The Coupling of Boundary Element and Finite Element Methods for a Nonlinear Exterior Boundary Value Problem[1])

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Dedicated to Prof. Dr. Dr. h. c. Solomon G. Mikhlin on the occasion of his 80th birthday

In der Arbeit werden die Methoden der beschränkten und der finiten Elemente gekoppelt und auf das äußere Dirichletsche Problem für eine Klasse nichtlinearer elliptischer Gleichungen zweiter Ordnung in Divergenzform angewandt. Besondere Betonung erfährt die Formulierung in Variationsform und die mathematische Begründung der Kopplungsprozedur. Insbesondere werden hinreichende Bedingungen an die Koeffizienten der Gleichung angegeben, unter denen mit Hilfe der Methode der monotonen Operatoren die Existenz einer Lösung und deren Unität folgen.

Работа посвящена применению соединения методов ограниченных и финитных элементов к внешней задаче Дирихле для одного класса нелинейных эллиптических уравнений второго порядка в дивергентной форме. Особое значение придается вариационной формулировке и математическому обоснованию соединительной процедуры. В частности, даются достаточные условия на коэффициенты уравнения из которых методом монотонных операторов следуют существование и единственность решения.

This paper is concerned with an application of the coupling of the boundary element and the finite element methods to an exterior Dirichlet problem for a class of nonlinear second-order elliptic equations in divergence form. Emphasis will be placed upon the variational formulation and the mathematical foundations of the coupling procedure. In particular, sufficient conditions are given for the coefficients of the equation from which existence and uniqueness results are established by the theory of monotone operators.

1. Introduction

Both the boundary element and the finite element methods are now recognized as general approximation processes which are applicable to a wide variety of engineering problems. The boundary element method is better suited to problems in which the domain extends to infinity but is usually confined to regions in which the governing equations are linear and homogeneous. On the other hand, the finite element method is restricted to problems in bounded domains but is applicable to problems in which the material properties are not necessarily homogeneous and nonlinearity may occur. Therefore, these two methods are complementary to each other, and thus, in recent years there have been increasing efforts to develop variational procedures especially for exterior problems by taking the advantages of the individual method. This leads us to the coupling of the boundary element and the finite element methods (see Hsiao [6]). Noticeable success of this approach may be found in COSTABEL [2], HAN [5], HSIAO and PORTER [8], JOHNSON and NEDELEC [10], MACCAMY and MARIN [11],

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and Wendland [16, 17], to name a few. However, all these works deal with only *linear problems*. The purpose of this paper is to discuss the feasibility of applying this approach to genuine *nonlinear problems*.

Specifically, we let Ω_0 be a bounded, simply connected domain in $\mathbf{R}^3$ with smooth boundary Γ_0. Let Ω^- be the annular region bounded by Γ_0 and another smooth closed surface Γ. We denote by Ω^+ the complement of $\Omega_0 \cup \overline{\Omega}^-$. For any function v on $\Omega^- \cup \Omega^+$, we write $v^\pm$ for its limits on Γ from $\Omega^\pm$. Then given smooth functions f on Ω^- and g on Γ_0, we consider the boundary value problem: Find u such that

$$-\sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x)\big) = f(x) \text{ in } \Omega^-; \qquad \Delta u = 0 \text{ in } \Omega^+,$$

$$u = g \text{ on } \Gamma_0,$$

$$u^- = u^+, \qquad \sum_{i=1}^{3} a_i\big(x, \nabla u^-(x)\big) n_i = \left(\frac{\partial u}{\partial n}\right)^+ (x) \text{ on } \Gamma, \tag{1.1}$$

$$u(x) = O(|x|^{-1}) \qquad \text{as } |x| \to +\infty,$$

where $\mathrm{n} = (n_1, n_2, n_3)$ is the outer normal to Γ. The nonlinear coefficients a_i will satisfy certain regularity conditions to be specified later. Explicit examples of non-linearities of this type generally appear in some subsonic flow problems (see, e.g. Feistauer [3]). Our goal here is to solve the problem (1.1) by the coupling procedure as in Han [5] and Costabel [2] for linear problems.

The plan of this paper is a follows. In Section 2, we will convert (1.1) to a nonlocal boundary problem in Ω^-. This contains a nonlocal boundary condition which relates the unknown function and its normal derivative on the boundary Γ from Green's formula. In Section 3, we give a weak formulation for the nonlocal boundary problem and reduce it to an equivalent operator equation form. Existence and uniqueness of the solution of this operator equation will be established in Section 4 by the theory of monotone operators. Finally in Section 5, we present a Galerkin procedure for the operator equation and provide an error estimate of Cea's type.

2. The nonlocal boundary problem

To reduce (1.1) to a nonlocal boundary problem, we need some results from potential theory. Let $\gamma(x, y) = (4\pi |x - y|)^{-1}$ be the fundamental solution for the three-dimensional Laplacian. Then from Green's theorem we have the representation

$$u(x) = \int_\Gamma u^+(y) \frac{\partial}{\partial n_y} \gamma(x, y) \, ds_y - \int_\Gamma \left(\frac{\partial u}{\partial n}\right)^+ (y) \gamma(x, y) \, ds_y, \qquad x \in \Omega^+ \tag{2.1}$$

from which we arrive at the integral equations on Γ:

$$u^+ = \left(\frac{1}{2} \mathbf{I} + \mathbf{K}\right) u^+ - \mathbf{V}(\partial u/\partial n)^+,$$

$$(\partial u/\partial n)^+ = -\mathbf{W} u^+ + \left(\frac{1}{2} \mathbf{I} - \mathbf{K}'\right)(\partial u/\partial n)^+. \tag{2.2}$$

Here $\mathbf{K}, \mathbf{V}, \mathbf{W}$ and $\mathbf{K}'$ are the boundary integral operators of the *double, simple, hyper-singular* and *adjoint* of the *double layer potentials*, respectively: For $x \in \Gamma$, they are

defined by

$$\mathbf{K}\sigma(x) = \int_\Gamma \sigma(y) \frac{\partial}{\partial n_y} \gamma(x, y)\, ds_y, \qquad \mathbf{V}\sigma(x) = \int_\Gamma \gamma(x, y)\, \sigma(y)\, ds_y,$$

$$\mathbf{W}\sigma(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \sigma(y) \frac{\partial}{\partial n_y} \gamma(x, y)\, ds_y, \qquad (2.3)$$

$$\mathbf{K}'\sigma(x) = \int_\Gamma \frac{\partial}{\partial n_x} \gamma(x, y) \cdot \sigma(y)\, ds_y.$$

These are four basic boundary integral operators (see MICHLIN [13]) whose properties will be discussed later. Now we use interface conditions from (1.1). We have $u^+ = u^-$, and if we set $\sum_{i=1}^{3} a_i(x, \nabla u^-(x))\, n_i = \sigma(x)$, then $(\partial u/\partial n)^+ = \sigma$. We substitute into (2.2) to obtain, on Γ,

$$\left(\frac{1}{2}\,\mathbf{I} - \mathbf{K}\right) u^- + \mathbf{V}\sigma = 0,$$

$$\sum_{i=1}^{3} a_i\big(\cdot, \nabla u^-(\cdot)\big)\, n_i = -\mathbf{W}u^- + \left(\frac{1}{2}\,\mathbf{I} - \mathbf{K}'\right) \sigma. \qquad (2.4)$$

These formulae lead us to the *nonlocal boundary problem*: Find (u, σ) such that

$$-\sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x)\big) = f(x) \quad \text{in } \Omega^-; \qquad u|_{\Gamma_\bullet} = g,$$

$$\sum_{i=1}^{3} a_i\big(\cdot, \nabla u^-(\cdot)\big)\, n_i = -\mathbf{W}u^- + \left(\frac{1}{2}\,\mathbf{I} - \mathbf{K}'\right) \sigma \qquad \text{on } \Gamma, \qquad (2.5)$$

$$\left(\frac{1}{2}\,\mathbf{I} - \mathbf{K}\right) u^- + \mathbf{V}\sigma = 0 \qquad \text{on } \Gamma.$$

It is easy to see from the above derivation that the *nonlocal boundary problem* (2.5) is equivalent to the problem (1.1) in the sense that if (u, σ) satisfies (2.5) and if we set, from Green's formula (2.1),

$$u(x) = \int_\Gamma u^-(y) \frac{\partial}{\partial n_y} \gamma(x, y)\, ds_y - \int_\Gamma \sigma(y)\, \gamma(x, y)\, ds_y, \qquad x \in \Omega^+, \qquad (2.6)$$

then u satisfies (1.1) with $\sum_{i=1}^{3} a_i\big(x, \nabla u^-(x)\big)\, n_i = \sigma(x)$ on Γ. Of course, the statement here can be made more precise if one introduces appropriate solution spaces (see GATICA [4]). We remark that the boundary conditions on Γ are nonlocal conditions, since the values u^- over the entire boundary Γ are needed in order to compute $\sum_{i=1}^{3} a_i\big(x, \nabla u^-(x)\big)\, n_i$ at a single point $x \in \Gamma$. Clearly, one may also consider other types of nonlocal conditions, and in principle, the nonlocal boundary problem (2.5) can be treated numerically by any conventional scheme, since it is a problem over the finite region. We will adopt the Galerkin procedure and lead to the coupling of the boundary element and finite element methods.

3. The weak formulation

In what follows, for m integer, s real, let $H^m(\Omega^-)$ and $H^s(\Gamma)$ denote the usual Sobolev spaces equipped with norms $\|\cdot\|_{H^m(\Omega^-)}$ and $\|\cdot\|_{H^s(\Gamma)}$, respectively. Furthermore, for $m > 0$, let $|\cdot|_{H^m(\Omega^-)}$ denote the seminorm,

$$|u|_{H^m(\Omega^-)} = \left\{ \sum_{|\alpha|=m} \int_{\Omega^-} |D^\alpha u|^2 \, dx \right\}^{1/2} \tag{3.1}$$

and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product,

$$\langle \sigma, \chi \rangle = \int_\Gamma \sigma(t) \, \chi(t) \, dt \qquad \forall (\sigma, \chi) \in H^s(\Gamma) \times H^{-s}(\Gamma). \tag{3.2}$$

We also introduce the subspace $H^1_{\Gamma_0}(\Omega^-)$ of $H^1(\Omega^-)$ defined by

$$H^1_{\Gamma_0}(\Omega^-) = \{ v \in H^1(\Omega^-) : v|_{\Gamma_0} = 0 \}. \tag{3.3}$$

It is easy to verify that $|v|_{H^1(\Omega^-)}$ is equivalent to the norm $\|v\|_{H^1(\Omega^-)}$ for all $v \in H^1_{\Gamma_0}(\Omega^-)$, i.e., there exists a constant $C > 0$ such that

$$|v|_{H^1(\Omega^-)} \leqq \|v\|_{H^1(\Omega^-)} \leqq C \, |v|_{H^1(\Omega^-)} \qquad \forall v \in H(\Omega^-)^1_{\Gamma_0}. \tag{3.4}$$

For smooth Γ, the following results are well known (see, e.g., HSIAO [7], and HSIAO and WENDLAND [9]).

Lemma 1: *For C^∞ boundary Γ, the boundary integral operators defined by (2.3)*

$$\mathbf{V}: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \qquad \mathbf{K}: H^{s+1/2}(\Gamma) \to H^{s+3/2}(\Gamma),$$

$$\mathbf{K}': H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \qquad \mathbf{W}: H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$$

are continuous for any $s \in \mathbf{R}$. Moreover, there are constants $\mu > 0$ and $\nu \geqq 0$ such that

$$\langle \mathbf{V}\sigma, \sigma \rangle \geqq \mu \|\sigma\|^2_{H^{-1/2}(\Gamma)} \qquad \forall \sigma \in H^{-1/2}(\Gamma) \tag{3.5}$$

and

$$\langle \sigma, \mathbf{W}\sigma \rangle \geqq \mu \|\sigma\|^2_{H^{1/2}(\Gamma)} - \nu \|\sigma\|^2_{H^0(\Gamma)} \qquad \forall \sigma \in H^{1/2}(\Gamma). \tag{3.6}$$

In fact, from (3.5) it is not difficult to show that

$$\langle \sigma, \mathbf{W}\sigma \rangle \geqq 0 \qquad \forall \sigma \in H^{1/2}(\Gamma). \tag{3.7}$$

These results will be needed later.

Now for the weak formulation, we multiply the partial differential equation in (2.5) by any function $v \in H^1_{\Gamma_0}(\Omega^-)$ and apply the divergence theorem to yield

$$\sum_{i=1}^{3} \int_{\Omega^-} a_i\big(x, \nabla u(x)\big) \frac{\partial v}{\partial x_i} \, dx + \left\langle v^-, \mathbf{W}u^- - \left(\frac{1}{2} \mathbf{I} - \mathbf{K}' \right) \sigma \right\rangle = \int_{\Omega^-} fv \, dx. \tag{3.8}$$

Similarly, we multiply the boundary integral equation in (2.5) by any test function $\lambda \in H^{-1/2}(\Gamma)$ and integrate over Γ to obtain

$$\langle \mathbf{V}\sigma, \lambda \rangle + \left\langle \left(\frac{1}{2} \mathbf{I} - \mathbf{K} \right) u^-, \lambda \right\rangle = 0. \tag{3.9}$$

Equations (3.8) and (3.9) then lead us to the *weak formulation*: *Given* $(f, g) \in H^{-1}(\Omega^-) \times H^{1/2}(\Gamma)$, *find* $(u, \sigma) \in H^1(\Omega^-) \times H^{-1/2}(\Gamma)$ *such that* $u - \bar{g} \in H^1_{\Gamma_o}(\Omega^-)$ *and* (u, σ) *satisfies*

$$\sum_{i=1}^{3} \int_{\Omega^-} a_i(x, \nabla u(x)) \frac{\partial v}{\partial x_i} dx + \mathbf{B}((u, \sigma); (v, \lambda)) = \int_{\Omega^-} fv \, dx \qquad (3.10)$$

for all $(v, \lambda) \in H^1_{\Gamma_o}(\Omega^-) \times H^{-1/2}(\Gamma)$, *where* $\bar{g} \in H^1(\Omega^-)$ *is an extension of* g *with* $\bar{g}|_{\Gamma_o} = g$, *and* $\mathbf{B}(\cdot, \cdot)$ *is the bilinear form defined by*

$$\mathbf{B}((u, \sigma), (v, \lambda)) = \langle v^-, \mathbf{W}u^- \rangle - \left\langle v^-, \left(\frac{1}{2} \mathbf{I} - \mathbf{K}'\right) \sigma \right\rangle$$

$$+ \langle \mathbf{V}\sigma, \lambda \rangle + \left\langle \left(\frac{1}{2} \mathbf{I} - \mathbf{K}\right) u^-, \lambda \right\rangle \qquad (3.11)$$

for all $(u - \bar{g}, \sigma), (v, \lambda) \in H^1_{\Gamma_o}(\Omega^-) \times H^{-1/2}(\Gamma)$.

In order to reduce (3.10) to an equivalent operator equation form, we need now to impose some conditions on the nonlinear coefficients a_i. We assume that $a_i \colon \Omega^- \times \mathbf{R}^3 \to \mathbf{R}$ and satisfies the conditions:

(H.1) *Carathéodory conditions*: The function $a_i(\cdot, \alpha)$ is measurable in Ω^- for all $\alpha \in \mathbf{R}^3$ and $a_i(x, \cdot)$ is continuous in $\mathbf{R}^3$ for almost all $x \in \Omega^-$.

(H.2) *Growth condition*: There exists $\phi_i \in L^2(\Omega^-)$, $i = 1, 2, 3$, such that

$$|a_i(x, \alpha)| \leqq C\{1 + |\alpha|\} + |\phi_i(x)|$$

for all $\alpha \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$. Here and in the sequel, C is a generic constant.

We also define the Hilbert space $\mathbf{H} = H^1_{\Gamma_o}(\Omega^-) \times H^{-1/2}(\Gamma)$ with the product norm $\|(w, \sigma)\|_{\mathbf{H}} = \{\|w\|^2_{H^1(\Omega^-)} + \|\sigma\|^2_{H^{-1/2}(\Gamma)}\}^{1/2}$. Further, let $\mathbf{H}^*$ be the dual of $\mathbf{H}$, with the norm defined by, $\|\cdot\|_{\mathbf{H}^*} = \sup \{[\cdot, (v, \lambda)]/\|(v, \lambda)\|_{\mathbf{H}} : 0 \neq (v, \lambda) \in \mathbf{H}\}$, where $[\cdot, \cdot]$ denotes the duality pairing on $\mathbf{H}^* \times \mathbf{H}$. Then, by (H.1) and (H.2) we can introduce a nonlinear operator $\mathbf{T} \colon \mathbf{H} \to \mathbf{H}^*$ defined by

$$[\mathbf{T}(w, \sigma), (v, \lambda)] = \sum_{i=1}^{3} \int_{\Omega^-} a_i(x, \nabla(w + \bar{g})(x)) \frac{\partial v}{\partial x_i} dx + \mathbf{B}((w; \sigma), (v, \lambda))$$
$$(3.12)$$

for all $(w, \sigma), (v, \lambda) \in \mathbf{H}$. Consequently, if we set, in the weak formulation, $w := u - \bar{g} \in H^1_{\Gamma_o}(\Omega^-)$, alternatively, we may write (3.10) in the form of an operator equation for the unknown (w, σ):

$$\mathbf{T}(w, \sigma) = \mathbf{F}, \qquad (3.13)$$

where $\mathbf{F} \in \mathbf{H}^*$ is defined by

$$\mathbf{F}(v, \lambda) = \int_{\Omega^-} fv \, dx - \mathbf{B}((\bar{g}, 0), (v, \lambda)). \qquad (3.14)$$

We remark that the bilinear form $\mathbf{B}$ is bounded as a consequence of Lemma 1 and the trace theorem. The operator equation form (3.13) then allows us to dicuss the solvability of the variational equation (3.10) by the theory of monotone operators.

4. Existence and uniqueness

To apply the theory of monotone operators, following NEČAS [14] and ODEN [15], we need to make further assumptions on the nonlinear coefficients a_i in (3.10). We consider the following conditions:

(H.3) *Coerciveness condition*: There exist a constant $C_1 > 0$ and a function C_2 $\in L^1(\Omega^-)$ such that $\sum\limits_{i=1}^{3} a_i(x, \boldsymbol{\alpha}) \, \alpha_i \geqq C_1 \, |\boldsymbol{\alpha}|^2 - C_2(x)$ for all $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$.

(H.4) *Monotony condition*:

$$\sum_{i=1}^{3} \big(a_i(x, \boldsymbol{\alpha}) - a_i(x, \boldsymbol{\alpha}')\big) (\alpha_i - \alpha_i') \geqq 0$$

for all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$.

(H.5) *Strict monotony condition*:

$$\sum_{i=1}^{3} \big(a_i(x, \boldsymbol{\alpha}) - a_i(x, \boldsymbol{\alpha}')\big)/(\alpha_i - \alpha_i') > 0$$

for all $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$.

(H.6) *Strongly monotone condition*: There exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{3} \big(\partial a_i(x, \boldsymbol{\alpha})/\partial \alpha_j\big) \, \xi_i \xi_j \geqq \mu \sum_{i=1}^{3} \xi_i^2$$

for all $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$.

Clearly these conditions are not mutually exclusive. Hence our main results can be summarized in the following theorems depending on the assumptions on a_i.

Theorem 1: *Suppose that the coefficients a_i satisfy the assumptions* (H.1)−(H.4). *Let* T: $\mathbf{H} \to \mathbf{H}^*$ *be the operator defined by* (3.12) *and let* $\mathbf{F} \in \mathbf{H}^*$ *be the bounded linear functional defined by* (3.14). *Then there exists a solution* $(w, \sigma) \in \mathbf{H}$ *of the equation* (3.13), *and the solution is unique if* (H.5) *is satisfied.*

Theorem 2: *Under the assumption* (H.1) *and* (H.2) *the operator* T *is continuous and bounded. Furthermore, if* (H.6) *holds, then* T *is strongly monotone; i.e., there exists a constant $\mu > 0$ such that*

$$[\mathbf{T}(w, \sigma) - \mathbf{T}(v, \lambda), (w, \sigma) - (v, \lambda)] \geqq \mu \, \|(w, \sigma) - (v, \lambda)\|_{\mathbf{H}}^2 \tag{4.1}$$

for all $(w, \sigma), (v, \lambda) \in \mathbf{H}$, *and hence* T *is coercive on* $\mathbf{H}$; *i.e.,*

$$\lim_{\|(w,\sigma)\|_{\mathbf{H}} \to +\infty} [\mathbf{T}(w, \sigma), (w, \sigma)]/\|(w, \sigma)\|_{\mathbf{H}} = +\infty. \tag{4.2}$$

We remark that Theorem 2 implies that the equation (3.13) has a unique solution. The proofs of these theorems are tedious and lengthy, but the arguments here are straightforward if one is familiar with the theory of monotone operators (see, e.g., NEČAS [14]). To facilitate the proofs, let us first make some observations. We note that the boundary operators corresponding to $\mathbf{B}$ in the definition of the nonlinear operator T of (3.12) possess all the nice properties. In particular, we see that in addition to being bounded,

$$\big|\mathbf{B}\big((w, \sigma), (v, \lambda)\big)\big| \leqq C \, \|(w, \sigma)\|_{\mathbf{H}} \, \|(v, \lambda)\|_{\mathbf{H}} \tag{4.3}$$

for all $(w, \bar{\sigma})$, $(v, \lambda) \in \mathbf{H}$, $\mathbf{B}$ satisfies the inequality,

$$\mathbf{B}\big((w, \sigma), (w, \sigma)\big) \geqq \mu \, \|\sigma\|^2_{H^{-1/2}(\Gamma)} \qquad \forall (w, \sigma) \in \mathbf{H}, \tag{4.4}$$

which follows from (3.11), (3.5) and (3.7). Hence the operator $\mathbf{T}$ is completely domi-
nated by the nonlinear term in $\mathbf{T}$, i.e., the *Nemytsky operator* $\mathbf{A}_i$, defined by

$$(\mathbf{A}_i w)\,(x) = a_i\big(x, \nabla(w + \bar{g})\,(x)\big). \tag{4.5}$$

The assumptions (H.1) and (H.2) simply imply that $\mathbf{A}_i$ is a continuous map from
$H^1(\Omega^-)$ into $L^2(\Omega^-)$ and the inequality

$$\|\mathbf{A}_i w\|^2_{L^2(\Omega^-)} \leqq C\{\mathrm{vol}\,(\Omega^-) + |w + \bar{g}|^2_{H^1(\Omega^-)} + \|\phi_i\|^2_{L^2(\Omega^-)}\} \tag{4.6}$$

holds for all $w \in H^1_{\Gamma_s}(\Omega^-)$. As a consequence of (4.3), (4.6), we have the following
lemma.

Lemma 2: *Under the assumptions* (H.1) *and* (H.2), *the operator* $\mathbf{T} \colon \mathbf{H} \to \mathbf{H}^*$ *has the
following properties*:
  (i) $\mathbf{T}$ *is bounded.*
  (ii) *There exists a constant* $M_0 > 0$ *such that* $\|\mathbf{A}_i w\|_{L^2(\Omega^-)} \leqq C \, \|(w, \sigma)\|_{\mathbf{H}}$ *for all*
     $(w, \sigma) \in \mathbf{H}$, $\|(w, \sigma)\|_{\mathbf{H}} \geqq M_0$.
  (iii) $\mathbf{T}$ *is continuous.*

It is easy to verify that the constant M_0 in (ii) is given by

$$M_0 = \left\{ \mathrm{vol}\,(\Omega^-) + |\bar{g}|^2_{H^1(\Omega^-)} + \sum_{i=1}^{3} \|\phi_i\|^2_{L^2(\Omega^-)} \right\}^{1/2}.$$

The assumption (H.3) naturally gives the coerciveness property of $\mathbf{T}$. (4.2) as can
be seen in the following. We may write from (3.13),

$$\sum_{i=1}^{3} \int_{\Omega^-} a_i\big(x, \nabla(w + \bar{g})\,(x)\big) \frac{\partial w}{\partial x_i}\, dx$$

$$= \sum_{i=1}^{3} \int_{\Omega^-} a_i\big(x, \nabla(w + \bar{g})\,(x)\big) \frac{\partial}{\partial x_i}\,(w + \bar{g})\, dx$$

$$- [\mathbf{T}(w, \sigma), (\bar{g}, 0)] + \mathbf{B}\big((w, \sigma), (\bar{g}, 0)\big),$$

and hence from (H.3) together with (4.3) and (ii) in Lemma 2, we have

$$\sum_{i=1}^{3} \int_{\Omega^-} a_i\big(x, \nabla(w + \bar{g})\,(x)\big) \frac{\partial w}{\partial x_i}\, dx$$

$$\geqq C_1 \, |w + \bar{g}|^2_{H^1(\Omega^-)} - \|C_2\|_{L^2(\Omega^-)} - C \, \|(w, \sigma)\|_{\mathbf{H}} \, \|\bar{g}\|_{H^1(\Omega^-)}$$

for all $(w, \sigma) \in \mathbf{H}$ with $\|(w, \sigma)\|_{\mathbf{H}} \geqq M_0$. Consequently, we deduce from (3.12), (3.4)
and (4.4),

$$\frac{[\mathbf{T}(w, \sigma), (w, \sigma)]}{\|(w, \sigma)\|_{\mathbf{H}}} \geqq \tilde{C}_1 \, \|(w, \sigma)\|_{\mathbf{H}} - C \, \|\bar{g}\|_{H^1(\Omega^-)} - \|C_2\|_{L^2(\Omega^-)} \, \|(w, \sigma)\|_{\mathbf{H}}^{-1}$$

for all $(w, \sigma) \in \mathbf{H}$ with $\|(w, \sigma)\|_{\mathbf{H}} \geqq M_0$. This proves that $\mathbf{T}$ is coercive on $\mathbf{H}$.

Similarly, it is not difficult to verify that

$$[\mathbf{T}(w, \sigma) - \mathbf{T}(v, \lambda), (w, \sigma) - (v, \lambda)]$$
$$= \mathbf{B}\big((w, \sigma) - (v, \lambda), (w, \sigma) - (v, \lambda)\big)$$
$$+ \sum_{i=1}^{3} \int_{\Omega^-} \{a_i(x, \nabla\bar{w}(x)) - a_i(x, \nabla\bar{v}(x))\} \left\{\frac{\partial\bar{w}}{\partial x_i} - \frac{\partial\bar{v}}{\partial x_i}\right\} dx \qquad (4.7)$$

with $\bar{w} = w + \tilde{g}$ and $\bar{v} = v + \tilde{g}$. Because of (4.4), the monotonicity conditions will be determined by the second term on the right-hand side of the above equality and thus by the conditions (H.4) and (H.5).

To complete the proof of Theorem 1, we remark that the continuity of $\mathbf{T}$ implies the hemicontinuity of $\mathbf{T}$, i.e., the mapping

$$\mathbf{R} \ni t \rightarrow \big[\mathbf{T}\big((w, \sigma) + t(p, \delta)\big), (v, \lambda)\big] \in \mathbf{R}$$

is continuous for all $(w, \sigma), (p, \delta), (v, \lambda) \in \mathbf{H}$. Moreover, $\mathbf{T}$ has the property (see Nečas [14: Theorem 3.3.14]):

(M) *Let $\{(w_n, \sigma_n)\}$ be a sequence of $\mathbf{H}$. Suppose that (w_n, σ_n) converges weakly to (w, σ), and $\mathbf{T}(w_n, \sigma_n)$ converges weakly to $\mathbf{F}$ with* $\limsup_{n\to+\infty} [\mathbf{T}(w_n, \sigma_n), (w_n, \sigma_n)] \leq [\mathbf{F}, (w, \sigma)]$, *then* $\mathbf{T}(w, \sigma) = \mathbf{F}$.

We also note that the continuity of $\mathbf{T}$ implies, clearly, the demicontinuity of $\mathbf{T}$, namely, *if (w_n, σ_n) converges to (w, σ), then $\mathbf{T}(w_n, \sigma_n)$ converges weakly to $\mathbf{T}(w, \sigma)$*. Now, property (M), coerciveness, demicontinuity and boundedness of $\mathbf{T}$ implies that $\mathbf{T}$ is surjective (see Nečas [14, Theorem 3.3.6]). The rest of the proof follows easily ∎

To complete the proof of Theorem 2, it remains to show that under (H.6), $\mathbf{T}$ satisfies the strongly monotone condition (4.1), since the coerciveness of $\mathbf{T}$ is a simple consequence of (4.1). We proceed as follows: We note that we may write

$$a_i(x, \nabla\bar{w}(x)) - a_i(x, \nabla\bar{v}(x)) = \int_0^1 \sum_{j=1}^3 \frac{\partial}{\partial\alpha_j} a_i(x, \boldsymbol{\alpha}(x; t)) \left\{\frac{\partial\bar{w}}{\partial x_j} - \frac{\partial\bar{v}}{\partial x_j}\right\} dt$$

with $\boldsymbol{\alpha}(x, t) = \nabla\bar{v}(x) + t(\nabla\bar{w}(x) - \nabla\bar{v}(x))$. Then (H.6) yields

$$\sum_{i=1}^3 \int_{\Omega^-} \{a_i(x, \nabla\bar{w}(x)) - a_i(x, \nabla\bar{v}(x))\} \left\{\frac{\partial\bar{w}}{\partial x_i} - \frac{\partial\bar{v}}{x_i}\right\} dx$$

$$\geq \mu \sum_{i=1}^3 \int_{\Omega^-} \left\{\frac{\partial\bar{w}}{\partial x_i} - \frac{\partial\bar{v}}{\partial x_i}\right\}^2 dx.$$

The result then follows from (3.4), (4.4) and (4.7) ∎

5. Galerkin approximations

To formulate the Galerkin approximations of the solution of (3.15), we let $\{\mathbf{H}_h\}_{h\in S}$ be a family of finite-dimensional subspaces of $\mathbf{H}$ such that $\cup \{\mathbf{H}_h : h \in S\}$ is dense in $\mathbf{H}$, where the parameter h is in an index set S and represents, without loss of generality, a measure of the size of the corresponding finite elements. Then the *Galerkin approxi-*

mation of the solution (w, σ) *of* (3.15) is defined as an element $(w^h, \sigma^h) \in \mathbf{H}_h$ satisfying the Galerkin equations

$$[\mathbf{T}(w^h, \sigma^h), (v^h, \lambda^h)] = [\mathbf{F}, (v^h, \lambda^h)] \tag{5.1}$$

for all $(v^h, \lambda^h) \in \mathbf{H}_h$. Clearly, the existence, uniqueness as well as the convergence of the Galerkin approximations (w^h, σ^h) of (5.1) depend heavily on the assumptions of the nonlinear coefficients a_i in T. From Theorem 1 in Section 4, we know at least that there exists a unique solution $(w, \sigma) \in \mathbf{H}$ of (3.13), provided (H.5) is fulfilled in addition to the assumptions (H.1)−(H.3). In this case, one can show that there exists a unique Galerkin approximation (w^h, σ^h) which has a subsequence converging only weakly to the exact solution (w, σ) of (3.13). On the other hand, the weak convergence of the Galerkin approximations can be improved by the strongly monotone condition (4.1). However, in order to obtain some kind of rate of convergence, an additional condition will be needed in contrast to the linear problems (see MICHLIN [12]). For this purpose, one will introduce a Lipschitz condition below. We recall that $\mathbf{T}: \mathbf{H} \to \mathbf{H}^*$ is *Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|\mathbf{T}(w, \sigma) - \mathbf{T}(v, \lambda)\|_{\mathbf{H}^*} \leq k \, \|(w, \sigma) - (v, \lambda)\|_{\mathbf{H}} \tag{5.2}$$

for all $(w, \sigma), (v, \lambda) \in \mathbf{H}$. As will be seen, a sufficient condition on the nonlinear coefficients a_i to ensure (5.2) is the following one:

(H.7) *Lipschitz condition*: The partial derivatives $\partial a_i(x, \boldsymbol{\alpha})/\partial \alpha_j$ $(i, j = 1, 2, 3)$ satisfy the Carathéodory conditions (H.1) and there exists a constant $C_0 > 0$ such that $|\partial a_i(x, \boldsymbol{\alpha})/\partial \alpha_j| \leq C_0 (i, j = 1, 2, 3)$ for all $\boldsymbol{\alpha} \in \mathbf{R}^3$ and for almost all $x \in \Omega^-$.

To establish (5.2), it suffices to show that the Nemytsky operator $\mathbf{A}_i$ defined by (4.5) is Lipschitz continuous on $H^1(\Omega^-)$: We denote that under conditions (H.1) and (H.2), $\mathbf{A}_i$ is a continuous map from $H^1(\Omega^-)$ into $L^2(\Omega^-)$. For $\bar{w} = w + \bar{g}$ and $\bar{v} = v + \bar{g} \in H^1(\Omega^-)$, we have by the definition of $\mathbf{A}_i$,

$$\|\mathbf{A}_i w - \mathbf{A}_i v\|_{L^2(\Omega^-)}^2 := \int_{\Omega^-} \left| a_i(x, \nabla \bar{w}(x)) - a_i(x, \nabla \bar{v}(x)) \right|^2 dx$$

$$= \int_{\Omega^-} \left| \sum_{j=1}^{3} \int_0^1 \frac{\partial}{\partial \alpha_j} a_i(x, \boldsymbol{\alpha}(x, t)) \left\{ \frac{\partial \bar{w}}{\partial x_j} - \frac{\partial \bar{v}}{\partial x_j} \right\} dt \right|^2 dx,$$

where $\boldsymbol{\alpha}(x, t) = \nabla \bar{v}(x) + t(\nabla \bar{w}(x) - \nabla \bar{v}(x))$. It follows from (H.7) that

$$\|\mathbf{A}_i w - \mathbf{A}_i v\|_{L^2(\Omega^-)}^2 \leq 3 C_0^2 \, |\bar{w} - \bar{v}|_{H^1(\Omega^-)}^2 \leq C \, \|w - v\|_{H^1(\Omega^-)}^2$$

which proves that $\mathbf{A}_i: H^1(\Omega^-) \to L^2(\Omega^-)$ is Lipschitz continuous.

The following result concerning the error estimates of the Galerkin approximations can now be easily established.

Theorem 3: *Suppose that the assumptions* (H.1), (H.2), (H.6) *and* (H.7) *are fulfilled. Let* $(w, \sigma) \in \mathbf{H}$ *and* $(w^h, \sigma^h) \in \mathbf{H}_h$ *be the unique solutions of the equations* (3.13) *and* (5.1), *respectively. Then, there exists a constant* $C > 0$ *independent of* h *such that*

$$\|(w, \sigma) - (w^h, \sigma^h)\|_{\mathbf{H}} \leq C \inf_{(v^h, \lambda^h) \in \mathbf{H}_h} \|(w, \sigma) - (v^h, \lambda^h)\|_{\mathbf{H}}. \tag{5.3}$$

We remark that as in the case for linear problems, this simple, yet crucial estimate (5.3) shows that the problem of estimating the error between the solution (w, σ) and the Galerkin approximation (w^h, σ^h) is reduced to a problem in the approximation

theory. We further comment that for the numerical implementations, one generally does not solve (5.1) exactly. This leads us to consider the following modification of (5.1). To this end, let $T_h : H_h \to H_h^*$ be an operator that approximates T on the subspace H_h, and let $F_h \in H_h^*$ be an approximation of F on H_h. Then we redefine a Galerkin approximation of the solution (w, σ) of (3.13) as an element $(w_h, \sigma_h) \in H_h$ (if it exists) such that

$$[T_h(w_h, \sigma_h), (v^h, \lambda^h)]_h = [F_h(v^h, \lambda^h)]_h \tag{5.4}$$

for all $(v^h, \lambda^h) \in H_h$, where $[\cdot, \cdot]_h$ denotes the duality pairing on $H_h^* \times H_h$. As far as the existence of a solution of (5.4) concerns, it suffices, from Theorem 1, to assume that T_h is a continuous, monotone and coercive operator on H_h. Since H_h is finite-dimensional, the monotonicity of T_h implies that T_h is bounded (see Oden [15: Theorem 27.3]). Similarly, from Theorem 2, we see that there exists a unique solution of (5.4) if T_h is continuous and strongly monotone on H_h. We now summarize these results in the following theorem.

Theorem 4: *Suppose that the coefficients a_i satisfy the assumptions* (H.1), (H.2), (H.6) *and* (H.7). *Let $F_h \in H_h^*$ be an approximation of F, and let $T_h : H_h \to H_h^*$ be an approximate operator of T with the properties:*

(i) T_h *is continuous.*

(ii) T_h *is uniformly strongly monotone, i.e., there exist constants $h_0 > 0$ and $\mu_0 > 0$, independent of h, such that*

$$[T_h(u_h, \varkappa_h) - T_h(v^h, \lambda^h), (u_h, \varkappa_h) - (v^h, \lambda^h)]_h \geqq \mu_0 \, \|(u_h, \varkappa_h) - (v^h, \lambda^h)\|_H^2$$

for all $(u_h, \varkappa_h), (v^h, \lambda^h) \in H_h$ and for all $h \in (0, h_0)$.

Then (3.13) *has a unique solution $(w, \sigma) \in H$ and* (5.4) *has a unique solution $(w_h, \sigma_h) \in H_h$. Moreover, the following estimates hold for all $h \in (0, h_0)$:*

$$\|(w, \sigma) - (w_h, \sigma_h)\|_H \leqq C \left\{ \sup_{\substack{(z^h, \delta^h) \in H_h \\ (z^h, \delta^h) \neq 0}} \frac{|[F_h, (z^h, \delta^h)]_h - [F, (z^h, \delta^h)]|}{\|(z^h, \delta^h)\|_H} \right.$$

$$+ \inf_{(v^h, \lambda^h) \in H_h} \left(\|(w, \sigma) - (v^h, \lambda^h)\|_H \right.$$

$$\left. \left. + \sup_{\substack{(z^h, \delta^h) \in H_h \\ (z^h, \delta^h) \neq 0}} \frac{|[T(v^h, \lambda^h), (z^h, \delta^h)] - [T_h(v^h, \lambda^h), (z^h, \delta^h)]_h|}{\|(z^h, \delta^h)\|_H} \right) \right\}$$

$$\tag{5.5}$$

where $C > 0$ is a constant independent of h.

The proof of the inequality (5.5) follows in the same manner as for the linear problems (see, e.g., Ciarlet [1]). We omit the details.

To conclude this paper, we remark that in Gatica [4], an explicit operator T_h will be given to possess the properties of (i) and (ii) in Theorem 4, and specific error estimates based on (5.5) will be also available for a family of finite element subspaces $\{H_h\}_{h \in (0, h_0)}$.

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