(V)$. Conversely, if* $\operatorname{alg}(V, V_{-1})$ *decomposes, then* (13) *holds with* $M = \|S\|$.

Proof: Define a linear mapping S on $\operatorname{alg}^0(V, V_{-1})$ by

$$S\colon \textstyle\sum_i \prod_j R_{ij}(V) \mapsto (\sum_i \prod_j R_{ij})(V) \in L^0(V).$$

S is well-defined and bounded by (13). Since $S(R) = R$ for $R \in L^0(V)$, the mapping S is a continuous projection operator mapping $\mathrm{alg}^0(V, V_{-1})$ onto $L^0(V)$. Its continuous extension onto the whole algebra $\mathrm{alg}(V, V_{-1})$ will again be denoted by S. Thus, $\mathrm{alg}(V, V_{-1}) = \mathrm{im}\, S \dotplus \ker S$. Obviously, $\mathrm{im}\, S = L(V)$, and it only remains to show that $\ker S = QC(V)$. By definition, $QC(V) \subseteq \ker S$. Now let $A \in \ker S$. Approximate A by A_n, $A_n \in \mathrm{alg}^0(V, V_{-1})$. Then $\|S(A_n)\| = \|S(A_n) - S(A)\| \to 0$. Since $S(A_n) - A_n \in QC^0(V)$, we get that A is in $QC(V)$ ∎

Remarks: (i) It is easy to see that the following is sufficient for (13): There is a family $\{W_n\}_{n=0}^{\infty}$ of left-invertible operators with left inverses $W_n^{(-1)}$ such that

1. $VW_n = W_nV$, $V_{-1}W_n^{(-1)} = W_n^{(-1)}V_{-1}$ for all $n \in \mathbf{Z}^+$,
2. $\sup_n \|W_n\| < \infty$, $\sup_n \|W_n^{(-1)}\| < \infty$,
3. $\|W_n^{(-1)}KW_n\| \to 0$ as $n \to \infty$ for any $K \in QC(V)$.

In particular, if $\sup_{n\in\mathbf{Z}} \|V_n\| < \infty$, we may take $\{V_n\}_{n\in\mathbf{Z}^+}$ as the family $\{W_n\}_{n\in\mathbf{Z}^+}$.

(ii) Obviously, $(\sum_i \prod_j R_{ij})(V) - \sum_i \prod_j R_{ij}(V) \in QC^0(V)$. Since $QC^0(V)$ is dense in $QC(V)$, we see that (13) is equivalent to the assertion $\|R(V)\| \leqq M \left\|\pi(R(V))\right\|$ for each polynomial R, i.e $\mathrm{alg}^\pi(V, V_{-1})$ is V-dominating. So we arrived at the following:

The algebra $\mathrm{alg}(V, V_{-1})$ *decomposes into* $L(V) \dotplus QC(V)$ *if and only if* $\mathrm{alg}^\pi(V, V_{-1})$ *is* V-*dominating. Moreover, if* $D = \mathrm{alg}^\pi(V, V_{-1})$ *is* V-*dominating, then* $L(V) = L_D(V)$.

(iii) By (ii) we can endow $L(V)$ with an equivalent norm and by a multiplication $\circ$ which makes $L(V)$ into a commutative Banach algebra $(L(V), \circ)$ which is isomorphic (and the isomorphism is continuous) to the quotient algebra $\mathrm{alg}^\pi(V, V_{-1})$. Obviously, $(L(V), \circ)$ is V-dominating and $L_{(L(V),\circ)}(V) = L(V)$.

6. Toeplitz operators on $l^{p,\gamma}$

We conclude these remarks by the following concrete realization, which is thought as an illustration of the general statements from the preceding sections. Let $X = l^{p,\gamma}$ $(p \geqq 1, \gamma \in \mathbf{R})$ be the Banach space of all sequences $x = (x_0, x_1, \ldots)$ of complex numbers with the norm

$$\|x\|_{p,\gamma} = \left(\sum_{k=0}^{\infty} |x_k|^p (k+1)^{p\gamma}\right)^{1/p}$$

and consider the operators V and V_{-1} on X given by

$$Vx = (0, x_0, x_1, \ldots), \qquad V_{-1}x = (x_1, x_2, \ldots).$$

For $n \in \mathbf{Z}^+$,

$$\|V_n\| = \begin{cases} (1+n)^\gamma & \text{if } \gamma \geqq 0, \\ 1 & \text{if } \gamma < 0, \end{cases} \qquad \|V_{-n}\| = \begin{cases} (1+n)^{-\gamma} & \text{if } \gamma \leqq 0, \\ 1 & \text{if } \gamma > 0. \end{cases}$$

In particular, the operators V and V_{-1} are subject of the hypothesis (H). The operators in $L(V)$ are called Toeplitz operators. By Theorem 2, each Toeplitz operator is uniquely determined by its symbol. As it is usual, if $A \in L(V)$ and if a is the symbol of A, we shall write $T(a)$ instead of A. The Banach spaces $l^{p,\gamma}$ and $l^{p,0}$ are isometrically isomorphic, and the isometry is given by

$$\Lambda\colon l^{p,0} \to l^{p,\gamma}, \qquad (x_k)_{k=0}^{\infty} \mapsto \left(x_k(k+1)^{-\gamma}\right)_{k=0}^{\infty}.$$

In the sequel we have to distinguish between Toeplitz operators on $l^{p,\gamma}$ and on $l^{p,0}$, both generated by the same symbol. Let us agree to designate the operators on the space $l^{p,0}$ (without weight γ) by a prime.

Proposition 7: *We have the inclusions,*

$$\Lambda^{-1}V_{\pm 1}\Lambda - V'_{\pm 1} \in QC(V'), \quad \Lambda V'_{\pm 1}\Lambda^{-1} - V_{\pm 1} \in QC(V).$$

Proof: We only demonstrate the first assertion. The matrix representation of $\Lambda^{-1}V\Lambda - V'$ in the canonical basis of $l^{p,0}$ is

$$\begin{pmatrix} 0 & 0 & 0 & \dots \\ \frac{2^\gamma}{1^\gamma} - 1 & 0 & 0 & \dots \\ 0 & \frac{3^\gamma}{2^\gamma} - 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

So the proof follows immediately from the characterization of the quasicommutator ideal given in Proposition 4/c) by invoking a simple norm estimation ∎

Notice that for $p \neq 1$ the operators $\Lambda^{-1}V_{\pm 1}\Lambda - V'_{\pm 1}$ are compact (this is due to Corollary 1). It is easy to see that this holds even for $p = 1$.

As a consequence of Proposition 7 we find that $\Lambda^{-1}P(V)\,\Lambda = P(V') + T'$ with $T' \in QC(V')$ for each polynomial P. As in the proof of Proposition 4, there is an $n \in \mathbf{Z}^+$ such that, for each $T' \in QC^0(V')$, $V'_{-n}\big(P(V') + T'\big)\,V_n = P(V')$. Since the norms $\|V'_{\pm n}\|$ are equal to 1 for $n \in \mathbf{Z}^+$, we conclude that

$$\|P(V')\| \leqq \|P(V') + T'\| \qquad \text{for all } T' \in QC(V'). \tag{14}$$

Corollary: *Let $T(a)$ be a Toeplitz operator on $l^{p,\gamma}$. Then a is also a symbol of a Toeplitz operator on $l^{p,0}$. Moreover, $\Lambda^{-1}T(a)\,\Lambda - T'(a) \in QC(V')$ and $\|T'(a)\| \leqq \|T(a)\|$.*

The proof follows immediately from (14) ∎

Theorem 10: *Let $T(a)$ be a Toeplitz operator on $l^{p,\gamma}$. Then $T(a)$ is at least one-sided invertible if and only if its symbol does not degenerate on* $\mathbf{T}$. *If this condition is fulfilled, then the one-sided invertibility of $T(a)$ corresponds to the winding number of a.*

Proof outline: If $T(a)$ is invertible, then, by Theorem 6, its symbol a must be invertible. Now let a be invertible. Consider the operator $T'(a)$ defined on $l^{p,0}$ by the same symbol a. By Remark (ii) we conclude that $T'(a)$ is at least one-sided invertible. On the other hand, Theorem 8 yields that $T(a)$ is a Φ-operator with ind $T(a) =$ $-$wind a. To finish the proof it remains to show that the kernels of $T(a)$ and of $T'(a)$ coincide ∎

Now we turn our attention to the structure of the algebra alg (V, V_{-1}) generated by V and V_{-1} on $l^{p,\gamma}$.

Proposition 8: *The algebra* alg (V, V_{-1}) *decomposes into $L(V) \dot{+} QC(V)$ if and only if $\gamma = 0$.*

Proof: It is easy to see that $\|\pi(V_n)\| = 1$ for all n, p and γ (cp. [5]). Comparing this with the norms of V_n quoted above we find that the projection S defined in Section 5 is bounded if and only if $\gamma = 0$ ∎

Now fix $p = 1$ and let $\gamma \neq 0$. Since $\|\pi(V_n)\| = 1$, the symbol algebra alg$^\pi$ (V, V_{-1}) contains a copy of the Wiener algebra consisting of all functions $a(t) = \sum_{i\in\mathbf{Z}} a_i t^i$

$(t \in \mathbf{T})$ with $\sum_{i \in \mathbf{Z}} |a_i| < \infty$. On the other hand, the matrix representation

$$T(a) \sim \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

of the operator $T(a)$ in the canonical basis of $l^{1,\gamma}$ shows that the sequence $(a_0, a_1, \dots)$ must necessarily belong to $l^{1,\gamma}$. Hence, there are symbols the corresponding operator of which is not the sum of a Toeplitz operator and an operator in $QC(V)$ (which has the zero symbol by definition). Such operators can be found among the operators of the form $\Lambda T'(a) \Lambda^{-1}$ where $T'(a)$ runs through the set $L(V')$ of all Toeplitz operators on $l^{p,0}$ as the following proposition indicates.

Proposition 9: *Let V and V' be the shift operators on $l^{p,\gamma}$ and $l^{p,0}$ $(p \geqq 1, \gamma \in \mathbf{R})$, respectively. Then*

a) $\operatorname{alg}(V, V_{-1}) = \operatorname{alg}(\Lambda V' \Lambda^{-1}, \Lambda V'_{-1} \Lambda^{-1})$,

b) *the algebras* $\operatorname{alg}^\pi(V, V_{-1}) = \operatorname{alg}(V, V_{-1})/QC(V)$ *and* $(L(V'), \circ)$ *(on $l^{p,0}$) are isomorphic.*

Proof: a) By Proposition 7, $\Lambda V'_{\pm 1} \Lambda^{-1} \in \operatorname{alg}(V, V_{-1})$, and so $\operatorname{alg}(\Lambda V' \Lambda^{-1}, \Lambda V'_{-1} \Lambda^{-1}) \subseteqq \operatorname{alg}(V, V_{-1})$. Analogously, $\operatorname{alg}(\Lambda^{-1} V \Lambda, \Lambda^{-1} V_{-1} \Lambda) \subseteqq \operatorname{alg}(V', V'_{-1})$, and since the algebras $\operatorname{alg}(\Lambda V' \Lambda^{-1}, \Lambda V'_{-1} \Lambda^{-1})$ and $\operatorname{alg}(V', V'_{-1})$ and the algebras $\operatorname{alg}(\Lambda^{-1} V \Lambda, \Lambda^{-1} V_{-1} \Lambda)$ and $\operatorname{alg}(V, V_{-1})$ are obviously isomorphic we get a).

b) Since $I - \Lambda V' \Lambda^{-1} \Lambda V'_{-1} \Lambda^{-1} = P_1$, the quasicommutator ideals $QC(V, V_{-1})$ of $\operatorname{alg}(V, V_{-1})$ and $QC(\Lambda V' \Lambda^{-1}, \Lambda'_{-1} \Lambda^{-1})$ of $\operatorname{alg}(\Lambda V' \Lambda^{-1}, \Lambda V'_{-1} \Lambda^{-1})$ coincide. Thus, the following algebras are isomorphic to each other:

$$\begin{aligned} \operatorname{alg}(V, V_{-1})/QC(V) &\cong \operatorname{alg}(\Lambda V' \Lambda^{-1}, \Lambda V'_{-1} \Lambda^{-1})/QC(\Lambda V' \Lambda^{-1}) \\ &\cong \operatorname{alg}(V', V'_{-1})/QC(V') \cong (L(V'), \circ), \end{aligned}$$

where the last isomorphy follows from the fact that $\operatorname{alg}(V', V'_{-1})$ decomposes ∎

The somewhat unexpected result of Proposition 9 is that the symbol algebras do not depend on γ (only on p). Hint: Proposition 9 does not mean that each symbol of a Toeplitz operator on $l^{p,0}$ is again a symbol of a Toeplitz operator on $l^{p,\gamma}$! In case $p = 2$ we can complete this picture as follows.

Proposition 10: *Let $p = 2$, $\gamma \in \mathbf{R}$. Then the algebra* $\operatorname{alg}(V, V_{-1})/QC(V)$ *is a C^*-algebra which is isomorphic to the algebra $C(\mathbf{T})$ of all continuous functions on the unit circle $\mathbf{T}$.*

Proof: Denote by $(\cdot, \cdot)_0$ the usual inner product on the Hilbert space $l^{2,0}$. The Banach space $l^{2,\gamma}$ $(\gamma \neq 0)$ can be made into a Hilbert space on defining an inner product by $(x, y)_\gamma = (\Lambda^{-1} x, \Lambda^{-1} y)_0$. Since

$$\begin{aligned} (x, Vy)_\gamma &= (\Lambda^{-1} x, \Lambda^{-1} V y)_0 = (\Lambda^{-1} x, \Lambda^{-1} V \Lambda \Lambda^{-1} y)_0 \\ &= \big(\Lambda^{-1} x, (V' + T') \Lambda^{-1} y\big)_0 \qquad (\text{with } T' \in QC(V')) \\ &= \big((V'_{-1} + T'') \Lambda^{-1} x, \Lambda^{-1} y\big)_0 \qquad (\text{with } T'' = (T')^* \in QC(V')) \\ &= \big(\Lambda^{-1} \Lambda (V'_{-1} + T'') \Lambda^{-1} x, \Lambda^{-1} y\big)_0 \\ &= \big(\Lambda^{-1} (V_{-1} + T) x, \Lambda^{-1} y\big)_0 \qquad (\text{with } T \in QC(V)) \\ &= \big((V_{-1} + T) x, y\big)_\gamma \end{aligned}$$

we see that $V^* - V_{-1} \in QC(V)$ and, analogously, we get $V_{-1}^* - V \in QC(V)$. Hence, $\text{alg}\,(V, V_{-1})$ is a C^*-algebra, and it remains to verify that the mapping $\mathcal{L}(l^{2,0}) \ni A \mapsto \Lambda A \Lambda^{-1} \in \mathcal{L}(l^{2,\gamma})$ is a $*$-isomorphism. This statement follows by similar conclusions as we have used above. In fact, we get $B^* = \Lambda^{-1}(\Lambda^{-1}B\Lambda)^* \Lambda$ for $B \in \text{alg}\,(V, V_{-1})$ where the star on the left denotes the adjoint with respect to $(\cdot, \cdot)_\gamma$ and the star on the right refers to the usual adjoint on $l^{2,0}$. Thus, the C^*-algebras $\text{alg}^\pi\,(V, V_{-1})$ and $\text{alg}^\pi\,(\Lambda V'\Lambda^{-1}, \Lambda V'_{-1}\Lambda^{-1})$ are star isomorphic, and the proof is complete since the latter is isomorphic to $C(\mathbb{T})$ by standard arguments (cp. Section 5) ∎

Our final goal is the finite section method for operators in $\text{alg}\,(V, V_{-1})$. Put $P_n = I - V_n V_{-n}$ and assume that $P_n \to I$ strongly. We say that the finite section method applies to $A \in \text{alg}\,(V, V_{-1})$ if there is an n_0 such that the equation $P_nAP_nx_n = P_ny$ has a unique solution $x_n \in \text{im}\, P_n$ for each $y \in X$ and for each $n \geqq n_0$ and if these solutions x_n converge in the norm of X to a solution x of the equation $Ax = y$. It is well known (Theorem 2.1 in [1: Chap. II, § 2]) that the finite section method applies to A if and only if the operator A is invertible and if the sequence $\{P_nAP_n\}$ is stable, i.e. there must exist an n_0 such that the operators $P_nAP_n|_{\text{im}\,P_n}$ are invertible for $n \geqq n_0$ and $\sup_{n \geqq n_0} \|(P_nAP_n|_{\text{im}\,P_n})^{-1}\| < \infty$, or, equivalently, if A is invertible and the sequence $\{Q_nA^{-1}Q_n\}$ is stable.

Theorem 11: *Let $A \in \text{alg}\,(V, V_{-1}) \subseteq \mathcal{L}(l^{p,\gamma})$. Then the finite section method applies to A if and only if the operator A is invertible on $l^{p,\gamma}$.*

Proof: Let A be invertible. Then, by Theorem 6, $A^{-1} \in \text{alg}\,(V, V_{-1}) = \text{alg}\,(\Lambda V'\Lambda^{-1}, \Lambda V'_{-1}\Lambda^{-1})$. Hence, $\Lambda^{-1}A^{-1}\Lambda \in \text{alg}\,(V', V'_{-1})$. Since the latter algebra decomposes we find that $\Lambda^{-1}A^{-1}\Lambda = T'(a) + K'$ with $T'(a) \in L(V')$, $K' \in QC(V')$ and $a = (\text{smb}\, A)^{-1}$. Hence,

$$A^{-1} = \Lambda T'(a)\Lambda^{-1} + K \tag{15}$$

with some $K \in QC(\Lambda V'\Lambda^{-1}) = QC(V)$. Multiply (15) from both sides by $Q_n = V_nV_{-n} = \Lambda V_n'V'_{-n}\Lambda^{-1}$ to find

$$\begin{aligned} Q_nA^{-1}Q_n &= \Lambda V_n'V'_{-n}T'(a)\, V_n'V'_{-n}\Lambda^{-1} + Q_nKQ_n \\ &= \Lambda V_n'T'(a)\, V'_{-n}\Lambda^{-1} + Q_nKQ_n. \end{aligned}$$

The invertibility of A implies (Theorem 6) that $\text{wind}\, a = 0$. Since the algebra $\text{alg}\,(V', V'_{-1})$ decomposes, the operator $T'(a)$ must be invertible (in $\text{alg}\,(V', V'_{-1})$). Consequently, the operators $\Lambda V_n'T'(a)\, V'_{-n}\Lambda^{-1}|_{\text{im}\,Q_n}$ are invertible, and the norms of their inverses $\Lambda V_n'(T'(a))^{-1}\, V'_{-n}\Lambda^{-1}|_{\text{im}\,Q_n}$ are uniformly bounded. Moreover, by Proposition 4, $\|Q_nKQ_n\| \to 0$ as $n \to \infty$, and these two facts lead to the stability of $\{Q_nA^{-1}Q_n\}$ as desired ∎

Remark: We emphasize that our approach to the theory of Toeplitz operators with continuous symbols on $l^{p,\gamma}$ also applies (with minor modifications) to Wiener-Hopf operators as well as to operators in finite differences on weighted L^p-spaces.

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