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The Explicit Solution of Elastodynamical Diffraction Problems by Symbol Factorization

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Dedicated to the occasion of the 80th birthday of Salomon G. Mikhlin on April 23, 1988

Es wird eine Klasse linearer Randwertprobleme betrachtet, die zeitharmonische elastische Wellenausbreitungen im Außengebiet einer Halbebene im $\mathbb{R}^3$ beschreiben. Die explizite Lösung ist möglich durch Wiener-Hopf-Faktorisierung bestimmter Typen von nichtrationalen 3×3 Matrixfunktionen, die als Fourier-Symbole der Randintegralgleichungen erscheinen.

Рассматривается класс линейных краевых задач описывающих гармонические по времени упругие распространения волн во внешней области полуплоскости в $\mathbb{R}^3$. Явное решение возможно посредством факторизации Винера-Хопфа определённого типа нерациональных функций от 3×3 матриц возникающих как символы Фурье уравнений с интегралами по границе.

A class of linear boundary value problems is considered due to time-harmonic elastic wave propagation in the exterior domain of a half-plane in $\mathbb{R}^3$. The explicit solution is obtained from a Wiener-Hopf factorization of specific types of non-rational 3×3 matrix functions, which occur as Fourier symbols in the corresponding boundary integral equations.

Introduction. Since S. G. MIKHLIN introduced the concept of the symbol of a singular integral operator 50 years ago [24], mathematicians working in various fields realized the importance of the fact that the structure of problems governed by convolutional type equations reflects in properties of the Fourier symbol function or matrix function, respectively. Wiener-Hopf equations and systems of them represent one of those fields. Their nature and explicit solution is directly connected with the factorization of the Fourier symbol, see the famous papers by I. GOHBERG and M. G. KREIN [8] up to the recent monographs by S. G. MIKHLIN and S. PRÖSSDORF [25] and others [13, 16, 27].

The problems treated here yield symbols in a particular algebra of non-rational matrix functions, for which we present a constructive factorization procedure. The basic ideas differ completely from those which are used for rational matrix functions, see [4, 5, 7, 9].

We shall concentrate on four boundary value problems which have been posed by V. D. KUPRADZE [15], but like to mention that the method applies also to other boundary value and transmission problems, see [21, 29, 30] for admissible boundary operators and [1–3, 6, 10, 14, 17, 18, 31] for background.

1. Formulation of the problems. Let $\Sigma = \{x \in \mathbb{R}^3 : x_1 > 0, x_3 = 0\}$, $\Omega = \mathbb{R}^3 - \bar{\Sigma}$, and boundary data $g^\pm$ be given in the vector Sobolev space $H^{-1/2}(\Sigma)^3$. We look for a weak solution $u \in H^1(\Omega)^3$ of

$$Lu = \left(\Delta + \frac{\lambda + \mu}{\mu} \operatorname{grad} \operatorname{div} + \frac{\omega^2 \varrho}{\mu}\right) u = 0 \quad \text{in } \Omega, \tag{1}$$

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$$V^{\pm}u = g^{\pm} \quad \text{on } \Sigma^{\pm}, \tag{2}$$

where λ, μ, ω, ϱ are known constants, which satisfy $\mu, \varrho > 0$, $\lambda + 2\mu > 0$, $\operatorname{Re}\omega$, $\operatorname{Im}\omega > 0$ [1]. The boundary data $V^{\pm}u$ which are always considered as functions defined on the full plane $(x_1, x_2) \in \mathbb{R}^2$ are given on the banks of Σ either by the values of the traction vector

$$\begin{aligned} u_T^{\pm} &= (Tu)^{\pm} = Tu|_{x_3=\pm 0} \\ &= \lambda n \operatorname{div} u + 2\mu \frac{\partial u}{\partial n} + \mu(n \times \operatorname{curl} u)|_{x_3=\pm 0} = g^{\pm} \end{aligned} \tag{2.I}$$

or the Dirichlet data (column) vector

$$u_0^{\pm} = (u_{01}^{\pm}, u_{02}^{\pm}, u_{03}^{\pm})^* = u|_{x_3=\pm 0} = g^{\pm} \tag{2.II}$$

or certain combinations of them

$$(u_{01}^{+}, u_{02}^{+}, u_{T3}^{+})^* = g^{+} \tag{2.III}$$

or

$$(u_{T1}^{\pm}, u_{T2}^{\pm}, u_{03}^{\pm})^* = g^{\pm} \tag{2.IV}$$

on $\Sigma^{\pm}$, respectively. For Dirichlet data, the given components naturally are assumed to belong to the trace space $H^{1/2}(\Sigma)$ instead of $H^{-1/2}(\Sigma)$. Moreover, it is well known that the jumps

$$f_0 = [u]_0 = u_0^{+} - u_0^{-} \in H^{1/2}(\mathbb{R}^2)^3, \quad f_T = [Tu]_0 = u_T^{+} - u_T^{-} \in H^{-1/2}(\mathbb{R}^2)^3 \tag{3}$$

of a solution of (1) across the plane $x_3 = 0$ are zero for $x_1 < 0$, in other words: the jumps across the boundary $[u]_\Sigma \in H^{1/2}(\Sigma)^3$, $[Tu]_\Sigma \in H^{-1/2}(\Sigma)^3$ are extendable by zero within the Cauchy data spaces $H^{\pm 1/2}(\mathbb{R}^2)^3$ (the manifold $\Sigma \subset \mathbb{R}^3$ is identified with a subset of $\mathbb{R}^2$). This operator theoretically important fact can be seen as a compatibility condition for the data and is often formulated as

$$f_0 \in \tilde{H}^{1/2}(\Sigma)^3, \quad f_T \in \tilde{H}^{-1/2}(\Sigma)^3 \tag{4}$$

meaning column vector functions with components in the closed subspaces of $H^{\pm 1/2}(\mathbb{R}^2)$ distributions supported on $\bar{\Sigma}$.

Therefore it makes sense to reformulate the boundary conditions (2) by use of the jumps (3), and, for symmetry, the sums of the data

$$\{u\}_0 = u_0^{+} + u_0^{-}, \quad \{Tu\}_0 = u_T^{+} + u_T^{-}. \tag{5}$$

So formulae (2) are transferred into transmission conditions where one of the data sets

$$f_T = [Tu]_0 \in \tilde{H}^{-1/2}(\Sigma)^3, \quad \{Tu\}_0 \in H^{-1/2}(\mathbb{R}^2)^3, \tag{6.I}$$

$$f_0 = [u]_0 \in \tilde{H}^{1/2}(\Sigma)^3, \quad \{u\}_0 \in H^{1/2}(\mathbb{R}^2)^3, \tag{6.II}$$

$$\begin{pmatrix} [u]_{01} \\ [u]_{02} \\ [Tu]_{03} \end{pmatrix} \in \tilde{H}^{1/2}(\Sigma)^2 \times \tilde{H}^{-1/2}(\Sigma), \quad \begin{pmatrix} \{u\}_{01} \\ \{u\}_{02} \\ \{Tu\}_{03} \end{pmatrix} \in H^{1/2}(\mathbb{R}^2)^2 \times H^{-1/2}(\mathbb{R}^2), \tag{6.III}$$

$$\begin{pmatrix} [Tu]_{01} \\ [Tu]_{02} \\ [u]_{03} \end{pmatrix} \in \tilde{H}^{-1/2}(\Sigma)^2 \times \tilde{H}^{1/2}(\Sigma), \quad \begin{pmatrix} \{Tu\}_{01} \\ \{Tu\}_{02} \\ \{u\}_{03} \end{pmatrix} \in H^{-1/2}(\mathbb{R}^2)^2 \times H^{1/2}(\mathbb{R}^2) \tag{6.IV}$$

is given on Σ. We denote by $\mathscr{P}_l$, $l = \mathrm{I, II, III, IV}$, the boundary value problem that corresponds to (1) and (6.l). The Dirichlet problem $\mathscr{P}_{\mathrm{II}}$ has already been treated by a simplified approach in [23]. The present more rigorous calculus also works (up to the explicit factorization) for boundary and transmission problems where $V^{\pm}u$ in (2) represent two arbitrary linear combinations of Dirichlet, Neumann ($u_1 = \partial u/\partial x_3$), and traction data

$$\begin{pmatrix} u_{01}^+ \\ u_{02}^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_{03}^+ \end{pmatrix}, \begin{pmatrix} u_{11}^+ \\ u_{12}^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_{13}^+ \end{pmatrix}, \begin{pmatrix} u_{T1}^+ \\ u_{T2}^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_{T3}^+ \end{pmatrix}, \begin{pmatrix} u_{01}^- \\ u_{02}^- \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ u_{T3}^- \end{pmatrix} \tag{7}$$

given on the banks of Σ. In regard of the equivalence to a Wiener-Hopf system the method applies also to (i) arbitrary plane Lipschitz domains (cracks) Σ, (ii) different media filling the half-spaces $x_3 > 0$ and $x_3 < 0$, respectively, and (iii) a second pair of conditions of type (7) instead of (3) on the complementary half-plane $\mathbb{R}^2 - \overline{\Sigma}$ as well, see analogue investigations for the Helmholtz equation [20–22, 28].

2. Representation of u by data on the plane $x_3 = 0$. We consider now the half-space $\Omega^+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$, a solution $u^+ \in H^1(\Omega^+)^3$ of (1), $Lu^+ = 0$ in Ω^+, the resulting Dirichlet data $u_0^+ = u^+|_{x_3=+0} \in H^{1/2}(\mathbb{R}^2)^3$ due to the trace theorem, which yields continuous dependence $u^+ \to u_0^+$, and ask for the inverse relation $u_0^+ \to u^+$, which means correct solution of the Dirichlet problem for (1) in Ω^+. We use the notation $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x' = (x_1, x_2)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi^2 = \xi_1^2 + \xi_2^2$, and, for brevity,

$$\hat{\varphi}_j(\xi) = F_{x'\mapsto\xi}\, \varphi_j(x') = \int_{\mathbb{R}^2} e^{ix'\xi} \varphi_j(x')\, dx',$$

$$t_j = t_j(\xi) = (\xi^2 - k_j^2)^{1/2}, \quad k_1^2 = \frac{\omega^2\varrho}{\lambda + 2\mu}, \quad k_2^2 = \frac{\omega^2\varrho}{\mu}$$

with $t_j \to +\infty$ as $\xi_1 \to +\infty$ and vertical branch cuts connecting $\pm i(\xi_2^2 - k_j^2)^{1/2}$ over ∞, I_+ denotes the characteristic function of $\mathbb{R}_+ = (0, \infty)$.

Proposition 1: *The general solution $u^+ \in H^1(\Omega^+)^3$ of the elastodynamical equations (1) in Ω^+ reads*

$$u^+(x) = F^{-1}_{\xi\mapsto x'} \begin{bmatrix} \hat{\varphi}_1(\xi)\, e^{-t_2(\xi)x_3} + \dfrac{i\xi_1}{t_1(\xi)}\, \hat{\varphi}_3(\xi)\, e^{-t_1(\xi)x_3} \\ \hat{\varphi}_2(\xi)\, e^{-t_2(\xi)x_3} + \dfrac{i\xi_2}{t_1(\xi)}\, \hat{\varphi}_3(\xi)\, e^{-t_1(\xi)x_3} \\ -\left(\dfrac{i\xi_1}{t_2(\xi)}\, \hat{\varphi}_1(\xi) + \dfrac{i\xi_2}{t_2(\xi)}\, \hat{\varphi}_2(\xi)\right) e^{-t_2(\xi)x_3} + \hat{\varphi}_3(\xi)\, e^{-t_1(\xi)x_3} \end{bmatrix} I_+(x_3)$$

or briefly (dropping the dependence on x' and ξ)

$$u^+ = F^{-1}\Phi_1 \cdot \begin{pmatrix} \hat{\varphi}_1\, e^{-t_2x_3} \\ \hat{\varphi}_2\, e^{-t_2x_3} \\ \hat{\varphi}_3\, e^{-t_1x_3} \end{pmatrix} I_+(x_3), \quad \Phi_1(\xi) = \begin{bmatrix} 1 & 0 & \dfrac{i\xi_1}{t_1} \\ 0 & 1 & \dfrac{i\xi_2}{t_1} \\ \dfrac{-i\xi_1}{t_2} & \dfrac{-i\xi_2}{t_2} & 1 \end{bmatrix} \tag{8}$$

where the column vector $\varphi^+ = (\varphi_1, \varphi_2, \varphi_3)^*$ satisfies

$$u_0^+ = B_1\varphi^+ = F^{-1}\Phi_1 \cdot F\varphi^+ \in H^{1/2}(\mathbb{R}^2)^3. \tag{9}$$

Proof: By use of the two-dimensional Fourier transformation with respect to x_1 and x_2, $D_j = \partial/\partial x_j = F^{-1}(-\mathrm{i}\xi_j)\cdot F$, the operator $L = L(D_1, D_2, D_3)$ can be written as

$$L = F^{-1}_{\xi\mapsto x'}\left\{\left(D_3^2 - \xi^2 + \frac{\omega^2\varrho}{\mu}\right) I - \frac{\lambda+\mu}{\mu}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \mathrm{i}\xi_1 D_3 \\ \xi_1\xi_2 & \xi_2^2 & \mathrm{i}\xi_2 D_3 \\ \mathrm{i}\xi_1 D_3 & \mathrm{i}\xi_2 D_3 & -D_3^2 \end{pmatrix}\right\} F_{x'\mapsto\xi}$$

on a (dense) subspace $S(\Omega^+)$ of smooth rapidly decreasing functions, where I is a suitably sized unit matrix. Abbreviating $\nu = (\lambda + \mu)/\mu$ and $k_2^2 = \omega^2\varrho/\mu$ we look for solutions of the homogeneous system of ordinary differential equations

$$L(-\mathrm{i}\xi_1, -\mathrm{i}\xi_2, D_3)\, u^{+\wedge}(\xi_1, \xi_2, x_3)$$
$$= \begin{pmatrix} D_3^2 - (\xi^2 + \nu\xi_1^2 - k_2^2) & -\nu\xi_1\xi_2 & -\mathrm{i}\nu\xi_1 D_3 \\ -\nu\xi_1\xi_2 & D_3^2 - (\xi^2 + \nu\xi_2^2 - k_2^2) & -\mathrm{i}\nu\xi_2 D_3 \\ -\mathrm{i}\nu\xi_2 D_3 & -\mathrm{i}\nu\xi_2 D_3 & (1+\nu)D_3^2 - (\xi^2 - k_2^2) \end{pmatrix}$$
$$\times\, u^{+\wedge}(\xi_1, \xi_2, x_3) = 0. \tag{10}$$

The *ansatz* $u^{+\wedge}(\xi, x_3) = \varphi(\xi)\,\mathrm{e}^{-t(\xi)x_3}$, $x_3 > 0$, with a parameter-dependent vector $\varphi(\xi)$ leads to the solvability condition

$$\det\begin{pmatrix} c & 0 & -\mathrm{i}\xi_1 c/t \\ 0 & c & -\mathrm{i}\xi_2 c/t \\ \mathrm{i}\nu\xi_1 t & \mathrm{i}\nu\xi_2 t & (1+\nu)\, t^2 - (\xi^2 - k_2^2) \end{pmatrix} = 0,$$

where $c = c(\xi) = t^2(\xi) - \xi^2 + k_2^2$. This yields $t = t_1$ or $t = t_2$ and the solutions (8) with arbitrary $\varphi_j \in S(\mathbb{R}^2)$.

According to the trace theorem for $H^1(\Omega^+)$ and the density of $S(\mathbb{R}^2)$ in $H^{1/2}(\mathbb{R}^2)$ one may extend this formula immediately to data $\varphi^+ \in H^{1/2}(\mathbb{R}^2)^3$, since the Dirichlet data u_0^+ result from φ^+ by the action of the pseudo-differential operator B_1 of order zero, see (8), (9).

Conversely we have

$$\varphi^{+\wedge} = \Phi_1^{-1}\cdot u_0^{+\wedge},$$

$$\Phi_1^{-1}(\xi) = \frac{1}{t_1t_2 - \xi^2}\begin{pmatrix} t_1t_2 - \xi_2^2 & \xi_1\xi_2 & -\mathrm{i}\xi_1 t_2 \\ \xi_1\xi_2 & t_1t_2 - \xi_1^2 & -\mathrm{i}\xi_2 t_2 \\ \mathrm{i}\xi_1 t_1 & \mathrm{i}\xi_2 t_1 & t_1t_2 \end{pmatrix}, \tag{11}$$

where, despite of the boundedness of $\Phi_1(\xi)$, $\xi \in \mathbb{R}^2$, the matrix elements can grow like $O(\xi^2)$ as $|\xi| \to \infty$ according to

$$\tau(\xi) = \xi^2 - t_1t_2 \to (k_1^2 + k_2^2)/2, \qquad |\xi| \to \infty. \tag{12}$$

This means that Dirichlet data $u_0^+ \in H^{1/2}(\mathbb{R}^2)^3$ yield only ansatz data $\varphi \in H^{-3/2}(\mathbb{R}^2)^3$ in general, but, however, the corresponding half-space solution u^+ is still in $H^1(\Omega^+)^3$ according to an asymptotic cancellation of higher order terms

$$u^+(x) = F^{-1}_{\xi\mapsto x'}\,\Phi_1(\xi)\begin{pmatrix} \mathrm{e}^{-t_1x_3} & 0 & 0 \\ 0 & \mathrm{e}^{-t_2x_3} & 0 \\ 0 & 0 & \mathrm{e}^{-t_1x_3} \end{pmatrix}\Phi_1^{-1}(\xi)\, 1_+(x_3)\, u_0^{+\wedge}(\xi)$$
$$= F^{-1}_{\xi\mapsto x'}\left\{\mathrm{e}^{-t_2x_3} I + \frac{\mathrm{e}^{-t_1x_3} - \mathrm{e}^{-t_2x_3}}{\xi^2 - t_1t_2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & -\mathrm{i}\xi_1 t_2 \\ \xi_1\xi_2 & \xi_2^2 & -\mathrm{i}\xi_2 t_2 \\ -\mathrm{i}\xi_1 t_1 & -\mathrm{i}\xi_2 t_1 & -t_1t_2 \end{pmatrix}\right\}$$
$$\times\, 1_+(x_3)\, u_0^{+\wedge}(\xi) \tag{13}$$

which easily shows $\|u^+\|_{H^1(\Omega^+)^3} \leq \text{const} \cdot \|u_0^+\|_{H^{1/2}(\mathbb{R}^2)^3}$. Thus we have existence of a solution u^+ and continuous dependence on u_0^+.

Proving uniqueness we show that $u^+ \in H_0^1(\Omega^+)^3$ and $Lu^+ = 0$ imply $u^+ = 0$ in Ω^+. In this case one may Fourier transform the differential equations with respect to all three variables, since $H_0^1(\Omega^+)^3$ is a subspace of $H^1(\mathbb{R}^3)^3$, obtaining (10) with D_3 replaced by $-i\xi_3$. The resulting matrix can be inverted, which yields $u^+ = 0$ after inverse Fourier transformation ∎

Remark: It is possible to express the general solution of $Lu^+ = 0$ in Ω^+ in terms of several other data on $x_3 = +0$ (see Chapter 3). The "ansatz data representation" (8) gives the simplest formulae in a sense and it includes the physically important decomposition $u^+ = u_s^+ + u_p^+$ into shear and pressure waves, which corresponds to curl $u_s^+ = 0$, div $u_p^+ = 0$ where $\varphi_1 = \varphi_2 = 0$; $\varphi_3 = 0$ hold, respectively.

In contrast to other elliptic boundary value problems, e.g. for the Helmholtz equation [21, 29, 30], which are also governed by coupled systems of Wiener-Hopf equations, the dependence of u^+ on the (exponential) ansatz data φ^+ (instead of u_0^+ or others) is not a topological mapping (since Φ_1^{-1} is unbounded). This fact is important if we look for well-posed formulations of elastodynamical boundary or transmission problems, and motivates the preference over the "s-p-decomposition" in this paper.

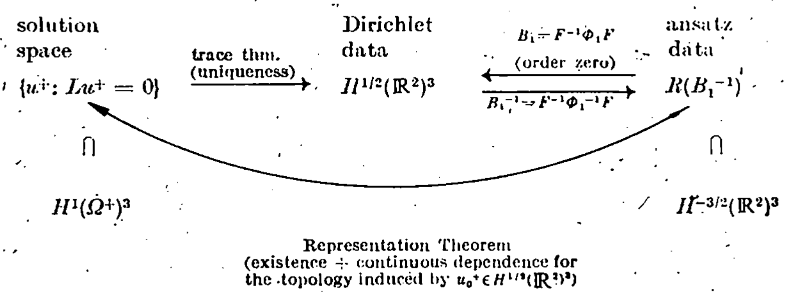

Figure 1: The choice of function spaces

Corollary 1: *The solution of $Lu^- = 0$ in $H^1(\Omega^-)^3$, $\Omega^- = \{x \in \mathbb{R}^3 : x_3 < 0\}$, reads*

$$u^- = F^{-1}\Phi_2 \begin{pmatrix} e^{t_1 x_3} & 0 & 0 \\ 0 & e^{t_1 x_3} & 0 \\ 0 & 0 & e^{t_1 x_3} \end{pmatrix} \Phi_2^{-1} 1_+(x_3)\, u_0^{-\wedge} \tag{14}$$

with Dirichlet data $u_0^- \in H^{1/2}(\mathbb{R}^2)^3$ on $x_3 = 0$ and

$$\Phi_2(\xi) = \begin{bmatrix} 1 & 0 & -\dfrac{i\xi_1}{t_1} \\ 0 & 1 & -\dfrac{i\xi_2}{t_1} \\ \dfrac{i\xi_1}{t_2} & \dfrac{i\xi_2}{t_2} & 1 \end{bmatrix} = M\Phi_1(\xi)\, M, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{15}$$

Again, the dependence $u_0^- \to u^-$, $H^{1/2}(\mathbb{R}^2)^3 \to \{u^- \in H^1(\Omega^-)^3 : Lu^- = 0\}$ is bijective, and the ansatz functionals $\varphi^- = B_2^{-1}u_0^- = F^{-1}\Phi_2^{-1} \cdot Fu_0^-$ are not necessarily in $H^{1/2}(\mathbb{R}^2)^3$ but in a (strange) non-closed subspace of $H^{-3/2}(\mathbb{R}^2)^3$.

Corollary 2: *The general solution* $u \in H^1(\Omega)^3$ *of* $Lu = 0$ *in* $\Omega = \mathbb{R}^3 - \bar{\Sigma}$ *is given by* $u = u^\pm$ *in* $\Omega^\pm$, *see formulae* (8), (9) *and* (14), *respectively, iff those half-space solutions satisfy*

$$f_0 = u_0^+ - u_0^- = 0, \qquad f_T = (Tu)^+ - (Tu)^- = 0 \quad \text{on } \mathbb{R}^2 - \Sigma. \tag{16}$$

This is also a consequence of Proposition 1. The traction jump condition can be replaced by the Neumann jump condition $f_1 = u_1^+ - u_1^- = 0$ on $\mathbb{R}^2 - \Sigma$ (which leads to equivalent but slightly less esthetic formulae).

3. The calculus of boundary operators and their Fourier symbol matrix functions. We study further relations between boundary data on $x_3 = \pm 0$ of half-space solutions of $Lu^\pm = 0$ in Ω^+ or Ω^-, respectively. From the preceeding formulae the following data are in 1-1-correspondence and related by convolution (translation invariant) operators $B_j = F^{-1}\Phi_j \cdot F$:

$$\begin{aligned}
&B_1: \varphi^+ \mapsto u_0^+, \quad B_2: \varphi^- \mapsto u_0^-, \quad B_3: \varphi^+ \mapsto u_1^+,\\
&B_4: \varphi^- \mapsto u_1^-, \quad B_5: \varphi^+ \mapsto u_T^+, \quad B_6: \varphi^- \mapsto u_T^-,\\
&B_7: \varphi^+ \mapsto (u_{01}^+, u_{02}^+, u_{T3}^+)^*, \quad B_8: \varphi^- \mapsto (u_{01}^-, u_{02}^-, u_{T3}^-)^*\\
&B_9: \varphi^+ \mapsto (u_{T1}^+, u_{T2}^+, u_{03}^+)^*, \quad B_{10}: \varphi^- \mapsto (u_{T1}^-, u_{T2}^-, u_{03}^-)^*.
\end{aligned} \tag{17}$$

It is easy to see that Φ_{2j} follows from Φ_{2j-1} by replacing t_1 by $-t_1$ and t_2 by $-t_2$. We now list all these matrices ($\xi \in \mathbb{R}^2$):

$$\Phi_1(\xi) = \begin{bmatrix} 1 & 0 & \dfrac{i\xi_1}{t_1} \\ 0 & 1 & \dfrac{i\xi_2}{t_1} \\ -\dfrac{i\xi_1}{t_2} & -\dfrac{i\xi_2}{t_2} & 1 \end{bmatrix}, \qquad \Phi_2 = M\Phi_1 M, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\Phi_3 = -\Phi_1 D, \qquad D = \begin{pmatrix} t_2 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \qquad \Phi_4 = \Phi_2 D = M\Phi_1 M D,$$

$$\Phi_5 = -\mu \begin{bmatrix} \dfrac{t_2^2 + \xi_1^2}{t_2} & \dfrac{\xi_1\xi_2}{t_2} & 2i\xi_1 \\ \dfrac{\xi_1\xi_2}{t_2} & \dfrac{t_2^2 + \xi_2^2}{t_2} & 2i\xi_2 \\ -2i\xi_1 & -2i\xi_2 & \dfrac{t_2^2 + \xi^2}{t_1} \end{bmatrix}, \qquad \Phi_6 = -M\Phi_5 M, \tag{18}$$

$$\Phi_7 = \begin{pmatrix} \Phi_1 \text{ rows} \\ \hline \Phi_5 \text{ row} \end{pmatrix} = \begin{bmatrix} 1 & 0 & \dfrac{i\xi_1}{t_1} \\ 0 & 1 & \dfrac{i\xi_2}{t_1} \\ 2i\mu\xi_1 & 2i\mu\xi_2 & -\mu\dfrac{t_2^2 + \xi^2}{t_1} \end{bmatrix}, \qquad \Phi_8 = \begin{pmatrix} \Phi_2 \text{ rows} \\ \hline \Phi_6 \text{ row} \end{pmatrix} = \Phi_7 M,$$

$$\Phi_9 = \begin{pmatrix} \Phi_5 \text{ rows} \\ \hline \Phi_1 \text{ row} \end{pmatrix} = \begin{pmatrix} -\mu \dfrac{t_2^2 + \xi_1^2}{t_2} & -\mu \dfrac{\xi_1\xi_2}{t_2} & -2i\mu\xi_1 \\ -\mu \dfrac{\xi_1\xi_2}{t_2} & -\mu \dfrac{t_2^2 + \xi_2^2}{t_2} & -2i\mu\xi_2 \\ -\dfrac{i\xi_1}{t_2} & -\dfrac{i\xi_2}{t_2} & 1 \end{pmatrix},$$

$$\Phi_{10} = \begin{pmatrix} \Phi_6 \text{ rows} \\ \hline \Phi_2 \text{ row} \end{pmatrix} = -\Phi_9 M.$$

Remark: One observes many common (more or less relevant) properties of these matrix functions

$$\Phi(\xi) = (\theta_{jk}(\xi_1, \xi_2))_{j,k=1,2,3} \in \mathbb{C}(\xi_1, \xi_2, t_1(\xi), t_2(\xi))^{3\times 3}$$

which are rational in ξ_1, ξ_2, t_1, t_2. There is, for instance, a certain symmetry in ξ_1 and ξ_2, which we briefly describe, putting $\theta_{jk}(\xi_2, \xi_1) = \theta_{jk}^{\times}(\xi_1, \xi_2)$, by $\theta_{11}^{\times} = \theta_{22}, \theta_{12}^{\times} = \theta_{21}, \theta_{12}^{\times} = \theta_{23}, \theta_{31}^{\times} = \theta_{32}$ so that 5 of 9 entries already describe the matrix:

$$\Phi = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{12}^{\times} & \theta_{11}^{\times} & \theta_{13}^{\times} \\ \theta_{31} & \theta_{31}^{\times} & \theta_{33} \end{pmatrix} \tag{19}$$

due to physical isotropy in ξ_1, ξ_2 (tangential Σ direction). The following three results are easily proved.

Lemma 1: *Matrix functions of symmetry type* (19) *form an algebra* $\mathcal{A}^{\times}$. *Linear combinations of boundary data mentioned in* (7) *depend on* φ^+ *and* φ^- *by operators* $A = F^{-1}\Phi \cdot F$ *with* $\Phi \in \mathcal{A}^{\times}$.

Further function theoretic (holomorphy) and operator theoretic (mapping) properties will be analyzed later. Here we present some algebraic insights, which are most useful for explicit factorization, and introduce for this purpose

$$\begin{aligned} \xi^0 &= \frac{1}{|\xi|}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad \xi^{0*} = \frac{1}{|\xi|}(\xi_1, \xi_2), \\ R_1(\xi) &= \frac{1}{\xi^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}, \qquad R_2(\xi) = \frac{1}{\xi^2}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix} \end{aligned} \tag{20}$$

where $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$ for $\xi \in \mathbb{R}^2$.

Lemma 2: R_j *are complementary projection matrices of rank* 1, *they are rational functions, symmetric and real-valued for* $\xi \in \mathbb{R}^2$, *i.e.*

$$\begin{aligned} &R_j^2 = R_j, \qquad R_1 + R_2 = I, \\ &R_1 = S^*\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S, \quad S = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix} = S^{-1*} \\ &R_j^* = R_j, \quad \operatorname{Im} R_j(\xi) = 0 \quad \textit{for} \quad \xi \in \mathbb{R}^2. \end{aligned} \tag{21}$$

Furthermore the vector ξ^0 *satisfies*

$$\begin{aligned} &\xi^{0*}\xi^0 = |\xi^0|^2 = 1, \qquad \xi^0\xi^{0*} = R_1, \\ &R_1\xi^0 = \xi^0, \qquad R_2\xi^0 = 0, \qquad \xi^{0*}R_1 = \xi^{0*}, \qquad \xi^{0*}R_2 = 0, \end{aligned} \tag{22}$$

in particular, it is an eigenvector of R_j, $j = 1, 2$. Finally there hold in 3×3 matrix notation

$$\begin{pmatrix} 0 & 0 & i\xi_1 \\ 0 & 0 & i\xi_2 \\ -i\xi_1 & -i\xi_2 & 0 \end{pmatrix}^2 = \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & 0 \\ \xi_1\xi_2 & \xi_2^2 & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}$$

and, more generally, in block matrix form

$$\left(\begin{array}{c|c} 0 & i\xi^0 \\ \hline -i\xi^{0*} & 0 \end{array}\right)^n = \begin{cases} \left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & 1 \end{array}\right), & \dfrac{n}{2} \in \mathbb{N}, \\ \left(\begin{array}{c|c} 0 & i\xi^0 \\ \hline -i\xi^{0*} & 0 \end{array}\right), & \dfrac{n-1}{2} \in \mathbb{N}. \end{cases} \tag{23}$$

Lemma 3: 1. *All matrix functions of the form*

$$\Theta = \left(\begin{array}{c|c} aR_1 + bR_2 & ic\xi^0 \\ \hline -id\xi^{0*} & e \end{array}\right) \tag{24}$$

with scalar functions a, b, c, d, e of the variable $\xi \in \mathbb{R}^2$ form an algebra $\mathcal{A}$.

2. *The product of θ_1, $\theta_2 \in \mathcal{A}$ reads (with suggestive numbering)*

$$\Theta_1\Theta_2 = \left(\begin{array}{c|c} (a_1a_2 + c_1d_2) R_1 + b_1b_2R_2 & i(a_1c_2 + c_1e_2)\xi^0 \\ \hline -i(d_1a_2 + e_1d_2)\xi^{0*} & d_1c_2 + e_1e_2 \end{array}\right). \tag{25}$$

3. *The determinant and inverse of Θ have the form*

$$\det \Theta = b(ae - cd),$$

$$\Theta^{-1} = \left(\begin{array}{c|c} \dfrac{e}{ae - cd} R_1 + \dfrac{1}{b} R_2 & \dfrac{-c}{ae - cd} i\xi^0 \\ \hline \dfrac{-d}{ae - cd} (-i\xi^{0*}) & \dfrac{a}{ae - cd} \end{array}\right). \tag{26}$$

Remarks: 1. For the proof it is convenient to show firstly by use of Lemma 2

$$\det (aR_1 + bR_2) = ab,$$

$$\Theta = \left(\begin{array}{c|c} aR_1 + 1 \cdot R_2 & ic\xi^0 \\ \hline -id\xi^{0*} & e \end{array}\right) \cdot \left(\begin{array}{c|c} 1 \cdot R_1 + bR_2 & 0 \\ \hline 0 & 1 \end{array}\right) \tag{27}$$

which two factors commute. So the b term can be handled as an isolated scalar (or diagonal) factor and the rest is governed by 2×2 matrix computational rules. **2.** All the 3×3 matrices Φ_j (as well as I, M, D and Φ_j^{-1}) have the form (24). **3.** According to the importance of rationality (and for brevity) we shall write (putting $\tilde{c} = c/|\xi|$ and $\tilde{d} = d/|\xi|$)

$$\Theta = \left(\begin{array}{c|c} aR_1 + bR_2 & \tilde{c}\mathrm{i}\xi \\ \hline \tilde{d}\mathrm{i}\xi^* & e \end{array}\right)$$

which is a rational matrix function, if the coefficients are rational. The product formula (25) is then changed into

$$\Theta_1\Theta_2 = \left(\begin{array}{c|c} (a_1a_2 - \tilde{c}_1\tilde{d}_2\xi^2) R_1 + b_1b_2R_2 & (a_1\tilde{c}_2 + \tilde{c}_1e_2)\,\mathrm{i}\xi \\ \hline (\tilde{d}_1a_2 + e_1\tilde{d}_2)\,\mathrm{i}\xi^* & e_1e_2 - \tilde{d}_1\tilde{c}_2\xi^2 \end{array}\right). \tag{28}$$

It is now very easy to compute the inverse symbol matrix functions due to Φ_j, $j = 1, \ldots, 10$.

Example: Write and compare with (11)

$$\Phi_1 = \begin{pmatrix} 1 & 0 & \dfrac{\mathrm{i}\xi_1}{t_1} \\ 0 & 1 & \dfrac{\mathrm{i}\xi_2}{t_1} \\ -\dfrac{\mathrm{i}\xi_1}{t_2} & -\dfrac{\mathrm{i}\xi_2}{t_2} & 1 \end{pmatrix} = \left(\begin{array}{c|c} 1 \cdot R_1 + 1 \cdot R_2 & \mathrm{i}\dfrac{|\xi|}{t_1}\xi^0 \\ \hline -\mathrm{i}\dfrac{|\xi|}{t_2}\xi^{0*} & 1 \end{array}\right),$$

$$\det \Phi_1 = b(ae - cd) = 1 \cdot (1 - \xi^2/t_1t_2) = (t_1t_2 - \xi^2)/t_1t_2,$$

$$\Phi_1^{-1} = \left(\begin{array}{c|c} \dfrac{t_1t_2}{t_1t_2 - \xi^2} R_1 + R_2 & \mathrm{i}\dfrac{-|\xi|\,t_2}{t_1t_2 - \xi^2}\xi^0 \\ \hline -\mathrm{i}\dfrac{-|\xi|\,t_1}{t_1t_2 - \xi^2}\xi^{0*} & \dfrac{t_1t_2}{t_1t_2 - \xi^2} \end{array}\right) = \frac{1}{t_1t_2 - \xi^2}\left(\begin{array}{c|c} t_1t_2I - \xi^2R_2 & -t_2\,\mathrm{i}\xi \\ \hline t_1\,\mathrm{i}\xi^* & t_1t_2 \end{array}\right).$$

Corollary 3: *The inverse matrices due to* (18) *read*

$$\begin{aligned} \Phi_5^{-1} &= \frac{-1}{\mu r}\left(\begin{array}{c|c} t_2(t_2^2 + \xi^2) R_1 + \dfrac{r}{t_2} R_2 & -2\mathrm{i}\xi \\ \hline 2\mathrm{i}\xi^* & t_1(t_2^2 + \xi^2) \end{array}\right), \\ \Phi_7^{-1} &= \frac{-1}{\mu k_2^2}\left(\begin{array}{c|c} \mu(t_2^2 + \xi^2) R_1 - \mu k_2^2R_2 & \mathrm{i}\xi \\ \hline 2\mu t_1\mathrm{i}\xi^* & -t_1 \end{array}\right), \\ \Phi_9^{-1} &= \frac{1}{\mu k_2^2}\left(\begin{array}{c|c} t_2R_1 - \dfrac{k_2^2}{t_2} R_2 & 2\mu t_2\mathrm{i}\xi \\ \hline \mathrm{i}\xi^* & -\mu(t_2^2 + \xi^2) \end{array}\right), \end{aligned} \tag{29}$$

where the Rayleigh function r [1] *occurs:*

$$r(\xi) = (t_2^2 + \xi^2)^2 - 4\xi^2t_1t_2 = 4\xi^2(\xi^2 - t_1t_2) - k_2^2(4\xi^2 - k_2^2). \tag{30}$$

The remaining matrices can be written as

$$\Phi_2^{-1} = M\Phi_1^{-1}M\,, \quad \Phi_3^{-1} = -D^{-1}\Phi_1^{-1}\,, \quad \Phi_4^{-1} = MD^{-1}\Phi_1^{-1}M\,,$$
$$\Phi_6^{-1} = -M\Phi_5^{-1}M\,, \quad \Phi_8^{-1} = M\Phi_7^{-1}\,, \quad \Phi_{10}^{-1} = -M\Phi_9^{-1}. \tag{31}$$

4. Equivalent Wiener-Hopf systems. We now define the 6×6 convolution operator matrices

$$B_l = F^{-1}\Psi_l \cdot F\,, \quad l = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\,, \tag{32}$$

by the following data relations for solutions of (1) firstly $B_\mathrm{I} = B_+B_-^{-1}$ given by

$$f = \begin{pmatrix} f_0 \\ f_T \end{pmatrix} = \begin{pmatrix} u_0^+ - u_0^- \\ u_T^+ - u_T^- \end{pmatrix} \xrightarrow{B_-^{-1}} \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \xrightarrow{B_+} \begin{pmatrix} u_T^+ - u_T^- \\ u_T^+ + u_T^- \end{pmatrix} = \begin{pmatrix} [Tu]_0 \\ \{Tu\}_0 \end{pmatrix} \tag{33.I}$$

or, instead of the last vector,

$$\begin{pmatrix} [u_0] \\ \{u\}_0 \end{pmatrix}, \quad \begin{pmatrix} [u]_{01} \\ [u]_{02} \\ [Tu]_{03} \\ \{u\}_{01} \\ \{u\}_{02} \\ \{Tu\}_{03} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} [Tu]_{01} \\ [Tu]_{02} \\ [u]_{03} \\ \{Tu\}_{01} \\ \{Tu\}_{02} \\ \{u\}_{03} \end{pmatrix}, \tag{33.II—IV}$$

respectively, according to the four canonical problems in Chapter 1. More precisely we consider B_l as the translation invariant extension on $H^{1/2}(\mathbb{R}^2)^3 \times H^{-1/2}(\mathbb{R}^2)^3$ instead of acting on vectors supported on Σ.

The first three rows of these correspondences are always decoupled, since $\Psi = \Psi_\mathrm{I},\ldots$ contains exactly one 1 in each of them. The remaining relations yield, after an elementary rearrangement, a 3×3 system of Wiener-Hopf equations with symbol $\Phi = \Phi_\mathrm{I}, \ldots$, say. In order to discuss the continuous dependence on all given data, we continue considering the full 6×6 operator matrix

$$\underline{W} = \chi_\Sigma \cdot B|_{\tilde{H}^{1/2}(\Sigma)^3 \times \tilde{H}^{-1/2}(\Sigma)^3} : f \to h \tag{34}$$

where $B = B_+B_-^{-1}$ stands for one of the convolution operators B_l in (32), $h = h_l$ denotes the restriction of the corresponding right hand side of (33.I—IV) on Σ and $W = W_l$ acts into a vector Sobolev space with components in $H^{\pm 1/2}(\Sigma)$ dependent on the type of the problem $\mathcal{P} = \mathcal{P}_l$, $l = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ (or others, see (7)), respectively.

Theorem 1: 1. *Problem $\mathcal{P}$ is equivalent to a 6×6 system of Wiener-Hopf equations $\underline{W}f = h$ where*

$$\underline{W} : \tilde{H}^{1/2}(\Sigma)^3 \times \tilde{H}^{-1/2}(\Sigma)^3 \to \mathop{\times}_{j=1}^{3} \tilde{H}^{s_j}(\Sigma) \times \mathop{\times}_{j=1}^{3} H^{s_j}(\Sigma)$$

is linear continuous and

$$(s_1, s_2, s_3) = \begin{cases} (-1/2, -1/2, -1/2) \\ (1/2, 1/2, 1/2) \\ (1/2, 1/2, -1/2) \\ (-1/2, -1/2, 1/2). \end{cases} \tag{35.I—IV}$$

2. *The reduced 3×3 Wiener-Hopf operator*

$$W = \chi_\Sigma \cdot A|_{\ldots} : \mathop{\times}_{j=1}^{3} \tilde{H}^{r_j}(\Sigma) \to \mathop{\times}_{j=1}^{3} H^{s_j}(\Sigma) \tag{36}$$

corresponding to the latter rearranged three rows, is of normal type, i.e.

$$A = F^{-1}\Phi \cdot F : \times H^{r_j}(\mathbb{R}^2) \to \times H^{s_j}(\mathbb{R}^2) \tag{37}$$

acts bijectively where $r_j + s_j = 0$, $j = 1, 2, 3$, holds for each of the four canonical problems.

Proof: 1. A solution u of $\mathcal{P}$ is given by Corollary 2 in dependence of the ansatz data or, see (33), in terms of the jumps f, since B_- is a 1-1-mapping, cf. the subsequent Lemma 4. Conversely, a solution of $Wf = h$ yields ansatz data $\varphi = B_-f$ and a solution u of $\mathcal{P}$.

2. Since the reduction to a 3×3 system and the space setting are obvious, we only have to prove the bijectivity of A, i.e.

$$\begin{gathered}\det \Phi(\xi) \neq 0, \qquad \xi \in \mathbb{R}^2,\\ \Phi(\xi) = \left(O(|\xi|^{r_k - s_j})\right)_{j,k=1,2,3}, \qquad \Phi^{-1}(\xi) = \left(O(|\xi|^{s_k - r_j})\right)_{j,k=1,2,3}.\end{gathered} \tag{38}$$

This is not trivial because of the unboundedness of Φ_1^{-1}, but it follows most evidently from the explicit matrix function representations given later in (43), (46), (47) ∎

Corollary 4: *$\mathcal{P}$ is well-posed for all data, iff W is invertible. Then the solution u is given for instance in terms of the Dirichlet data by* (13), (14) *where*

$$\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} = \frac{1}{2}\left(\begin{array}{c|c} I & I \\ \hline -I & I \end{array}\right) B_{\mathrm{II}}^{-1} f, \qquad f = \underline{W}^{-1} h = \left(\begin{array}{c|c} I & 0 \\ \hline -W^{-1}W_{21} & W^{-1} \end{array}\right) h \tag{39}$$

with a certain 3×3 block W_{21} of $\underline{W}$. All dependences are continuous in the sense of

$$\begin{gathered} h \mapsto f \mapsto u_0 \mapsto u, \\ \begin{pmatrix} \displaystyle\mathop{\times}_{j=1}^{3} \tilde{H}^{s_j}(\Sigma) \\ \displaystyle\mathop{\times}_{j=1}^{3} H^{s_j}(\Sigma) \end{pmatrix} \to \begin{pmatrix} \tilde{H}^{1/2}(\Sigma)^3 \\ \tilde{H}^{-1/2}(\Sigma)^3 \end{pmatrix} \to H^{1/2}(\mathbb{R}^2)^6 \to H^1(\Omega)^3. \end{gathered} \tag{40}$$

Remark: All single scalar Wiener-Hopf operators have the form $\chi_\Sigma \cdot F^{-1}\sigma \cdot F|\ldots : \tilde{H}^r(\Sigma) \to H^s(\Sigma)$ with $|r| = |s| = 1/2$. So they are of order -1, 1 or 0 in the sense of pseudo-differential operators and correspond with weakly singular (L^1 convolution type), differential and hypersingular, or unit operators (times constant), respectively [6].

The operator theoretic structure of the systems can be analyzed in advance and very detailed after lifting W on $L^2(\Sigma)^3$ by Bessel-potential operators [21], a transformation from $\mathbb{R}$ on the unit circle (Cayley transformation or stereographic projection [24]), and by use of the theory of Cauchy type singular integral equations [25]. We refer to [21, Section 3] for details. It turns out, for instance, that (37) is necessary for the Fredholm property of W, which is equivalent to the invertibility [27]. The partial winding numbers of the lifted symbol determinants are always zero for the canonical problems, since these determinants are even functions in ξ_1 and ξ_2. But the elements of the lifted symbol may have jumps at $\xi_1 \to \pm\infty$ for further problems, see (7), which then corresponds to higher singularities of ∇u at $x = 0$, see [29, 30]. Here we concentrate on the explicit factorization of the (unlifted) symbols.

5. Related symbols. We are going to determine the Fourier symbol matrix functions Ψ_{B_-} of B_- in (33.I), its inverse $\Psi_{B_-}^{-1}$, the 6×6 matrices $\Psi = \Psi_l$, $l = \mathrm{I, II, III, IV}$, from (32) as well as the 3×3 blocks of the reduced versions $\Phi = \Phi_l$ in (37).

Lemma 4: *The symbol of B_- and its inverse read*

$$\Psi_{B_-} = P_{36} \begin{pmatrix} \Phi_7 & 0 \\ 0 & \Phi_9 \end{pmatrix} \begin{pmatrix} I & -M \\ I & M \end{pmatrix}, \qquad \Psi_{B_-}^{-1} = \frac{1}{2} \begin{pmatrix} I & I \\ -M & M \end{pmatrix} \begin{pmatrix} \Phi_7^{-1} & 0 \\ 0 & \Phi_9^{-1} \end{pmatrix} P_{36}, \tag{41}$$

where P_{36} denotes the permutation matrix for the exchange of the 3^{rd} and the 6^{th} row.

Proof: By definition one obtains in block matrix notation

$$\Psi_{B_-} = \begin{pmatrix} \Phi_1 & -\Phi_2 \\ \Phi_5 & -\Phi_6 \end{pmatrix} = \begin{pmatrix} \Phi_1 & -M\Phi_1 M \\ \Phi_5 & M\Phi_5 M \end{pmatrix},$$

see (18), where the second block column coincides with the first one up to certain signs, which fact we describe symbolically by

$$\Psi_{B_-} = \left(\begin{array}{c|ccc} & - & - & + \\ \Phi_1 & - & - & + \\ & + & + & - \\ \hline & + & + & - \\ \Phi_5 & + & + & - \\ & - & - & + \end{array}\right) = P_{36} \left(\begin{array}{c|ccc} & - & - & + \\ \Phi_7 & - & - & + \\ & - & - & + \\ \hline & + & + & - \\ \Phi_9 & + & + & - \\ & + & + & - \end{array}\right).$$

The rest of the proof is obvious ∎

Proposition 2: *The full symbol of the pure traction problem $\mathcal{P}_{\mathrm{I}}$ reads*

$$\Psi_{\mathrm{I}} = \begin{pmatrix} 0 & I \\ \Phi_{\mathrm{I}} & \Phi_{\mathrm{I}}' \end{pmatrix}, \tag{42.I}$$

where

$$\Phi_{\mathrm{I}} = \frac{\mu}{k_2^2} \left(\begin{array}{c|c} \frac{r}{t_2} R_1 - k_2^2 t_2 R_2 & 0 \\ \hline 0 & \frac{r}{t_1} \end{array}\right), \qquad \Phi_{\mathrm{I}}' = \frac{1}{k_2^2} \left(\begin{array}{c|c} 0 & \frac{\sigma}{t_2}\, \mathrm{i}\xi \\ \hline -\frac{\sigma}{t_1}\, \mathrm{i}\xi^* & 0 \end{array}\right),$$

$$\sigma = \sigma(\xi) = t_2^2 + \xi^2 - 2t_1 t_2 = 2\tau - k_2^2 \tag{43.I}$$

(*remember* $t_j = (\xi^2 - k_j^2)^{1/2}$, $\tau = \xi^2 - t_1 t_2$, $r = (t_2^2 + \xi^2)^2 - 4\xi^2 t_1 t_2 = 4\xi^2 \tau - k_2^2 \times (4\xi^2 - k_2^2)$).

Proof: Formulae (32), (33.I), (17) and (41) yield

$$\Psi_{\mathrm{I}} = \Psi_{B_+} \Psi_{B_-}^{-1} = \begin{pmatrix} \Phi_5 & -\Phi_6 \\ \Phi_5 & \Phi_6 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} I & I \\ -M & M \end{pmatrix} \begin{pmatrix} \Phi_7^{-1} & 0 \\ 0 & \Phi_9^{-1} \end{pmatrix} P_{36}$$

which can be simplified by use of $\Phi_6 = -M\Phi_5 M$, $M^2 = I$ to

$$\Psi_{\mathrm{I}} = \frac{1}{2} \begin{pmatrix} I - M & I + M \\ I + M & I - M \end{pmatrix} \begin{pmatrix} \Phi_5 \Phi_7^{-1} & 0 \\ 0 & \Phi_5 \Phi_9^{-1} \end{pmatrix} P_{36} = P_{36} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \Phi_5 \Phi_7^{-1} & 0 \\ 0 & \Phi_5 \Phi_9^{-1} \end{pmatrix} P_{36}.$$

The 3×3 blocks are easily obtained by the aim of Lemma 3 from (18) and (29):

$$\Phi_5\Phi_7^{-1} = -\mu \left(\begin{array}{c|c} \dfrac{t_2^2+\xi^2}{t_2} R_1 + t_2 R_2 & 2\mathrm{i}\xi \\ \hline -2\mathrm{i}\xi^* & \dfrac{t_2^2+\xi^2}{t_1} \end{array}\right)$$

$$\times \frac{-1}{\mu k_2^2} \left(\begin{array}{c|c} \mu(t_2^2+\xi^2)\, R_1 - \mu k_2^2 R_2 & \mathrm{i}\xi \\ \hline 2\mu t_1\, \mathrm{i}\xi^* & -t_1 \end{array}\right)$$

$$= \frac{1}{k_2^2} \left(\begin{array}{c|c} \dfrac{\mu r}{t_2} R_1 - \mu k_2^2 t_2 R_2 & \dfrac{\sigma}{t_2}\, \mathrm{i}\xi \\ \hline 0 & k_2^2 \end{array}\right), \tag{44}$$

$$\Phi_5\Phi_9^{-1} = \Phi_5 \cdot \frac{1}{\mu k_2^2} \left(\begin{array}{c|c} t_2 R_1 - \dfrac{k_2^2}{t_2} R_2 & 2\mu t_2 \mathrm{i}\xi \\ \hline \mathrm{i}\xi^* & -\mu(t_2^2+\xi^2) \end{array}\right)$$

$$= \frac{1}{k_2^2} \left(\begin{array}{c|c} k_2^2 I & 0 \\ \hline -\dfrac{\sigma}{t_1}\, \mathrm{i}\xi^* & \mu\, \dfrac{r}{t_1} \end{array}\right),$$

The rest of the proof consists in the exchange of some rows and columns in the last formula for Ψ_{I} ∎

Remark: The pure Dirichlet problem $\mathscr{P}_{\mathrm{II}}$ has already been solved in [23] by a modified approach, which could also be used to treat the traction problem. In our opinion, the present more rigorous calculus shows clearer how the mixed type boundary symbols Φ_7 and Φ_9 come into the game — even for the pure problems $\mathscr{P}_{\mathrm{I}}$ and $\mathscr{P}_{\mathrm{II}}$.

A completely analogue calculation for the Dirichlet problem yields the corresponding formulae (just replace Φ_5 by Φ_1 and Φ_6 by Φ_2 in the last proof):

$$\Psi_{\mathrm{II}} = \left(\begin{array}{c|c} I & 0 \\ \hline \Phi'_{\mathrm{II}} & \Phi_{\mathrm{II}} \end{array}\right), \tag{42.II}$$

where

$$\Phi_{\mathrm{II}} = \frac{-1}{\mu k_2^2} \left(\begin{array}{c|c} \frac{\tau}{t_1} R_1 + \frac{k_2^2}{t_2} R_2 & 0 \\ \hline 0 & \frac{\tau}{t_2} \end{array} \right), \qquad \Phi'_{\mathrm{II}} = \frac{1}{k_2^2} \left(\begin{array}{c|c} 0 & -\frac{\sigma}{t_1}\, \mathrm{i}\xi \\ \hline \frac{\sigma}{t_2}\, \mathrm{i}\xi^* & 0 \end{array} \right) \tag{43.II}$$

cf. [23, formula (27)].

For the complexity of the explicit factorization of the reduced symbol matrix $\Phi = \Phi_{\mathrm{I}}$, its block structure ($c = d = 0$) turns out to be most important (the same form we obtained for Φ_{II}). Note that this simplification is not given for the other two cases.

Proposition 3: *The two mixed type problems $\mathcal{P}_{\mathrm{III}}$ and $\mathcal{P}_{\mathrm{IV}}$ are governed by the reduced 3×3 symbols*

$$\Phi_{\mathrm{III}} = \Phi_7 \Phi_9^{-1}, \qquad \Phi_{\mathrm{IV}} = \Phi_9 \Phi_7^{-1} \tag{45}$$

respectively, presented below.

Proof: By analogy to the last proof there follows with $\Phi_8 = \Phi_7 M$, $\Phi_{10} = -\Phi_9 M$, see (18),

$$\Psi_{\mathrm{III}} = \Psi_{B_+} \Psi_{B_-}^{-1} = \left(\begin{array}{c|c} \Phi_7 & -\Phi_8 \\ \hline \Phi_7 & \Phi_8 \end{array} \right) \cdot \frac{1}{2} \left(\begin{array}{c|c} \Phi_7^{-1} & \Phi_9^{-1} \\ \hline -M\Phi_7^{-1} & M\Phi_9^{-1} \end{array} \right) P_{36}$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \Phi_7 \Phi_9^{-1} \end{array} \right) P_{36},$$

$$\Psi_{\mathrm{IV}} = \left(\begin{array}{c|c} \Phi_9 & -\Phi_{10} \\ \hline \Phi_9 & \Phi_{10} \end{array} \right) \cdot \frac{1}{2} \left(\begin{array}{c|c} \Phi_7^{-1} & \Phi_9^{-1} \\ \hline -M\Phi_7^{-1} & M\Phi_9^{-1} \end{array} \right) = \left(\begin{array}{c|c} 0 & I \\ \hline \Phi_9 \Phi_7^{-1} & 0 \end{array} \right) P_{36}.$$

The reduced symbols read, see (18), (29), (28)

$$\Phi_{\mathrm{III}} = \Phi_7 \Phi_9^{-1}$$

$$= \left(\begin{array}{c|c} I & \frac{1}{t_1}\, \mathrm{i}\xi \\ \hline 2\mu\, \mathrm{i}\xi^* & -\mu \frac{t_2^2 + \xi^2}{t_1} \end{array} \right) \cdot \frac{1}{\mu k_2^2} \left(\begin{array}{c|c} t_2 R_1 - \frac{k_2^2}{t_2} R_2 & 2\mu t_2 \mathrm{i}\xi \\ \hline \mathrm{i}\xi^* & -\mu(t_2^2 + \xi^2) \end{array} \right)$$

$$= \frac{-1}{k_2^2 t_1} \left(\begin{array}{c|c} \frac{\tau}{\mu} R_1 + \frac{k_2^2}{\mu} \frac{t_1}{t_2} R_2 & \sigma\, \mathrm{i}\xi \\ \hline \sigma\, \mathrm{i}\xi^* & -\mu r \end{array} \right) \tag{46}$$

and analogously, or just by inversion, see (26),

$$\Phi_{\mathrm{IV}} = \Phi_9\Phi_7^{-1} = \frac{1}{k_2^2 t_2}\left(\begin{array}{c|c} \mu r R_1 - \mu k_2^2 t_2^2 R_2 & \sigma\,\mathrm{i}\xi \\ \hline \sigma\,\mathrm{i}\xi^* & -\dfrac{\tau}{\mu} \end{array}\right) \tag{47}$$

due to $\det \Phi_7\Phi_9^{-1} = -(k_2^4/\mu)\, t_1$ ∎

6. Explicit solution of the traction problem. In this chapter we construct the (continuous) inverse W^{-1} of the reduced Wiener-Hopf operator (36) that was needed to represent the solution of problem $\mathcal{P}_{\mathrm{II}}$ in Corollary 4. This will be done in two steps.

First we look for a strong Wiener-Hopf factorization [27]

$$\Phi = \Phi_-\Phi_+ \tag{48}$$

into two factors with the following properties:

(i) $\Phi_\pm(\cdot, \xi_2)$ are continuous, invertible 3×3 matrix functions on $\mathbb{R}$ for almost every $\xi_2 \in \mathbb{R}$ (for all but $\xi_2 = 0$);

(ii) $\Phi_+^{\pm 1}(\cdot, \xi_2)$ and $\Phi_-^{\pm 1}(\cdot, \xi_2)$ possess holomorphic extensions into the upper or the lower complex half-plane $\mathbb{C}_+$ or $\mathbb{C}_-$, respectively, and are continuous on the closures $\overline{\mathbb{C}_\pm} = \mathbb{R} \cup \mathbb{C}_\pm$ for a.e. $\xi_2 \in \mathbb{R}$;

(iii) the factors and their inverses admit asymptotic estimates as $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2} \to +\infty$ such that the (lifted [21]) matrices

$$\Phi_{0-} = \left(t_{2-}^{2s_j}\delta_{jk}\right) \cdot \Phi_-, \qquad \Phi_{0+} = \Phi_+ \cdot \left(t_{2+}^{-2r_j}\delta_{jk}\right)$$

with $t_{2\pm}^2(\xi) = \xi_1 \pm \mathrm{i}(\xi_2^2 - k_2^2)^{1/2}$, and their inverses are essentially bounded (except at $\xi_2 = 0$) with respect to $\xi \in \mathbb{R}^2$. Note that this is more than is needed in the classical (function theoretic) Wiener-Hopf procedure [19, 26, 32], which requires only algebraic growth at infinity and admits a finite number of zeros and poles in $\mathbb{C}_\pm$, in order to find the explicit solution of a single problem (instead of the inverse W^{-1} which additionally yields the correctness of $\mathcal{P}$ and a priori estimates of the solution in terms of the data).

Secondly we prove that (48) with properties (i)–(iii) leads to an operator (theoretic) factorization of the basic convolution operator

$$A = A_-A_+, \qquad A_\pm = F^{-1}\Phi_\pm \cdot F \tag{49}$$

with respect to the pair of Sobolev spaces $H^{\pm 1/2}(\mathbb{R}^2)$ and appropriate projectors P_1, P_2, which enables us to present W^{-1} in the form of a general Wiener-Hopf operator inverse [27].

Lemma 5: *Consider a 2×2 matrix function*

$$G = aR_1 + bR_2 \tag{50}$$

where $a(\cdot, \xi_2)$, $b(\cdot, \xi_2)$ are regular elements in the Wiener algebra $\mathcal{W} = \mathbb{C} + FL^1(\mathbb{R})$ and (for simplicity) even functions in the first variable for a.e. $\xi_2 \in \mathbb{R}$. Let the factorizations of $a = a_-a_+$ be defined by

$$a_\pm(\xi) = \sqrt{a(\infty)}\, \exp\left\{F_{x_1\mapsto\xi_1} I_\pm(x_1) \cdot F^{-1}_{\xi_1\mapsto x_1} \log \frac{a(\xi)}{a(\infty)}\right\} \tag{51}$$

and of $b = b_- b_+$ by analogy, be uniformly bounded with respect to (a.e.) $\xi_2 \in \mathbb{R}$, i.e. $a_\pm^{\pm 1}, b_\pm^{\pm 1} \in L^\infty(\mathbb{R}^2)$. Further let $\lambda: \mathbb{R} \to \mathbb{C}$ be defined by

$$\lambda(\xi_2) = \frac{b_+(\xi_1, \xi_2)}{a_+(\xi_1, \xi_2)}\bigg|_{\xi_1 = \mathrm{i}|\xi_2|}, \qquad \xi_2 \in \mathbb{R},$$

be measurable and essentially bounded.

Then G admits a strong Wiener-Hopf factorization in the sense of (48) *(as 2×2 matrices now with $r_j = s_j = 0$) given by*

$$G = G_- G_+ \left(a_- R_1 + \frac{b_-}{\lambda} R_2\right) R_- R_+ \left(a_+ R_1 + \frac{b_+}{\lambda} R_2\right), \tag{52}$$

where $R_\pm$ represent factors of the matrix function

$$R(\xi, \lambda) = R_1(\xi) + \lambda^2 R_2(\xi) = R_-(\xi, \lambda)\, R_+(\xi, \lambda)$$

$$= \frac{1}{1+\lambda^2} \begin{pmatrix} \dfrac{\xi_1 - \mathrm{i}\lambda^2 |\xi_2|}{\xi_1 - \mathrm{i}\,|\xi_2|} & \pm\mathrm{i} \\ \pm\mathrm{i}\, \dfrac{\lambda^2 \xi_1 - \mathrm{i}\,|\xi_2|}{\xi_1 - \mathrm{i}\,|\xi_2|} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\xi_1 + \mathrm{i}\lambda^2 |\xi_2|}{\xi_1 + \mathrm{i}\,|\xi_2|} & \mp\mathrm{i}\, \dfrac{\lambda^2 \xi_1 + \mathrm{i}\,|\xi_2|}{\xi_1 + \mathrm{i}\,|\xi_2|} \\ \mp\mathrm{i}\lambda^2 & \lambda^2 \end{pmatrix} \tag{53}$$

with $\pm\mathrm{i} = \mathrm{i} \cdot \operatorname{sgn} \xi_2$.

Proof: The factorization $G = (a_- R_1 + b_- R_2)(a_+ R_1 + b_+ R_2)$ obviously has all desired properties but simple poles at $\xi_1 = \pm\mathrm{i}\,|\xi_2|$, see (20). These cancel out, if the coefficients coincide at the corresponding point (consider the Laurent series), which then happens simultaneously in both factors according to the symmetry in ξ_1 of the factors (51) of an even function a. Otherwise, introducing λ, we observe pole cancellation in the outer factors of (52). The remaining factorization of the rational (in ξ_1) matrix function R is simply done by a standard technique [5, 23] ∎

Remark: Non-even coefficients would yield different λ_1, λ_2 in the plus/minus correction terms, but this generalization is not needed here.

Proposition 4: *Let $\Phi = \Phi_{\mathrm{I}}$ be given by the reduced 3×3 symbol* (43) *due to the traction problem $\mathcal{P}_2$. Then a strong factorization in the sense of* (48) *with $r_j = 1/2$ and $s_j = -1/2$, $j = 1, 2, 3$, is given by*

$$\Phi = \Phi_- \Phi_+ = t_{2-} \left(\begin{array}{c|c} G_- & 0 \\ \hline 0 & e_- \end{array}\right) \left(\begin{array}{c|c} G_+ & 0 \\ \hline 0 & e_+ \end{array}\right) t_{2+}, \tag{54}$$

$$t_{2\pm}(\xi) = \left(\xi_1 \pm \mathrm{i}(\xi_2^2 - k_2^2)^{1/2}\right)^{1/2},$$

$$G = aR_1 + bR_2 = \frac{\mu}{k_2^2} \frac{r}{t_2^2} R_1 - \mu R_2, \qquad e = \frac{\mu}{k_2^2} \frac{r}{t_1 t_2} = \frac{t_2}{t_1} a$$

and the formulae of the last lemma.

Proof: The 3×3 block matrices treated here form a commutative subalgebra of $\mathcal{A}$ (due to $c = d = 0$, see (24) etc.). The factorization problem decouples into one for a scalar (lifted) function e in the Wiener algebra and one for a 2×2 matrix function, which we investigated before. In total we have to factor the Rayleigh function r, square roots t_j and (possibly) one rational (in ξ_1) matrix function of the type (53), according to the only non-commutative computation in this procedure. Considerations concerned with the orders r_j and s_j are obvious from Theorem 1 and formula (54) where order $t_{2\pm} = 1/2$ and the block matrices are bounded invertible ∎

Theorem 2: *The inverse 3×3 Wiener-Hopf operator due to* (36) *and $\mathcal{P}_{II}$ reads*

$$W^{-1} = A_+^{-1}\chi_\Sigma \cdot A_-^{-1} l_{\mathrm{odd}}|_{H^{-1/2}(\Sigma)^3} \tag{55}$$

and maps onto $\tilde{H}^{1/2}(\Sigma)^3$, where $A_\pm^{-1} = F^{-1}\Phi_\pm^{-1}F$ are taken from (54) *and where l_{odd} denotes odd extension with respect to ξ_1 from Σ onto the full plane $\mathbb{R}^2$ (or any other continuous extension operator).*

The proof is based on the philosophy of asymmetric general Wiener-Hopf operators [27, 28]. Consider $\tilde{W} = P_2A|_{P_1X}$ where $A : X \to Y$ is a linear bijection between Banach spaces, P_1 and P_2 are (continuous) projection operators on X and Y, respectively. It is known that the invertibility of $\tilde{W}$ is equivalent to a strong Wiener-Hopf factorization of $A = A_-A_+$ into invertible operators $A_+ : X \to Z$, $A_- : Z \to Y$ with a suitable intermediate Banach space Z, such that, for an appropriate projector P on Z,

$$A_+P_1X = PZ, \qquad A_-(I - P)\,Z = (I - P_2)\,Y \tag{56}$$

are satisfied. The inverse of $\tilde{W}$ then reads $\tilde{W}^{-1} = A_+^{-1}PA_-^{-1}|_{P_2Y}$. In our situation we identify, see (36) or (43), $X = H^{1/2}(\mathbb{R}^2)^3$, $Y = H^{-1/2}(\mathbb{R}^2)^3$, $Z = H^0(\mathbb{R}^2)^3 = L^2(\mathbb{R}^2)^3$, $A = F^{-1}\Phi \cdot F$, $A_\pm = F^{-1}\Phi_\pm \cdot F$ and, since $H^{-1/2}(\Sigma)$ is not a subspace of $H^{-1/2}(\mathbb{R}^2)$, we consider $\tilde{W} = l_{\mathrm{odd}}W = l_{\mathrm{odd}}\chi_\Sigma \cdot A|_{\tilde{H}^{-1/2}(\Sigma)^3}$ which is equivalent to W and maps into the subspace of $H^{-1/2}(\mathbb{R}^2)^3$ distributions, which are odd in ξ_1. Putting $P = \chi_\Sigma\cdot$, $P_2 = l_{\mathrm{odd}}\chi_\Sigma\cdot$ and $P_1 = I - l_{\mathrm{even}}\chi_{\mathbb{R}^2-\bar{\Sigma}}\cdot$ one arrives at $\tilde{W} = P_2A|_{P_1X}$ and obtaines also the above-mentioned factor properties (56), see [22, 28] for more details ∎

Remark: For the proof of Theorem 2 one can also lift the problem on $L^2(\mathbb{R}_+)^3$ by Bessel-potential operators, see [21, Proposition 3.1], and treat the equivalent symmetric Wiener-Hopf operator acting between L^2 spaces [25].

7. The factoring procedure for the symbols of the mixed problems. Consider again

$$\Phi_{III} = \Phi_7\Phi_9^{-1} = \Phi_{IV}^{-1} = \frac{-1}{k_2^2t_1}\left(\begin{array}{c|c} \frac{\tau}{\mu}R_1 + \frac{k_2^2}{\mu}\frac{t_1}{t_2}R_2 & \sigma\,\mathrm{i}\xi \\ \hline \sigma\mathrm{i}\xi^* & -\mu r \end{array}\right). \tag{46}$$

It is sufficient to factor only Φ_{III} according to a symmetry argument for Φ_{IV} (exchange of left and right factorizations).

We are going to present a constructive method, which also applies to other non-rational 3×3 matrix functions correspondent to elastodynamical boundary value and transmission problems, see (7). The basic idea is to treat the matrix (46), after removing the R_2 term, like a 2×2 (block) matrix by our method presented in the last section. Since it is not possible to write this matrix in paired (block) form similar to (50) with projection matrices in the algebra $\mathcal{A}$ (the proof is left to the reader), one needs some preliminary transformations. These are modifications of tricks, which are common for 2×2 matrix functions of Khrapkov type

$$G = a_1Q_1 + a_2Q_2 \tag{57}$$

with rational matrix functions Q_j and non-rational coefficients a_j [11, 12, 30].

After some elementary transformations for eliminating the constant factors and lifting Φ_{III} on the L^2 space level, we start with the equivalent bounded invertible symbol

$$\Phi_0 = -\mu k_2^2 \begin{pmatrix} t_{1-} & 0 & 0 \\ 0 & t_{1-} & 0 \\ 0 & 0 & \dfrac{1}{\mu t_{1-}} \end{pmatrix} \Phi_{\mathrm{III}} \begin{pmatrix} t_{1+} & 0 & 0 \\ 0 & t_{1+} & 0 \\ 0 & 0 & \dfrac{1}{\mu t_{1+}} \end{pmatrix}$$

$$= \left[\begin{array}{c|c} \tau R_1 + k_2^2 \dfrac{t_1}{t_2} R_2 & \dfrac{\sigma}{\xi_1 + \mathrm{i}w_1} \mathrm{i}\xi \\ \hline \dfrac{\sigma}{\xi_1 - \mathrm{i}w_1} \mathrm{i}\xi^* & \dfrac{-r}{t_1^2} \end{array}\right], \tag{58}$$

where $t_{1\pm} = (\xi_1 \pm \mathrm{i}w_1)^{1/2}$, $w_1 = (\xi_2^2 - k_1^2)^{1/2}$ with $\mathrm{Im}\, w_1 > 0$ for $\xi_2 \in \mathbb{R}$. Note that $\Phi_0(\cdot, \xi_2) \in \mathcal{W}^{3\times 3}$ for any fixed $\xi_2 \in \mathbb{R}$.

The second step consists in a decomposition of the algebra into a direct sum

$$\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \tag{59}$$

of subalgebras of matrix functions (24) with $b = 0$ and $a = c = d = e = 0$, respectively. These obviously represent algebras of singular 3×3 matrix functions with unit elements and we try to factor the first component

$$\Phi_{01} = \left[\begin{array}{c|c} \tau R_1 & \dfrac{\sigma}{\xi_1 + \mathrm{i}w_1} \mathrm{i}\xi \\ \hline \dfrac{\sigma}{\xi - \mathrm{i}w_1} \mathrm{i}\xi^* & \dfrac{-r}{t_1^2} \end{array}\right] \tag{60}$$

in $\mathcal{A}_1$ like a 2×2 matrix function. So we write it, in the (block) form of (57) remembering $\tau = \xi^2 - t_1 t_2$, $\sigma = 2\tau - k_2^2$, $r = 4\xi^2\tau - k_2^2(4\xi^2 - k_2^2) = \sigma(4\xi^2 - k_2^2) - \tau(4\xi^2 - 2k_2^2)$ as

$$\Phi_{01} = \tau \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & \dfrac{4\xi^2 - 2k_2^2}{t_1^2} \end{array}\right] + \sigma \left[\begin{array}{c|c} 0 & \dfrac{1}{\xi_1 + \mathrm{i}w_1} \mathrm{i}\xi \\ \hline \dfrac{1}{\xi_1 - \mathrm{i}w_1} \mathrm{i}\xi^* & -\dfrac{4\xi^2 - k_2^2}{t_1^2} \end{array}\right].$$

with rational matrix functions and scalar non-rational coefficients. We factor the first rational matrix function elementary (up to poles of R_1) into

$$\left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & \dfrac{4\xi^2 - 2k_2^2}{t_1^2} \end{array}\right] = \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & 2\,\dfrac{\xi_1 - iw_2}{\xi_1 - iw_1} \end{array}\right] \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & 2\,\dfrac{\xi_1 + iw_2}{\xi_1 + iw_1} \end{array}\right]$$

$$= Q_{1-}Q_{1+}$$

where $w_2 = (\xi_2^2 - k_2^2/2)^{1/2}$, $\operatorname{Im} w_2 > 0$ and obtain

$$\Phi_{01} = Q_{1-}\{\tau I_{\mathcal{A}_1} + \sigma Q_{1-}^{-}Q_2Q_{1+}^{-}\}\, Q_{1+}, \qquad I_{\mathcal{A}_1} = \left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & 1 \end{array}\right); \tag{61}$$

$$Q_{1-}^{-}Q_2Q_{1+}^{-} = \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & \dfrac{1}{2}\,\dfrac{\xi_1 - iw_1}{\xi_1 - iw_2} \end{array}\right] \times Q_2 \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & \dfrac{1}{2}\,\dfrac{\xi_1 + iw_1}{\xi_1 + iw_2} \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & \dfrac{1}{2}\,\dfrac{1}{\xi_1 - iw_2}\, i\xi \\ \hline \dfrac{1}{2}\,\dfrac{1}{\xi_1 + iw_2}\, i\xi^* & -\dfrac{\xi^2 - k_2^2/4}{\xi^2 - k_2^2/2} \end{array}\right].$$

The term $\tilde{\Phi}_{01}$ in braces can be written in block commutant form [11, 12, 30]

$$\tilde{\varphi}_{01} = \left(\tau - \frac{\sigma}{2}\,\frac{\xi^2 - k_2^2/4}{\xi^2 - k_2^2/2}\right) I_{\mathcal{A}_1} + \frac{\sigma}{2}\, C,$$

where

$$C = \left(\begin{array}{c|c} \dfrac{\xi^2 - k_2^2/4}{\xi^2 - k_2^2/2} R_1 & \dfrac{1}{\xi_1 + \mathrm{i}w_2} \mathrm{i}\xi \\ \hline \dfrac{1}{\xi_1 - \mathrm{i}w_2} \mathrm{i}\xi^* & -\dfrac{\xi^2 - k_2^2/4}{\xi^2 - k_2^2/2} \end{array}\right), \quad C^2 = \frac{k_2^4/16}{(\xi^2 - k_2^2/2)^2} I_{\mathcal{A}_1} = q^2 I_{\mathcal{A}_1}.$$

Fortunately q^2 is the square of a rational function, which implies that

$$\mathcal{R}_1 = \frac{1}{2}\left(I + \frac{1}{q} C\right), \quad \mathcal{R}_2 = \frac{1}{2}\left(I - \frac{1}{q} C\right)$$

are complementary projection matrices in $\mathcal{A}_1$ with rational entries. This enables us to write $\tilde{\Phi}_{01}$ in paired form as the 2×2 blocks $aR_1 + bR_2$ before, see (24), and to follow those ideas using the computational rules of Lemma 3. We have

$$\begin{aligned} \mathcal{R}_1 &= \frac{2}{k_2^2} \left(\begin{array}{c|c} \xi^2 R_1 & (\xi_1 - \mathrm{i}w_2)\, \mathrm{i}\xi \\ \hline (\xi_1 + \mathrm{i}w_2)\, \mathrm{i}\xi^* & -(\xi^2 - k_2^2/2) \end{array}\right), \\ \mathcal{R}_2 &= \frac{2}{k_2} \left(\begin{array}{c|c} -(\xi^2 - k_2^2/2) R_1 & -(\xi_1 - \mathrm{i}w_2)\, \mathrm{i}\xi \\ \hline -(\xi_1 + \mathrm{i}w_2)\, \mathrm{i}\xi^* & \xi^2 \end{array}\right), \end{aligned} \tag{62}$$

$$\mathcal{R}_1 + \mathcal{R}_2 = I_{\mathcal{A}_1}, \quad \mathcal{R}_1 - \mathcal{R}_2 = \frac{1}{q} C,$$

$$\begin{aligned} \tilde{\Phi}_{01} &= a_1 I_{\mathcal{A}_1} + a_2 C = a_1(\mathcal{R}_1 + \mathcal{R}_2) + a_2 q(\mathcal{R}_1 - \mathcal{R}_2) \\ &= (a_1 + a_2 q)\, \mathcal{R}_1 + (q_1 - a_2 q)\, \mathcal{R}_2 = b_1 \mathcal{R}_1 + b_2 \mathcal{R}_2, \end{aligned} \tag{63}$$

$$b_1 = a_1 + a_2 q = \tau - \frac{\sigma}{2} \frac{\xi^2 - k_2^2/4}{\xi^2 - k_2^2/2} + \frac{\sigma}{2} \frac{k_2^2/4}{\xi^2 - k_2^2/2} = \tau - \frac{\sigma}{2} = \frac{k_2^2}{2},$$

$$b_2 = a_1 - a_2 q = \tau - \frac{\sigma}{2} \frac{\xi^2}{\xi^2 - k_2^2/2} = \frac{k_2^2}{2} + \frac{\sigma}{2} \frac{k_2^2/2}{\xi^2 - k_2^2/2}.$$

Obviously, b_1 is a constant and $b_2(\cdot, \xi_2)$ can be factored with respect to the first variable into $b_2 = b_{2-} b_{2+}$ as a regular Wiener algebra element with vanishing winding number. One obtains the following result.

Theorem 3: *A Wiener-Hopf factorization of Φ_{III} in the sense of* (48) *with properties* (i)—(iii) *up to poles (of R_j and $\mathcal{R}_2$) and algebraic growth at infinity (of $\mathcal{R}_j$) is given by* (58) *and*

$$\Phi_0 = Q_{-1}(b_{1-}\mathcal{R}_1 + b_{2-}\mathcal{R}_2 + b_{3-}\mathcal{R}_3) \cdot (b_{1+}\mathcal{R}_1 + b_{2+}\mathcal{R}_2 + b_{3+}\mathcal{R}_3)\, Q_{1+}, \tag{64}$$

where $b_{j-}b_{j+} = b_j$ are strongly factored in the Wiener algebra (with fixed $\xi_2 \in \mathbb{R}$)

$$b_1 = \frac{k_2^2}{2}, \quad b_2 = \frac{b_2^2}{2}\left(1 + \frac{\sigma/2}{\xi^2 - k_2^2/2}\right), \quad b_3 = k_2^2\frac{t_1}{t_2},$$

$Q_{1\pm}$, $\mathcal{R}_1$, $\mathcal{R}_2$ are defined in (61), (62) *and $\mathcal{R}_3$ by* $\mathcal{R}_3 = I_{\mathcal{A}_1} = \left(\begin{array}{c|c} R_2 & 0 \\ \hline 0 & 0 \end{array}\right)$.

Remark: A strong factorization can be obtained as a modification of (64) by analogy to the arguments in Lemma 5.

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