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Some Properties of Pseudo-Differential Operators with Amplitude $Q(x, y, \xi)$

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In der Arbeit werden lineare Pseudodifferentialoperatoren Q mit Amplitude $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ betrachtet, wobei G eine offene Menge des $\mathbb{R}^n$ ist. Es wird ein Kriterium überprüft, welches garantiert, daß Q stetig von $C_0^\infty(G)$ in $C^\infty(G)$ abbildet und daß der stetige formal Transponierte $Q': C_0^\infty(G) \to C^\infty(G)$ von Q existiert. Ferner wird eine Bedingung formuliert, unter der die Erweiterung $\mathbf{Q}$ des Operators Q von $E'(G)$ nach $D'(G)$ pseudolokal ist. Schließlich wird die Zerlegung $Q = \tilde{Q} + R_l$ betrachtet, wobei $\tilde{Q}$ einen eigentlichen Träger hat und pseudolokal ist und R_l von $E'(G)$ in $C^\infty(G)$ abbildet.

В работе рассматриваются линейные псевдодифференциальные операторы Q с амплитудой $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$, причем G — открытое множество в $\mathbb{R}^n$. Проверяется критерий, который гарантирует что Q отображает непрерывно $C_0^\infty(G)$ в $C^\infty(G)$ и что непрерывный формально сопряженный $Q': C_0^\infty(G) \to C^\infty(G)$ существует. Далее, сформулируется условие, при котором расширение $\mathbf{Q}$ отображения Q от $E'(G)$ в $D'(G)$ псевдолокально. Наконец, разложение $Q = \tilde{Q} + R_l$ рассматривается, где $\tilde{Q}$ имеет собственный носитель и псевдолокально, а R_l отображает $E'(G)$ в $C^\infty(G)$.

The paper considers linear pseudo-differential operators Q with amplitude $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$, where G is an open set in $\mathbb{R}^n$. A criterion, which guarantees that Q maps continuously $C_0^\infty(G)$ into $C^\infty(G)$ and that the continuous formal transpose $Q': C_0^\infty(G) \to C^\infty(G)$ of Q exists, is verified. Furthermore, a condition, under which the extension $\mathbf{Q}$ of Q from $E'(G)$ to $D'(G)$ is pseudolocal, is expressed. Also, the decomposition $Q = \tilde{Q} + R_l$ where $\tilde{Q}$ is properly supported (and pseudolocal) and R_l maps $E'(G)$ into $C_\infty(G)$, is considered.

1. Introduction

Consider linear pseudo-differential operator Q defined by

$$(Q\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi, \qquad x \in G \tag{1.1}$$

for $\varphi \in C_0^\infty(G)$, where G is an open set in $\mathbb{R}^n$. It holds a well-known calculus of this kind of operators, when the amplitude $Q(\cdot, \cdot, \cdot)$ of Q is well-behaved (cf. [4, pp. 112 to 172], [2], [7, pp. 63–96] and [8, pp. 36–59], for example). We shall show some rules of this calculus in the case when the amplitude $Q(\cdot, \cdot, \cdot)$ is not demanded to belong to specific class of amplitudes.

Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ such that for each compact set $K \subset G \times G$ and for each $\alpha, \beta \in \mathbb{N}_0^n$ there exist constants $C_{\alpha,\beta,K} > 0$ and $N_{\alpha,\beta,K} \in \mathbb{R}$ so that

$$\sup_{(x,y) \in K} |(D_x^\alpha D_y^\beta Q)(x, y, \xi)| \leqq C_{\alpha,\beta,K} (1 + |\xi|)^{N_{\alpha,\beta,K}} \quad \forall \xi \in \mathbb{R}^n. \tag{1.2}$$

When $N_{\alpha,\beta,K} = N_{\alpha,K} + \delta_{\alpha,K} |\beta|$, with $\delta_{\alpha,K} < 1$, Q maps $C_0^\infty(G)$ continuously into $C^\infty(G)$ (Theorem 2.1). Supposing that $N_{\alpha,\beta,K} = N_K + \delta_K |\alpha + \beta|$, with $\delta_K < 1$, we show the existence of the continuous formal transpose $Q': C_0^\infty(G) \to C^\infty(G)$ of Q.

Assume that for each compact set $K \subset G \times G$ and for each $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exist constants $C_{\alpha,\beta,\gamma,K} > 0$ and $N_{\alpha,\beta,\gamma,K} \in \mathbb{R}$ such that

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x, y, \xi)| \leqq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N_{\alpha,\beta,\gamma,K}} \quad \forall \xi \in \mathbb{R}^n. \tag{1.3}$$

Supposing that $N_{\alpha,\beta,\gamma,K} = N_{\alpha,K} + \delta_{\alpha,K} |\beta| - \varrho_{\alpha,\beta,K} |\gamma|$, with $\delta_{\alpha,K} < 1$ and $\varrho_{\alpha,\beta,K} > 0$, we establish that $(x - y)^\gamma T = T_\gamma$, where T is the Schwartz kernel of Q and T_γ is the distribution induced by the function $t_\gamma(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi$. In the case when $N_{\alpha,\beta,\gamma,K} = N_K + \delta_K |\alpha + \beta| - \varrho_{\alpha,\beta,K} |\gamma|$, with $\delta_K < 1$ and $\varrho_{\alpha,\beta,K} > 0$ we show the inclusion sing supp $\mathbf{Q}u \subset$ sing supp u for $u \in E'(G)$, where $\mathbf{Q}$ is the continuous extension of Q. Furthermore, in this case we show that Q can be decomposed in the form $Q = \tilde{Q} + R_l$, where $\tilde{Q}$ satisfies the estimate (1.3) with

$$N_{\alpha,\beta,\gamma,K} = N_K + \delta_K |\alpha + \beta| - \Big(\min_{u \leqq \alpha,\, v \leqq \beta} \{\varrho_{u,v,K}\}\Big) |\gamma|,$$

$\tilde{Q}$ is properly supported, R_l satisfies (1.3) with

$$N_{\alpha,\beta,\gamma,K} = N_K - (\min \{\varrho_{u,v,K}\})\, l + \delta_K |\alpha + \beta| - \Big(\min_{u \leqq \alpha, v \leqq \beta} \{\varrho_{u,v,K}\}\Big) |\gamma|$$

and R_l maps $E'(G)$ into $C^\infty(G)$. Furthermore, we verify the inclusion sing supp $\tilde{\mathbf{Q}}u \subset$ sing supp u for $u \in D'(G)$, where $\tilde{\mathbf{Q}}: D'(G) \to D'(G)$ is the continuous extension of $\tilde{Q}$.

In the study of pseudo-differential operators the presentation of the operator Q in the form (1.1) implies in many cases simplicity for the calculus; for example for the existence and for the form of the formal transpose $Q': C_0^\infty(G) \to C^\infty(G)$. The existence of the formal transpose is a necessary property of Q in the study of realizations of Q, that is, in the study of closable and closed extensions of Q. Pseudo-differential operators cannot often (without further study) be written in the usual form

$$(Q\psi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} L(x, \xi)\, (F\psi)(\xi)\, e^{i(\xi,x)}\, d\xi$$

in the way that the symbol $L(\cdot, \cdot)$ would be sufficient regular; for instance in the investigation of the decomposition $Q = \tilde{Q} + R_l$, where $\tilde{Q}$ is properly supported and where R_l is regularizing. For the operator of the form (1.1) this decomposition is often possible to see. The concept of the properly supported operator is needed in the study of extensions $\mathbf{Q}: D'(G) \to D'(G)$ of Q.

2. General notations and the operator Q

2.1. Let G be an open set in $\mathbb{R}^n$. For the definition of spaces $D(G)$, $D'(G)$, $C^\infty(G)$, $E'(G)$, S and S' we refer to the monograph [5, pp. 1—53]. Let K be a compact subset of G (we denote $K \subset_K G$). Then $D(K)$ is the subspace of $D(G)$ defined by $D(K) = \{\varphi \in D(G) \mid \operatorname{supp} \varphi \subset K\}$. We recall that $D(G)$ is the space $C_0^\infty(G)$ of test functions equipped with the standard inductive limit topology. Furthermore, $C^\infty(G)$ is a Frechet space, when the topology in $C^\infty(G)$ is defined by the semi-norms $p_{m,K}(\psi) = \sup_{|\alpha| \leqq m} \big(\sup_{x\in K} |(D^\alpha \psi)(x)|\big)$, where $m \in \mathbb{N}_0$ and $K \subset_K G$. A linear operator $L: D(G \to C^\infty(G)$ is continuous if and only if its restriction $L_K := L|_{D(K)}: D(K) \to C^\infty(G)$ is continuous for each $K \subset_K G$. Here $D(K)$ is equipped with the topology included by the topology of $C^\infty(G)$. Hence a linear operator $L: D(G) \to C^\infty(G)$ is continuous if and only if the following criterion holds: Let $K' \subset_K G$; then for each $m \in \mathsf{N}_0$ and $K \subset_K G$ one finds constants $C > 0$ and $m' \in \mathsf{N}_0$ such that $p_{m,K}(L\psi) \leqq C p_{m',K'}(\psi)$ for all $\psi \in D(K')$ (cf. [9, p. 64 and p. 128]).

2.2. Suppose that $Q(\cdot, \cdot, \cdot)$ is a function $G \times G \times \mathsf{R}^n \to \mathsf{C}$. In the sequel we consider linear operators Q which are defined by the requirement

$$(Q\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathsf{R}^n} \left(\int_{\mathsf{R}^n} Q(x, y, \xi)(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi, \tag{2.1}$$

for $\varphi \in C_0^\infty(G)$. Our first step is to yield a condition, which implies that Q maps the space $D(G)$ continuously into $C^\infty(G)$.

Theorem 2.1: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathsf{R}^n)$ such that for each compact set $K \subset_K G \times G$ and for each $(\alpha, \beta) \in \mathsf{N}_0^n \times \mathsf{N}_0^n$ there exist constants $C_{\alpha,\beta,K} > 0$, $N_{\alpha,K} \in \mathsf{R}$ and $\delta_{\alpha,K} < 1$ with which the estimate*

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta Q)(x, y, \xi)| \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{N_{\alpha,K} + \delta_{\alpha,K}|\beta|} \tag{2.2}$$

holds for all $\xi \in \mathsf{R}^n$. Then Q maps $D(G)$ continuously into $C^\infty(G)$.

Proof: A. Let $f: G \times \mathsf{R}^n \to \mathsf{C}$ be a function such that

$$f(x, \xi) = \int_G Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy.$$

Then for each $\alpha \in \mathsf{N}_0^n$, the partial derivative $(D_x^\alpha f)(x, \xi)$ exists for all $x \in G$ and the mapping $x \to (D_x^\alpha f)(x, \xi)$ is continuous: Let x be in G and let ε be a positive number such that $B(x, 2\varepsilon) \subset G$. Then one has $K := \overline{B(x, \varepsilon)} \times \operatorname{supp} \varphi \subset_K G \times G$, where $\varphi \in C_0^\infty(G)$. In view of (2.2) we obtain for all $(z, y) \in K$

$$|D_z^\alpha(Q(z, y, \xi)\, \varphi(y)\, e^{i(z-y,\xi)})| \leqq \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} |\xi^{\alpha-\gamma} (D_z^\gamma Q)(z, y, \xi)|\, |\varphi(y)|$$

$$\leqq \sum_{\gamma \in \alpha} \binom{\alpha}{\gamma} C_{\gamma,0,K}(1 + |\xi|)^{N_{\alpha,K} + |\alpha-\gamma|}\, |\varphi(y)|. \tag{2.3}$$

Hence the Mean Value Theorem and the Lebesgue Dominated Convergence Theorem imply that $(D_x^\alpha f)(x, \xi)$ exists and that

$$(D_x^\alpha f)(x, \xi) = \int_G D_x^\alpha\big(Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\big)\, dy. \tag{2.4}$$

Furthermore, the Lebesgue Dominated Convergence Theorem and (2.3) imply that the function $x \to (D_x^\alpha f)(x)$ is continuous.

B. For each $\alpha, \beta \in \mathsf{N}_0^n$ and $z \in \overline{B(x, \varepsilon)}$ one obtains by (2.6) and by partial integration

$$|\xi^\beta (D_z^\alpha f)(z, \xi)|$$

$$= \left| \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} \int_G (D_z^\gamma Q)(z, y, \xi)\, \varphi(y)\, \xi^{\alpha-\gamma+\beta}\, e^{i(z-y,\xi)}\, dy \right|$$

$$= \left| \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \int_G D_y^{\alpha-\gamma+\beta}\big((D_z^\gamma Q)(z, y, \xi)\, \varphi(y)\big)\, e^{i(z-y,\xi)}\, dy \right|$$

$$\le \sum_{\gamma \le \alpha} \sum_{\tau \le \alpha-\gamma+\beta} \binom{\alpha}{\gamma} \binom{\beta}{\tau} \int_G |(D_z^\gamma D_y^\tau Q)(z, y, \xi)|\, |(D_y^{\alpha-\gamma+\beta-\tau}\varphi)(y)|\, dy$$

$$\le \sum_{\gamma \le \alpha} \sum_{\tau \le \alpha-\gamma+\beta} \binom{\alpha}{\gamma} \binom{\beta}{\tau} C_{\gamma,\tau,K} \int_G |(D_y^{\alpha-\gamma+\beta-\tau}\varphi)(y)|\, dy\, (1 + |\xi|)^{N_{\gamma,K}+\delta_{\gamma,K}|\tau|}$$

$$\le C_{\alpha,\beta,K,\varphi}(1 + |\xi|)^{M_{\alpha,K}+\delta'_{\alpha,K}|\beta|}, \tag{2.5}$$

where $C_{\alpha,\beta,K,\varphi} > 0$ is a suitable constant, $M_{\alpha,K} = \max_{\gamma \le \alpha} \{N_{\gamma,K} + \delta_{\gamma,K}\, |\alpha - \gamma|\}$ and where $\delta'_{\alpha,K} = \max_{\gamma \le \alpha} \{\delta_{\gamma,K}\} < 1$. Hence for each $l \in \mathbf{N}$ there exist constants $C_{\alpha,l,K,\varphi} > 0$ such that

$$\sup_{z \in \overline{B(x,\varepsilon)}} |(D_z^\alpha f)(z, \xi)| \le C_{\alpha,l,K,\varphi}(1 + |\xi|)^{M_{\alpha,K}+(\delta'_{\alpha,K}-1)l} \quad \forall \xi \in \mathbf{R}^n. \tag{2.6}$$

In virtue of (2.3)—(2.4) and the Lebesgue Dominated Convergence Theorem one sees that the function $\xi \to (D_z^\alpha f)(z, \xi)$ is continuous. Hence choosing $l \in \mathbf{N}$ such that $M_{\alpha,K} + (\delta'_{\alpha,K} - 1)\, l \le -(n + 1)$ we obtain by (2.6) that the function $\xi \to (D_z^\alpha f)(z, \xi)$ is integrable. Now, let $g : G \to \mathbf{C}$ be the function defined by $y(x) = \int_{\mathbf{R}^n} f(x, \xi)\, d\xi$. Then the estimate (2.6), the Mean Value Theorem and the Lebesgue Dominated Convergence Theorem imply that $(D_x^\alpha g)(x)$ exists and that

$$(D_x^\alpha g)(x) = \int_{\mathbf{R}^n} (D_x^\alpha f)(x, \xi)\, d\xi. \tag{2.7}$$

Furthermore, the estimate (2.6) and the Lebesgue Dominated Convergence Theorem imply that the mapping $x \to (D_x^\alpha g)(x)$ is continuous. This shows that $Q\varphi \in C^\infty(G)$ and that

$$D_x^\alpha(Q\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left(\int_G D_x^\alpha\big(Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\big)\, dy \right) d\xi.$$

C. Let $K', K'' \subset_K G$ and let $m \in \mathbf{N}_0$. Define $K \subset_K G \times G$ by $K = K'' \times K'$. In virtue of (2.7) one sees that for all $x \in K''$ and $\varphi \in D(K')$

$$|\xi^\beta (D_x^\alpha f)(x, \xi)|$$

$$\le \sum_{\gamma \le \alpha} \sum_{\tau \le \alpha-\gamma+\beta} \binom{\alpha}{\gamma} \binom{\beta}{\tau} C_{\gamma,\tau,K}(1 + |\xi|)^{N_{\gamma,K}+\delta_{\gamma,K}|\tau|} \int_G |D_y^{\alpha-\gamma+\beta-\tau}\varphi(y)|\, dy$$

$$\le C'_{\alpha,\beta,K}(1 + |\xi|)^{M_{\alpha,K}+\delta'_{\alpha,K}|\beta|}\, m(K') \sup_{\substack{|\omega| \le |\alpha+\beta| \\ y \in K'}} |(D_y^\omega \varphi)(y)|,$$

where $C'_{\alpha,\beta,K} \geqslant 0$ is a suitable constant and where $M_{\alpha,K} = \max_{\gamma \le \alpha} \{N_{\gamma,K} + \delta_{\gamma,K}|\alpha - \gamma|\}$. $m(K')$ denotes the Lebesgue measure of K'. Hence for each $l \in \mathbf{N}$ one finds a constant $C_{\alpha,l,K} > 0$ such that

$$\sup_{x \in K''} |(D_x^\alpha f)(x, \xi)| \le C_{\alpha,l,K} \sup_{\substack{|\omega| \le |\alpha|+l \\ y \in K'}} |(D_y^\omega \varphi)(y)|\, (1 + |\xi|)^{M_{\alpha,K}+(\delta'_{\alpha,K}-1)l}. \tag{2.8}$$

Choosing $l \in \mathbb{N}$ such that $M_{\alpha,K} + (\delta'_{\alpha,K} - 1)\, l \leqq -(n+1)$ we obtain by (2.7) and by (2.8) that

$$\sup_{x \in K''} |D_x^\alpha(Q\varphi)(x)| \leqq \frac{1}{(2\pi)^n} C_{\alpha,l,K} \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{-(n+1)} \, d\xi \right) p_{|\alpha|+l,K'}(\varphi),$$

and then Q is a continuous operator $D(G) \to C^\infty(G)$ ∎

Remark: In view of (2.7) one has

$$D_x^\alpha(Q\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G D_x^\alpha(Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}) \, dy \right) d\xi.$$

2.3. Suppose that Q is a linear operator $C_0^\infty(G) \to C^\infty(G)$. We say that the formal transpose of Q exists, if one finds a linear operator $Q': C_0^\infty(G) \to C^\infty(G)$ such that

$$(Q\varphi)(\psi) := \int_G (Q\varphi)(x)\, \psi(x)\, dx = \varphi(Q'\psi) \qquad \forall \varphi, \psi \in C_0^\infty(G).$$

The operator Q' is called a *formal transpose* of Q. The following theorem yields a sufficient condition for the existence of Q'.

Theorem 2.2: *Suppose that* $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ *such that for each compact set* $K \subset_K G \times G$ *and for each* $(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ *there exist constants* $C_{\alpha,\beta,K} > 0$, $N_K \in \mathbb{R}$ *and* $\delta_K < 1$ *with which the estimate*

$$\sup_{(x,y) \in K} |(D_x^\alpha D_y^\beta Q)(x, y, \xi)| \leqq C_{\alpha,\beta,K} (1 + |\xi|)^{N_K + \delta_K |\alpha + \beta|} \tag{2.9}$$

holds for all $\xi \in \mathbb{R}^n$. *Then the formal transpose* Q' *of the operator* Q *defined by* (2.2) *exists and*

$$(Q'\psi)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G Q(x, y, -\xi)\, \psi(x)\, e^{i(y-x,\xi)} \, dx \right) d\xi. \tag{2.10}$$

Proof: A. In virtue of Theorem 2.1 the operator Q' defined by (2.10) maps $C_0^\infty(G)$ into $C^\infty(G)$. Furthermore, the operator Q defined by (2.2) maps $C_0^\infty(G)$ into $C^\infty(G)$. We have to establish that $(Q\varphi)(\psi) = \varphi(Q'\psi)$ for all $\varphi, \psi \in C_0^\infty(G)$.

B. Let ϕ be in C_0^∞ such that $\phi(\xi) = 1$ for all $|\xi| \leqq 1$ and define $\phi_j \in C_0^\infty$ by $\phi_j(\xi) = \phi(\xi/j)$. Furthermore, let $r: G \times \mathbb{R}^n \to \mathbb{C}$ be a function such that $r(y, \xi) = \int_G Q(x, y, -\xi)\, \psi(x)\, e^{-i(x,\xi)} \, dx$. For each $\alpha \in \mathbb{N}_0^n$ and for all $y \in \operatorname{supp} \varphi$ we obtain

$$\begin{aligned}
|\xi^\alpha r(y, \xi)| &= \left| \int_G Q(x, y, -\xi)\, \psi(x)\, (-D_x)^\alpha (e^{-i(x,\xi)}) \, dx \right| \\
&= \left| \int_G D_x^\alpha \big(Q(x, y, -\xi)\, \psi(x)\big)\, e^{-i(x,\xi)} \, dx \right| \\
&\leqq \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} \int_G |(D_x^{\alpha-\gamma} Q)(x, y, -\xi)|\, |(D_x^\gamma \psi)(x)| \, dx \\
&\leqq \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} C_{\alpha-\gamma,0,K} (1 + |\xi|)^{N_K + \delta_K |\alpha-\gamma|} \int_G |(D_x^\gamma \psi)(x)| \, dx,
\end{aligned}$$

where $K := \operatorname{supp} \psi \times \operatorname{supp} \varphi$, and then for each $l \in \mathbf{N}$ there exists a constant $C_{l,K,\psi} > 0$ such that

$$|r(y, \xi)| \leqq C_{l,K,\psi}(1 + |\xi|)^{N_K + (\delta_K - 1)l} \qquad \forall (y, \xi) \in \operatorname{supp} \varphi \in \mathbf{R}^n. \tag{2.11}$$

Hence (choosing $l \in \mathbf{N}$ such that $N_K + (\delta_K - 1) l \leqq -(n + 1)$) the Lebesgue Dominated Convergence Theorem implies that

$$(Q_j'\psi)(y) := \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \phi_j(\xi)\, r(y, \xi)\, e^{i(y,\xi)}\, d\xi \to (Q'\psi)(y) \qquad \forall y \in \operatorname{supp} \varphi. \tag{2.12}$$

Let $s: G \times \mathbf{R}^n \to \mathbf{C}$ be a function such that $s(x, \xi) = \int_G Q(x, y, \xi)\, \varphi(y)\, e^{i(-y,\xi)}\, dy$. Then one verifies as above that for each $l \in \mathbf{N}$ there exists a constant $C_{l,K,\varphi} > 0$ such that

$$|s(x, \xi)| \leqq C_{l,K,\varphi}(1 + |\xi|)^{N_K + (\delta_K - 1)l} \qquad \forall (x, \xi) \in \operatorname{supp} \psi \times \mathbf{R}^n. \tag{2.13}$$

Hence (choosing $l \in \mathbf{N}$ so that $N_K + (\delta_K - 1) l \leqq -(n + 1)$) the Lebesgue Dominated Convergence Theorem implies that

$$\begin{aligned}(Q_j\varphi)(x) &:= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \phi_j(\xi)\, s(x, \xi)\, e^{i(x,\xi)}\, d\xi \\ &\to \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} s(x, \xi)\, e^{i(x,\xi)}\, d\xi = (Q\varphi)(x) \qquad \forall x \in \operatorname{supp} \psi.\end{aligned} \tag{2.14}$$

D. The function $(x, y, \xi) \to Q(x, y, -\xi)\, \psi(x)\, \phi_j(\xi)\, \varphi(y)\, e^{i(y-x,\xi)}$ belongs to $L^1(G \times G \times \mathbf{R}^n)$. Hence due to the Fubini Theorem one gets

$$\begin{aligned}(Q_j'\psi)(\varphi) &= \int_G \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \phi_j(\xi) \left(\int_G Q(x, y, -\xi)\, \psi(x)\, e^{i(y-x,\xi)}\, dx\right) d\xi\right) \varphi(y)\, dy \\ &= \int_G \psi(x) \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \phi_j(\xi) \left(\int_G Q(x, y, -\xi)\, \varphi(y)\, e^{i(y-x,\xi)}\, dy\right) d\xi\right) dx \\ &= \psi(Q_j\varphi).\end{aligned} \tag{2.15}$$

In virtue of (2.11) and (2.13) (where we choose $l \in \mathbf{N}$ so that $N_K + (\delta_K - 1) l \leqq -(n + 1)$) one finds that

$$|(Q_j'\psi)(y)| \leqq \frac{C_{l,K,\psi}}{(2\pi)^n} \int_{\mathbf{R}^n} (1 + |\xi|)^{-(n+1)}\, d\xi < \infty, \qquad y \in \operatorname{supp} \varphi$$

and

$$|(Q_j\varphi)(x)| \leqq \frac{C_{l,K,\varphi}}{(2\pi)^n} \int_{\mathbf{R}^n} (1 + |\xi|)^{-(n+1)}\, d\xi < \infty, \qquad x \in \operatorname{supp} \psi.$$

Hence the convergences (2.12), (2.14), the relation (2.15) and the Lebesgue Dominated Convergence Theorem imply that $(Q\varphi)(\psi) = \lim_{j\to\infty} (Q_j\varphi)(\psi) = \lim_{j\to\infty} \varphi(Q_j'\psi) = \varphi(Q'\psi)$, which completes the proof ∎.

Remark: As in the proof of Theorem 2.1/Part C, one sees that the formal transpose Q' given by (2.10) is continuous $D(G) \to C^\infty(G)$.

Corollary: *Suppose that $L(\cdot, \cdot) \in C^\infty(G \times \mathbb{R}^n)$ such that for each compact set $K \subset_K G$ and for each $\alpha \in \mathbb{N}_0^n$ there exist constants $C_{\alpha,K} > 0$, $N_K \in \mathbb{R}$ and $\delta_K < 1$ with which the estimate $\sup_{x \in K} |(D_x^\alpha L)(x, \xi)| \leqq C_{\alpha,K}(1 + |\xi|)^{N_K + \delta_K|\alpha|}$ for all $\xi \in \mathbb{R}^n$ holds. Then the operator L defined by*

$$(L\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} L(x, \xi)(F\varphi)(\xi) e^{i(\xi, x)} d\xi$$

maps $D(G)$ continuously into $C^\infty(G)$. Furthermore, the continuous formal transpose $L' : D(G) \to C^\infty(G)$ exists and L' is given by

$$(L'\psi)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} L(y, -\xi)(F\psi)(\xi) e^{i(\xi, y)} d\xi.$$

Here F denotes the Fourier transform $S \to S$.

Proof: Let $Q(x, y, \xi) = L(x, \xi)$. Then $Q(\cdot, \cdot, \cdot)$ obeys the estimate (2.9). Furthermore, one has

$$(Q\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G L(x, \xi) \varphi(y) e^{-i(y,\xi)} dy \right) e^{i(x,\xi)} d\xi = (L\varphi)(x)$$

and

$$(Q'\psi)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G L(x, -\xi) \psi(x) e^{-i(x,\xi)} dx \right) e^{i(y,\xi)} d\xi = (L'\psi)(y)$$

and then the proof is ready ∎

3. The pseudolocal property

3.1. Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ satisfies the estimate (2.3). Let Q (for $\varphi \in C_0^\infty(G)$) be defined by (2.2). In view of Theorem 2.1 we know that Q is a continuous operator $D(G) \to C^\infty(G)$. Hence due to the Schwarz Kernel Theorem one finds an unique distribution $G \in D'(G \times G)$ such that

$$(Q\varphi)(\psi) = T(\varphi \times \psi) \quad \text{for all } \varphi, \psi \in C_0^\infty(G),$$

where $\varphi \times \psi \in C_0^\infty(G \times G)$ is defined by $(\varphi \times \psi)(x, y) = \varphi(x)\psi(y)$.

Lemma 3.1: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ such that for each $K \subset_K G \times G$ and for each $(\alpha, \beta, \gamma) \in \mathbb{N}_0^{3n}$ one finds constants $C_{\alpha,\beta,\gamma,K} > 0$, $N_{\alpha,\beta,K} \in \mathbb{R}$ and $\varrho_{\alpha,\beta,K} > 0$ such that the estimate*

$$\sup_{(x,y) \in K} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x, y, \xi)| \leqq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N_{\alpha,\beta,K} - \varrho_{\alpha,\beta,K}|\gamma|} \quad \forall \xi \in \mathbb{R}^n \tag{3.2}$$

is valid. Then the function $t_\gamma : G \times G \to \mathbb{C}$ defined by

$$t_\gamma(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi) e^{i(x-y,\xi)} d\xi \tag{3.3}$$

lies in $C^j(G \times G)$, when $\gamma \in \mathbb{N}_0^n$ so that $j + \max_{|\alpha+\beta| \leqq j} \{N_{\alpha,\beta,K}\} - \left(\min_{|\alpha+\beta| \leqq j} \{\varrho_{\alpha,\beta,K}\}\right)|\gamma| \leqq -(n + 1)$.

Proof: Let (x, y) be in $G \times G$ and let ε be a positive number so that $B((x, y), 2\varepsilon) \subset G \times G$. Then one has $K := \overline{B((x, y), \varepsilon)} \subset_K G \times G$ and so by (3.2) one gets for $|\alpha + \beta| \leqq j$

$$\sup_{(u,v)\in K} \left|D_u{}^\alpha D_v{}^\beta\left((D_\xi{}^\gamma Q)(u, v, \xi)\, e^{i(u-v,\xi)}\right)\right|$$

$$\leqq \sum_{\tau\leqq\alpha} \sum_{\omega\leqq\beta} \binom{\alpha}{\tau}\binom{\beta}{\omega} \sup_{(u,v)\in K} |(D_u{}^{\alpha-\tau} D_v{}^{\beta-\omega} D_\xi{}^\gamma Q)(u, v, \xi)\, \xi^{\tau+\omega}| \tag{3.4}$$

$$\leqq \sum_{\tau\leqq\alpha} \sum_{\omega\leqq\beta} \binom{\alpha}{\tau}\binom{\beta}{\omega} C_{\alpha-\tau,\beta-\omega,\gamma,K}(1 + |\xi|)^{N_{j,K}-\varrho_{j,K}|\gamma|+j},$$

where $N_{j,k} = \max_{|\alpha+\beta|\leqq j} \{N_{\alpha,\beta,K}\}$ and $\varrho_{j,k} = \min_{|\alpha+\beta|\leqq j} \{\varrho_{\alpha,\beta,K}\} > 0$. When $\gamma \in \mathsf{N}_0{}^n$ such that $j + N_{j,K} - \varrho_{j,K}|\gamma| \leqq -(n + 1)$ we see by (3.4) that

$$\sup_{(u,v)\in K} |D_u{}^\alpha D_v{}^\beta\left((D_\xi{}^\gamma Q)(u, v, \xi)\, e^{i(u-v,\xi)}\right)| \leqq C'_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{-(n+1)}$$

for each $|\alpha + \beta| \leqq j$. Hence the Mean Value Theorem and the Lebesgue Dominated Convergence Theorem imply that $(D_x{}^\alpha D_y{}^\beta t_\gamma)(x, y)$ exists and that

$$(D_x{}^\alpha D_y{}^\beta t_\gamma)(x, y) = \frac{1}{(2\pi)^n} \int_{\mathsf{R}^n} D_x{}^\alpha D_y{}^\beta\left((D_\xi{}^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\right) d\xi.$$

One also sees that $D_x{}^\alpha D_y{}^\beta t_\gamma$ is continuous, which completes the proof ∎

In the sequel we denote by T_γ the distribution included by t_γ, that is,

$$T_\gamma(\varphi) = \int_{G\times G} t_\gamma(x, y)\, \varphi(x, y)\, dx\, dy.$$

Theorem 3.2: *Suppose that $Q(\cdot,\cdot,\cdot) \in C^\infty(G \times G \times \mathsf{R}^n)$ such that for each $K \subset_K G \times G$ and for each $(\alpha, \beta, \gamma) \in \mathsf{N}_0{}^{3n}$ one finds constants $C_{\alpha,\beta,\gamma,K} > 0$, $N_{\alpha,K}$, $\delta_{\alpha,K} < 1$ and $\varrho_{\alpha,\beta,K} \geqq 0$ with which the estimate*

$$\sup_{(x,y)\in K} |(D_x{}^\alpha D_y{}^\beta D_\xi{}^\gamma Q)(x, y, \xi)| \leqq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N_{\alpha,K}+\delta_{\alpha,K}|\beta|-\varrho_{\alpha,\beta,K}|\gamma|} \tag{3.5}$$

is valid. Then the distribution kernel T of Q satisfies the relation $(y - x)^\gamma T = T_\gamma$, when $\gamma \in \mathsf{N}_0{}^n$ such that

$$N_{0,K} - \varrho_{0,0,K}|\gamma| \leqq -(n + 1). \tag{3.6}$$

Proof: A. In virtue of Theorem 2.1 Q maps continuously $D(G)$ into $C^\infty(G)$ and due to Lemma 3.1 T_γ is a continuous function when $\gamma \in \mathsf{N}_0{}^n$ obeys (3.6). It sufficies to establish the relation $((y - x)^\gamma T)(\varphi \times \psi) = T_\gamma(\varphi \times \psi)$ for all $\varphi, \psi \in C_0{}^\infty(G)$ (cf. [5, p. 127]). Furthermore we get

$$((x - y)^\gamma T)(\varphi \times \psi)$$

$$= \sum_{\tau\leqq\gamma} \binom{\gamma}{\tau} (-1)^{|\tau|} (y^\tau x^{\gamma-\tau})(\varphi \times \psi)$$

$$= \sum_{\tau\leqq\gamma} \binom{\gamma}{\tau} (-1)^{|\tau|} T((y^\tau\varphi) \times (x^{\gamma-\tau}\psi)) = \sum_{\tau\leqq\gamma} \binom{\gamma}{\tau} (-1)^{|\tau|} (Q(y^\tau\varphi))(x^{\tau-\gamma}\psi)$$

$$= \sum_{\tau \leq \gamma} \binom{\gamma}{\tau} (-1)^{|\tau|} \int_G \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \left(\int_G Q(x, y, \xi)\, y^\tau \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) \right) d\xi\, x^{\gamma-\tau} \psi(x)\, dx$$

$$= \int_G \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \left(\int_G \sum_{\tau \leq \gamma} \binom{\gamma}{\tau} (-1)^{|\tau|} y^\tau x^{\gamma-\tau} Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) \right) d\xi\, \psi(x)\, dx$$

$$= \int_G \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G (x-y)^\gamma Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi\, \psi(x)\, dx. \tag{3.7}$$

Let $\phi_j \in C_0^\infty$ be as in the proof of Theorem 2.3 and let $q_\gamma: G \times \mathbb{R}^n \to \mathbb{C}$ be the function $q_\gamma(x, \xi) = \int_G (x-y)^\gamma Q(x, y, \xi)\, \varphi(y)\, e^{i(-y,\xi)}\, dy$. For each $\beta \in \mathbb{N}_0^n$ and for all $x \in \operatorname{supp} \psi$ we get

$$\begin{aligned} |\xi^\beta q_\gamma(x, \xi)| &= \left| \int_G D_y^\beta\big((x-y)^\gamma Q(x, y, \xi)\, \varphi(y)\big)\, e^{i(-y,\xi)}\, dy \right| \\ &\leq \sum_{u \leq \beta} \sum_{v \leq \beta - u} \binom{\beta}{u} \binom{\beta-u}{v} \int_G |D_y^u\big((x-y)^\gamma\big)\, (D_y^v Q)(x, y, \xi)\, (D_y^{\beta-u-v}\varphi)(y)|\, dy \\ &\leq \sum_{u \leq \beta} \sum_{v \leq \beta - u} \binom{\beta}{u} \binom{\beta-u}{v} \sup_{(x,y) \in K} |D_y^u (x-y)^\gamma| \int_G |(D_y^{\beta-u-v}\varphi)(y)|\, dy \\ &\quad \times C_{0,v,0,K} (1 + |\xi|)^{N_{0,K} + \delta_{0,K}|\beta|} \\ &\leq C_{\beta,\varphi,K} (1 + |\xi|)^{N_{0,K} + \delta_{0,K}|\beta|} \end{aligned}$$

with suitable constant $C_{\beta,\varphi,K} > 0$, where $K = \operatorname{supp} \psi \times \operatorname{supp} \varphi$. Hence for each $l \in \mathbb{N}$ one finds $C_{l,\varphi,K} > 0$ so that $|q_\gamma(x, \xi)| \leq C_{l,\varphi,K} (1 + |\xi|)^{N_{0,K} + (\delta_{0,K} - 1) l}$ for all $x \in \operatorname{supp} \psi$ and $\xi \in \mathbb{R}^n$. Choose $l \in \mathbb{N}$ such that $N_{0,K} + (\delta_{0,K} - 1)\, l \leq -(n+1)$. Then the Lebesgue Dominated Convergence Theorem implies that

$$(Q_{\gamma,j}\varphi)(x) := \int_{\mathbb{R}^n} \phi_j(\xi)\, q_\gamma(x, \xi)\, e^{i(x,\xi)}\, d\xi \to \int_{\mathbb{R}^n} q_\gamma(x, \xi)\, e^{i(x,\xi)}\, d\xi. \tag{3.8}$$

B. Since the function $(y, \xi) \to (x-y)^\gamma Q(x, y, \xi)\, \phi_j(\xi)\, \varphi(y)\, e^{i(x-y,\xi)}$ lies in $L^1(G \times \mathbb{R}^n)$ we get due to the Fubini Theorem

$$\begin{aligned} (Q_{\gamma,j}\varphi)(x) &= \int_{\mathbb{R}^n} \left(\int_G (x-y)^\gamma Q(x, y, \xi)\, \phi_j(\xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi \\ &= \int_G \left(\int_{\mathbb{R}^n} Q(x, y, \xi)\, \phi_j(\xi)\, (x-y)^\gamma e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \\ &= (-1)^{|\gamma|} \int_G \left(\int_{\mathbb{R}^n} D_\xi^\gamma\big(Q(x, y, \xi)\, \phi_j(\xi)\big)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \\ &= (-1)^{|\gamma|} \int_G \left(\int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, \phi_j(\xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \\ &\quad + (-1)^{|\gamma|} \sum_{u < \gamma} \binom{\gamma}{u} \int_G \left(\int_{\mathbb{R}^n} (D_\xi^u Q)(x, y, \xi)\, (D_\xi^{\gamma-u}\phi_j)(\xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \\ &=: I_1 + I_2. \end{aligned}$$

C. The first integral I_1 is converging to

$$(-1)^{|\gamma|} \int_G \left(\int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy\,. \tag{3.9}$$

Since $\phi_j(\xi) \to 0$ with $j \to \infty$ and since for all $(x, y) \in K := \operatorname{supp} \psi \times \operatorname{supp} \varphi$ one has

$$\begin{aligned} |(D_\xi^\gamma Q)(x, y, \xi])\, \phi_j(\xi)\, e^{i(x-y,\xi)}| &\leqq \sup_{\xi \in \mathbb{R}^n} |\phi(\xi)|\, C_{0,0,\gamma,K}(1 + |\xi|)^{N_{0,K} - \varrho_{0,0,K}|\gamma|} \\ &\leqq \sup |\phi(\xi)|\, C_{0,0,\gamma,K}(1 + |\xi|)^{-(n+1)}, \end{aligned} \tag{3.10}$$

one sees due to the Lebesgue Dominated Convergence Theorem that

$$\int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, \phi_j(\xi)\, e^{i(x-y,\xi)}\, d\xi \to \int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi$$

for all $y \in \operatorname{supp} \varphi$ and $x \in \operatorname{supp} \psi$. In addition, one has by (3.10)

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, \phi_j(\xi)\, e^{i(x-y,\xi)}\, d\xi \right| \\ &\leqq \sup |\phi(\xi)|\, C_{0,0,\gamma,K} \int_{\mathbb{R}^n} (1 + |\xi|)^{-(n+1)}\, d\xi \end{aligned}$$

for all $y \in \operatorname{supp} \varphi$. Hence the Lebesgue Dominated Convergence Theorem implies that I_1 is tending to (3.9).

D. We consider now the integral I_2. Due to the Fubini Theorem we obtain

$$\begin{aligned} &\int_G \left(\int_{\mathbb{R}^n} (D_\xi^u Q)(x, y, \xi)\, (D_\xi^{\gamma-u}\phi_j)(\xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \\ &= \int_{\mathbb{R}^n} \left(\int_G (D_\xi^u Q)(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) (D_\xi^{\gamma-u}\phi_j)(\xi)\, d\xi\,. \end{aligned} \tag{3.11}$$

For each $\beta \in \mathbb{N}_0^n$ and $(x, y) \in K = \operatorname{supp} \psi \times \operatorname{supp} \varphi$ we get

$$\begin{aligned} &\left| \xi^\beta \left(\int_G (D_\xi^u Q)(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) \right| \\ &\leqq \sum_{v \leq \beta} \binom{\beta}{v} \int_G |(D_y^v D_\xi^u Q)(x, y, \xi)|\, |(D_y^{\beta-v}\varphi)(y)|\, dy \\ &\leqq \sum_{v \leq \beta} \binom{\beta}{v} C_{0,v,u,K}(1 + |\xi|)^{N_{0,K} + \delta_{0,K}|v| - \varrho_{0,v,K}|u|} \cdot \int_G |(D_y^{\beta-v}\varphi)(y)|\, dy \\ &\leqq C_{\beta,u,K,\varphi}(1 + |\xi|)^{N_{0,K} + \delta_{0,K}|\beta|} \end{aligned}$$

with suitable constant $C_{\beta,u,K,\varphi} > 0$ (note that $\varrho_{0,v,K} > 0$ for each $v \leqq \beta$). Hence one finds a constant $C_{u,K,\varphi} > 0$ such that

$$\left| \int_G (D_\xi^u Q)(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right| \leqq C_{u,K,\varphi}(1 + |\xi|)^{-(n+1)}$$

for all $(x, y) \in K$ and $\xi \in \mathbb{R}^n$. Since $(D_y^{\gamma-u}\phi_j)(\xi) = j^{-|\gamma-u|}(D_y^{\gamma-u}\phi)(\xi/j)$ one obtains by (3.11)

$$\begin{aligned} &\left| \int_G \left(\int_{\mathbb{R}^n} (D_\xi^u Q)(x, y, \xi)\, (D_\xi^{\gamma-u}\phi_j)(\xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy \right| \\ &\leqq C_{u,K,\varphi} \int_{\mathbb{R}^n} (1 + |\xi|)^{-(n+1)}\, d\xi \sup_{\xi \in \mathbb{R}^n} |(D_\xi^{\gamma-u}\phi)(\xi)| \left(\frac{1}{j} \right)^{|\gamma-u|} \to 0 \text{ with } j \to \infty. \end{aligned}$$

E. In view of the parts B, C and D one obtains that

$$(Q_{\gamma,j}\varphi)(x) \to (-1)^{|\gamma|} \int_G \left(\int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy$$

and then by (3.8)

$$\int_{\mathbb{R}^n} \left(\int_G (x-y)^\gamma Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi$$

$$= (-1)^{|\gamma|} \int_G \left(\int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi \right) \varphi(y)\, dy.$$

This implies by (3.7) that

$$\left((x-y)^\gamma T\right)(\varphi \times \psi)$$

$$= (-1)^{|\gamma|} \int_{G\times G} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (D_\xi^\gamma Q)(x, y, \xi)\, e^{i(x-y,\xi)}\, d\xi \right) \psi(x)\, \varphi(y)\, dx\, dy$$

$$= (-1)^{|\gamma|} T_\gamma(\psi \times \varphi),$$

as desired ∎

Denote by D the diagonal of $G \times G$, that is, $D = \{(x, y) \in G \times G \mid x = y\}$.

Corollary: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ satisfies the estimate (3.5). Then the distribution kernel T of Q obeys $T|_{G\times G \setminus D} \in C^\infty(G \times G \setminus D)$.*

Proof: Let $j \in \mathbb{N}_0$. Choose $\gamma \in \mathbb{N}_0^n$ so that

$$j + \max_{|\alpha+\beta|\leq j} \{N_{\alpha,K} + \delta_{\alpha,K} |\beta|\} - \left(\min_{|\alpha+\beta|\leq j} \{\varrho_{\alpha,\beta,K}\} \right) |\gamma| \leqq -(n+1).$$

Then $t_\gamma \in C^j(G \times G)$ (Lemma 3.1). Due to Theorem 3.2 one has

$$T|_{G+G\setminus D} = \left((x-y)^{-\gamma} T_\gamma\right)|_{G+G\setminus D}$$

and then $t_\gamma/(x-y)^\gamma \in C^j(G \times G)$ induces the distribution $T|_{G\times G\setminus D}$. Thus $T|_{G\times G\setminus D} \in C^j(G \times G \setminus D)$ for each $j \in \mathbb{N}_0$ ∎

3.2. Suppose that Q is a continuous mapping $D(G) \to C^\infty(G)$ and that the continuous formal transpose $Q'\colon D(G) \to C^\infty(G)$ exists. Then one is able to define an extension $\mathbf{Q}\colon E'(G) \to D'(G)$ of Q by

$$(\mathbf{Q}u)(\varphi) = u(Q'\varphi) \qquad \text{for } \varphi \in C_0^\infty(G).$$

$\mathbf{Q}$ is continuous, when $E'(G)$ is equipped with the locally convex topology defined by the semi-norms $P_\psi(u) = |u(\psi)|$, $\psi \in C^\infty(G)$ and when $D'(G)$ is equipped with the locally convec topology defined by the semi-norms $P_\varphi(u) = |u(\varphi)|$, $\varphi \in C_0^\infty(G)$.

Let u be in $D'(G)$. The singular support of u is denoted by sing supp u (cf. [5, p. 42]). We show

Theorem 3.3: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ such that for each $K \subset_K G \times G$ and for each $(\alpha, \beta, \gamma) \in \mathbb{N}_0^{3n}$ one finds constants $C_{\alpha,\beta,\gamma,K} > 0$, $N_K \in \mathbb{R}$, $\delta_K < 1$ and $\varrho_{\alpha,\beta,K} > 0$ with which the estimate*

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x, y, \xi)| \leqq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N_K + \delta_K|\alpha+\beta| - \varrho_{\alpha,\beta,K}|\gamma|} \tag{3.12}$$

is valid. Then the inclusion sing supp $\mathbf{Q}u \subset$ sing supp u *for* $u \in E'(G)$ *holds.*

Proof: In view of Theorem 2.3 the continuous formal transpose $Q' : D(G) \to C^\infty(G)$ exists, and so we can form the extension $\mathbf{Q}: E'(G) \to D'(G)$. Furthermore, due to Corollary of Theorem 3.2 the kernel T of Q obeys $T|_{G\times G\setminus D} \in C^\infty(G \times G \setminus D)$. Denote by $t \in C^\infty(G \times G \setminus D)$ the function which induces the distribution $T|_{G\times G\setminus D}$. Suppose that $x \notin \operatorname{sing\,supp} u$. Choose $\varepsilon > 0$ so that $u|_{B(x,3\varepsilon)} \in C^\infty(B(x, 3\varepsilon))$. Let $\phi \in C_0^\infty(B(x, 3\varepsilon))$ such that $\phi(x) = 1$, for $x \in \overline{B(x, 2\varepsilon)}$. Then one has $\mathbf{Q}u = \mathbf{Q}(\phi u) + \mathbf{Q}((1 - \phi) u)$. The term $\mathbf{Q}(\phi u) = Q(\phi u)$ belongs to $C^\infty(G)$. We establish that $\mathbf{Q}((1 - \phi) u) \in C^\infty(B(x, \varepsilon/2))$. Let $\{\varphi_n\} \subset C_0^\infty(G)$ be a sequence such that $\varphi_n \to u$ in $E'(G)$ (cf. [5, p. 89]). Since $\mathbf{Q}: E'(G) \to D'(G)$ is continuous we have for all $\varphi \in C_0^\infty(B(x, \varepsilon/2))$:

$$\begin{aligned}
&[\mathbf{Q}((1 - \phi) u)] (\varphi) \lim_{n\to\infty} [Q((1 - \phi) \varphi_n)] (\varphi) \\
&= \lim_{n\to\infty} T((1 - \phi) \varphi_n \times \varphi) \\
&= \lim_{n\to\infty} \int_{G\times G\setminus D} t(x, y) ((1 - \phi) \varphi_n) (x) \varphi(y)\, dy\, dx \\
&= \lim_{n\to\infty} \int_{B(x,\varepsilon/2)} \left(\int_{G\setminus B(x,\varepsilon)} ((1 - \phi) \varphi_n) (x)\, t(x, y)\, dx \right) \varphi(y)\, dy \\
&= \int_{B(x,\varepsilon/2)} ((1 - \phi) u) (t(\cdot, y)) \varphi(y)\, dy,
\end{aligned}$$

since $t \in C^\infty((G \setminus B(x, \varepsilon)) \times B(x, \varepsilon/2)$ (cf. [5, p. 132]). The function $y \to ((1 - \phi) u) (t(\cdot, y))$ lies in $C^\infty(B(x, \varepsilon/2))$ and then the proof is ready ∎

4. The decomposition of Q

4.1. Let G be an open set in $\mathbb{R}^n$ and let D be the diagonal of $G \times G$. Then one has, as well-known

Lemma 4.1: *There exists a function $h \in C^\infty(G \times G)$ and an open neighbourhood $U \subset G \times G$ of D such that $h(x, y) = 1$, for all $(x, y) \in U$, and for each compact set $K' \subset_K G$ there exists a compact set $K'' \subset_K G$ so that* $\operatorname{supp} h(x, \cdot) \subset K''$, *for all $x \in K'$, and* $\operatorname{supp} h(\cdot, y) \subset K''$, *for all $y \in K'$.*

Proof: Let U and V be subsets of $G \times G$ such that

$$U = \{(x, y) \in G \times G \mid |x - y| + |1/d(x, \partial G) - 1/d(y, \partial G)| < 1\}$$

and

$$V = \{(x, y) \in G \times G \mid |x - y| + |1/d(x, \partial G) - 1/d(y, \partial G)| < 2\}.$$

Then one finds a function $h \in C^\infty(G \times G)$ such that $h(x, y) = 1$ for all $(x, y) \in U$ and $\operatorname{supp} h \subset V$ (cf. [3, 16.4.3.]). This completes the proof ∎

Suppose that $Q(\cdot, \cdot, \cdot)$ satisfies (3.7). We decompose $Q(\cdot, \cdot, \cdot)$ as follows:

$$Q(\cdot, \cdot, \cdot) = hQ(\cdot, \cdot, \cdot) + (1 - h) Q(\cdot, \cdot, \cdot) =: \tilde{Q}(\cdot, \cdot, \cdot) + R(\cdot, \cdot, \cdot). \tag{4.1}$$

Lemma 4.2: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbb{R}^n)$ satisfies the estimate* (3.5). *Then for each $\gamma \in \mathbb{N}_0^n$ one has $R = R_\gamma$, where the amplitude $R_\gamma(\cdot, \cdot, \cdot)$ of R_γ is given by*

$$R_\gamma(x, y, \xi) = \begin{cases} \left(\dfrac{(1 - h(x, y))}{(x - y)^\gamma} \right) (D_\xi^\gamma Q)(x, y, \xi), & \text{for } x \neq y \\ 0, & \text{for } x = y. \end{cases}$$

Proof: Denote by H the function $1 - h$. Then we obtain

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\gamma R)(x, y, \xi)| \leqq \sum_{u\leqq\alpha} \sum_{v\leqq\beta} \binom{\alpha}{u}\binom{\beta}{v} (D_x^{\alpha-u} D_y^{\beta-v} H)(x, y)$$

$$\leqq \sum_{u\leqq\alpha} \sum_{v\leqq\beta} \binom{\alpha}{u}\binom{\beta}{v} \sup_{(x,y)\in K} |(D_x^{\alpha-u} D_y^{\beta-v} H)(x, y)| \sup_{(x,y)\in K} |(D_x^u D_y^v D_\xi^\gamma Q)(x, y, \xi)|$$

$$\leqq \sum_{u\leqq\alpha} \sum_{v\leqq\beta} \binom{\alpha}{u}\binom{\beta}{v} \sup_{(x,y)\in K} |(D_x^{\alpha-u} D_y^{\beta-v} H)(x, y)|$$

$$\times C_{u,v,\gamma,K}(1 + |\xi|)^{N_{u,K}+\delta_{u,K}|v|-\varrho_{u,v,K}|\gamma|}$$

$$\leqq C'_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N'_{\alpha,K}+\delta'_{\alpha,K}|\beta|-\varrho'_{\alpha,\beta,K}|\gamma|}, \tag{4.2}$$

where $C'_{\alpha,\beta,\gamma,K} > 0$ and where $N'_{\alpha,K} = \max_{u\leqq\alpha} \{N_{u,K}\}$, $\delta'_{\alpha,K} = \max_{u\leqq\alpha} \{\delta_{u,K}\}$ and $\varrho'_{\alpha,\beta,K} = \min_{u\leqq\alpha, v\leqq\beta} \{\varrho_{u,v,K}\}$. Hence in virtue of Theorem 3.2 the distribution kernel T_R of R obeys the relation

$$(x - y)^{\gamma_0} T_R = T_{R,\gamma_0}, \tag{4.3}$$

when $\gamma_0 \in \mathbb{N}_0^n$ so that $N_{0,K} - \varrho_{0,0,K}|\gamma_0| = N'_{0,K} - \varrho'_{0,0,K}|\gamma_0| \leqq -(n + 1)$. Here T_{R,γ_0} is a distribution induced by the function

$$t_{R,\gamma_0}(x, y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} H(x, y)(D_\xi^{\gamma_0} Q)(x, y, \xi) e^{i(x-y,\xi)} d\xi = H(x, y) t_{\gamma_0}(x, y),$$

where t_γ is defined by (3.3). One sees easily that $T_{R|U} = T_{R,\gamma_0|U} = 0$. Let H_{γ_0} be the function

$$H_{\gamma_0} = \begin{cases} H(x, y)/(x - y)^{\gamma_0} & \text{for } x \neq y \\ 0 & \text{for } x = y. \end{cases} \tag{4.4}$$

Then H_{γ_0} lies in $C^\infty(G \times G)$ and by (4.3) the distribution T_R is induced by the function $H_{\gamma_0} t_{\gamma_0}$. Hence one has

$$(2\pi)^n (R\varphi)(\psi)$$
$$= T_R(\varphi \times \psi)$$
$$= \int_{G\times G} H_{\gamma_0}(x, y) \left(\int_{\mathbb{R}^n} (D_\xi^{\gamma_0} Q)(x, y, \xi) \varphi(y) e^{i(x-y,\xi)} d\xi\right) \varphi(y) \psi(x) dy\, dx$$
$$= \int_G \left(\int_{\mathbb{R}^n} \left(\int_G H_{\gamma_0}(x, y)(D_\xi^{\gamma_0} Q)(x, y, \xi) \varphi(y) e^{i(x-y,\xi)} dy\right) d\xi\right) \psi(x) dx$$

for all $\varphi, \psi \in C_0^\infty(G)$. Hence one has

$$(R\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_G H_{\gamma_0}(x, y)(D_\xi^{\gamma_0} Q)(x, y, \xi) e^{i(x-y,\xi)} dy\right) d\xi = (R_{\gamma_0}\varphi)(x).$$

Similarly one sees that $R_{\gamma'} = R_{\gamma_0}$ for each $\gamma' \in \mathbb{N}_0^n$, which proves the assertion ∎

Remark: **A.** Assume that the estimate (3.5) holds. Using the notation (4.4) one sees that the amplitude of R_γ can be written in the form $R_\gamma(x, y, \xi) = H_\gamma(x, y)(D_\xi^\gamma Q)(x, y, \xi)$. Performing the computation as in (4.2) one has

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\tau R_\gamma)(x, y, \xi)| \leqq C_{\alpha,\beta,\tau,\gamma,K}(1 + |\xi|)^{N'_{\alpha,K}+\delta'_{\alpha,K}|\beta|-\varrho'_{\alpha,\beta,K}|\tau|-\varrho'_{\alpha,\beta,K}|\gamma|},$$

where $N'_{\alpha,K} = \max_{u\leqq\alpha} \{N_{u,K}\}$, $\delta'_{\alpha,K} = \max_{u\leqq\alpha} \{\delta_{u,K}\}$ and $\varrho'_{\alpha,\beta,K} = \min_{u\leqq\alpha, v\leqq\beta} \{\varrho_{u,v,K}\}$.

In the case when there exists $N \in \mathbf{R}$, $\delta < 1$ and $\varrho > 0$ such that $N_{\alpha,K} \leqq \delta |\alpha|$, $\delta_{\alpha,K} \leqq \delta$ and $\varrho_{\alpha,\beta,K} \geqq \varrho$ one sees that for each $l \in \mathbf{N}$ one finds R_l so that $R \doteq R_l$ and

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\tau R_l)(x, y, \xi)| \leqq C_{\alpha,\beta,\tau,l}(1 + |\xi|)^{-l+\delta|\alpha+\beta|-\varrho|\tau|}$$

(choose $\gamma \in \mathbf{N}_0^n$ so that $-\varrho |\gamma| \leqq -l$ and that $N - \varrho |\gamma| \leqq -(n + 1)$).

B. Suppose that the estimate (3.5) is valid. Then for each $l \in \mathbf{N}$ one finds an operator R_l so that $R = R_l$ and $\sup_{(x,y)\in K} |R_l(x, y, \xi)| \leqq C_{l,K}(1 + |\xi|)^{-l}$. It sufficies to select $\gamma \in \mathbf{N}_0^n$ such that $N_{0,K} - \varrho_{0,0,K} |\gamma| \leqq -l$ and choose $R_l := R_\gamma$.

Since $R_\gamma = T_{H_\gamma t_\gamma}$ for $\gamma \in \mathbf{N}_0^n$; $N_{0,K} - \varrho_{0,0,K} |\gamma| \leqq -(n + 1)$, one has

Corollary: *Suppose that $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathbf{R}^n)$ satisfies the estimate (3.5). Then the distribution kernel T_R of R is induced by the function $H_\gamma t_\gamma$, where $\gamma \in \mathbf{N}_0^n$ so that $N_{0,K} - \varrho_{0,0,K} |\gamma| \leqq -(n + 1)$.*

Let $k_s\colon \mathbf{R}^n \to \mathbf{R}$; $s \in \mathbf{R}$ be defined by $k_s(\xi) = (1 + |\xi|^2)^{s/2}$. Define the norm $\|\cdot\|_s$ in $C_0^\infty(G)$ by

$$\|\varphi\|_s = \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |(F\varphi)(\xi)\, k_s(\xi)|^2 \, d\xi\right)^{1/2}.$$

For $l \in \mathbf{N}_0$ one has by the Parseval formula

$$\|\varphi\|_l \leqq C_l \sum_{|\alpha|\leqq l} \|D^\alpha \varphi\|_0, \qquad \|\varphi\|_0 := \|\varphi\|_{L_2(G)}.$$

Let H_s denote the Hörmander space B_{2,k_s} (cf. [6, p. 7]) and let $H_s^C(G) := E'(G) \cap H_s$. Denote by $C^l(G)$ the linear space of all l times continuous be differentiable functions $f\colon G \to \mathbf{C}$. Then $C^l(G)$ is a Frechet space, when one equips it with the locally convex topology defined by the semi-norms $q_K(f) = \sup_{x\in K, |\alpha|\leqq l} |D^\alpha f(x)|$, $K \subset_K G$ (cf. [9, pp. 85—89]).

Lemma 4.3: *Suppose that $\Phi \in C^{l+j}(G \times G)$ and that $u \in H_{-l}^C(G)$. Then there exist a sequence $\{\varphi_n\} \subset C_0^\infty(G)$ and an element $f_{\Phi,u} \in C^j(G)$ such that*

$$\|\varphi_n - u\|_{-l} \to 0 \text{ and } \sup_{y\in K, |\beta|\leqq j} \left|D_y^\beta\left[\varphi_n\big(\Phi(\cdot, y)\big) - f_{\Phi,u}(y)\right]\right| \to 0 \text{ for } K \subset_K G.$$

Proof: **A.** Choose $\theta \in C_0^\infty(G)$ so that $\theta(x) = 1$ for all $x \in G'$ where G' is an open set of G such that $\overline{G'} \subset_K G$ and that $\operatorname{supp} u \subset G'$. Since $u \in H_{-l}$ one finds a sequence $\{\psi_n\} \subset C_0^\infty(\mathbf{R}^n)$ such that $\|\psi_n - u\|_{-l} \to 0$ with $n \to \infty$. Then one has $\|\theta\psi_n - u\|_{-l} = \|\theta\psi_n - \theta u\|_{-l} \to 0$. We choose $\varphi_n = \theta\psi_n$.

B. Let $\beta \in \mathbf{N}_0^n$ such that $|\beta| \leqq j$ and let $K \subset_K G$. Choose $\theta' \in C_0^\infty(G)$ such that $\theta'\theta = \theta$. Then we obtain

$$\sup_{y\in K} \left|D_y^\beta\left[\varphi_n\big(\Phi(\cdot, y)\big) - \varphi_m\big(\Phi(\cdot, y)\big)\right]\right| = \sup_{y\in K} \left|\big((\varphi_n - \varphi_m)\big)\big(\theta'(D_y^\beta\Phi)(\cdot, y)\big)\right|$$

$$\leqq \|\varphi_n - \varphi_m\|_{-l} \sup_{y\in K} \|\theta'(D_y^\beta\Phi)(\cdot, y)\|_l$$

$$\leqq \|\varphi_n - \varphi_m\|_{-l}\, C_l \sum_{|\alpha|\leqq l} \sup_{y\in K} \left\|D_x^\alpha\big(\theta'(D_y^\beta\Phi)(\cdot, y)\big)\right\|_0$$

$$\leqq \|\varphi_n - \varphi_m\|_{-l}\, C_l \sum_{|\alpha|\leqq l} \sum_{\tau\leqq\alpha} \binom{\alpha}{\tau} \sup_{y\in K,\ x\in\operatorname{supp}\theta'} |(D_x^\tau D_y^\beta\Phi)(x, y)|\, \|D_x^{\alpha-\tau}\theta'\|_0,$$

where we used the elementary fact that $D_y^\beta(\varphi_n(\Phi(\cdot, y))) = \varphi_n((D_y^\beta\Phi)(\cdot, y))$. Hence $\{\varphi_n(\Phi(\cdot, y))\}$ is a Cauchy sequence in $C^j(G)$ and so one finds an element $f_{\Phi,u} \in C^j(G)$ such that $\sup_{y\in K, |\beta|\leq j} |D_y^\beta[\varphi_n(\Phi(\cdot, y)) - f_{\Phi,u}(y)]| \to 0$ with $n \to \infty$, which proves the assertion ∎

Theorem 4.4: *Suppose that the estimate (3.12) is valid. Then the operator Q can be decomposed in the form*

$$Q = \tilde{Q} + R_l, \tag{4.5}$$

where $\tilde{Q}$ and R_l satisfy

1° *for each $K \subset_K G \times G$ and $(\alpha, \beta, \gamma) \in \mathsf{N}_0^{3n}$ the estimate*

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\gamma \tilde{Q})(x, y, \xi)| \leq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{N_K+\delta_K|\alpha+\beta|-\varrho'_{\alpha,\beta,K}|\gamma|}$$

holds.

2° *$\tilde{Q}$ is properly supported (for the terminology cf.* [2, p. 177]).

3° *for each $K \subset_K G \times G$ and $(\alpha, \beta, \tau) \in \mathsf{N}_0^{3n}$ the estimate*

$$\sup_{(x,y)\in K} |(D_x^\alpha D_y^\beta D_\xi^\tau R_l)(x, y, \xi)| \leq C_{\alpha,\beta,\tau,K}(1 + |\xi|)^{N_K-\varrho'_{\alpha,\beta,K}l+\delta_K|\alpha+\beta|,\varrho'_{\alpha,\beta,K}|\tau|}$$

holds.

4° $\mathsf{R}_l(u) \in C^\infty(G)$ *for all $u \in E'(G)$. Here* $\varrho'_{\alpha,\beta,K} = \min_{u\leq\alpha, v\leq\beta} \{\varrho_{u,v,K}\}$ *and* $\mathsf{R}_l: E'(G) \to D'(G)$ *is the extension of R_l.*

Proof: **A.** We decompose $Q(\cdot, \cdot, \cdot)$ as in (4.1). Then $\tilde{Q}(\cdot, \cdot, \cdot)$ obeys the estimate (1°) (cf. (4.2)). Let $K' \subset_K G$; then by Lemma 4.1 one finds $K'' \subset_K G$ so that supp $h(\cdot, y) \subset K''$ for all $y \in K'$. Hence for all $x \in G \setminus K''$ and $\varphi \in D(K')$ we obtain

$$(\tilde{Q}\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathsf{R}^n} \left(\int_{K'} h(x, y)\, Q(x, y, \xi)\, \varphi(y)\, e^{i(x-y,\xi)}\, dy \right) d\xi = 0. \tag{4.6}$$

Similarly one sees that for each compact set $K' \subset G$ there exists $K'' \subset_K G$ such that $(\tilde{Q}\varphi)(x) = 0$ for all $x \in K'$ when $\varphi \in C_0^\infty(G)$ so that $\varphi(y) = 0$ for $y \in K''$. This proves 2°.

B. Let $N \in \mathsf{N}$ and let $\gamma \in \mathsf{N}_0^n$ so that $|\gamma| = l$. Then $R_l := R_\gamma$ satisfies the estimate (4.6). In virtue of Lemma 4.2 one has $R_l = R$ and so the relation (4.5) holds. We have to show that $\mathsf{R}_l u \in C^j(G)$ for each $j \in \mathsf{N}$. In virtue of the Paley-Wiener Theorem (cf. [5, p. 181]) u belongs to $H^C_{-m}(G)$ with some $m \in \mathsf{N}$. Choose $\mu \in \mathsf{N}_0^n$ such that $m + j + N_K + \delta_K(l + m) - \left(\min_{|\alpha+\beta|\leq j+m} \{\varrho_{\alpha,\beta,K}\}\right)|\mu| \leq -(n + 1)$. Then the function t_μ defined by (3.3) belongs to $C^{l+j}(G \times G)$ (Lemma 3.1). In virtue of the Corollary from Lemma 4.2 the distribution kernel T_{R_l} of R_l is induced by the function $H_\mu t_\mu$, which is a $C^{m+j}(G \times G)$-function. Let $\{\varphi_n\} \subset C_0^\infty(G)$ be a sequence such that

$$\|\varphi_n - u\|_{-m} \to 0 \text{ and } \sup_{y\in K, |\beta|\leq j} |D_y^\beta[\varphi_n((H_\mu t_\mu(\cdot, y)) - f_{H_\mu t_\mu,u}(y)]| \to 0.$$

Then one sees (since $\mathsf{R}_l: E'(G) \to D'(G)$ is continuous)

$$(\mathsf{R}_l u)(\varphi) = \lim_{n\to\infty} (R_l\varphi_n)(\varphi) = \lim_{n\to\infty} T_{R_l}(\varphi_n \times \varphi)$$
$$= \lim_{n\to\infty} (H_\mu t_\mu)(\varphi_n \times \varphi) = \lim_{n\to\infty} \int_G \varphi_n\big((H_\mu t_\mu)(\cdot, y)\big)\,\varphi(y)\,dy$$
$$= \int_G f_{H_\mu t_\mu, u}(y)\,\varphi(y)\,dy,$$

and then $R_l u$ is induced by the function $f_{H_\mu t_\mu, u} \in C^l(G)$, which proves the assertion ∎

Since $\tilde{Q}$ satisfies 1° of Theorem 4.4 we know by Theorem 2.3 that the continuous formal transpose $\tilde{Q}': D(G) \to C^\infty(G)$ exists and

$$(\tilde{Q}'\psi)(y) = \frac{1}{(2\pi)^n} \int_{\mathsf{R}^n} \left(\int_G \tilde{Q}(x, y, -\xi)\, \psi(x)\, e^{i(y-x,\xi)}\, dx \right) d\xi.$$

Since $Q(\cdot, \cdot, \cdot) = h(\cdot, \cdot)\, Q(\cdot, \cdot, \cdot)$ one sees that $\tilde{Q}'$ is also properly supported.

Lemma 4.5: *Suppose that the estimate* (3.12) *is valid. Then the operators* $\tilde{Q}$ *and* $\tilde{Q}': D(G) \to D(G)$ *are continuous.*

Proof: Since $D(G)$ is bornological it sufficies to verify that $\tilde{Q}$ and $\tilde{Q}'$ map bounded sets into bounded sets (cf. [10, pp. 46–47]). Since $\tilde{Q}$ and $\tilde{Q}': D(G) \to C^\infty(G)$ are continuous, they are also bounded. Hence one easily sees that $\tilde{Q}$ and $\tilde{Q}'$ are bounded (we recall that $B \subset D(G)$ is bounded if and only if there exists $K \subset_K G$ so that $B \subset D(K)$ and B is bounded in $D(K)$) ∎

Since $\tilde{Q}': D(G) \to D(G)$ is continuous we can define a continuous extension $\tilde{\mathsf{Q}}$: $D'(G) \to D'(G)$ of Q as $(\tilde{\mathsf{Q}}u)(\varphi) = u(\tilde{Q}'\varphi)$ for all $\varphi \in C_0^\infty(G)$. We have

Theorem 4.6: *Suppose that the estimate* (3.12) *is valid. Then the inclusion* sing supp $\tilde{\mathsf{Q}}u \subset$ sing supp u *for* $u \in D'(G)$ *holds.*

Proof: Suppose that $x \notin$ sing supp u. Choose $\varepsilon > 0$ such that $u|_{B(x,2\varepsilon)} \in C^\infty\big(B(x, 2\varepsilon)\big)$. Furthermore choose $\phi \in C_0^\infty\big(B(x, 2\varepsilon)\big)$ so that $\phi(y) = 1$ for all $y \in \overline{B(x, \varepsilon)}$. Then $\phi\varphi \in D(\operatorname{supp}\phi)$ and then one finds a compact set K of G such that $\tilde{Q}'(\phi\varphi) \in D(K)$ for all $\varphi \in C_0^\infty(G)$. Let G' be an open set of G such that $K \subset G'$ and $\overline{G}' \subset_K G$ and choose $\psi \in C_0^\infty(G)$ such that $\psi(x) = 1$ for all $x \in \overline{G}'$. Then one sees

$$\big(\phi(\tilde{\mathsf{Q}}u)\big)(\varphi) = u\big(\tilde{Q}'(\phi\varphi)\big) = u\big(\psi\tilde{Q}'(\phi\varphi)\big) = \big(\phi\tilde{\mathsf{Q}}(\psi u)\big)(\varphi) \qquad \forall \varphi \in C_0^\infty(G),$$

and then $\phi\tilde{\mathsf{Q}}u = \phi\tilde{\mathsf{Q}}(\psi u)$. In view of Theorem 3.3 we get

$$\operatorname{sing\,supp}(\phi\tilde{\mathsf{Q}}u) = \operatorname{sing\,supp}\big(\phi\tilde{\mathsf{Q}}(\psi u)\big)$$
$$\subset \operatorname{sing\,supp}\tilde{\mathsf{Q}}(\psi u) \subset \operatorname{sing\,supp}(\psi u) \subset \operatorname{sing\,supp} u.$$

Hence there exists $0 < \varepsilon' \leqq \varepsilon$ such that $\tilde{\mathsf{Q}}u|_{B(x,\varepsilon')} = \phi\tilde{\mathsf{Q}}u|_{B(x,\varepsilon')} \in C^\infty\big(B(x, \varepsilon')\big)$, as desired ∎

4.2. Suppose that (Φ, ϕ) forms a pair of weight functions $G \times \mathsf{R}^n \to \mathsf{R}$ in the sense of Beals and Fefferman (cf. [2, p. 176 and p. 162], cf. also [1]). Furthermore, let

$\int_{\Phi,\phi}^{M,m}(G \times G)$ denote the class of symbols $Q \in C^\infty(G \times G \times \mathsf{R}^n)$ such that

$$|D_x^\alpha D_y^\beta D_\xi^\gamma Q(x, y, \xi)| \leqq C_{\alpha,\beta,\gamma,K} \Phi^{M-|\gamma|}(x, \xi) \, \phi^{m-|\alpha+\beta|}(x, \xi), \tag{4.7}$$

for all $((x, y), \xi) \in K \times \mathsf{R}^n$, where $K \subset_K G \times G$. Denote by $p_1, p_2 \colon G \times G \to G$ the projections $p_1((x, y)) = x$ and $p_2((x, y)) = y$ and write $K' = p_1(K)$, $K'' = p_2(K)$. Then one sees due to the property (i) of weight functions Φ and ϕ

$$\Phi^{M-|\gamma|}(x, \xi) \leqq \max \{c_{K'}^{M-|\gamma|}, C_{K'}^{M-|\gamma|}\} \, (1 + |\xi|)^{|M|+|\gamma|}$$

and

$$\phi^{m-|\alpha+\beta|}(x, \xi) \leqq \max \{c_{K''}^{m-|\alpha+\beta|}, C_{K''}^{m-|\alpha+\beta|}\} \, (1 + |\xi|)^{(1-\varepsilon_{K''})(|m|+|\alpha+\beta|)}$$

with some positive constants $c_{K'}, C_{K'}, c_{K''}, C_{K''}$ and $\varepsilon_{K''}$. Hence the estimate (1.2) (= (3.5)) holds with $N_{\alpha,\beta,K} = |M| + (1 - \varepsilon_{K''}) |m| + (1 - \varepsilon_{K''}) |\alpha + \beta|$.

Assume that Φ obeys with $\varrho_{K'} > 0$ the condition

$$\Phi(x, \xi) \geqq c_{K'}(1 + |\xi|)^{\varrho_{K'}} \text{ for all } (x, \xi) \in K' \times \mathsf{R}^n, \text{ where } K' \subset_K G. \tag{4.8}$$

Then one sees

$$\Phi^{M-|\gamma|}(x, \xi) \leqq \begin{cases} C_{K'}^{M-|\gamma|}(1 + |\xi|)^{(M-|\gamma|)} & \text{if } M - |\gamma| \geqq 0 \\ c_{K'}^{M-|\gamma|}(1 + |\xi|)^{\varrho_{K'} M - \varrho_{K'}|\gamma|} & \text{if } M - |\gamma| < 0 \end{cases}$$

and then

$$\Phi^{M-|\gamma|}(x, \xi) \leqq \max \{C_{K'}^{M-|\gamma|}, c_{K'}^{M-|\gamma|}\} \, (1 + |\xi|)^{|M| - \varrho_{K'}|\gamma|}.$$

Thus the estimate (1.3) (= (3.12)) holds with $N_{\alpha,\beta,\gamma,K} = |M| + (1 - \varepsilon_{K''}) |m| + (1 - \varepsilon_{K''}) |\alpha + \beta| - \varrho_{K'} |\gamma|$.

Corollary: *Suppose that (Φ, ϕ) forms (locally) a pair of weight functions and that Φ satisfies* (4.8). *Let $Q(\cdot, \cdot, \cdot) \in C^\infty(G \times G \times \mathsf{R}^n)$ such that* (4.7) *is valid. Then the operator Q (defined by* (2.2)) *maps $D(G)$ continuously into $C^\infty(G)$, the continuous formal transpose $Q' \colon D(G) \to C^\infty(G)$ exists and the inclusion* sing supp $Qu \subset$ sing supp u *for $u \in E'(G)$ holds. Furthermore, Q can be decomposed in the form $Q = \tilde{Q} + R_l$ where $\tilde{Q}$ is properly supported, R_l satisfies the estimate* (3.12) *with*

$$N_{\alpha,\beta,\gamma,K} = |M| + (1 - \varepsilon_{K''}) |m| - \varrho_{K'} l + (1 - \varepsilon_{K''}) |\alpha + \beta| - \varrho_{K'} |\gamma|$$

and the extension R_l maps $E'(G)$ into $C^\infty(G)$. Also the inclusion sing supp $\tilde{Q}u \subset$ sing supp u *for $u \in D'(G)$ holds.*

Particularly the symbols $L(\cdot, \cdot) \in \int_{\Phi,\phi}^{M,m}(G)$ of the Beals and Fefferman type (cf. [2, p. 177]) satisfy the estimate (4.7).

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