Zeitschrift für Analysis
und ihre Anwendungen
Bd. 8 (5) 1989, S. 425–430

A Note on the Strong-Operator Topology of Countably Generated $\mathcal{O}$-Vector Spaces

K. Schmüdgen

Es werden die abzählbar erzeugten geschlossenen $\mathcal{O}$-Vektorräume $\mathcal{A}$ durch eine notwendige und hinreichende Bedingung charakterisiert, für die die starke Operatorentopologie die feinste lokalkonvexe Topologie auf $\mathcal{A}$ ist.

Даётся необходимое и достаточное условие для счётно порождённых замкнутых $\mathcal{O}$-векторных пространств $\mathcal{A}$ для которых сильная операторная топология является сильнейшей локально выпуклой топологией на $\mathcal{A}$.

The countably generated closed $\mathcal{O}$-vector spaces $\mathcal{A}$ for which the strong operator topology is the finest locally convex topology on $\mathcal{A}$ are characterized by a necessary and sufficient condition.

The objective of the present paper is to prove a theorem which characterizes those countably generated closed $\mathcal{O}$-vector spaces $\mathcal{A}$ contained in $\mathcal{L}^+(\mathcal{D})$ for which the strong operator topology on $\mathcal{A}$ coincides with the finest locally convex topology on $\mathcal{A}$. This theorem generalizes the assertion of Theorem 3 in [1]. Examples and applications of this result are given in Section 2 of [1]. Further, our proof as given below fills a gap contained in the proof of Theorem 3 in [1].

We collect the terminology used in this paper (see e.g. [2]). Let $\mathcal{D}$ be a linear subspace of a Hilbert space $\mathcal{H}$. An *$\mathcal{O}$-vector space* $\mathcal{A}$ on $\mathcal{D}$ is a linear subspace of closable linear operators with domain $\mathcal{D}$ such that $\mathcal{A}$ contains the identity map I of $\mathcal{D}$. The *graph topology* $t_{\mathcal{A}}$ is the locally convex topology on $\mathcal{D}$ generated by the seminorms $\|\cdot\|_a := \|a\cdot\|$, $a \in \mathcal{A}$. Let $\mathcal{D}(\hat{\mathcal{A}})$ be the completion of the locally convex space $\mathcal{D}[t_{\mathcal{A}}]$ considered as a linear subspace of $\cap \{\mathcal{D}(\bar{a}) : a \in \mathcal{A}\}$. The $\mathcal{O}$-vector space $\hat{\mathcal{A}} := \{\bar{a} \upharpoonright \mathcal{D}(\hat{\mathcal{A}}) : a \in \mathcal{A}\}$ is called the *closure of* $\mathcal{A}$. By $\mathcal{L}^+(\mathcal{D})$ we mean the set of all closable linear operators a with domain $\mathcal{D}$ such that $a\mathcal{D} \subseteqq \mathcal{D}$, $\mathcal{D} \subseteqq \mathcal{D}(a^*)$ and $a^*\mathcal{D} \subseteqq \mathcal{D}$. The *strong operator topology* $\sigma^{\mathcal{D}}$ on $\mathcal{A}$ is the locally convex topology defined by the family of seminorms $\|\cdot\|_\varphi := \|\cdot\varphi\|$, $\varphi \in \mathcal{D}$. The finest locally convex topology on $\mathcal{A}$ is denoted by τ_{st}.

The result mentioned above is the following

Theorem: *Let $\mathcal{A}$ be an $\mathcal{O}$-vector space on $\mathcal{D}$ with countable algebraic basis. Suppose that $\mathcal{A} \subseteqq \mathcal{L}^+(\mathcal{D})$. Then the following two statements are equivalent:*

(i) *The strong operator topology on $\hat{\mathcal{A}}$ is equal to the finest locally convex topology τ_{st} on $\hat{\mathcal{A}}$.*

(ii) *For each continuous seminorm p on $\mathcal{D}[t_{\mathcal{A}}]$ the vector space $\mathcal{A}^p := \bigcup_{\lambda>0} \{a \in \mathcal{A} : \|a\varphi\| \leqq \lambda p(\varphi)$ for all $\varphi \in \mathcal{D}\}$ is finite-dimensional.*

Proof: Upon replacing $\mathcal{A}$ by its closure $\hat{\mathcal{A}}$ we can assume without loss of generality that $\mathcal{A}$ is already closed. First we verify the implication (i) $\Rightarrow$ (ii). Assume to the contrary that the space $\mathcal{A}^p$ is infinite-dimensional for some continuous seminorm p

on $\mathcal{D}[t_{\mathcal{A}}]$. Then there is a linearly independent subset $\{a_n : n \in \mathbf{N}\}$ of $\mathcal{A}^p$ such that $\|a_n\varphi\| \leqq p(\varphi)$ for $n \in \mathbf{N}$ and all $\varphi \in \mathcal{D}$. Since $\sigma^{\mathcal{D}} = \tau_{\text{st}}$ on $\mathcal{A}$ by (i), for each seminorm q on $\mathcal{A}$ there are vectors $\psi_1, \ldots, \psi_k \in \mathcal{D}$ such that $q(a) \leqq \|a\psi_1\| + \ldots + \|a\psi_k\|$ for all $a \in \mathcal{A}$. If we take a seminorm q such that $q(a_n) > n$ for $n \in \mathbf{N}$, we obtain the desired contradiction.

Now we turn to the proof of (ii) $\Rightarrow$ (i) which is the main assertion of the theorem. The proof will be divided into five steps. In case where $\mathcal{A}$ is finite-dimensional the assertion follows immediately from

Statement 1: *For each finite-dimensional O-vector space $\mathcal{B}$ on $\mathcal{D}$ there are finitely many vectors $\xi_1, \ldots, \xi_l \in \mathcal{D}$ such that $x \to \max\{\|x\xi_1\|, \ldots, \|x\xi_l\|\}$ defines a norm on $\mathcal{B}$.*

Proof: Let $|||\cdot|||$ be a norm on $\mathcal{B}$ and let $\mathcal{S}$ be the unit sphere of $(\mathcal{B}, |||\cdot|||)$. For $x \in \mathcal{S}$ there is a $\xi_x \in \mathcal{D}$ such that $x\xi_x \neq 0$. By the compactness of $\mathcal{S}$, the open covering $\mathcal{V}^x := \{y \in \mathcal{S} : y\xi_x \neq 0\}$, $x \in \mathcal{S}$, of $\mathcal{S}$ has a finite subcover, say $\mathcal{V}^{x_1}, \ldots, \mathcal{V}^{x_m}$. Letting $\xi_j := \xi_{x_j}$ for $j = 1, \ldots, m$, the assertion follows ∎

Statement 2: *There is a sequence $(a_n; n \in \mathbf{N}_0)$ of operators from $\mathcal{A}$ with $a_0 = I$ such that*

the graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ is generated by the seminorms

$$p_n(\varphi) := \|a_0\varphi\| + \ldots + \|a_n\varphi\|, \quad n \in \mathbf{N}_0; \tag{1}$$

$$\sup\{p_n(\varphi)\, p_{n-1}(\varphi)^{-1} : \varphi \in \mathcal{D}\} = +\infty \text{ for } n \in \mathbf{N}; \tag{2}$$

$$\sup\{p_n(\varphi)\,(p_{n-1}(\varphi) + \|x\varphi\|)^{-1} : \varphi \in \mathcal{D}\} < +\infty \text{ for each } x \in \mathcal{A}^{p_n} \setminus \mathcal{A}^{p_{n-1}} \text{ and } n \in \mathbf{N}. \tag{3}$$

Proof: First we construct inductively a sequence $(a_n; n \in \mathbf{N}_0)$ such that (1) and (2) are fulfilled. Since $\mathcal{A}$ has a countable basis, there is a sequence $(b_n; n \in \mathbf{N})$ of operators from $\mathcal{A}$ such that $t_{\mathcal{A}}$ is generated by the seminorms $\|\cdot\|_{b_n}$, $n \in \mathbf{N}$. Let $a_0 := I$. If $a_0, \ldots, a_n$ are already chosen, let m be the smallest natural number for which $\sup\{\|b_m\varphi\|\, p_n(\varphi)^{-1} : \varphi \in \mathcal{D}\} = +\infty$. Such an $m \in \mathbf{N}$ exists, since otherwise $\mathcal{A} = \mathcal{A}^{p_n}$ and $\mathcal{A}$ would be finite-dimensional. Defining $a_{n+1} := b_m$, (2) is satisfied in the case of $n + 1$.

Next we show how in addition (3) can be fulfilled. Suppose that the supremum in (3) is infinite for some $x \in \mathcal{A}^{p_k} \setminus \mathcal{A}^{p_{k-1}}$ and $n = k \in \mathbf{N}$. Then we replace (a_n) by the new sequence $(a_0, \ldots, a_{k-1}, x, a_k, \ldots)$. To verify (2) for the latter, it suffices to do this for $n = k$ and for $n = k + 1$. From $x \notin \mathcal{A}^{p_{k-1}}$ it follows (2) in case $n = k$. Since the supremum in (3) is infinite for x and $n = k$, we get (2) for $n = k + 1$. Since all spaces $\mathcal{A}^{p_n}$, $n \in \mathbf{N}_0$, are finite-dimensional, by an inductive argument this procedure can be continued until (3) is satisfied ∎

We keep the sequence $(a_n; n \in \mathbf{N})$ from Statement 2 fixed. Let $\mathcal{A}_0 := \mathcal{A}^{p_0}$. For $n \in \mathbf{N}$, we choose a linear subspace $\mathcal{A}_n$ of the (finite-dimensional!) vector space $\mathcal{A}^{p_n}$ such that $\mathcal{A}^{p_n}$ is the direct sum of $\mathcal{A}_n$ and $\mathcal{A}^{p_{n-1}}$. Now fix $n \in \mathbf{N}_0$. Let $\{a_{n1}, \ldots, a_{nd_n}\}$ be a base of the vector space $\mathcal{A}_n$. Without loss of generality we assume that $\|a_{nr}\varphi\| \leqq p_n(\varphi)$ for $\varphi \in \mathcal{D}$, $r = 1, \ldots, d_n$. Define a norm $|||\cdot|||$ on $\mathcal{A}_n$ by $\left|\left|\left| \sum_r \lambda_r a_{nr} \right|\right|\right| := \sum_r |\lambda_r|$, where $\lambda_1, \ldots, \lambda_{d_n} \in \mathbf{C}$. Clearly, $\mathcal{A}_0 \equiv \mathcal{A}^{\|\cdot\|}$ is the vector space of all bounded operators contained in $\mathcal{A}$ and $\|x_0\| \leqq |||x_0|||$ for all $x_0 \in \mathcal{A}_0$.

Because of (1), $\mathcal{A}$ is the algebraic direct sum of the vector spaces $\mathcal{A}_n$, $n \in \mathbf{N}_0$. That is, each $x \in \mathcal{A}$ can be written in a unique way as a finite sum $x = \sum x_n$ with $x_n \in \mathcal{A}_n$.

For each positive sequence $\gamma = (\gamma_n; n \in \mathbf{N}_0)$, we define a seminorm q_γ on $\mathcal{A}$ by $q_\gamma(x) := (\sum |\gamma_n| \, |||x_n|||^2)^{1/2}$ for $x = \sum x_n$. Since the topology τ_{st} on $\mathcal{A}$ is, of course, generated by the family of seminorms q_γ, it suffices to show for each positive sequence $\gamma = (\gamma_n; n \in \mathbf{N})$ there are vectors $\varphi_1, \ldots, \varphi_l \in \mathcal{D}$, $l \in \mathbf{N}$, such that $q_\gamma(x) \leqq \max \{\|x\varphi_1\|, \ldots, \|x\varphi_l\|\}$ for all $x \in \mathcal{A}$. From now on let $\gamma = (\gamma_n; n \in \mathbf{N}_0)$ be fixed.

Statement 3: *For each $n \in \mathbf{N}$ there is an $\varepsilon_n > 0$ such that*

$$\|x\varphi\| \geqq |||x||| \left(\varepsilon_n p_n(\varphi) - p_{n-1}(\varphi)\right) \;\forall\, x \in \mathcal{A}_n \text{ and } \varphi \in \mathcal{D}. \tag{4}$$

Proof: Assume that the assertion is false for some $n \in \mathbf{N}$. Then there are operators $x_k \in \mathcal{A}_n$ and vectors $\varphi_k \in \mathcal{D}$ such that

$$\|x_k\varphi_k\| < |||x_k||| \left(\frac{1}{k} p_n(\varphi_k) - p_{n-1}(\varphi_k)\right) \quad \text{for } k \in \mathbf{N}. \tag{5}$$

After norming if necessary we can assume that $p_n(\varphi_k) = |||x_k||| = 1$ for all $k \in \mathbf{N}$. Then, by (5), $\|x_k\varphi_k\| \to 0$ if $k \to \infty$ and $p_{n-1}(\varphi_k) \leqq 1/k$ for $k \in \mathbf{N}$. By the compactness of the unit sphere of the finite-dimensional space $(\mathcal{A}_n, |||\cdot|||)$, there is a subsequence of (x_k) converging to some $x \in \mathcal{A}_n$, $x \neq 0$. For simplicity we assume that already $\lim |||x_k - x||| = 0$. From $\|(x_k - x)\varphi_k\| \leqq |||x_k - x||| \max \|a_{n_r}\varphi_k\| < |||x_k - x||| \, p_n(\varphi_k) = |||x_k - x||| \to 0$ and $\|x_k\varphi_k\| \to 0$ if $k \to \infty$ we conclude that $\|x\varphi_k\| \to 0$ if $k \to \infty$. Therefore,

$$\sup_{k \in \mathbf{N}} p_n(\varphi_k) \left(p_{n-1}(\varphi_k) + \|x\varphi_k\|\right)^{-1} \geqq \sup_{k \in \mathbf{N}} 1 \cdot \left(\frac{1}{k} + \|\varphi_k\|\right)^{-1} = +\infty.$$

Since $x \in \mathcal{A}^{p_n} \setminus \mathcal{A}^{p_{n-1}}$, this contradicts (3). ∎

Statement 4: *There are numbers $\varepsilon_0 > 0$ and $l \in \mathbf{N}$, a double sequence $(\delta_{nm}; n, m \in \mathbf{N}_0)$ of positive numbers and sequences $(\varphi_{kn}; n \in \mathbf{N})$, $k = 1, \ldots, l$, of vectors from $\mathcal{D}$ such that for all $k = 1, \ldots, l$ and $n \in \mathbf{N}$ the following is satisfied:*

$$\max_{j=1,\ldots,l} \|x_0\varphi_{j0}\| \geqq \left(\varepsilon_0 + \sqrt{\delta_{00} + \gamma_0}\right) |||x_0||| \text{ for } x_0 \in \mathcal{A}_0; \tag{6}$$

$$p_n(\varphi_{kn}) = \varepsilon_n^{-1}\left(\sqrt{\delta_{nm} + \gamma_n} + 2\delta_{n-1,n-1}\right) + 1 + \sum_{m=1}^{n-1} p_n(\varphi_{km}) \leqq \delta_{nn}; \tag{7}$$

$$p_m(\varphi_{kn}) \leqq \delta_{nn}^{-1} 2^{-n} \leqq \varepsilon_0 2^{-n} \text{ if } m \in \mathbf{N}_0,\ m < n; \tag{8}$$

$$\delta_{mn} = \delta_{nm} \geqq 1 + \sum_{r,s<n} p_m(\varphi_{kr})\, p_n(\varphi_{ks}) \text{ if } m \in \mathbf{N}_0,\ m < n; \tag{9}$$

$$\langle x_r\varphi_{kn}, x_s\varphi_{km}\rangle = 0 \text{ if } m, s, r \in \mathbf{N}_0,\ m < n,\ r \leqq n,\ s \leqq n,\ x_r \in \mathcal{A}_r \text{ and } x_s \in \mathcal{A}_s; \tag{10}$$

$$D_n := \begin{vmatrix} \delta_{00} & -\delta_{01} & \cdots & -\delta_{0n} \\ -\delta_{10} & \delta_{11} & \cdots & -\delta_{1n} \\ \vdots & \vdots & & \vdots \\ -\delta_{n0} & -\delta_{n1} & \cdots & \delta_{nn} \end{vmatrix} > 0. \tag{11}$$

(Here we set $\delta_{-1,-1} = \sum_{m=1}^{n-1} \cdots = \sum_{r,s<n} \cdots = 0$ in case $n = 0$.)

Proof: Applying Statement 1 to the finite-dimensional vector space $\mathcal{A}_0$ there are unit vectors $\xi_1, \ldots, \xi_l \in \mathcal{D}$, $l \in \mathbf{N}$, such that $q_0(\cdot) := \max \{\|\cdot\xi_1\|, \ldots, \|\cdot\xi_l\|\}$ is a norm on $\mathcal{A}_0$. Thus there is an $\varepsilon_0 > 0$ such that $q_0(\cdot) \geqq \varepsilon_0|||\cdot|||$ on $\mathcal{A}_0$. Now take a positive number $\delta_{00} \geqq \varepsilon_0^{-1}$ such that $\delta_{00} \geqq \varepsilon_0^{-1}\sqrt{\delta_{00}+\gamma_0}+1 =: \lambda_0$ and set $\varphi_{k0} = \lambda_0\xi_k$, $k = 1, \dots, l$. Then (6) and (7) in case $n = 0$ are fulfilled.

We continue by induction. Let $n \in \mathbf{N}$. Suppose that δ_{ij} and φ_{kl} for $i, j, k, l \in \mathbf{N}$, $i \leqq n-1$, $j \leqq n-1$, $k \leqq l$ are chosen such that (6)—(11) are true in that case. First take numbers $\delta_{mn} = \delta_{nm}$, $m = 0, \dots, n-1$, such that (9) is satisfied. Since $D_{n-1} > 0$ by induction hypothesis, there is a $\delta_{nn} \geqq \varepsilon_0^{-1}$ such that $D_n > 0$ and, for $k = 1, \dots, l$,

$$\delta_{nn} \geqq \varepsilon_n^{-1}\left(\sqrt{\delta_{nn}+\gamma_n}+2\delta_{n-1,n-1}\right)+1+\sum_{m=1}^{n-1} p_n(\varphi_{km}) =: \lambda_{kn}.$$

Let $\mathcal{E}$ be the linear space of $x_r^+x_s\varphi_{km}$, where $x_r \in \mathcal{A}_r$, $x_s \in \mathcal{A}_s$, $m, s, r \in \mathbf{N}_0$, $m < n$, $r \leqq n$, $s \leqq n$. Since the vectors φ_{km} are in $\mathcal{D}$ and $\mathcal{A} \subseteqq \mathcal{L}^+(\mathcal{D})$, $\mathcal{E}$ is a well-defined finite-dimensional subspace of $\mathcal{D}$. We check that

$$\sup\{p_n(\varphi)\,p_{n-1}(\varphi)^{-1} : \varphi \in \mathcal{D}, \varphi \neq 0 \text{ and } \varphi \perp \mathcal{E}\} = +\infty. \tag{12}$$

Assume to the contrary that the supremum in (12) is finite, say equal to λ. Let e be the orthogonal projection with range $\mathcal{E}$. Since the operators $a_1, \dots, a_n$ are, of course, bounded on $\mathcal{E}$, there is a $\mu > 0$ such that $p_n(e\varphi) \leqq \mu\,\|\varphi\|$ for $\varphi \in \mathcal{D}$. Thus

$$\begin{aligned} p_n(\varphi) &\leqq p_n\big((I-e)\,\varphi\big) + p_n(e\varphi) \leqq \lambda p_{n-1}\big((I-e)\,\varphi\big) + p_n(e\varphi) \\ &\leqq \lambda p_{n-1}(\varphi) + (\lambda+1)\,\mu\,\|\varphi\| \leqq (\lambda+\lambda\mu+\mu)\,p_{n-1}(\varphi) \text{ for } \varphi \in \mathcal{D} \end{aligned}$$

which contradicts (2). Let $k \in \{1, \dots, l\}$. Because of (12), there is a vector $\varphi_{kn} \in \mathcal{D}$ such that $\varphi_{kn} \perp \mathcal{E}$ and $p_n(\varphi_{kn}) > \lambda_{kn}\delta_{nn}2^n p_{n-1}(\varphi_{kn})$. After multiplication by some constant we get $p_n(\varphi_{kn}) = \lambda_{kn}$. Since $p_m(\cdot) \leqq p_{n-1}(\cdot)$ for $m \leqq n-1$, we thus obtain (7) and (8) in case of n. (10) follows from $\varphi_{kn} \perp \mathcal{E}$ ∎

Let $k \in \{1, \dots, l\}$. From (1) and (7) it follows that $\left(\psi_{kn} := \sum_{m=0}^{n} \varphi_{kn}; n \in \mathbf{N}_0\right)$ is a Cauchy sequence in the complete locally convex space $\mathcal{D}[t_\mathcal{A}]$. Hence there is a vector $\varphi_k \in \mathcal{D}$ such that $\varphi_k = t_\mathcal{A} - \lim_{n\to\infty} \varphi_{kn}$. That is, we have $x\varphi_k = \sum_{n=0}^{\infty} x\varphi_{kn}$ for each $x \in \mathcal{A}$ in the norm of $\mathcal{H}$.

Statement 5: $\max_{k=1,\dots,l} \|x\|_{\varphi_k} \geqq q_\gamma(x)$ *for all* $x \in \mathcal{A}$.

Proof: First fix a $k \in \{1, \dots, l\}$. Suppose $m, n \in \mathbf{N}_0$, $m < n$. Using (7)—(10), we get for $i = 1, \dots, d_m$ and $j = 1, \dots, d_n$

$$\begin{aligned} &|\langle a_{mi}\varphi_k, a_{nj}\varphi_k\rangle| \\ &\leqq |\langle a_{mi}\varphi_{kn}, a_{nj}\varphi_{kn}\rangle| + \sum_{r,s<n} |\langle a_{mi}\varphi_{kr}, a_{nj}\varphi_{ks}\rangle| + \sum_{r\geqq n+1} |\langle a_{mi}\varphi_{kr} a_{nj}\varphi_{kr}\rangle| \\ &\leqq p_m(\varphi_{kn})\,p_n(\varphi_{kn}) + \sum_{r,s<n} p_m(\varphi_{kr})\,p_n(\varphi_{ks}) + \sum_{r\geqq n+1} p_m(\varphi_{kr})\,p_n(\varphi_{kr}) \\ &\leqq \delta_{nn}^{-1}\delta_{nn}2^{-n} + \sum_{r,s<n} \cdots + \sum_{r\geqq n+1} 2^{-r}2^{-r} \leqq \delta_{mn}. \end{aligned}$$

By the definition of the norm $|||\cdot|||$, this yields

$$\begin{aligned} |\langle x_m\varphi_k, x_n\varphi_k\rangle| &\leqq |||x_m|||\;|||x_n|||\max_{i,j} |\langle a_{mi}\varphi_k, a_{nj}\varphi_k\rangle| \\ &\leqq \delta_{mn}\;|||x_m|||\;|||x_n||| \quad \text{for } x_m \in \mathcal{A}_m \text{ and } x_n \in \mathcal{A}_n. \end{aligned} \tag{13}$$

By symmetry, (13) is true for all $m, m \in \mathbf{N}$ with $n \neq m$. Let $n \in \mathbf{N}_0$. Since $\delta_{rr} \geqq 1$ and hence $p_n(\varphi_{kr}) \leqq 2^{-r}$ for $r > n$, it follows from (8) and (7) that

$$|p_n(\varphi_k) - p_n(\varphi_{kn})| \leqq \sum_{r=n+1}^{\infty} p_n(\varphi_{kr}) + \sum_{m=0}^{n-1} p_n(\varphi_{km}) \leqq 1 + \sum_{m=0}^{n-1} p_n(\varphi_{km})$$

and $p_n(\varphi_k) \leqq p_n(\varphi_{kn}) + \delta_{nn} \leqq 2\delta_{nn}$. By this and (7),

$$\begin{aligned} &\varepsilon_n p_n(\varphi_k) - p_{n-1}(\varphi_k) \\ &\geqq \varepsilon_n \left\{ p_n(\varphi_{kn}) - 1 - \sum_{m=1}^{n-1} p_n(\varphi_{km}) \right\} - 2\delta_{n-1,n-1} = \sqrt{\delta_{nn} + \gamma_n} \end{aligned}$$

for $n \in \mathbf{N}$. Combining the latter with (4), we get

$$\|x_n \varphi_k\|^2 \geqq (\delta_{nn} + \gamma_n) \, |||x_n|||^2 \text{ for } x_n \in \mathcal{A}_n \text{ and } n \in \mathbf{N}. \tag{14}$$

Let $x_0 \in \mathcal{A}_0$. Since $\|x_0\| \leqq |||x_0|||$, (8) yields

$$\begin{aligned} \|x_0 \varphi_k\| &\geqq \|x_0 \varphi_{k0}\| - \sum_{m=1}^{\infty} \|x_0 \varphi_{km}\| \\ &\geqq \|x_0 \varphi_{k0}\| - \sum_{m=1}^{\infty} \|x_0\| \, \varepsilon_0 2^{-m} \geqq \|x_0 \varphi_{k0}\| - \varepsilon_0 \, |||x_0||| \end{aligned}$$

for $k = 1, \ldots, l$. Combined with (6), this gives

$$\max_{k=1,\ldots,l} \|x_0 \varphi_k\|^2 - \gamma_0 \, |||x_0|||^2 \geqq \delta_{00} \, |||x_0|||^2. \tag{15}$$

Now let $x \in \mathcal{A}$. We write x as a finite sum $x = \sum x_n$ with $x_n \in \mathcal{A}_n$. By (13) and (14), we have for $k = 1, \ldots, l$

$$\begin{aligned} \|x\|_{\varphi_k}^2 - q_\gamma(x)^2 &= \sum_n \|x_n \varphi_k\|^2 - \gamma_n \, |||x_n|||^2 + \sum_{m \neq n} |\langle x_m \varphi_k, x_n \varphi_k \rangle| \\ &\geqq \|x_0 \varphi_k\|^2 - \gamma_0 \, |||x_0|||^2 + \sum_{n=1}^{\infty} \delta_{nn} \, |||x_n|||^2 - \sum_{m \neq n} \delta_{mn} \, |||x_m||| \, |||x_n|||. \end{aligned}$$

Therefore, by (15),

$$\left(\max_{k=1,\ldots,l} \|x\|_{\varphi_k} \right)^2 - q_\gamma(x)^2 \geqq \sum_n \delta_{nn} \, |||x_n|||^2 - \sum_{m \neq n} \delta_{mn} \, |||x_m||| \, |||x_n|||.$$

From (11) it follows that the latter is non-negative.

Remark: The preceding proof actually shows the following: Let $\mathcal{A}$ be as in the Theorem and assume that (ii) is fulfilled. If the (finite-dimensional) vector space $\mathcal{A}^{\|\cdot\|}$ of all bounded operators in $\mathcal{A}$ has a separating set consisting of l vectors $\varphi_1, \ldots, \varphi_l \in \mathcal{D}$ (i.e., if $x \in \mathcal{A}^{\|\cdot\|}$ satisfies $x\varphi_k = 0$ for $k = 1, \ldots, l$, then $x = 0$), then for each seminorm q on $\mathcal{A}$ there are l vectors $\varphi_1, \ldots, \varphi_l \in \mathcal{D}$ such that $q(x) \leqq \max\{\|x\|_{\varphi_1}, \ldots, \|x\|_{\varphi_l}\}$ for all $x \in \mathcal{A}$. In particular, if the multiples of the identity are the only bounded operators in $\mathcal{A}$, then already one vector $\varphi_1 \in \mathcal{D}$ is sufficient.

REFERENCES

[1] SCHMÜDGEN, K.: Two theorems about topologies on countably generated Op*-algebras Acta Math. Acad. Sci. Hungar. **35** (1980), 139–150.
[2] SCHMÜDGEN, K.: Unbounded Operator Algebras and Representations. Berlin: Akademie-Verlag 1989

Manuskripteingang: 03. 03. 1988

VERFASSER:

Prof. Dr. KONRAD SCHMÜDGEN
Sektion Mathematik der Karl-Marx-Universität
Karl-Marx-Platz
DDR-7010 Leipzig