A Quadrature-Based Approach to Improving the Collocation Method for Splines of Even Degree¹)

IAN.H. SLOAN and W. L. WENDLAND

Dedicated to Prof. Dr. S. G. Mikhlin on the occasion of his 80th birthday

Die "Qualokationsmethode" ist eine kürzlich entstandene durch numerische Quadratur modifizierte Kollokationsmethode. Sie wird hier auf eine Klasse von Randintegralgleichungen angewendet, wobei Spline-Funktionen stückweiser Polynome von geradem Grad als Ansatzfunktionen benutzt werden. Die analysierten Probleme sind von der Form $(L + K) u = f$, wobei L ein Faltungsoperator mit geradem Symbol und K ein Operator mit stärkeren Glättungseigenschaften als L ist. Wir zeigen, wie für einen Ansatzraum der Dimension n spezielle $2n$ -Punkt-Quadraturformeln konstruiert werden können, welche die komponierten Zweipunktformeln von Gauß verallgemeinern, so daß eine stabile konvergente Qualokationsmethode entsteht, deren Konvergenzordnung in geeigneten Sobolev-Räumen negativer Ordnung höher ist als die der gewöhnlichen Mittelpunktkollokation, die kürzlich von Saranen analysiert. wurde. Mit Hilfe der Technik glatter Störungen wird darüber hinaus hier auch die Analysis der Qualokationsmethoden mit Splines ungeraden Grades, die von Sloan entwickelt wurde, verallgemeinert.

Развитый недавно метод квалокации является методом коллокации, модифицированным квадратурными формулами. Этот метод здесь применяется к одному классу граничных интегральных уравнений, причём в качестве базисных функций используют полиномиальные сплайн-функций чётной степени. Рассматриваемые задачи имеют вид $(L + K)u = f$ где L оператор в свёртках с чётным симболом, а K оператор, обладающий более сильным свойством сглаживания чем L. Мы покажем, как можно построить для базисного пространства размерности п 2п-точечные квадратурные формулы, обобщающие составные двух точечные формулы Гаусса, чтобы получить устойчивый и сходящийся метод квалокации. Порядок сходимости этого метода в подходящих пространствах Соболева отрицательного порядка выше чем в случае обычной коллокации в средней точке, изучаемой недавно Saranen. Кроме того обобщается с помощью техники гладких возмущений анализ методов кралокации со сплайн-функциями иечётной степени, развитый Sloan.

The "qualocation" method, a recently proposed quadrature-based extension of the collocation method, is here applied to a class of boundary integral equations, using an even degree spline trial space on a uniform partition. The problems handled are of the form $(L+K)u = j$, where L is a convolutional operator with even symbol, and K is an operator with a greater smoothing effect than L. For a trial space of dimension n, it is shown that a certain $2n$ -point quadrature rule, which is a generalization of the repeated 2-point Gauss rule, gives a stable qualocation method, and yields an order of convergence, in suitable negative norms, two powers of h higher than achieved by the mid-point collocation method in the recent analysis of Saranen. The treatment of the smooth perturbation covers also the earlier analysis of the odd degree spline case by Sloan.

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1. Introduction

In a previous paper [14] a geieral class of methods ('qualocation methods') was proposed for solving a special class of boundary integral equations.

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Lu = f.
$$

There a specific method was developed for a particular class of strongly elliptic / boundary integral equations with even principal symbol, for trial spaces of *odd* degree smoothest splines on- a uniform mesh. Here we consider, for a slightly more general class of prpblems, the case of *even* degree smoothest splines; for example piecewise-constant functions, or piecewise-quadratics with continuous first deriv-362 LAN H. SLOAN and W. L. WENDLAND

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In brief, a qualocation method requires the choice of a 'trial' space *Sh,* here a spline space; a 'test' space T_h of the same finite dimension n_h ; and a quadrature rule Q_h of

the form $Q_h g = \sum_{l=1}^{m_h} w_l g(t_l)$, with $m_h \ge n_h$. The method is: find $u_h \in S_h$ such that

$$
Q_h(\bar{\chi}_h L u_h) = Q_h(\bar{\chi}_h f) \qquad \forall \chi_h \in T
$$

It is shown in [14] that if $m_h \ge n_h$. The method is: find $u_h \in S_h$ such that
 $Q_h(\bar{\chi}_h L u_h) = Q_h(\bar{\chi}_h f)$ $\forall \chi_h \in T_h$. (1.2)

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with trial space S_h and collocation points $t_1, ..., t_{n_h}$. Thus in general the method may be thought of as a quadrature-based generalization of the collocation method; hence the name 'qualocation'. It may also be thought of as a semi-discrete version of the Petrov-Galerkin method, or a generalization of quadrature formula methods [9]. The essential point is, that the freedom that exists in the design of the quadrature rule may be exploited to *improve the order of convergence*, especially in 'negative norms', over that achievable with the collocation method. Hence, qualocatiori can provide higher order convergence than collocation. As we shall see, the order of convergence can even exceed 'that of the superconvergent Galerkin or corresponding . Galerkin with trial space S_h and collocation points t_1, \ldots, t_n ,
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As in [14], the functions u and f in dodifferential operator of real order symbo

$$
Lu(x) = \hat{u}(0) + \sum_{k \neq 0} |k|^{\beta} \hat{u}(k) e^{2\pi i kx}, \quad \text{where } \hat{u}(k) = \int_{0}^{1} u(x) e^{-2\pi i kx} dx; \quad (1.3)
$$

and then in Section 6 consider-smooth perturbations $(L + K)u$ of (1.3). As discussed in [14], the logarithmic potential on a circle is obtained by setting $\beta = -1$ in (1.3); and the choice $\beta = 0$ makes L the identity, in which case the collocation method $\text{L}u(x) = \hat{u}(0) + \sum_{k=0} |k|$
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 $Lu(x) = \hat{u}(0) + \sum_{i\neq 0} |k|^{\beta} \hat{u}(k) e^{2\pi i kx}$, where $\hat{u}(k)$ Section 6 consider smooth perturbations $(L + K) u$ of (1.3). As discussed
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As indicated above, in this paper we choose the trial space S_h to be $S_h^d \subset C^{d-1}$, the space of smoothest splines of even degree *d,* subordinate to the uniform mesh spacing As indicated above, in this paper we choose the trial space S_h to b

space of smoothest splines of even degree d, subordinate to the unifo
 $h = 1/n_h = 1/n$. The breakpoints are chosen to be
 $\{jh : j = 0, 1, ..., n' - 1\}$.

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\{jh\colon j=0,\,1,\,...,\,n^t-1\}.
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As in [14], the test space T_h is taken to be the trigonometric space

$$
T_h = \text{span} \{e^{2\pi i p x} : -n/2 < p \leq n/2, p \in \mathbb{Z}\}.
$$

An Approach to Improving the Collocation Method 363

(In practice a slight modification of T_h , as in Section 5 of [14], is useful, to guarantee that *^U^h* is real). This choice of test space gives high orders of convergence for the Petrov-Galerkin method, according to the analysis of [2]. There, however, quadrature was not considered.

The most interesting question concerns the choice of the quadrature rule Q_h . If we choose a rule with *n* points (i.e. one point per sub-interval) then, as noted already, the method is equivalent to a collocation method. Hence the simplest non-trivial qualocation method that respects-the rotational symmetry of the test and trial spaces uses two points per sub-interval, and may be expressed as ice a slight modification of T_n , as in Section 5 of [14]
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$$
Q_h g = \frac{1}{n} \sum_{j=0}^{n-1} \left[w_1 g \left(\frac{j+\varepsilon_1}{n} \right) + w_2 g \left(\frac{j+\varepsilon_2}{n} \right) \right], \tag{1.6}
$$

where $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, and $w_1 + w_2 = 1$.

The question, then, is how to choose the free parameters ε_1 , ε_2 and w_2 in (1.6), in order to obtain high orders of convergence. In [141, for the case of splines of odd degree, we made the simple choice $\varepsilon_1 = 0$, $\varepsilon_2 = 1/2$, leaving only the weight w_2 to be fixed. However, for the splines of even degree it seems less obvious that the breakpoints are desirable quadrature points. Consequently, in the present work we avoid $Q_h g = \frac{1}{n} \sum_{j=0}^{n} \left[w_1 g \left(\frac{1-e_1}{n} \right) + w_2 g \left(\frac{1-e_2}{n} \right) \right],$
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order to obtain high orders of convergence. In [14

a priori assumptions about ε_1 and ε_2 .
The main results of the paper are contained in Theorem 2.1 in the next section. There are three parts of the theorem, the successive parts being concerned with 'increasing, levels of specialization in the choice of the quadrature parameters.

The first part of the theorem states a comforting result; that almost every sensible choice of the quadrature parameters yields a stable method that has the same optimalorder rate of convergence as predicted by SARANEN and WENDLAND [11] and ARNOLD and WENDLAND [4] for the mid-point collocation method and a more general class of. operators L. The only restrictions are that one of the quadrature-point parameters, say ε_2 , lies in the interior of the interval [0, 1] and has an associated positive weight, and that the other weight is non-negative. For the general choice of quadrature parameters the highest rate of convergence predicted by the first part of Theorem 2.1, as in [4] for the collocation method, is Figure 1: The highest rate of
 *u*_h - $u||_{\beta} \leq C h^{d+1-\beta}$ theorem, the successive parts being concerned with

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||u_h - u||_{\beta} \leq C h^{d+1-\beta} ||u||_{d+1}.
$$
 (1.7)

Here
$$
||u||_s
$$
 denotes the norm in the periodic Sobolev space H^s , which can be defined by
\n
$$
||u||_s^2 = |u(0)|^2 + \sum_{k \neq 0} |k|^{2s} |u(k)|^2,
$$
\n(1.8)

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ere as elsewhere in the paper, denotes a generic constant, which may take
values at its differ and *C*, here as elsewhere in the paper, denotes a generic constant, which may take different values at its different occurrences. *As* a special case of Theorem 2.1(i), we recover (by setting $w_1 = 0$, and $\varepsilon_2 = \varepsilon$) a known result (see Theorem 3 of SCHMIDT, $[13]$, namely that the *e*-collocation method of $[13]$, which is the collocation method with the collocation points $\{(j + \varepsilon) h : j = 0, 1, ..., n-1\}$, is stable for the operator (1.3) , and yields the order of convergence (1.7), for all values of ε in the open interval $(0, 1)$.

The second part of Theorem 2.1 asserts that a quadrature rule that is.'symmetric about the mid-point' can yield a higher order of convergence, in an appropriate negative norm. Precisely, the error bound-with the highest possible 'power of *h* now becomes

$$
|u_h - u||_{s-1} \leq Ch^{d+2-\beta} ||u||_{d+2}.
$$

Thus one extra power of *h* is achieved. However, this is at the expense of a more restrictive smoothness requirement on u on the right of (1.9) , and a 'more-negative'

norm on the left. For the special case of mid-point collocation (i.e. $\varepsilon_1 = \varepsilon_2 = 1/2$) the ¹

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 14D result (1.9) has already been obtained by SARANEN in [10]. Thus the second part of Theorem 2.1 may be thought of as a generalization of Saranen's result, to more general symmetric qualocation methods. 1364 IAN H. SLOAN and W. L. WENDLAND

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possibility of an improved approximation) is the last: for this asserts that one particular choice of symmetric quadrature rule can lead to yet two more powers of h in appropriate cases. In addition to the bounds (1.7) and (1.9), for this special choice 364 IAN H. SLOAN an
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$$
||u_h - u||_{\beta - 3} \leq C h^{d + 4 - \beta} ||u||_{d + 4}.
$$
\n(1.10)

For example, for the piecewise-constant trial space (i.e. $d = 0$) and the logarithmic potential (i.e. $\beta = -1$) we obtain $O(h^5)$ convergence, provided $u \in H^4$, and provided we are willing to look at érrors in the H^{-4} norm. A striking reflection is that the Galerkin method for the same problem yields at best $O(h^3)$ convergence $-$ see, for example, [7]. On the other hand it should be said that the Galerkin result imposes a weaker smoothness requirement on u (namely $u \in H$ ¹), and is achieved in a much less-negative norm (namely H^{-2}). α potential (i.e. $\beta = -\alpha$
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Theorem 2.1 is proved in Sections 3 and 4.
The special quadrature parameters in the third part of the theorem, yielding the higher rate of convergence shown in (1.10), are defined as follows: $w_1 = w_2 = 1/2$, $\epsilon_1 = x_0$, and $\epsilon_2 = 1 - x_0$, where x_0 is the unique zero in the interval (0, 1/2) of the function

$$
G_{\gamma}(x)=\sum_{n=1}^{\infty}n^{-\gamma}\cos 2\pi nx,
$$

with $\gamma = d + 2 - \beta$. The properties of this function are discussed in Section 5. Here we record some values of $x_0(y)$ so that they are available for application.
Table 1. Least positive zeros of the function G_y defined by (1.11)

 (1.11)

Table 1. Least positive zeros of the function G_r defined by (1.11)

The following remarks may help to give some insight into the quadrature parameters in Table 1. First, for $\gamma = 2$ the points x_0 and $1 - x_0$ are the abscisses of the 2-point Gauss rule, shifted to the interval [0, 1]. The value $\gamma = 2$ is the appropriate value of γ if we have $\beta = 0$, in which case *L* is the identity operator, and $d = 0$. Now in this-ease the integrals to be approximated, if we adopt the- Petrov-Galerkin view, are discontinuous piecewie-smooth functions, and for such functions the two-point Gauss rule would seem to be an ideal choice. Second, if *d* is very large, or if the order β has a large negative value, then the function Lu_h becomes a very

smooth periodic function, for which the equal weight and nearly equally-spaced rule that is achieved when γ is large would seem entirely appropriate. The other values in Table 1 may be thought of as intermediate between these two extremes.

The treatment of the perturbations $L + K$ in Section 6 is expressed in sufficiently general terms to cover not only the present method but also that in [14], and other qualocation methods. An important consequence of the perturbation result in Section 6 is that the conclusions expressed in the following theorem, and in the corresponding result in [14], hold not only for uniform meshes on circles, but also for smoothly varying meshes on' smooth closed curves (see ['4]). Hence, the qualocation methods could be applied to all the examples given in [6], and would converge with the high orders predicted here and in [14].

Acknowledgements: The authors express their 'gratitude to G. C. Chandler for providing us with the nontrivial entries of Table 1, and to G. Brown and D. Wilson of the University of New South Wales for proving for us, in [5], that G_r in (1.11) has a unique zero on $(0, 1/2)$. Figure 2.1 and in [14]
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2. The main result

Theorem 2.1: Let β be a real number, and let L be as in (1.3). Let $d > \beta - 1/2$ be a, *non-negative even integer, and let* $S_h = S_h^d$ be the space of smoothest splines of degree d. *with breakpoints (1.4). Let* T_A *be the trigonometric function space given by (1.5), and let* 2. The main result

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Assume in addition that s'and t are real numbers satisfying $s \leq t$ *,* $s < d + 1/2$ *,* $\beta + 1/2 < t$, and that the solution u of (1.1) belongs to H^t .

(i) For $\beta \leq s$ and $t \leq d + 1$, there holds the estimate

$$
||u_h - u||_s \leq C h^{t-s} ||u||_t.
$$
 (2.1)

(ii) If the quadrature rule is symmetric, in the sense that $w_1 = w_2 = 1/2$ *, and* $\varepsilon_2 = 1/2$ (i) For $\beta \leq s$ and $t \leq d+1$, there ho
 $||u_h - u||_s \leq Ch^{t-s} ||u||_t$.

(ii) If the quadrature rule is symmetrical $1 - \varepsilon_1$, then there holds the estimate $||u_h - u||_s \le$
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$$
||u_h - u||_s \leq Ch^{\min(t-s, t-\beta, d+1-\delta, d+2-\beta)} ||u||_t.
$$
\n(2.2)

(iii) If $w_1 = w_2 = 1/2$ and $\varepsilon_2 = 1 - \varepsilon_1$, and if ε_1 is the least positive zero of the (ii) *If the quadrature*
 $1 - \varepsilon_1$, then there holds
 $||u_h - u||_s \leq 0$

(iii) *If* $w_1 = w_2 = 1$,
 function $G_{d+2-\beta}$, where *t* $\|u_h - u\|_s \leq Ch^{\min(d-s,t-\beta)}$
 $\|u_h - u\|_s \leq Ch^{\min(d-s,t-\beta)}$

(iii) *If* $w_V = w_2 = 1/2$ and $\varepsilon_2 =$
 function $G_{d+2-\beta}$, where
 $G_{\gamma}(x) = \sum_{n=1}^{\infty} n^{-\gamma} \cos 2\pi nx$,
 then there holds the estimate
 $\|u_h - u\|_s \leq Ch^{\min(t-s,t-\beta)}$

\n- \n 1. Then the quadratic equation (1.2), with
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f
$$
 any continuous function, is solvable for $u_h \in S_{h}$.\n
\n- \n 2.1. Then the quadratic equation:\n $u_h \in S_{h}$.\n
\n- \n 3.2.1. If $h \geq h$ is a addition that s and t are real numbers satisfying\n $s \leq t$,\n $s < d + 1/2$,\n $s < t$,\n and that the solution u of (1.1) belongs to H^t .\n
\n- \n 2.1. If $h \geq s$ and $t \leq d + 1$, there holds the estimate\n $||u_h - u||_s \leq Ch^{min(t-s,t-\beta,d+1-s,d+2-\beta)} ||u||_t$.\n (2.2)\n $|u_l - u||_s \leq Ch^{min(t-s,t-\beta,d+1-s,d+2-\beta)} ||u||_t$.\n (2.2)\n $w_l = w_2 = 1/2$ and $\varepsilon_2 = 1 - \varepsilon_1$,\n and if ε_1 is the least positive zero of the\n $G_{d+2-\beta}$,\n where\n $G_{\gamma}(x) = \sum_{n=1}^{\infty} n^{-y} \cos 2\pi nx$,\n $G_{\gamma}(x) = \sum_{n=1}^{\infty} \frac{1}{2} \sin(1 - \varepsilon, \varepsilon$

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$$
||u_h - u||_s \leq Ch^{\min(t - s, t - \beta, d + 1 - s, d + 4 - \beta)} ||u||_t.
$$
\n(2.4)

then there holds the estimate
 $||u_h - u||_s \leq Ch^{min(t-s,t-\beta,d+1-s,d+4-\beta)} ||u||_t.$ (2.4)

Remark 1: Without loss we may restrict *6* and *t* in (2.2) by $\beta - 1 \leq s \leq t \leq d + 2$ and in

(2.4) by $\beta - 3 \leq s \leq t \leq d + 4$.

 $||u_h - u||_s \leq Ch^{\min(l-s, l-\beta, d+1-s, d+2-\beta)} ||u||_t$.

(iii) If $w_1 = w_2 = 1/2$ and $\varepsilon_2 = 1 - \varepsilon_1$, and if ε_1 is the least positive zero-
 $U_n(x) = \sum_{n=1}^{\infty} n^{-\gamma} \cos 2\pi nx$,

then there holds the estimate
 $||u_h - u||_s \leq Ch^{\min(l-s, l-\beta$ Remark 2: Since the condition $d > \beta - 1/2$, which is modelled on [4], is not strong enough to guarantee absolute convergence of the Fourier series for *LU,* (see Section 3), care is needed in defining the sense of convergence. We shall always understand the convergence to be in the conventional sense for the pointwise convergence of Fourier series, namely g, where
 $=\sum_{n=1}^{\infty} n^{-\gamma} \cos 2\pi nx,$ (2.3)

the estimate
 $u||_s \leq Ch^{min(t-s,t-\beta,d+1-s,d+4-\beta)} ||u||_t.$ (2.4)

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(2.5)

(2.5)

(2.5)

(2.5)

(2.5)

(2 *k* 1: Without loss we may restrict s and t in (2.2) by β
 $-3 \leq s \leq t \leq d + 4$.
 k 2: Since the condition $d > \beta - 1/2$, which is modelle

ce absolute convergence of the Fourier series for Lu_h the sense of convergence

$$
\sum_{k} = \lim_{L \to \infty} \sum_{|k| < L} \tag{2.5}
$$

3. Proof of Theorem 2.1 – first stage

Following [4] and [14], it is convenient to define $A_n = \{p \in \mathbb{Z} : -n/2 < p \leq n/2\}$.

Then the test space T_h becomes $T_h = \text{span }\{\varphi_p : p \in A_n\}$, with $\varphi_p(x) = e^{2\pi i px}$, $p \in \mathbb{Z}$,

an span $\{\varphi_p: p \in A_n\}$, with $\varphi_p(x) = e^{2\pi i px}$, $p \in \mathbb{Z}$, and the method becomes: find $u_h \in S_h^d$ such that *g* [4] and [14], it is conver
 i test space T_h becomes T_h

method becomes: find $u_h \in Q_h(\overline{\varphi}_pLu_h) = Q_h(\overline{\varphi}_pLu)$,

th thand side of which we h FILAND

P **E** $\left(\frac{p}{p}\right)^2$

P = span $\{\varphi_p : p \in \Lambda_n\}$
 $\mathcal{P} \in \Lambda_n$,

ave replaced f by

converges absoluted and we have llowing [4] and [14], it is convenient to define $A_n = \{p \in \mathbb{Z} : -n\}$

en the test space T_h becomes $T_h = \text{span } \{\varphi_p : p \in A_n\}$, with $\varphi_p(x)$

d the method becomes: find $u_h \in S_h^d$ such that
 $Q_h(\overline{\varphi}_p Lu_h) = Q_h(\overline{\varphi}_p Lu), \q$

$$
Q_h(\bar{\varphi}_p L u_h) = Q_h(\bar{\varphi}_p L u), \qquad p \in A_n, \tag{3.1}
$$

in the right-hand side of which we have replaced \int by Lu .

d the method becomes: ind $u_h \in S_h^{\infty}$ such that
 $Q_h(\bar{\varphi}_p Lu_h) = Q_h(\bar{\varphi}_p Lu)$, $p \in A_n$,

the right-hand side of which we have replaced *j* by *Lu*.

As in [14], the sum (1.3) for *Lu(x)* converges absolutely because β $-1/2 < t$, thus the right-hand side of (1.3) is well defined, and we have th thand side of which we have the thand side of (1.3) for $Lu(x)$ contracts that the sum (1.3) is well defined $Q_h(\bar{p}_pLu) = \hat{u}(0) Q_h(\bar{p}_p) + \sum_{k \in \mathbb{Z}_p^*}$ from the side of which we have replarition to the sum (1.3) for $Lu(x)$ converged and side of (1.3) is well defined, and $(\overline{\varphi}_pLu) = \hat{u}(0) Q_h(\overline{\varphi}_p) + \sum_{k \in \mathbb{Z}^*} |k|^{\beta} \hat{u}(k)$
 $(\overline{\varphi}_pLu) = \hat{u}(0) Q_h(\overline{\varphi}_p) + \sum_{k \in \mathbb$ $\frac{1}{2}$ $\frac{1}{2}$ in the right-hand side of which we have replaced f by Lu.

As in [14], the sum (1.3) for Lu(x) converges absolutely because $\beta + 1/2 <$

the right-hand side of (1.3) is well defined, and we have
 $Q_h(\bar{p}_pLu) = \hat{u}(0) Q_h(\bar{p}_$

$$
Q_h(\overline{\varphi}_p Lu) = \hat{u}(0) Q_h(\overline{\varphi}_p) + \sum_{k \in \mathbb{Z}^*} |k|^\beta \hat{u}(k) Q_h(\overline{\varphi}_p \varphi_k),
$$

where $\mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$. By direct application of (1.6), or by symmetry, we find \mathbf{v} wh

$$
Q_h(\bar{\varphi}_p \varphi_k) = 0 \quad \text{if} \quad k \equiv p(\text{mod } n). \tag{3.2}
$$

.For the non-vanishing quadrature sums we define,

$$
l = Q_h(\bar{\varphi}_p \varphi_{p+\mathbf{f}n}) = Q_h(\varphi_{ln}), \tag{3.3}
$$

and find, using the quadrature rule (1.6),

$$
e_i = w_1 e^{2\pi i l e_1} + w_2 e^{2\pi i l e_1}.
$$

where
$$
Z^* = Z \setminus \{0\}
$$
. By direct application of (1.6), or by symmetry, we find $Q_h(\bar{p}_p q_k) = 0$ if $k \equiv p \pmod{n}$. (3.2) For the non-vanishing quadrature sums we define $e_l = Q_h(\bar{p}_p q_{p+1n}) = Q_h(p_{ln})$, and find, using the quadrature rule (1.6), $e_l = w_l e^{2\pi l t_l} + w_2 e^{2\pi l t_l}$. (3.4) Then the right-hand side of (3.1) becomes\n
$$
\frac{1}{Q_h(\bar{p}_p L u)} = \begin{cases} \frac{\partial(0) + \sum_{l \in Z} |\ln|^p \hat{u}(l n) e_l}{l n!} & p = 0, \\ |\ln|^p \hat{u}(p) + \sum_{l \in Z} |\ln|^p \hat{u}(l n) e_l, & p \in A_n^* \end{cases}
$$
\nwhere $A_n^* = A_n \setminus \{0\}$. Now consider the left-hand side of (3.1). The analogue of (1.3) is $L u_h(x) = \hat{u}_h(0) + \sum_{k \in Z} |k|^p \hat{u}_h(k) e^{2\pi l x}$. (3.6) Because u_h is a spline of even degree d, its Fourier coefficients satisfy the following recurrence relation (see [4, equation (2.7)], with appropriate phase adjustment to cater for the different choice of breaking): $(p + ln)^{d+1} \hat{u}_h(p + ln) = p^{d+1} \hat{u}_h(p), p, l \in \mathbb{Z}$. Thus\n
$$
\hat{u}_h(p + ln) = \begin{cases} 0, & p = 0, l \in \mathbb{Z}^*, \\ \frac{p^{d+1}}{(p + ln)^{d+1}} \hat{u}_h(p), & p \in A_n^*, l \in \mathbb{Z}, \end{cases}
$$
\nand we may write (3.6) as

•

Now consider the left-hand side of (3.1) . The analogue of (1.3) is

$$
Lu_h(x) = \hat{u}_h(0) + \sum_{k \in \mathbf{Z}^*} |k|^\beta \, \hat{u}_h(k) \, \mathrm{e}^{2\pi \mathrm{i} \, kx} \,. \tag{3.6}
$$

Because u_h is a spline of even degree *d*, its Fourier coefficients satisfy the following
recurrence relation (see [4, equation (2.7)], with appropriate phase adjustment to
cater for the different choice of breakpoints) recurrence relation (see $[4,$ equation (2.7)], with appropriate phase adjustment to cater for the different choice of breakpoints):

$$
(p + ln)^{d+1} \hat{u}_h(p + ln) = p^{d+1} \hat{u}_h(p), p, l \in \mathbb{Z}.
$$

$$
\hat{u}_h(p + ln) = \begin{cases} 0, & p = 0, l \in \mathbb{Z}^*, \\ \frac{p^{d+1}}{1 + \frac{
$$

$$
\overset{.}{(3.7)}
$$

(3.4)

and we may write (3.6) as

Thus

Because
$$
u_h
$$
 is a spline of even degree d, its Fourier coefficients satisfy the following
\nrecurrence relation (see [4, equation (2.7)], with appropriate phase adjustment to
\ncater for the different choice of breaking) :
\n
$$
(p + ln)^{d+1} \hat{u}_h(p + ln) = p^{d+1} \hat{u}_h(p), p, l \in \mathbb{Z}.
$$
\nThus
\n
$$
\hat{u}_h(p + ln) =\begin{cases}\n0, & p = 0, l \in \mathbb{Z}^*, \\
\frac{p^{d+1}}{(p + ln)^{d+1}} \hat{u}_h(p), & p \in A_n^*, l \in \mathbb{Z},\n\end{cases}
$$
\n(3.7)
\nand we may write (3.6) as
\n
$$
Lu_h(x) = \hat{u}_h(0) + \sum_{p \in A_n^*} \hat{u}_h(p) [|p|^{\beta} e^{2\pi i px} + p^{d+1} \sum_{l \in \mathbb{Z}^*} \text{sign } l | p + ln|^{\beta - d - 1} e^{2\pi i (p + ln)x}].
$$
\n(3.8)
\nBecause we assume only $d > \beta - 1/2$, the sum over l is not absolutely convergent,
\nand therefore care is needed in forming the partial sums. The convergence is always

/

Because we assume only $d > \beta - 1/2$, the sum over *l* is not absolutely convergent, and therefore care is needed in forming the partial sums. The convergence is always 366 ${}^{'}$ Lev II. Stone and W. L. Westers
 3. Proof of Theorem 2.1 – first stage

Following (4) and (44), it is convergient to define $A_n - |x \in B_n - n| \leq p \leq n/2$,

and the nethed incomes in the $q \leq q^2$ such that
 α i understood in the sense of (2.5); that is, the symmetric partial sums are used. (Note that (3.6) and (3.8) are equivalent when interpreted in this sense, because the individual terms in the sum over l in the latter converge to zero.)

We now show that the series (3.8) for $Lu_h(x)$ converges for all real x. First, for $x_0 \in ((j - 1)/n, j/n)$, $j \in \mathbb{Z}$, we may argue as for the mid-point collocation method in $[4]$: Because $d > \beta - 1/2$, we see from (3.7) and (1.8) that $u_h \in H^{\beta}$. Because u_h is also smooth in a neighbourhood of x_0 , it then follows, by application of Lemma 3.2 (c) of [4], that Lu_b is Hölder continuous in a neighbourhood of $x₀$. Thus, the Fourier series for Lu_h , which is the right-hand side of (3.8), converges at x_0 (in the sense of (2.5)) to $Lu_h(x_0)$. On the other hand, for $x = j/n$, $j \in \mathbb{Z}$, the right-hand side of (3.8) converges because the sum over p is over the finite set A_n^* and the sum over *l* can An Approach to Improving the Colloc

We now show that the series (3.8) for $Lu_h(x)$ converges for
 $x_0 \in ((j-1)/n, j/n), j \in \mathbb{Z}$, we may argue as for the mid-point $[4]$: Because $d > \beta - 1/2$, we see from (3.7) and (1.8) that $\sum_{i \in \mathbb{Z}^*} \text{EVALUATE:}$
 $\sum_{i \in \mathbb{Z}^*} \text{sign } l \cdot |p + ln|^{\beta - d - 1} e^{2\pi i (p + ln) / n}$
 $\sum_{i \in \mathbb{Z}^*} \text{sign } l \cdot |p +$ *i* use $d > \mu$
 j that Lu_h , which
 i $Lu_h(x_0)$. C
 i $Lu_h(x_0)$. C
 s because
 n (remem
 z sign l |
 $l \in \mathbb{Z}^*$
 $= e^{2\pi i p j/n}$ Filso smoot (c) of [4], and the series for the series for the series of the series w show that the series (3.8) for $Lu_h(x)$ converges for all real $-1/|n, j/n|$, $j \in \mathbb{Z}$, we may argue as for the mid-point collocation
use $d > \beta - 1/2$, we see from (3.7) and (1.8) that $u_h \in H^p$. If
the in a neighbourhood

also smooth in a neighborhood of
$$
x_0
$$
, it then follows, by application of Lemma 3.2
\n(c) of [4], that Lu_h is Hölder continuous in a neighbourhood of x_0 . Thus, the Fourier
\nseries for Lu_h , which is the right-hand side of (3.8), converges at x_0 (in the sense of
\n(2.5)) to $Lu_h(x_0)$. On the other hand, for $x = j/n$, $j \in \mathbb{Z}$, the right-hand side of (3.8)
\nconverges because the sum over p is over the finite set A_n^* and the sum over l can
\nbe written (remembering (2.5)!) as
\n
$$
\sum_{i\in \mathbb{Z}^*} \text{sign } l |p + ln|^{\beta - d - 1} e^{2n i (p + ln) j/n}
$$
\n
$$
= e^{2n i p j/n} \sum_{i=1}^{\infty} [(ln + p)^{-a} - (ln - p)^{-a}]
$$
\n
$$
= e^{2n i p j/n} \sum_{i=1}^{\infty} \left[\left(1 + \frac{y_p}{2l} \right)^{-a} - \left(1 - \frac{y_p}{2l} \right)^{-a} \right],
$$
\nwhere we have introduced
\n
$$
\alpha = d + 1 - \beta > 1/2
$$
\n(3.10)
\nand
\n
$$
y_p = 2p/n \in [-1, 1].
$$
\nSince the mean-value theorem gives
\n
$$
\left(1 + \frac{y}{2l} \right)^{-a} - \left(1 - \frac{y}{2l} \right)^{-a} = -\frac{\alpha}{2l} y \left[\left(1 + \frac{\theta y}{2l} \right)^{-a-1} + \left(1 - \frac{\theta y}{2l} \right)^{-a-1} \right],
$$
\n(3.11)

and''

$$
y_p=2p/n\in[-1,1].
$$

Since the mean-value theorem gives
\n
$$
\left(1+\frac{y}{2l}\right)^{-a}-\left(1-\frac{y}{2l}\right)^{-a}=-\frac{\alpha}{2l}y\left[\left(1+\frac{\theta y}{2l}\right)^{-a-1}+\left(1-\frac{\theta y}{2l}\right)^{-a-1}\right]
$$
\nwhere $0 < \theta(y) < 1$, it follows that $|(1 + y/2l)^{-a} - (1 - y/2l)^{-a}| \le C/l$. Thus that sum in (3.9) converges, and the convergence of the series for $Lu_h(x)$ has be

 C/l . Thus the last sum in (3.9) converges, and the convergence of the series for $Lu_h(x)$ has been proved for all x .

(It may be remarked that the convergence of the series for Lu_h at the breakpoints occurs in the present work, but not in general in [4], because we have here restricted attention to operators L of even 'symbol' - that is, the quantity that multiples $\hat{u}(k)$ in (1.3) is an even function Since the mean-value theorem gives
 $\left(1 + \frac{y}{2l}\right)^{-a} - \left(1 - \frac{y}{2l}\right)^{-a} = -\frac{\alpha}{2l}y\left[\left(1 + \frac{\partial y}{2l}\right)^{-a-1} + \left(1 - \frac{\partial y}{2l}\right)^{-a-1}\right],$

where $0 < \theta(y) < 1$, it follows that $|(1 + y/2l)^{-a} - (1 - y/2l)^{-a}| \le C/l$. Thus the last sum in

Since the series (3.8) for $Lu_h(x)$ converges for all x, it follows that the quadrature since the series (3.8) for $Lu_h(x)$ converges for all x, it follows that the quadrature
sum on the left-hand side of (3.1) is well defined for any choice of the quadrature
parameters. Now, using (3.2), (3.3), (3.8), (3.10) parameters. Now, using (3.2), (3.3), (3.8), (3.10) and (3.11) we obtain, for the left-
hand side of (3.1),
 $Q_b(\bar{\phi}_n L u_b) = \begin{cases} \hat{u}_b(0), & p = 0, - \end{cases}$ (3.12) (It may be remarked that the convergence of the series for Lu_h at the bre
the present work, but not in general in [4], because we have here restricted a
tors L of even 'symbol' - that is, the quantity that multiples $\hat{$ 3.8) for $Lu_h(x)$ converges for all x,

and side of (3.1) is well defined for

using (3.2), (3.3), (3.8), (3.10) and
 $=$ $\begin{cases} \hat{u}_h(0), & p = 0 \\ |p|^{\beta} \hat{u}_h(p) D(y_p), & p \in A_n \end{cases}$
 $+$ sign y $|v|^{\beta} \hat{R}(v)$ $\begin{array}{lll} \text{ature} \ \text{after} \ \text{left} & \frac{1}{2} \ \text{left} & \frac{1}{2} \ \text{(3.12)} & \end{array}$ $\frac{6}{6}$ $\frac{k}{k}$. Since
sum or
paramed hand si
where
where
and ay be remarked that the convergence of the series for Lu_h at the breakpoints occurs in
ent work, but not in general in [4], because we have here restricted attention to opera-
f even 'symbol' - that is, the quantity that (It may be remarked that the convergence of the series for *L*
the present work, but not in general in [4], because we have her
tors *L* of even 'symbol' - that is, the quantity that multiples a
of *k*.)
Since the series

$$
Q_h(\overline{\varphi}_p L u_h) = \begin{cases} \hat{u}_h(0), & p = 0, - \\ |p|^{\beta} \hat{u}_h(p) D(y_p), & p \in A_n^*, \end{cases}
$$
 (3.12)

Where

$$
D(y) = 1 + \text{sign } y \, |y|^a \, E(y), \qquad y \in [-1, 1], \tag{3.13}
$$

the present work, but not in general in [4], because we have here restricted attention to oper-
\ntors L of even 'symbol' – that is, the quantity that multiples
$$
\hat{u}(k)
$$
 in (1.3) is an even function
\nof k.)
\nSince the series (3.8) for $Lu_h(x)$ converges for all x, it follows that the quadrature
\nsum on the left-hand side of (3.1) is well defined for any choice of the quadrature
\nparameters. Now, using (3.2), (3.3), (3.8), (3.10) and (3.11) we obtain, for the left-
\nhand side of (3.1),
\n $Q_h(\bar{\varphi}_p Lu_h) =\begin{cases} \hat{u}_h(0), & p = 0, - \\ |p|^{\beta} \hat{u}_h(p) D(y_p), & p \in A_n^*, \end{cases}$ (3.12)
\nwhere
\n $D(y) = 1 + \text{sign } y |y|^{\alpha} E(y), \qquad y \in [-1, 1],$ (3.13)
\nand
\n $E(y) = \sum_{l \in \mathbb{Z}^*} \text{sign } l |y + 2l|^{-\alpha} e_l$
\n $= \sum_{l=1}^{\infty} [(2l + y)^{-\alpha} e_l - (2l - y)^{-\alpha} e_{-l}], \qquad y \in [-1, 1],$ (3.14)

(3.11)

368 -- IAN H. SLOAN and W. L. WENDLAND
where e_t is given by (3.4). Since the left-hand side of (3.12) is a well de where e_l is given by (3.4). Since the left-hand side of (3.12) is a well defined convergent series, the series $E(y)$ in the definition of $D(y)$ must also converge at least for $y = y_p$ $=2p/n$; i.e. it must converge for all non-zero rational numbers y in $(-1, 1]$. The following lemma establishes that in fact (3.14) converges, and therefore $E(y)$ and $D(y)$ are well defined, for all $y \in [-1, 1]$. (In this lemma the assumption on α is weakened to $\alpha > 0$, since this may be done without cost.) **H.** SLOAN and W. L. WENDLAND

ven by (3.4). Since the left-hand side of (3.12) is a well defined convergent

ries $E(y)$ in the definition of $D(y)$ must also converge at least for $y = y_p$

it must converge for all non-zero

Lemma 3.1: For $\alpha > 0$, *consider the series*

$$
\sum_{l=1}^{\infty} (2l+y)^{-s} c_l, \qquad y \in [-1,1],
$$

where $|c_i| \leq m$, with *m* independent of *l. If the series converges for some* $y_0 \in [-1, 1]$, *then it does so for all* $y \in [-1, 1]$ *, and the resulting function belongs to* $C^{\infty}[-1, 1]$.

Proof: The formal derivative of the series (3.15) is (absolutely) uniformly convergent, and can therefore be integrated term by term. Thus for $y \in [-1, 1]$ we have

$$
\leq m, with m independent of l. If the series converges for some $y_0 \in [-1, 1]$,
as so for all $y \in [-1, 1]$, and the resulting function belongs to $C^{\infty}[-1, 1]$.
: The formal derivative of the series (3.15) is (absolutely) uniformly conver-
can therefore be integrated term by term. Thus for $y \in [-1, 1]$ we have

$$
\int_{u}^{u} \left[-\alpha \sum_{i=1}^{\infty} (2l + t)^{-\alpha-1} c_i \right] dt = \sum_{i=1}^{\infty} [(2l + y)^{-\alpha} - (2l + y_0)^{-\alpha}] c_i.
$$
 (3.16)
$$

Denoting the series (3.15) by $F(y)$, we know that $\sum_{l=1}^{\infty} (2l + y_0)^{-a} c_l = F(y_0)$ is a conwhere $|c_i| \leq m$, with r
then it does so for all y
Proof: The formal
gent, and can therefor
 $\int_{\nu_1}^{\nu} \left[-\alpha \sum_{i=1}^{\infty} \left(\frac{\alpha}{\nu} \right) \right] d\nu_1$
Denoting the series (;
vergent sum, from w
expressed as the diffe vergent sum, from which it now follows that the right-hand side of (3.16) can be $\int_{u_{\bullet}}^{u} \left[-\alpha \sum_{l=1}^{\infty} (2l + t)^{-\alpha-1} c_l \right] dt = \sum_{l=1}^{\infty} [(2l + y)^{-\alpha} - (2l + y_0)^{-\alpha}] c_l.$ (3.16)
Denoting the series (3.15) by $F(y)$, we know that $\sum_{l=1}^{\infty} (2l + y_0)^{-\alpha} c_l = F(y_0)$ is a convergent sum, from which it now f also a convergent sum. Then from (3.16) we have $\mu \leq m$, with m inde

oes so for all $y \in [-1]$

f: The formal derived

d can therefore be in
 $\int_{t=1}^{y} \left[-\alpha \sum_{l=1}^{\infty} (2l + t) \right]$

ig the series (3.15) b

sum, from which if

ad as the difference

onvergent sum. The
 hen it does so for all y \in [-1, 1], *and the resulting function*
 Proof: The formal derivative of the series (3.15) is (a

gent, and can therefore be integrated term by term. Thus
 $\int_{\nu}^{\nu} \left[-\alpha \sum_{l=1}^{\infty} (2l +$ 1.15) by $F(y)$, we know that $\sum_{l=1}^{n} (2l +$
hich it now follows that the right-harence of two convergent series. Thus
n. Then from (3.16) we have
 $\sum_{l=1}^{\infty} (2l + y)^{-s-1} c_l$, $y \in (-1, 1)$,
cause the latter series is unif

$$
F'(y) = -\alpha \sum_{l=1}^{\infty} (2l+y)^{-\alpha-1} c_l, \quad y \in (-1,1),
$$

'remembering that because the latter series is uniformly convergent,, it defines a continuous function on $[-1, 1]$. Thus we have $F \in C¹[-1, 1]$. A similar (but simpler) argument now yields

$$
F''(y)=\alpha(\alpha+1)\sum_{l=1}^{\infty}(2l+y)^{-\alpha-2}c_l,
$$

and $F \in C^{2}[-1, 1]$; and so on. Thus $F \in C^{\infty}[-1, 1]$, and the result is proved **I**

Corollary 3.2: With $\alpha > 1/2$, $w_1, w_2 \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 \in [0, 1]$, the series expression' (3.14) *for E(y) converges for all y* \in $[-1, 1]$ *. Moreover, E* \in C^{∞} $[-1, 1]$ *. Thus D is a* $continuous$ function on $[-1, 1]$, and is smooth outside an arbitrary neighbourhood of *zero.* Corollary 3.2: With $\alpha > 1/2$, $w_1, w_2 \in \mathbb{R}$ and ε_1 ,

(3.14) for $E(y)$ converges for all $y \in [-1, 1]$. Moreover

continuous function on $[-1, 1]$, and is smooth outside

zero.

As in [4] and [14], the properties of

As in [4] and [14], the properties of the function *D* are crucial for both the stability and the rate of convergence of the method. To study the stability, it is convenient to *Z(y)* = I. + sign y l *yl*^l ^Zsign *ly + 21I* e2 " ¹ , *y* € [-1, 1]. (3.17)

$$
D(y) = w_1 Z_{\epsilon_1}(y) + w_2 Z_{\epsilon_2}(y), \qquad y \in [-1, 1];
$$

where

•

$$
Z_{\epsilon}(y) = 1 + \text{sign } y \, |y|^{\alpha} \sum_{l \in \mathbb{Z}^*} \text{sign } l \, |y + 2l|^{-\alpha} \, e^{2\pi \, |t|}, \qquad y \in [-1, 1]. \tag{3.17}
$$

 $D(y) = w_1 Z_{\epsilon_1}(y) + w_2 Z_{\epsilon_2}(y),$ $y \in [-1, 1],$

where
 $Z_{\epsilon}(y) = 1 + \text{sign } y |y|^{\alpha} \sum_{l \in \mathbb{Z}^*} \text{sign } l |y + 2l|^{-\alpha} e^{2\pi_1 \epsilon_l},$ $y \in [-1, 1].$ (3.17)

Since Z_{ϵ} is the special case of *D* obtained by setting $w_2 = 0$, $w_1 = 1$, **•** function on $[-1, 1]$, and is smooth outside an arbitrary neighbourhood of zero.

An Approach to Improving the Collocation Method 369

For the stability of the method, the requirement, as in [4] and [14], is that *D* be bounded away from zero. This will follow from the following lemma (the rather technical appearance of which is dictated by our desire to obtain explicit bounds for the stability constants).

Lemma 3.3: Let α and let Z_{ϵ} be the complex-valued function defined by (3.17).

(i) For arbitrary $\varepsilon \in (0, 1)$, Re $Z_{\varepsilon}(y) \ge (1 - 3^{-\alpha}) (1 - \max(\cos 2\pi \varepsilon, 0)) > 0$, $y \in [-1, 1].$

(ii) $For \epsilon = 0, Z_0(y) > 0, y \in (-1, 1), Z_0(\pm 1) = 0.$

Proof: For arbitrary $\varepsilon \in [0, 1]$ we have, from the definition (3.17),

Let
$$
z
$$
 is a positive constant, z is a positive constant, z .

\nLet z and z be the complex-valued function defined by (3, 1) for arbitrary $\varepsilon \in (0, 1)$, $\text{Re } Z_{\varepsilon}(y) \geq (1 - 3^{-a}) \left(1 - \max(\cos 2\pi \varepsilon, 0)\right)$

\n $y \in [-1, 1].$

\n(ii) For $\varepsilon = 0$, $Z_0(y) > 0$, $y \in (-1, 1)$, $Z_0(\pm 1) = 0$.

\nProof: For arbitrary $\varepsilon \in [0, 1]$ we have, from the definition (3.17), $\text{Re } Z_{\varepsilon}(y) = 1 + \text{sign } y \left|y\right|^\alpha \sum_{\varepsilon \in \mathbb{Z}^\ast} \text{sign } l \left|y + 2l\right|^{-a} \cos 2\pi \varepsilon l$

\n $= 1 + \text{sign } y \left|y\right|^\alpha \sum_{i=1}^\infty \left[(2l + y)^{-a} - (2l - y)^{-a}\right] \cos 2\pi \varepsilon l$, which is manifestly even. For $y \in [0, 1]$ we define $h_l(y) = y^{\varepsilon}[(2l + y)^{-a} - 1]$. Then for $y \in [0, 1]$ we have

\nRe $Z_{\varepsilon}(y) = 1 + \sum_{i=1}^\infty h_i(y) \cos 2\pi \varepsilon l \geq 1 + \sum_{i=2}^\infty h_i(y) + h_1(y) \cos 2\pi \varepsilon$

\nbecause $h_l(y) \leq 0$ and $\cos 2\pi \varepsilon l \leq 1$. As in (3.9), an application of the theorem shows that $\sum h_i(y)$ is absolutely and uniformly convergent in $[0, 1]$ over, because

\n $h_l'(y) = 2l\alpha y^{a-$

 $-(2l - y)^{-\alpha}$. Then for $y \in [0, 1]$ we have

$$
= 1 + \operatorname{sign} y \, |y|^{\alpha} \sum_{l=1}^{\infty} [(2l + y)^{-\alpha} - (2l - y)^{-\alpha}] \cos 2\pi \epsilon l,
$$

which is manifestly even. For $y \in [0, 1]$ we define $h_l(y) = y^{\epsilon}[(2l + y)^{-\alpha} - (2l - y)^{-\alpha}]$.
Then for $y \in [0, 1]$ we have

$$
\operatorname{Re} Z_{\epsilon}(y) = 1 + \sum_{l=1}^{\infty} h_l(y) \cos 2\pi \epsilon l \ge 1 + \sum_{l=2}^{\infty} h_l(y) + h_1(y) \cos 2\pi \epsilon, \qquad (3.18)
$$

because $h_l(y) \le 0$ and $\cos 2\pi \epsilon l \le 1$. As in (3.9), an application of the mean-value
theorem shows that $\sum h_l(y)$ is absolutely and uniformly convergent in [0, 1]. More-

 $h_i(y) \leq 0$ and $\cos 2\pi \epsilon \leq 1$. As in (3.9), an application of the measures that $\sum h_i(y)$ is absolutely and uniformly convergent in [0, 1] ause
 $h_i'(y) = 2\ell \alpha y^{\alpha-1}[(2l + y)^{-\alpha-1} - (2l' - y)^{-\alpha-1}] < 0$, $y \in (0, 1]$,

$$
h_l'(y) = 2l\alpha y^{\alpha-1}[(2l+y)^{-\alpha-1} - (2l'-y)^{-\alpha-1}] < 0, \qquad y \in (0,1],
$$

we see that $h₁(y)$ is a decreasing function on [0, 1], thus

Then for
$$
y \in [0, 1]
$$
 we have\n
$$
\text{Re } Z_i(y) = 1 + \sum_{l=1}^{\infty} h_l(y) \cos 2\pi \text{e} \ge 1 + \sum_{l=2}^{\infty} h_l(y) + h_1(y) \cos 2\pi \text{e}, \qquad (3.18)
$$
\nbecause $h_l(y) \le 0$ and $\cos 2\pi \text{e} \le 1$. As in (3.9), an application of the mean-value theorem shows that $\sum h_l(y)$ is absolutely and uniformly convergent in [0, 1]. Moreover, because\n
$$
h_l'(y) = 2\text{log}y^{-1}[(2l + y)^{-a-1} - (2l - y)^{-a-1}] < 0, \qquad y \in (0, 1],
$$
\nwe see that $h_l(y)$ is a decreasing function on [0, 1], thus\n
$$
\text{Re } Z_i(y) \ge 1 + \sum_{l=2}^{\infty} h_l(1) + h_1(1) \max(\cos 2\pi \text{e}, 0)
$$
\n
$$
= 1 - 3^{-a} - (1 - 3^{-a}) \max(\cos 2\pi \text{e}, 0), \qquad (3.19)
$$
\nand the proof of part (i) of the theorem is complete.\nFor the case $\epsilon = 0$ we see that $Z_0(y)$ is real, thus the conclusion arrived at so far can be expressed as $Z_0(y) \ge 0$. Now since $h_l(y) > h_l(1)$ for $y \in (0, 1]$, it follows from (3.18) and (3.19) that the inequality $Z_0(y) \ge 0$ is strict for $y \in (-1, 1)$. Finally, $Z_0(\pm 1) = 1 + \sum_{l=1}^{\infty} h_l(1) = 0$.\n\nRemark: The second part of the lemma was proved, in effect, in [4, Lemma 2.3] in a different context – see the case of the function g and splines of odd degree.\n\nCorollary 3.4: Let $\alpha > 1/2$.\n\n(i) With $w_1, w_2 \ge 0$, and arbitrary ϵ_1 , $\epsilon_$

and the proof of part (i) of the theorem is complete.

For the case $\varepsilon = 0$ we see that $Z_0(y)$ is real, thus the conclusion arrived at so far can be expressed as $Z_0(y) \ge 0$. Now since $h_i(y) > h_i(1)$ for $y \in (0, 1]$, it follows from can be expressed as $Z_0(y) \geq 0$. Now since $h_l(y) \geq 0$ is real, thus the conclusion arrived at so far can be expressed as $Z_0(y) \geq 0$. Now since $h_l(y) > h_l(1)$ for $y \in (0, 1]$, it follows from (3.18) and (3.19) that the i and the proof of part (i) of the theorem is complete.

For the case $\varepsilon = 0$ we see that $Z_0(y)$ is real, thus the conclusion arrived at so far

can be expressed as $Z_0(y) \ge 0$. Now since $h_i(y) > h_i(1)$ for $y \in (0, 1]$, it For the case $\varepsilon = 0$ we see that $Z_0(y)$ is real, thus the conclusion arr
can be expressed as $Z_0(y) \ge 0$. Now since $h_l(y) > h_l(1)$ for $y \in (0, 1]$, it
(3.18) and (3.19) that the inequality $Z_0(y) \ge 0$ is strict for $y \in ($ **and the proof of part (i) of the theorem is complete.**

For the case $\epsilon = 0$ we see that $Z_0(y)$ is real, thus the

can be expressed as $Z_0(y) \ge 0$. Now since $h_1(y) > h_1(1)$

(3.18) and (3.19) that the inequality $Z_0(y)$

$$
Z_0(\pm 1) = 1 + \sum_{l=1}^{\infty} h_l(1) = 0 \quad \blacksquare
$$

Remark: The second part of the lemma was proved, in effect, in [4, Lemma 2.3] in a different context $-$ see the case of the function g and splines of *odd* degree.

(i) With $w_1, w_2 \ge 0$, and arbitrary $\varepsilon_1, \varepsilon_2 \in [0, 1]$, the function D defined by (3.13) and *-* **(3.14)** *satisfies*

In be expressed as
$$
Z_0(y) \geq 0
$$
. Now since $h_i(y) > h_i(1)$ for $y \in (0, 1]$, it follows 18) and (3.19) that the inequality $Z_0(y) \geq 0$ is strict for $y \in (-1, 1)$. If $Z_0(\pm 1) = 1 + \sum_{i=1}^{\infty} h_i(1) = 0$ \bullet \bullet <math display="inline</p>

I

370 IAN H. SLOAN and W. L. WENDLAND

Remark: The first part of this result establishes the stability of the method under the conditions stated in the theorem (effectively, that the weights are non-negative and there is at least one quadrature point in the interior). The second establishes that the latter condition is necessary: in effect it expresses the well known fact that the method of collocation at the breakpoints is unstable in the case of even-degree splines. The first part of this result establishes the state
s stated in the theorem (effectively, that the
t least one quadrature point in the interior). T
molition is necessary: in effect it expresses the
location at the breakpo

Now we turn to the question of the rate of convergence. As in [4] and [14], the maximum rate of convergence is determined by the behaviour of the function $D(y)$ in the neighbourhood of 0.. Now we turn to the question of the rate of convergence. As in [4] and [14], one
aximum rate of convergence is determined by the behaviour of the function $D(y)$
the neighbourhood of 0.
Initially we allow the quadrature par

Initially we allow the quadrature parameters ε_1 , ε_2 to be chosen arbitrarily.

in the neighbourhood of 0.
 in the neighbourhood of 0.
 by (3.13) and (3.14) satisfies $|D(y) - 1| \le \alpha$, $w_1, w_2, \epsilon_1, \epsilon_2$, but not on y. $C |y|^{\alpha}, y \in [-1, 1],$ where C depends on α , w_1 , w_2 , ε_1 , ε_2 , but not on y.

Proof: This follows immediately from (3.13) and Corollary 3.2

Next, we consider the quadrature parameters to be chosen symmetrically, as in Next, we consider the quadrature parameters to be chosen symmetrically, as in part (ii) of the theorem: that is, $w_1 = w_2 = 1/2$, and $\varepsilon_2 = 1 - \varepsilon_1$. The special feature in this case is that the function $E(y)$ defined by (3.14) now vanishes at $y = 0$: in fact we have, from (3.4), $e_i = e_{-i} = \cos 2\pi \epsilon_1 i$, and hence **•** Froof: This follows immediately from (3.13) and Core

Next, we consider the quadrature parameters to be opart (ii) of the theorem: that is, $w_1 = w_2 = 1/2$, and $\varepsilon_2 =$

in this case is that the function $E(y)$ defined *E* turn to the question of the rate of convergence. As in
 F_1 rate of convergence is determined by the behaviour of
 g hbourhood of 0..
 g we allow the quadrature parameters ε_1 , ε_2 to be chosen
 α_1 x, w_1 , w_2 , ε_1 , ε_2 , but not on y.

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Next, we consider the quadrature parameters to be chosen synapt (ii) of the theorem: that is, $w_1 = w_2 = 1/2$

have, from (3.4),
$$
e_i = e_{-i} = \cos 2\pi e_i t
$$
, and hence
\n
$$
E(y) = \sum_{l=1}^{\infty} [(2l + y)^{-\alpha} - (2l - y)^{-\alpha}] \cos 2\pi e_i l,
$$
\n(3.20)
\nthe property $E(0) = 0$ is immediately apparent. By the mean-value theorem,
\nwrite
\n
$$
(2l + y)^{-\alpha} - (2l - y)^{-\alpha} = -\alpha y [(2l + \theta y)^{-\alpha-1} + (2l - \theta y)^{-\alpha-1}],
$$
\n(3.21)
\n $\alpha \theta(u) \le 1$, thus

so that the property $E(0) = 0$ is immediately apparent. By the mean-value theorem, we may write

$$
(2l + y)^{-\alpha} - (2l - y)^{-\alpha} = -\alpha y [(2l + \theta y)^{-\alpha - 1} + (2l - \theta y)^{-\alpha - 1}], \qquad (3.21)
$$

$$
|E(y)| \leq \sum_{l=1}^{\infty} |(2l + y)^{-\alpha} - (2l - y)^{-\alpha}|
$$

\n
$$
\leq \sum_{l=1}^{\infty} |\alpha| |y| |2(2l - 1)^{-\alpha - 1} = C |y|.
$$

Since D is related to E by (3.13), we obtain the following lemma.

Lem ma 3.6: With $\alpha > 1/2$, $w_1 = w_2 = 1/2$, $\varepsilon_2 = 1 - \varepsilon_1$, and $\varepsilon_1 \in [0, 1]$, the function Since *D* is related to *E* by (3.13), we obtain the following lemma.

Lemma 3.6: *With* $\alpha > 1/2$, $w_1 = w_2 = 1/2$, $\varepsilon_2 = 1 - \varepsilon_1$, and $\varepsilon_1 \in [0, 1]$, the function
 D defined by (3.13) satisfies $|D(y) - 1| \leq C |y|^{s+$ and ε_1 , but not on y.

The function $E(y)$ for the symmetric case can be subjected to a more precise analysis, leading to the parameter choice in part (iii) of the theorem. Replacing the $\left(\begin{array}{c} \frac{1}{2} & \frac{1}{2} \\ \frac{$

where
$$
0 < \theta(y) < 1
$$
, thus\n
$$
|E(y)| \leq \sum_{i=1}^{\infty} |(2l + y)^{-a} - (2l - y)^{-a}|
$$
\n
$$
\leq \sum_{i=1}^{\infty} |\alpha| |y| |2(2l - 1)^{-a-1} = C |y|.
$$
\nSince D is related to E by (3.13), we obtain the following lemma. Lemma 3.6: With $\alpha > 1/2$, $w_1 = w_2 = 1/2$, $\varepsilon_2 = 1 - \varepsilon_1$, and $\varepsilon_1 \in [0, 1]$, the function D defined by (3.13) satisfies $|D(y) - 1| \leq C |y|^{a+1}$, $y \in [-1, 1]$, where C depends on a_1 , but not on y .\n\nThe function $E(y)$ for the symmetric case can be subjected to a more precise analysis, leading to the parameter choice in part (iii) of the theorem. Replacing $(2l + y)^{-a} - (2l - y)^{-a}$.\n\n
$$
= -\alpha 2^{-a}l^{-a-1}y - \frac{\alpha(\alpha + 1)(\alpha + 2)}{6}y^3[(2l + 0y)^{-a-3} + (2l - \theta y)^{-a-3}],
$$
\nwhere $0 < \theta(y) < 1$, we obtain from (3.20).\n\n
$$
E(y) = -\alpha 2^{-a} y \sum_{i=1}^{\infty} l^{-a-1} \cos 2\pi \varepsilon_i l
$$

where $0 < \theta(y) < 1$, we obtain from (3.20)

the expression (3.21) by the higher-order version

\n
$$
(2l + y)^{-a} - (2l - y)^{-a}
$$
\n
$$
= -\alpha 2^{-a} l^{-a-1} y - \frac{\alpha(\alpha + 1)(\alpha + 2)}{6} y^3 [(2l + \theta y)^{-a-3} + (2l - \theta y)^{-a-3}],
$$
\n
$$
< \theta(y) < 1, \text{ we obtain from (3.20)}
$$
\n
$$
E(y) = -\alpha 2^{-a} y \sum_{i=1}^{\infty} l^{-a-1} \cos 2\pi \epsilon_i l
$$
\n
$$
- \frac{\alpha(\alpha + 1)(\alpha + 2)}{6} y^3 \sum_{i=1}^{\infty} [(2l + \theta y)^{-a-3} + (2l - \theta y)^{-a-3}] \cos 2\pi \epsilon_i l.
$$

If ϵ_f is chosen as in part (iii) of Theorem 2.1 then the first term of this expression vanishes, and We have From 1.1 then the first term of this express

eorem 2.1 then the first term of this express
 $y|^3 \sum_{l=1}^{\infty} 2(2l - 1)^{-a-3} = C |y|^3.$

lowing lemma.
 $y^a = 1/2$ $y^b = 1 - \varepsilon$, and ε , the least nost

An Approach to Improving the Collocation
hosen as in part (iii) of Theorem 2.1 then the first term a
and we have

$$
|E(y)| \le \frac{\alpha(\alpha+1)}{6} \frac{(\alpha+2)}{5} |y|^3 \sum_{l=1}^{\infty} 2(2l-1)^{-\alpha-3} = C |y|^3.
$$

Thus we obtain, using (3.13), the following lemma.

Lemma 3.7: *With* $\alpha > 1/2$, $w_1 = w_2 = 1/2$, $\varepsilon_2 = 1 - \varepsilon_1$, and ε_1 the least positive *zero of* G_{a+1} *, defined, by (2.3), the function D defined by (3.13) satisfies.* $|D(y) - 1|$ $\leq C |y|^{\alpha+3}$, where C depends on α , but not on y. An Approach to Improving the Colloc

If ϵ_f is chosen as in part (iii) of Theorem 2.1 then the first ter

vanishes, and we have
 $|E(y)| \le \frac{\alpha(\alpha + 1)(\alpha + 2)}{6} |y|^3 \sum_{i=1}^{\infty} 2(2l - 1)^{-a-3} = C |y|$

Thus we obtain, using (3.1 i, using (3.13), the following lemma.
 \therefore *With* $\alpha > 1/2$, $w_1 = w_2 = 1/2$, $\varepsilon_2 = 1 - \varepsilon_1$, and ε_1 the least positive

defined by (2.3), the function *D* defined by (3.13) satisfies $|D(y) - 1|$

ere *C* depends

In the sequel we make use of whichever of the three Lemmas $3.5-3.7$ is appro-

4. Proof of Theorem 2.1 - final stage

Now we are ready to establish the existence of u_h , and prove the estimates (2.1), (2.2) Now we are ready to establish the existence of u_h , and prove the estimates (2.1), (2.2)
and (2.4) for $||u_h - u||_s$. Recall that the qualocation method is expressed by (3.1).
With the aid of (3.5) and (3.12), this is expre

With the aid of (3.5) and (3.12), this is expressible as the set of equations.
\n
$$
u_h(0) = \hat{u}(0) + \sum_{l \in \mathbb{Z}^*} |l_n|^{\beta} \hat{u}(ln) e_l, \qquad p = 0,
$$
\n
$$
|p|^{\beta} \hat{u}_h(p) D(y_p) = |p|^{\beta} \hat{u}(p) + \sum_{l \in \mathbb{Z}^*} |p + ln|^{\beta} \hat{u}(p + ln) e_l, \qquad p \in \Lambda_n^*.
$$
\n(4.2)

$$
|p|^{\beta} \hat{u}_h(p) D(y_p) = |p|^{\beta} \hat{u}(p) + \sum_{l \in \mathbb{Z}^*} |p + ln|^{\beta} \hat{u}(p + ln) e_l, \qquad p \in \Lambda_n^* \ . \ (4.2)
$$

The subsequent analysis is almost the same as in $[14]$, thus we shall be brief. Since $D(y) = 0$ for $y \in [-1, 1]$, the above equations uniquely, determine $\hat{u}_h(p)$ for all $p \in A_n$. The recurrence relation (3.7) then determines all other Fourier coefficients of u_h . Thus the approximate solution u_h exists and is unique for every continuous right-hand side function f. To study the convergence of u_h to u , we investigate, for $s < d+1/2,$ ame as in [14], thus we shall
bove equations uniquely dete
then determines all other Fou
 $\frac{1}{h}$ exists and is unique for ev
e convergence of u_h to u , we
 $\sum_{k \in \mathbb{Z}^*} |k|^{2s} |\hat{u}_h(k) - \hat{u}(k)|^2$
 $2 \sum_{k \in \mathbb{Z}^*} |$ $(u_k, v) = 0$ for v_k
 $(v) = 0$ for v_k
 $(v) = 0$ for v_k
 $(v) = 0$ for v_k
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$

$$
|p|^{\beta} \hat{u}_{\Lambda}(p) D(y_{p}) = |p|^{\beta} \hat{u}(p) + \sum_{l \in \mathbb{Z}^{*}} |p + ln|^{\beta} \hat{u}(p + ln) e_{l}, \quad p \in \Lambda_{n}^{*}. \quad (4.2)
$$

The subsequent analysis is almost the same as in [14], thus we shall be brief.
Since $D(y) \neq 0$ for $y \in [-1, 1]$, the above equations uniquely determine $\hat{u}_{\Lambda}(p)$ for
all $p \in \Lambda_{n}$. The recurrence relation (3.7) then determines all other Fourier coefficients
of \hat{u}_{h} . Thus the approximate solution \hat{u}_{h} exists and is unique for every continuous
right-hand side function f . To study the convergence of \hat{u}_{h} to \hat{u} , we investigate, for
 $s < d + 1/2$,

$$
||u_{h} - u||_{s}^{2} = |d_{h}(0) - \hat{u}(0)|^{2} + \sum_{k \in \mathbb{Z}^{*}} |k|^{2s} |d_{h}(k) - \hat{u}(k)|^{2}
$$

$$
\leq |d_{h}(0) - \hat{u}(0)|^{2} + 2 \sum_{k \in \Lambda_{n}} |k|^{2s} |d_{h}(k)|^{2}
$$

$$
+ 2 \sum_{k \in \Lambda_{n}} |k|^{2s} |d_{h}(k)|^{2} + \sum_{p \in \Lambda_{n}} |p|^{2s} |d_{h}(p) - \hat{u}(p)|^{2}. \quad (4.3)
$$

For $u \in H^{t}$, and $\beta + 1/2 < t$, it follows as in [14] that the first term of (4.3) is bounded
by
 $|d_{h}(0) - \hat{u}(0)|^{2} \leq Ch^{2(t-\beta)} ||u||_{t}^{2}$.
(4.4)
Similarly, because $s \leq t$ we obtain as in [14] the bound

$$
2 \sum_{k \in \Lambda_{n}} |k|^{2s} |d(k)|^{2} \leq Ch^{2(t-\beta)} ||u||_{t}^{2}
$$

$$
4.5
$$

for the second term of (4.3). For the third term of (4.3) it is necessary, as in [14], to
first make use of the recurrence relation (3.7) to express $\hat{u}_{h}($

For $u \in H^{\iota}$, and $\beta\,+\,1/2\,<\iota$, it follows as in [14] that the first term of (4.3) is bounded by (4.3)
bounded
 (4.4) $u \in H^t$, and $\beta + 1/2 < t$
 $|\hat{u}_h(0) - \hat{u}(0)|^2 \leq C$

ilarly, because $s \leq t$ we
 $2 \sum_{k \in A_n} |k|^{2s} |\hat{u}(k)|^2 \leq C$

$$
|\hat{u}_h(0) - \hat{u}(0)|^2 \leq Ch^{2(t-\beta)} \|u\|_t^2. \tag{4.4}
$$

 $|u_h(0) - u(0)|^2 \leq C h^{2(t-\beta)} ||u||_t^2.$
Similarly, because $s \leq t$ we obtain as in [14] the bound

$$
2\sum_{k\in\Lambda_n} |k|^{2s} \, |u(k)|^2 \leq Ch^{2(l-s)} \, ||u||_l^2 \tag{4.5}
$$

for the second term of (4.3) . For the third term of (4.3) it is necessary, as in [14], to first make use of the recurrence relation (3.7) to express $\hat{u}_h(k)$ in terms of $\hat{u}_h(p)$, with $p \in A_n^*$; and then to use (4.2) to express $\hat{u}_h(p)$ in terms of Fourier coefficients of *u*. It is in the latter phase that it becomes essential to assume, as in the statement of the theorem, and as we shall assume fro It is in the 'latter phase that it becomes essential to assume, as in the statement of the theorem, and as we shall assume from now on, that at least one of ε_1 , ε_2 is in the *open interval* (0, 1), *and has associated with it a positive weight*. Then from Corollary 3.4 we have $|D(y)^{-1}| \leq C(\alpha)$, and as in [14] it follows that the third term of (4.3) has the bound .

$$
2\sum_{k\in\Lambda_n} |k|^{2s} |\alpha \hat{u}_h(k)|^2 = 2 \sum_{p\in\Lambda_n} \sum_{l\in\mathbb{Z}^*} |p|^{2s} |h|^{2s} |u_h(p+ln)|^2
$$

=
$$
2\sum_{p\in\Lambda_n} |p|^{2(d+1)} |\hat{u}_h(p)|^2 \sum_{l\in\mathbb{Z}^*} |p|^{2(s-d-1)}
$$

Because $s < d + 1/2$, the last sum can be estimated by

$$
\sum_{l\in\mathbf{Z}^*} |p+l n|^{2(s-d-1)} \leq C(s) n^{2(s-d-1)},
$$

see $[4]$. Next, we use (4.2) and obtain

$$
2 \sum_{k \in \Lambda_n} |k|^{2s} |\hat{u}_h(k)|^2
$$

\n
$$
\leq Cn^{2(s-d-1)} \sum_{p \in \Lambda_n^*} |p|^{2(d+1)} \left[|\hat{u}(p)|^2 + |p|^{-2\beta} \left(\sum_{k \in \mathbb{Z}^*} |p + ln|^{\beta} |\hat{u}(p + ln)| \right)^2 \right]
$$

\n
$$
\leq Cn^{2(s-t)} \sum_{p \in \Lambda_n^*} |p|^{2t} |\hat{u}(p)|^2
$$

\n
$$
\sqrt{Cn^{2(s-d-1)}} \sum_{p \in \Lambda_n^*} |p|^{2(d+1-\beta)} \sum_{m \in \mathbb{Z}^*} |p + mn|^{2(\beta-t)} \sum_{k \in \mathbb{Z}^*} |p + ln|^{2t} |\hat{u}(p + ln)|^2
$$

where $\tau = \min (d + 1, t)$.

Hence, because $\beta + 1/2 < t$,

$$
2 \sum_{k \in \Lambda_n} |k|^{2s} |\hat{u}_h(k)|^2
$$

\n
$$
\leq C n^{2(s-t)} \|u\|_{t}^{2} + C n^{2(s-d-1+\beta-t)} \sum_{k \in \Lambda_n} |p|^{2(d+1-\beta)} \sum_{l \in \mathbb{Z}^*} |p + ln|^{2l} |\hat{u}(p + ln)|^2
$$

\n
$$
\leq (C n^{2(s-t)} + C' n^{2(s-t)}) \|u\|_{t}^{2},
$$

or

$$
2\sum_{k\in\Lambda_n}|k|^{2s}|\hat{u}_h(k)|^2\leq Ch^{2(\tau-s)}\|u\|_{l}^2.
$$
\n(4.6)

Finally, we consider the fourth term of (4.3). This is the term that plays the crucial rôle in limiting the maximum attainable rate of convergence. Using (4.2) we have

$$
\hat{u}_h(p) - \hat{u}(p) = D(y_p)^{-1} \left[\left(1 - D(y_p) \right) \hat{u}(p) + |p|^{-\beta} \sum_{i \in \mathbb{Z}^*} |p| + \ln |\beta| \hat{u}(p + ln) e_i \right].
$$

Then with the aid of the stability property in Corollary 3.4, and also whichever of the three Lemmas 3.5 - 3.7 is appropriate, together with $y_p = 2p/n$, we obtain

$$
|\hat{u}_h(p) - \hat{u}(p)|^2 \leq C \left(\frac{p}{n}\right)^{2(d+1-\beta+r)} |\hat{u}(p)|^2
$$

+ C $|p|^{-2\beta} \left(\sum_{i \in \mathbf{Z}^*} |p| |\hat{u}(p+ln)|^2 \right).$

where $r = 0$, 1 or 3, depending on whether the quadrature parameters are as in part. (i), (ii) or (iii) of the theorem. It then follows, by the same argument as in $[14]$, that the fourth term of (4.3) satisfies

$$
\sum_{\beta \in \Lambda_n^*} |p|^{2s} |\hat{u}_h(p) - \hat{u}(p)|^2 \leq Ch^{2\min(t-s,t-\beta,d+1+r-\beta)} \|u\|_{l}^2.
$$
 (4.7)

An Approach to Improving the Collocation Method

Collecting together the separate bounds (4.4) , (4.5) , (4.6) and (4.7) , we obtain $||u_h - u||_s \leq Ch^{min(t-s,t-\beta,d+1-s,d+1+r-\beta)} ||u||_t,$

and the proof of Theorem 2.1 is completed by inserting in this bound the appropriate values of $r \Box$

5. The function G_r

Since the function G_r , defined by

$$
G_{\gamma}(x) = \sum_{n=1}^{\infty} n^{-\gamma} \cos 2\pi nx, \qquad x \in \mathbb{R}, \gamma > 1,
$$
 (5.1)

plays a key rôle in the theory, we note briefly some of its properties.

Since $y > 1$, the series (5.1) is (absolutely) uniformly convergent, and therefore G_x is a continuous, even, 1-periodic function on R. Moreover.

$$
G_{\gamma}(0) = \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} = \zeta(\gamma) > 0,
$$

\n
$$
G_{\gamma}\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\gamma}} = -(1 - 2^{1-\gamma})\zeta(\gamma) < 0,
$$

where ζ is the Riemann zeta function, and where the last step follows, for example, from JAHNKE and EMDE [8, p. 319]. Since the continuous function G_x changes sign on $(0, 1/2)$, it has at least one zero in that interval.

If γ is an even integer, then G_{γ} is closely related to a Bernoulli polynomial: in fact for $\gamma = 2, 4, ...$ we have [1, p. 805]

$$
G_{\gamma}(x) = (-1)^{1+\gamma/2} \frac{2^{\gamma-1} \pi^{\gamma}}{\gamma!} B_{\gamma}(x), \qquad x \in [0, 1].
$$

Here the uniqueness of the zero on $(0, 1/2)$ is apparent from the known behaviour of Bernoulli polynomials. The following result establishes that there is a unique zero on $(0, 1/2)$ for all real $\gamma \geq 1$.

Theorem 5.1 [5]: For $\gamma \ge 1$ the function G, defined by (5.1) is decreasing on (0, 1/2). and has a unique zero $x_0(\gamma) \in (0, 1/2)$, which satisfies

$$
\lim x_0(\gamma) = 1/4.
$$

6. Qualocation in the presence of a smoother perturbation

In this section we apply the same qualocation method to the more general pseudodifferential equation which can be written as perturbed equation.

$$
(L+K) u = I,
$$

where L is as in (1.3), and K is an operator having a greater smoothing effect than L .

The following result is stated initially in a more general form than we need for our present purposes. The proof is modelled on one used by ARNOLD and WENDLAND [4] for the case of the collocation method.

25 Analysis Bd. 8, Heft 4 (1989)

 (6.1)

Theorem 6.1: Let β be a real number, and assume that L is an isomorphism from. **HERE IS IN PROPERTY TO FALL THEOTEM CONSUMPLERED THEOTEM 6.1:** Let β be a real number, and assume that L is an isomorphism from H^s onto $H^{s-\beta}$ $\forall s \in \left[\beta-a, \mu+d+\frac{1}{2}\right)$, with μ , d and a as given below. Let μ *be such that K maps H^{5-p} boundedly into H^{5-β} for the above range of s values, and* assume $L + K$ is bijective from H^s onto $H^{s-\beta}$. Assume also that a given qualocation method, with trial space. S_h consisting of smoothest splines of degree $d > \beta - \mu$, and *with given test space* S_h consisting of smoothest splines of degree $d > \beta - \mu$, and with given test space T_h and quadrature rule Q_h , has the following property when applied to the equation **applied to the equation of** \mathcal{L}_E $\$ $\forall s \in [\beta - a, \mu + d + \frac{1}{2})$, with μ , d and a as given below.
 K maps $H^{s-\mu}$ boundedly into $H^{s-\beta}$ for the above range of s ve
 K is bijective from H^s onto $H^{s-\beta}$. Assume also that a given qu
 K is bijective Theorem 6.1: Let β be a real number, and assume that L is an isomorphism fF
 fF onto $H^{s-\beta} \forall s \in [\beta - a, \mu + d + \frac{1}{2})$, with μ , d and a as given below. Let $\mu >$

be such that K maps $H^{s-\mu}$ boundedly into $H^{$ $\begin{aligned}\n &\begin{aligned}\n &\text{P} \cdot \nabla \circ \xi \in \left[\beta - a, \mu + d + \frac{1}{2} \right], \text{ with } \mu, \text{ a and a as given beta}\n \end{aligned} \\
 &\text{if } K \text{ maps } H^{s-\mu} \text{ boundedly into } H^{s-\beta} \text{ for the above range of }\n \end{aligned} \\
 &\text{if } K \text{ is bijective from } H^s \text{ onto } H^{s-\beta}. \text{ Assume also that a give }\n \end{aligned} \\
 &\text{if } K \text{ is bijective from } H^s \text{ onto } H^{s-\beta}. \text{ Assume also that a give }\n \begin{aligned}\n &\text{$

$$
Lw = f, \qquad f \in H^{t-\beta}, \qquad w \in H^t: \qquad (6.2)
$$

namely that the qualocation approximation $w_h \in S_h$ *exists uniquely for* $h \leq h_0$, *with* h_0 *independent of f, and satisfies, for some fixed a* \geq 0, $h \leq h_0$,
 \therefore

$$
||w_b - w||_{\epsilon} \leq Ch^{\min(t - s, t - \beta, d + 1 - s, d + 1 + a - \beta)} ||w||_{t}
$$
\n(6.3)

$$
\beta - a \leq s \leq t, \qquad s < d + 1/2, \qquad \beta + 1/2 < t. \tag{6.4}
$$

Then the same iesult holds, provided t satisfies also

$$
\beta - a \leq s \leq t, \quad s < d + 1/2, \quad \beta + 1/2 < t.
$$
\n(6.4)

\nthe same result holds, provided t satisfies also

\n
$$
\mu + \beta - a \leq t < \mu + d + 1/2,
$$
\ne quadratic function method $\{S_h, T_h, Q_h\}$ applied to the equation

\n
$$
(L + K) u = f, \quad f \in H^{t-\beta}, \quad u \in H^t.
$$
\n(6.6)

for the qualocation method $\{S_h, T_h, Q_h\}$ *applied to the equation.*

$$
(L+K) u = f, \qquad f \in H^{t-\beta}, \qquad u \in H^t. \tag{6.6}
$$

(6.2)
 $w_h \in S_h$ exists uniquely for $h \leq h_0$, with h_0
 $d \ a \geq 0$,
 $1+a-\beta$. $||w||_t$

(6.3)
 $\beta + 1/2 < t$.

(6.4)

is also

(6.5)

ed to the equation
 $u \in H^t$.

(6.6)
 $u \in H^t$.

(6.6)

(6.6) Stated fully, the result is that the qualocation approximation $u_{h_j} \in S_h$ exists uniquely for *h* sufficiently small, and satisfies *hndependent of f, and satisfies, for some fixed* $a \ge 0$ *,
* $||w_h - w||_s \le Ch^{mind - s, t - \beta, d + 1 - s, d + 1 + a - \beta}$ *,* $||w||_t$ *

for all real s, t satisfying
* $\beta - a \le s \le t$ *,* $s < d + 1/2$ *,* $\beta + 1/2 < t$ *.

Then the same result holds, provided t s* $\beta - a \leq s \leq t$, $s < d + 1/2$, $\beta + 1/2 < t$.
 uen the same result holds, provided t satisfies also
 $\mu + \beta - a \leq t < \mu + d + 1/2$,
 the qualocation method $\{S_h, T_h, Q_h\}$ applied to the equation
 $(L + K) u = f$, $f \in H^{t-\beta}$, $u \in H^t$ Then the same result holds, provided *t* satisfies also $\mu + \beta - a \leq t < \mu + d + 1/2$,

for the qualocation method $\{S_h, T_h, Q_h\}$ applied to the equation
 $(L + K)u = f, \quad f \in H^{t-\beta}, \quad u \in H^t$.
 \bullet Stated fully, the result is that th $\mu + \beta - a \leq t < \mu + d + 1/2$,

for the qualocation method $\{S_h, T_h, Q_h\}$ applied to the equatio
 $(L + K) u = f, \quad f \in H^{t-\beta}, \quad u \in H^t$.

Stated fully, the result is that the qualocation approximation

h, sufficiently small, and satisf

$$
||u_h - u||_s \leq Ch^{\min(t - s, t - \beta, d + 1 - s, d + 1 + a - \beta)} ||u||_t
$$
 (6.7)

for all s, t satisfying (6.4) and (6.5).

Proof: Take $h \leq h_0$, and assume provisionally that u_h , a qualocation solution to (6.6), exists. Then by definition $u_h \in S_h$ satisfies, for arbitrary $\chi_h \in T_h$, *w*: Take $h \leq h_0$, and assume provids.
 Well $u_h \in S_h$ as
 $Q_h(\bar{z}_h(L + K)(u - u_h)) = 0$,
 $Q_h(\bar{z}_h L u_h) = Q_h(\bar{z}_h L u + K(u - u_h))$
 $w = u + L^{-1}K(u - u_h)$.

If from the assumption, since $\chi_h \in \mathbb{R}$ assumption, since $\chi_h \in \mathbb{R}$.

$$
Q_h(\bar{\chi}_h(L+K)(u-u_h))=0,
$$

$$
Q_h(\bar{\chi}_hLu_h)=Q_h(\bar{\chi}_hLu+K(u-u_h)])=Q_h(\bar{\chi}_hLw),
$$

where

,• - **S**

$$
w=u+L^{-1}K(u-u_h).
$$

It follows from the assumption, since $\chi_h \in T_h$ is arbitrary, that u_h is the unique qualo- $||u_h - u||_s \leq Chmint t-s.t-s.d+1+a-\beta ||u||_t$ (6.7)

for all s, it satisfying (6.4) and (6.5).

Proof: Take $h \leq h_0$, and assume provisionally that u_h , a qualocation solution to

(6.6), exists. Then by definition $u_h \in S_h$ satisfies, (6.8). Then the estimate (6.3) gives $\begin{array}{c}\n \text{cation} \\
 (6.8).\n \end{array}$ ws from the assumption,
pproximation to an equi-
hen the estimate (6.3) gi
 $\lceil |u - u_h + L^{-1}K(u - u)| \rceil$ or
 $Q_h(\bar{\chi}_h L u_h) = Q_h(\bar{\chi}_h [Lu + K(u - u_h)]) = Q_h(\bar{\chi}_h L w),$

where
 $w = u + L^{-1} K(u - u_h).$

It follows from the assumption, since $\chi_h \in T_h$ is arbitrary, that u_h is the

cation approximation to an equation of the form (6.2) for which th lluhI + *C* Hu - uhlhe. (6.10)

$$
||u - u_h + L^{-1}K(u - u_h)||_s
$$

\n
$$
\leq Ch^{\min(t-s, t-\beta, d+1-s, d+1+a-\beta}||u + L^{-1}K(u - u_h)||_t,
$$
\n(6.9)

. .' (6.8)

for s, t satisfying (6.4). Assume that t satisfies also (6.5). Then it follows from the

$$
\leq Ch^{\min(t-s,t-\beta,d+1+s,d+1+a-\beta)}\|u + L^{-1}K(u-u_h)\|_{t},
$$
\nfor s, t satisfying (6.4). Assume that t satisfies also (6.5). Then it follows from the mapping properties of L and K that\n
$$
\|u + L^{-1}K(u-u_h)\|_{t} \leq \|u\|_{t} + C \|K(u-u_h)\|_{t-\beta}
$$
\n
$$
\leq \|u\|_{t} + C \|u-u_h\|_{t-\rho}.
$$
\n(6.10)

We now obtain, using (6.9), (6.10) and the assumed mapping property of $K+\overset{\cdot}{L}$,

An Approach to Improving the Collocation Method 375
\nobtain, using (6.9), (6.10) and the assumed mapping property of
$$
K + L
$$
,
\n
$$
||u - u_h||_s \leq C ||(L + K) (u - u_h)||_{s-\beta}
$$
\n
$$
\leq C ||L^{-1}(L + K) (u - u_h)||_s = C ||(I + L^{-1}K) (u - u_h)||_s
$$
\n
$$
\leq Ch^{\min(t-s,t-\beta,d+1-s,d+1+a-\beta)}(||u||_t + ||u - u_h||_{t-\mu}).
$$
\n(6.11)
\nular, setting $s = t - \mu$ we obtain
\n
$$
||u - u_h||_{t-\mu} \leq Ch^{\min(u,t-\beta,\mu+d+1-t,d+1+a-\beta)}(||u||_t + ||u - u_h||_{t-\mu}).
$$
\nexponent of h is positive because of the assumptions (6.4) and (6.5), for h
\nly small (say $h \leq h_1$) we have
\n
$$
||u - u_h||_{t-\mu} \leq Ch^{\min(u,t-\beta,\mu+d+1-t,d+1+a-\beta)} ||u||_t.
$$
\nsecond term of (6.11) is of higher order than the first, and so (6.11) yields the
\nresult (6.7).

In particular, setting $s = t - \mu$ we obtain

$$
\equiv \frac{1}{2} \cos \left(\frac{\ln |\mathbf{a}|}{|\mathbf{a}|} \right)
$$

ular, setting $s = t - \mu$ we obtain

$$
||u - u_h||_{t-\mu} \leq Ch^{\min\{\mu, t-\beta, \mu+d+1-t, d+1+a-\beta\}}(||u||_t + ||u - u_h||_{t-\mu}).
$$

Since the exponent of h is positive because of the assumptions (6.4) and (6.5), for h sufficiently small $\|u - u_h\|_{t-\mu} \leq Ch^{\min(\mu, t-\beta, \mu+d)}$

Since the exponent of *h* is positive becomprision is a positive becomprision of $h \leq h_1$) we have
 $\|u - u_h\|_{t-\mu} \leq Ch^{\min(\mu, t-\beta, \mu+d)}$

Thus the second term of (6, 11)

$$
||u - u_h||_{t-u} \leq Ch^{\min(\mu, t-\beta, \mu+d+1-t, d+1+a-\beta)}||u||_{t}.
$$

Thus the second term of (6.11) is of higher order than the first, and so (6.11) yields the desired result (6.7).

To complete the proof, we now observe from (6.7) that the qualocation solution u_h is necessarily unique for $h \leq h_1$. Thus the corresponding linear equations are of full rank, from which it follows that the qualocation solution exists for every continuous function f -

It follows from the theorem that if μ (which is related to the smoothing property of K) is smaller than $a + 1/2$, then the maximum rate of convergence of $||u_h - u||_s$, for any value of *s*, is reduced from $O(h)^{(d+1+a-\beta)}$ to $O(h^{d+1/2+\mu-\epsilon-\beta})$, where $\epsilon > 0$ is arbitrary. On the other hand, if K is a perturbation arising from the mapping of the logarithmic potential from a C^{∞} curve to the unit circle, then μ can typically be taken arbitrarily large, and then no reduction in the maximum order will occur.

We now specialize to the case of even-degree splines, and the particular qualocation method of this paper.

Corollary 6.2: Let β , L, d, S_h, T_h and Q_h be as in Theorem 2.1. Let $t > \beta + 1/2$ and τ *the such that K maps* H^{s-n} *boundedly into* H^{s-p} *for these values of* s . Assume that $L + K$ is bijective from H^{s-p} for $s \in [1, 2, 3]$, $L = \frac{1}{2}$ and $L > 0$ be such that K maps H^{s-n} boundedly into H^{s-p} for and assume that $L + K$ is bijective from H^s onto $H^{s-\beta}$ for these values of s. Assume also $f \in H^{t-\beta}$, where $\mu + \beta - a \leq t < \mu + d + 1/2$. Let $u \in H^t$ be the unique solution of *partion in the other hand,* if K is a perturbation arising between the back of the other hand, if K is a perturbation arising taken arbitrarily large, and then no reduction in the max We now specialize to the case of

Then for h sufficiently small, the qualocation method ${S_h, T_h, Q_h}$ applied -to (6.1) *yields a unique approximation* $u_h \in S_h$, and *if s and t satisfy (6.4) (with a taking the values* 0, 1, 3 *respectively*), then the error norms $||u_h - u||$, satisfy the statements in parts (i), (ii) and (iii) of Theorem 2.1. boundary integral method. Math. Comp. 41 (1983), 383-397.

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the analogous to Corollary 6.2, hold is and i satisfy (6.4) (with a taking

es 0, 1, 3 respectively), then the error norms $||u_h - u||_s$

Before concluding, we note that Theorem 6.1 can be applied with equal effect to the case of the odd-degree spline qualocation method considered in [14]. Thus a result analogous to Corollary 6.2 holds also in that case. values 0, 1, 3 To

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FEFERENCES

[1] ABRAMOWITZ

Washington:

[2] ARNOLD, D.

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REFERENCES

- **[1] ABRAMOWITZ,** M., and L. A. STEOUN (eds.): Handbook of Mathematical Functions Washington:'U.S. Department of Commerce 1964.
- [2] ARNOLD, D.' N.: A spline-trigonometric Galerkin method and an exponentially convergent boundary integral method. Math. Comp. 41 (1983), 383-397.

376 IAN H. SLOAN and W. L. WENDLAND

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.
in Engine [3] ARNOLD, D. N., and W. L. WENDLAND: Collocation versus Galerkin procedures for bound ary integral equations. In: Boundary Element Methods in Engineering (ed.: C. A. Brebbia). Berlin: Springer-Verlag 1982, $18-33$. • [3] ARNOLD, D. N., and W. L. WENDLAND

[3] ARNOLD, D. N., and W. L. WENDLAND: Collocation versus Galer

• ary integral equations. In: Boundary Element Methods in Eng

bia). Berlin: Springer-Verlag 1982, 18-33.

[4] ARNOL 376 Lan H. SLOAN and W. L. V

[3] ARNOLD, D. N., and W. L. WEN

ary integral equations. In: Bou

bia). Berlin: Springer-Verlag 19

[4] ARNOLD, D. N., and W. L. WEN

elliptic equations on curves. Nu

[5] BROWN, G., and D. W [3] ARNOLD, D. N., and W. L. W.
ary integral equations. In: B
bia). Berlin: Springer-Verlag
[4] ARNOLD, D. N., and W. L. W.
elliptic equations on curves.
[5] BROWN, G., and D. WLSON:
of the Centre for Mathematic
appear.
[6

c'.

 \cdot

- [4] ARNOLD, D. N., and W. L. WENDLAND: The convergence of spline collocation for strongly elliptic equations on curves. Numer. Math. 47 (1985), 317–341.
- [5] BROWN, G., and D. WILSON: Trigonometric sums and polynomial zeros. In: Proceedings of theCentre for Mathematical Analysis. Canberra: Australian National University. To
- [6] HSIAO, G. C., Korp, P., and *W.* L. \VENDI,AND: Some applications of a Galerkin-collo- 'cation method for boundary integral equations of the first kind. Math. Meth. Appl. Sci:
- [7] HSIAO, G. C., and W. L. WENOLAND: The Aubin-Nitsche lemma for integral equations. J. Int. Equ. 3 (1981), 399–415.
[8] JAHNKE, E., and F. EMDE: Tables of Higher Functions. Teubner 1933.
-
- [3] ARNOLD, D. N., and W. L. WENDLAYD: Collocation versus Galerkin procedures for

any integral equations. In: Boundary Element Methods in Engineering (ed.; C. A

[4] ARNOLD, D. N., and W. L. WENDLAYD: The convergence of [9] RATHSFELD, A.: Quadraturformelverfahren für eindimensionale singuläre Integralgleichungen. In: Seminar Analysis, Operator Equations and Numerical Analysis 1985/86. Berlin: Karl-WeierstraB-Institut für Mathematik (1986), 147-186. $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$
 $\begin{bmatrix} 10 \\ 11 \end{bmatrix}$
 $\begin{bmatrix} 11 \\ 12 \end{bmatrix}$
	- [10] SARANEN, J.: The convergence 'of even degree spline collocation solution for potential problemsin smooth domains of the, plane. Numer. Math. **53** (1988), 499-512.
	- [11] SARANEN, J., and W. L. WENDLAND: On the asymptotic convergence of collocation methods with spline functions of even degree. Math. Comp. 45 (1985), $91-108$.
	- $[12]$ SARANEN, J., and W. L. WENDLAND: The Fourier series representation of pseudo-differproblems in smooth domains of the plane. Numer. Math. 53 (1988), 499-512.

	11) SARANEN, J., and W. L. WENDLAND: On the asymptotic convergence of collocation me

	thods with spline functions of even degree. Math. Comp. 45 (1
	- [13] SCHMIDT, G.: On spline collocation methods for boundary integral equations in the plane. Math. Meth. Appl. Sci. 7 (1985), 74–89. operators on closed curves. Complex Variables, Theory and Appl. S (198

	FDT, G.: On spline collocation methods for boundary integral equations in

	Meth. Appl. Sci. 7 (1985), 74 – 89.

	1. I. H.: A quadrature-based approach
- [14] SLOAN, I. H.: A quadrature-based approach to improving the collocation method. Numer. Math. 54 (1988), $41-56$. [14] SLOAN, I. H.: A quadrature-based approach to improving the

Math. 54 (1988), 41-56.

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