On the Semigroup Approach for the Optimal Control of Semilinear Parabolic **Equations Including Distributed and Boundary Control**

F. Tröltzsch

Es werden das Konzept'der milden Lösung und die Formel der Variation der Konstanten angewendet auf die Herleitung notwendiger Optimalitätsbedingungen erster Ordnung für Steuerprobleme bei semilinearen parabolischen Anfangs-Randwertaufgaben. Ein adjungiertes System wird mit Hilfe einer abstrakten Integralgleichung definiert und deren Lösung als milde Lösung einer adjungierten parabolischen Gleichung nachgewiesen.

Доказываются необходимые условия оптимальности первого порядка для проблем оптимального управления систем полулинейных нараболических уравнений в частных производных. Применяются концепция обобщенных решений и формула вариации постоянных. Определяется сопряженная система с помощью абстрактного интегрального уравнения, решение которого является обобщенным решением сопряженного параболического уравнения.

The concept of mild solutions and the variation of constants formula are applied to derive first-order necessary conditions for optimal control problems governed by semilinear parabolic initial-boundary value problems. An adjoint system is defined by means of an abstract integral equation, the solution of the latter being a mild solution of an adjoint parabolic equation.

1. Introduction

The aim of this paper is to apply semigroup methods to control problems governed by semilinear parabolic differential equations, which include both distributed and boundary controls. Much pioneering work on the treatment of inhomogeneous boundary conditions by strongly continuous semigroups has been done for linear boundary control by BALAKRISHNAN [2], FATTORINI [3], LASIECKA [8], and WASH-BURN [16]. It is rather obvious that the celebrated variation of constants formula discussed in these papers allows the treatment of non-linear boundary conditions, too. However the work in L_2 -spaces, which is sufficient for linear boundary control. systems, causes too restrictive assumptions on the non-linearities. In a recent publication by AMANN [1] the application of the variation of constants formula to nonlinear boundary conditions in \dot{W}_{p} ^{*}-spaces was considered. Stimulated by these results the author extended own results on non-linear boundary control, which were focused only on the W_2^s -case. In this way a satisfactory handling of non-linear boundary control systems is possible, in particular the consideration of states which are continuous both in time and space. This paper is to present the outcome of these investigations, thus filling in a gap in the author's book [14], where distributed controls were handled by a semigroup approach but boundary control systems were described by an integral equation with a Green function as kernel. The use of Green functions is, to a certain extent, equivalent to the application of strongly continuous semigroups, but the widely investigated semigroup theory makes the latter more favourable.

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$$
J(w, u) = \Phi(w(T, \cdot)) + \int_0^T \int f_1(t, x, w(t, x), u_1(t, x)) dx dt + \int_0^T \int f_2(t, x, w(t, x), u_2(t, x)) dS_x dt + \int_0^T \int f_2(t, x, w(t, x), u_2(t, x)) dS_x dt
$$
\nsubject to the parabolic semilinear initial-boundary value problem
\n
$$
w_t(t, x) = (Av)(t, x) + h(t, x, w(t, x), u_1(t, x)) \text{ in } (0, T] \times \Omega,
$$
\n
$$
w(0, x) = w_0(x) \text{ on } \Omega,
$$
\n
$$
w(t, x)/\partial n = g(t, x, w(t, x), u_2(t, x)) \text{ on } (0, T] \times T
$$
\nand to the constraints on the controls
\n
$$
u_i \leq u_i(t, x) \leq \overline{u}_i, \quad i = 1, 2.
$$
\n(1.3)
\nIn this paper we shall not admit state-constraints. The consideration of state-con-
\nstrains is connected with special investigations of adjoint operators, which would
\nexpected the size of this paper (see for instance TRÖLTZSCF [14]).

subject to the parabolic semilinear initial-boundary value problem $\begin{array}{c|c}\n\hline\n\text{32} & \text{W} \\
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$$
\nto the parabolic semilinear initial-boundary value problem
\n
$$
w_t(t, x) = (\Delta w) (t, x) + h(t, x, w(t, x), u_1(t, x)) \quad \text{in } (0, T] \times \Omega,
$$
\n
$$
w(0, x) = w_0(x) \quad \text{on } \Omega,
$$
\n
$$
\frac{\partial w(t, x)}{\partial n} = g(t, x, w(t, x), u_2(t, x)) \quad \text{on } (0, T] \times \Gamma
$$
\nthe constraints on the controls
\n
$$
u_i \leq u_i(t, x) \leq \overline{u}_i, \quad i = 1, 2.
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$$
u_i \leq u_i(t, x) \leq \overline{u}_i, \qquad i = 1, 2. \tag{1.3}
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 (1.2)

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In this paper we shall not admit state-constraints. The consideration of state-constraints is connected with special investigations of adjoint operators, which would $w(0, x) = w_0(x)$ on Ω ,
 $\partial w(t, x)/\partial n = g(t, x, w(t, x), u_2(t, x))$ on $(0, T] \times T$

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In this paper we shall not admit state-constraints. The considerations is exceed the size of this paper (see for instance Trönderson [14]).
In our problem we have the following fixed quantities: Real constants $T > 0$,

 $u_i \leq \overline{u}_i$ ($i = 1, 2$), and a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary *I* such that Ω is $w_i(t, x) = (A\omega)(t, x) + h(t, x, w(t, x), u_i(t, x))$ in $(0, T] \times \Omega$,
 $\omega(0, x) = w_0(x)$ on Ω ,
 $\partial w(t, x)/\partial n = g(t, x, w(t, x), u_2(t, x))$ on $(0, T] \times T$

and to the constraints on the controls
 $u_i \leq u_i(t, x) \leq \overline{u}_i$, $i = 1, 2$.

In this paper we sh locally at one side of Γ and Γ is sufficiently smooth, say of type C^2 . By Δ the Laplace operator and by $\partial w/\partial n$ the conormal derivative is denoted. Moreover, real functions' $f_1, h: [0, T] \times \overline{\Omega} \times \mathbb{R} \times [u_1, \overline{u}_1] \to \mathbb{R}$ and $f_2, g: [0, T] \times \Gamma \times \mathbb{R} \times [u_2, \overline{u}_2] \to \mathbb{R}$ with. appropriate differentiability properties are given, which will be specified later. Φ is a real Fréchet-differentiable functional on $L_p(Q)$, where p is chosen according to (2.10) . The *controls* u_1 (distributed control) and u_2 (boundary control) belong to $L_\infty(0,\,T\,;\Omega)$ and $L_\infty(0,\,T\,;\,\varGamma),$ respectively (by $L_\infty(0,\,T\,;\,D)$ we shall denote the space of bounded and measurable functions on $[0, T] \times D$. The function *w* is said to be *a state corresponding to* $u = (u_1, u_2)$ *. It is defined in the sense of mild solutions to (1.2)* (see Section 2) and belongs to $C[0,T; W_p^{\sigma}(\Omega)]$, where $W_p^{\sigma}(\Omega)$ is the usual Sobolev space of functions on *Q* with derivatives in $L_p(Q)$ and $C[0, T; X]$ is the space of continuous abstract functions from $[0, T]$ to X. Once and for all we fix p and σ such that (2.10), $n/p < \sigma < 1 + 1/p$, holds. In order to ensure the continuity of $w(t, \cdot)$ the (fixed) initial value $w_0(x)$ is supposed to belong to $W_p^o(\Omega)$. ble functions on $[0, T] \times D$. The function $= (u_1, u_2)$. It is defined in the sense of mild ings to $C[0, T; W_p^o(\Omega)]$, where $W_p^o(\Omega)$ is the with derivatives in $L_p(\Omega)$ and $C[0, T; X]$ tions from $[0, T]$ to X . Once and for

The functions f_{i} h, g depending on (t, x, w, u) are supposed to fulfil the following Carathéodory type condition: For fixed (t, x) they are continuously partially differentiable with respect to w and u, and for fixed (w, u) they and their derivatives are measurable with respect to (t, x) . Moreover these functions and their derivatives are suppose to be bounded if (w, u) runs through a bounde to be bounded if (w, u) runs through a bounded subset of \mathbb{R}^2 .

Throughout the paper the following notation is used, where $D = Q$ or $D = \Gamma$:

 $(X:$ Banach space, X^* : its dual space). If in the norms the underlying domain *D* is missing, then we mean $D = \Omega$. $\mathcal{L}(X, Y)$ is the Banach space of linear and continuous • operators from X to Y endowed with the uniform operator topology, $\mathcal{L}(X) = \mathcal{L}(X, X)$.

2. The variation of constants formula

Following the lines of $[1, 3, 8, 16]$ and others we introduce in this section the concept of mild solutions to (1.2). We define a linear operator A in $X = L_n(\Omega)$ by

$$
D(A) = \left\{ w \in W_p^2(\Omega) \middle| \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma \right\}, \quad Aw = -2w + bw \text{ on } D(A),
$$

where $b \in \mathbb{R}$ is supposed to be positive such that the resolvent $R(\lambda, A)$ exists in particular for all real $\lambda \geq 0$. A is closed and densely defined, and $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of operators in $\mathcal{L}(X)$. This is known for Dirichlet boundary conditions (see PAZY [11]) and extends to our case of Neumann boundary conditions by the results of STEWART [13]. We have $d S(t) w/dt$ $A = -AS(t)$ w and $S(t)$ w $\in D(A)$ for all $w \in X$ and $t > 0$. Moreover, the choice of b yields the existence of fractional powers A^{α} for $0 \leq \alpha \leq 1$, and

$$
A \circ S(t) w = S(t) A \circ w, \qquad w \in D(A^{\alpha}), \qquad (2.1)
$$

$$
||A^{\alpha}S(t) w||_p \leq ct^{-\alpha} ||w||_p
$$
 (2.2)

 $(t > 0, \alpha \in [0, 1])$. If h is sufficiently smooth and $w_0 \in X$, then

$$
w(t) = S(t) w_0 + \int_{0}^{t} S(t-s) h(s) ds
$$
 (2.3)

is a strong solution to the Cauchy problem $w'(t) + Aw(t) = h(t)$, $w(0) = w_0$ (including the homogeneous boundary condition $\partial w/\partial n = 0$ in the domain of A). After a couple of formal manipulations, which are clear for sufficiently smooth data, the inhomogeneous, boundary condition $(\partial w/\partial n)(t) = g(t), g: [0,T] \to L_p(\Gamma)$, can be handled by the variation of constants formula

$$
w(t) = S(t) w_0 + \int_0^t S(t-s) h(s) ds + \int_0^t A S(t-s) N g(s) ds, \qquad (2.4)
$$

where $N: L_p(\Gamma) \to W_p^s(\Omega)$, $s < 1 + 1/p$, assigns to $g \in L_p(\Gamma)$ the solution w of $\Delta w - bw = 0$ on Ω , $\partial w/\partial n = g$ on Γ . (2.5)

We refer to the discussions by FATTORINI [3] OF AMANN [1]. The idea behind (2.4) is to write $w(t) = w_1(t) + w_2(t)$, where w_1 fulfils the homogeneous boundary condition, $w_2(t)$ solves (2.5) for $g = g(t)$, and to apply (2.3) to the resulting system for w_1 . It should be remarked that in terms of the Green function γ .

$$
\mathscr{E}(x, y, t) = \sum_{i=1}^{\infty} v_n(x) v_n(y) \exp(-c_n t),
$$

\n
$$
\Delta v_n + bv_n = c_n v_n, \, \partial v_n/\partial n = 0, \text{ the expression (2.4) coincides with}
$$

\n
$$
w(t, x) = \int_{0}^{t} \mathscr{E}(x, y, t) w_0(y) dy + \int_{0}^{t} \int_{0}^{s} \mathscr{E}(x, y, t - s) h(s, y) dy ds
$$

\n
$$
+ \int_{0}^{t} \int_{0}^{s} \mathscr{E}(x, y, t - s) g(s, y) dS_y ds
$$
\n(2.6)

 (dS) : surface element on T). This can be proved after an integration by parts in the last term of (2.4) by means of Green's formula. Equation (2.6) was taken for the $\frac{1}{2}$

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• definition of generalized . solutions in several earlier papers on optimal control of parabolic equations, for instance by FRIEDMAN [6], SACHS [12], TRÖLTZSCH [14], and V. WOLFERSDORF [17]. **434** *F. TRÖLTZSCH*
 definition of generalized solutions in

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v. WOLFERSDORF [17].

Now we define transformations
 $G: [0, T] \times W_p^{\sigma-1/p}(T) \times L_\infty(T) \to L_\infty$
 $(H(t, w(\cdot), u(\cdot)))(x) = h(t, x, y)$ several earlier papers on optimal

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³ $H:[0, T] \times W_p^{\sigma}(\Omega) \times L_{\infty}(\Omega) \to L_{\infty}$

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Now we define transformations $H : [0, T] \times W_{p^{\sigma}}(\Omega) \times L_{\infty}(\Omega) \to L_{\infty}(\Omega)$ and

$$
\left(H(t, w(\cdot), u(\cdot))\right)(x) = h(t, x, w(x), u(x)) + bw(x),
$$

$$
\left(G(t; w(\cdot), u(\cdot))\right)(x) = g(t, x, w(x), u(x))
$$

(note that $w_t = Aw + h$ iff $w_t = -Aw + H$). Then any solution $w \in C[0, T; W_{p^{\sigma}}(\Omega)]$

of
 $w(t) = S(t) w_0 + \int_0^t S(t - s) H(s, w(s), u_1(s)) ds$ of

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\ndefinition of generalized solutions in several earlier papers on optimal control of parabolic equations, for instance by FRIEDMax [18], Arörrzscr [14], and
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\therefore \text{Worexsslope [17]} \qquad \text{Now we define transformations } H: [0, T] \times W_p^*(\Omega) \times L_{\infty}(\Omega) \rightarrow L_{\infty}(\Omega) \text{ and}
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$$
G:[0, T] \times W_p^{-1/p}(I) \times L_{\infty}(I) \rightarrow L_{\infty}(I) \text{ by } \qquad \text{(a)} \qquad (H(t, w(\cdot), u(\cdot))) (x) = \delta(t, x, w(x), u(x)) + bw(x),
$$
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$$
(G(t, w(\cdot), u(\cdot))) (x) = \delta(t, x, w(x), u(x)) + bw(x),
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$$
w(t) = S(t) w_0 + \int_{0}^{t} S(t-s) H(s, w(s), u_1(s)) ds
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+ \int_{0}^{t} A S(t-s) M G(s, rw(s), u_2(s)) ds
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+ \int_{0}^{t} A S(t-s) M G(s, rw(s), u_2(s)) ds
$$
\n
$$
= \int_{0}^{t} \delta(t) w_0 |_{p,t} \leq d \cdot e^{-s/2} ||w||_p,
$$
\n
$$
= \int_{0}^{t} \delta(t) w_0 |_{p,t} \leq d \cdot e^{-(t+t-\alpha)2} ||g||_p (T).
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$$
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$$
= \int_{0}^{t} \delta(t) w_0 |_{p,t}
$$

 $AS(t)$ N at $t = 0$. It was already proven by AMANN [1] that is: along the lines of [15]. It is known that $||w||_{p,s} \le c||A^{s/2+1/2}||w||_p$
 $||A \otimes (U) w||_{p,s} \le c||A^{s/2+1/2}||w||_{p,s}$
 $||A \otimes (U) w||_{p,s} \le d - 1 + 1/p$.
 $||A \otimes (U) w||_{p,s} \le d - 1 + 1/p$.
 $||A \otimes (U) w||_{p,s} \le d - 1 + 1/p$.
 $||A \otimes (U) w||_{p,s} \le d - 1 +$ is the trace op
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 $s/2$). Consequen
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$$
\|S(t) w\|_{p,s} \leq ct^{-s/2} \|w\|_p,\tag{2.8}
$$

$$
||AS(t) Ng||_{p,s} \leq ct^{-(1+(s-\epsilon)/2)} ||g||_p (T)
$$
\n(2.9)

We shall briefly illustrate corresponding estimations by means of fractional powers of *^A* For $i > 0$ and $0 < s < \varepsilon < 1 + 1/p$.

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along the lines of [15]. It is known that $||w||_{p,s} \le c ||A^{s/2}w$
 $t > 0, \alpha \in [0, 1]$ briefly illustrate corresponding estimations by means of fractional powers of A
nes of [15]. It is known that $||w||_{p,s} \leq c ||A^{s/2}w||_p$ on $D(A^{s/2})$. Consequently, for
0, 1]
^{[a}S(t) $w||_{p,s} \leq \bar{c} ||A^{s/2+s}S(t) w||_p \leq ct^{-(s+s/$

$$
||A^sS(t) w||_{p,s} \leq c ||A^{s/2+s}S(t) w||_p \leq ct^{-(\alpha+s/2)} ||w||_p
$$

with a generic constant *c*, by (2.2). Thus (2.8) follows for $\alpha = 0$. For $s < 1 + 1/p$, $s \neq 1$, the equality, $W_p^{s}(\Omega) = (L_p(\Omega), D(A))_{s/2,p}$ holds. We refer to the remarks by AMANN [1]. Here ($t > 0$, $\alpha \in [0, 1]$
 $||A^s S(t) w||_{p,s} \le c ||A^{s/2+s} S(t) w||_p \le ct^{-(\alpha+s/2)} ||w||_p$

with a generic constant c, by (2.2). Thus (2.8) follows for $\alpha = 0$. For $s < 1 + 1/p$, $s \ne 1$, the

equality $W_p^s(\Omega) = (L_p(\Omega), D(A))_{s/2,p}$ holds. We' r *AS(t)* $Ng||_{p,s} \leq ct^{-(1+(s-\epsilon)/2)} ||g||_p (T)$
for $t > 0$ and $0 < s < \epsilon < 1 + 1/p$.
We shall briefly illustrate corresponding estimations by meating the lines of [15]. It is known that $||w||_{p,s} \leq c ||A^{s/2}w||_p$ or $t > 0$, $\alpha \in [0, 1]$ (2)

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with some effort the some point of the same set of the same eric constant c, by (2.2). Thus (2.8) follows for $\alpha = 0$. For $s < 1$
 $V_p^s(\Omega) = (L_p(\Omega), D(A))_{s/2,p}$ holds. We' refer to the remarks by P

ootes the real interpolation functor. Then it can be shown with $(L_p(\Gamma), L_p(\Omega)), 0 < \varepsilon < s$ along the lines of [1]
 $t > 0, \alpha \in [0, 1]$
 $||A \cdot S(t) \cdot w||_p$

with a generic constant

equality $W_p s(Q) = ($
 $(\cdot, \cdot)_t, p$ denotes the r
 $A^{t/2} N \in \mathscr{L}(L_p(\Gamma), L_p)$
 $||AS(t) \cdot Ng||$
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Finally we note t

out the paper $\begin{array}{c}\n\text{all } t \\
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 $\alpha \in [0, 1]$
 $\alpha \in [0, 1]$
 $\|A^s S(t) w\|_{p,s} \leq c \|A^{s/2+s} S(t) w\|_p \leq c \|A^{s/2+s} S(t) w\|_{p,s}$

a generic constant c, by (2.2). Thus (2.8) follows for
 $\|W_p^s(Q) = (L_p(\Omega), D(A))_{s$ for $t > 0$ and $0 < s < \varepsilon < 1 + 1/p$.

We shall briefly illustrate corresponding estimations

along the lines of [15]. It is known that $||w||_{p,s} \leq c||A$
 $t > 0, \alpha \in [0, 1]$
 $||A^s S(t) w||_{p,s} \leq c||A^{s/2+s} S(t) w||_p \leq c t^{-(\alpha+s/2)}||$

wit II briefly illustrate corresponding estimations by means of fractional powers of Alines of [15]. It is known that $||w||_{p,s} \le c ||A^{s/2}w||_p$ on $D(A^{s/2})$. Consequently, for $[0,1]$
 $||A^{s}S(t) w||_{p,s} \le c ||A^{s/2+s}S(t) w||_p \le dt^{-(s+g/$ $\begin{minipage}{0.9\linewidth} \begin{tabular}{l} \hline \texttt{MAN} & \texttt{[1]}. \end{tabular} \end{minipage} \vspace{0.2cm} \begin{tabular}{l} \hline \texttt{MOM} & \texttt{[1]}. \end{tabular} \end{minipage} \vspace{0.2cm} \begin{tabular}{l} \hline \texttt{[1]}. \end{tabular} \end{minipage} \vspace{0.2cm} \begin{tabular}{l} \hline \texttt{[1]}. \end{tabular} \end{minipage} \vspace{0.2cm} \begin{tabular}{l} \hline \texttt{[1]}. \end{tabular} \end$

$$
||AS(t) Ng||_{p,s} \leq c ||A^{s/2+1-\epsilon/2}S(t) A^{\epsilon/2} Ng||_{p} \leq ct^{(-1+(s-\epsilon)/2)} ||g||_{p} (T),
$$

 $0 < s < \varepsilon < 1 + 1/p$, by (2.1) and (2.2).

Finally we note that $W_p^s(Q) \hookrightarrow C(\overline{\Omega})$ for $s > n/p$. Therefore we fix p and o through-
out the paper such that $p > n - 1$ and S

$$
n/p < \sigma < 1 + 1/p. \tag{2.10}
$$

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Then we can take $\varepsilon \in (\sigma, 1 + 1/p)$ so that $AS(t)$ *N* is only "weakly singular" from $L_n(\Gamma)$ to $W_n^{\sigma}(\Omega)$. $S(t)$ is by (2.8) weakly singular for $p > 1$.

3. Abstract setting and linearization.

By means of the semigroup approach discussed in the precedin section we can formulate the control problem $(1.1-3)$ in an abstract form, which covers many. other types of applications, too. In our presentation we shall confine ourselves to the model problem $(1.1-3)$ as a background, but the reader will observe that 4.14 P. Transmost α , α as a constraint in a second carbier propers on equivalent in the problem (1.1, and we have the model problem (1.1, α but the model problem (1.1, α but the model problem (1.1, α but th the method also extends to other problems. For instance, more general elliptic $n/p < \sigma < 1 + 1/p$.

Then 'we can take $\varepsilon \in (\sigma, 1 + 1/p)$ so that $AS(t)$ N is only $L_p(\Gamma)$ to $W_p^s(Q)$. $S(t)$ is by (2.8) weakly singular for $p > 1$.

3. Abstract setting and linearization.

By means of the semigroup appr

-. operators can be substituted for $-\Delta$, and systems of parabolic equations, more general types of. boundary, conditions or other functionals. instead of (1.1) can be treated as well. Of course, some work still remains to be done then: namely to deter-On the Semigroup Approach ... 435

operators can be substituted for $-\Delta$, and systems of parabolic equations, more

general types of boundary conditions or other functionals instead of (1.1) can be

treated as well. Of co On the Semigroup Approach ...

operators can be substituted for $-\Delta$, and systems of parabolic equations, m

general types of boundary conditions or other functionals instead of (1.1) can

treated as well. Of course, some *J(w, u)* = $\Phi(w(T))$ *-4, and systems of parabolic equations, more* ypes of boundary conditions or other functionals instead of (1.1) can be swell. Of course, some work still remains to be done then: namely to detertion in **Solution**

Suppose the sum operators can be sum general types of bout

treated as well. Of comme and to interpret

According to our number
 $J(w, u) = \Phi$

subject to
 $w(t) = S(t) v$

s well. Of course, some work still remains to be done then: namely to deter-
to interpret certain adjoint operators and systems.
ing to our notation the control problem (1.1-3) admits the form to mini-

$$
J(w, u) = \Phi(w(T)) + \int_0^T \{F^I(s, w(s), u_1(s)) + F^2(s, w(s), u_2(s))\} ds
$$
(3.1)
or

$$
w(t) = S(t) w_0 + \int S(t-s) H(s, w(s), u_1(s)) ds
$$

operators can be substituted for
$$
-A
$$
, and systems of parabolic equations, more
general types of boundary conditions or other functionals instead of (1.1) can be
treated as well. Of course, some work still remains to be done then: namely to deter-
mine and to interpretec ertain adjoint operators and systems.
According to our notation the control problem (1.1-3) admits the form to mini-
mize

$$
J(w, u) = \Phi(w(T)) + \int_{0}^{T} \{F^{t}(s, w(s), u_{1}(s)) + F^{2}(s, w(s), u_{2}(s))\} ds
$$
(3.1)
subject to

$$
w(t) = S(t) w_{0} + \int_{0}^{t} S(t-s) H(s, w(s), u_{1}(s)) ds + \int_{0}^{t} AS(t-s) M G(s, \tau w(s), u_{2}(s)) ds + \int_{0}^{t} AS(t-s) M G(s, \tau w(s), u_{2}(s)) ds,
$$
(3.2)

$$
u_{i} \in U_{1}^{sd}, t \in [0, T], \text{ where } U_{i}^{ad} \text{ are the convex and closed sets of } U_{1} = L_{\infty}(0, T; \Omega)
$$

and $U_{2} = L_{\infty}(0, T; \Gamma)$, respectively, defined by (1.3), and the state w is from

$$
W = C[0, T; W_{p}(2)].
$$
 The functionals F^{1} and F^{2} are defined by

$$
F^{1}(t, w, u) = \int_{0}^{t} f_{1}(t, x, w(x), u(x)) dx, \qquad (w \in W_{p}(Q), u \in L_{\infty}(\Omega)),
$$

$$
F^{2}(t, w, u) = \int_{0}^{t} f_{2}(t, x, w(x), u(x)) ds, \qquad (w \in W_{p}^{*}e^{-1/p}(\Gamma), u \in L_{\infty}(\Gamma)).
$$

In all that follows let $(w^{0}, u_{1}^{0}, u_{2}^{0})$ be a locally optimal triple for (3.1)–(3.3). This means $J(w^{0}, u_{1}^{0}, u_{2}^{0}) \leq J(w, u_{1}, u_{2})$ for all (w, u_{1}, u_{2}) satisfying (3.1–3) and being contained in an open ball around $(w^{0}, u_{1}^{0}, u_{2}^{0})$ in $W \times U_{1} \times U_{2}$. Later we shall

 $u_i \in U_i^{\text{ad}}, t \in [0, T]$, where U_i^{ad} are the convex and closed sets of $U_1 = L_{\infty}(0, T; \Omega)$ and $U_2 = L_{\infty}(0, T; \Gamma)$, respectively, defined by (1.3), and the state w is from $W = C[0, T; W_{p}(\Omega)]$. The functionals F^1 and F^2 are defined by

$$
u_i \in U_i^{\text{ad}}, t \in [0, T], \text{ where } U_i^{\text{ad}} \text{ are the convex and closed sets of } U_1 = L_{\infty}(0, T; \Omega) \n\text{ and } U_2 = L_{\infty}(0, T; \Gamma), \text{ respectively, defined by (1.3), and the state } w \text{ is from}^2 \nW = C[0, T; W_p^o(\Omega)]. \text{ The functionals } F^1 \text{ and } F^2 \text{ are defined by} \nF^1(t, w, u) = \int_{\Omega} f_1(t, x, w(x), u(x)) dx \qquad (w \in W_p^o(\Omega), u \in L_{\infty}(\Omega)),
$$
\n
$$
\sum_{r=0}^{\infty} F^2(t, w, u) = \int_{\Gamma} f_2(t, x, w(x), u(x)) dS_x \qquad (w \in W_p^o - 1/p(\Gamma), u \in L_{\infty}(T)).
$$
\nIn all that follows let (w^0, u_1^0, u_2^0) be a locally optimal triple for (3.1)–(3.3). This means $J(w^0, u_1^0, u_2^0) \leq J(w, u_1, u_2)$ for all (w, u_1, u_2) satisfying (3.1–3) and being

In all that follows let $(w^0, u_1^{\prime 0}, u_2^{\prime 0})$ be a *locally opti* $\bigcap_{\Gamma} F^2(t, w, u) = \int_{\Gamma} f_2(t, x, w(x), u(x)) dS_x \qquad (w \in W_p^{\sigma-1/p}(\Gamma), u \in L_\infty(\Gamma)).$
In all that follows let (w^0, u_1^0, u_2^0) be a *locally optimal triple* for $(3.1) - (3.3)$. This means $J(w^0, u_1^0, u_2^0) \leq J(w, u_1, u_2)$ for all $(w, u_$ $F^1(t, w, u) = \int f_1(t, x, w(x), u(x)) dx$ $(w \in W_p^{\sigma}(\Omega), u \in L_{\infty}(\Omega)),$
 $\int F^2(t, w, u) = \int f_2(t, x, w(x), u(x)) dS_x$ $(w \in W_p^{\sigma-1/p}(\Gamma), u \in L_{\infty}(\Gamma)).$

In all that follows let (w^0, u_1^0, u_2^0) be a locally optimal triple for $(3.1) - (3.3)$. This means various partial Fréchet-derivatives of F^t , H , and G at the fixed triple $(w⁰, u₁⁰, u₂⁰)$, which will be indicated by appropriate subscripts. For instance, the partial derivatives of F^1 at the fixed element $(w, u) \in W_p(\Omega) \times L_\infty(\Omega)$ with respect to w and *u* are: denoted by $F_w^1(t, w, u)$ and $F_u^1(t, w, u)$ (*t* fixed). These derivatives exist due to the. Carathéodory type assumptions (this follows from KRASNOSELSKII a.o. [7] after embedding $W_p^{\sigma}(\Omega)$ into $L_{\infty}(\Omega)$). Inserting $w = w^0(t)$, $u = u_1^0(t)$ in these derivatives $F^1(t, w, u) = \int f_1(t, u, u)$
 \bullet $F^2(t, w, u) = \int f_2(t, u, u)$

In all that follows let $(w^0, u_1^0$

means $J(w^0, u_1^0, u_2^0) \leq J(w, u)$

contained in an open ball arc

various partial Fréchet-deriva

which will be indicated by any
 $W = \frac{1}{2}$ For (u^0, u_1^0, u_2^0) , the partial derives spect to w and u and v and by $F_w^1(t, w, u)$ and $F_u^1(t, w, u)$ (t fixed). These derivatives
dory type assumptions (this follows from KRASNOSELSK
ig $W_p^{\sigma}(\Omega)$ into $L_{\infty}(\Omega)$). Inserting $w = w^0(t), u = u_1^0(t)$ in t
for short
 $F_w^1(t) = F_w^1(t, w^0(t), u_1^0(t)),$

$$
F_w^{-1}(t) = F_w^{-1}(t, w^0(t), u_1^0(t)), \qquad F_u^{-1}(t) = F_u^{-1}(t, w^0(t), u_1^0(t))
$$

Analogously $F_u^2(t)$, $F_u^2(t)$, $H_w(t)$, $H_u(t)$, $G_w(t)$, and $G_u(t)$ are defined. As a conclusion from the Caratheodory conditions we can regard these quantities as abstract functi- $\min\limits_{\mathbf{p}}\min\{0,T\}$ with values $\inf L_{\infty}(\Omega), L_{\infty}(\Omega), L_{\infty}(T), L_{\infty}(T),$ $\mathcal{F}\big(L_p(\Omega)\big), \mathcal{F}\big(L_p(\Omega)\big), \mathcal{F}\big(L_p(T)\big),$ $\mathcal{L}(L_p(T))$, respectively, which are bounded and measurable with respect to t. For *example, the mapping* $H_w(t)$ is defined by $(H_w(t) w(\cdot)) (x) = h_w(t, x, w^0(t, x), u_1^0(t, x))$ $\times w(x)$, and h_w is bounded and measurable with respect to t and x. Hence $H_w(t)$ $\in \mathcal{L}(L_{\mathfrak{a}}(\Omega))$ for all $1 \leq \alpha \leq \infty$ (t fixed), and the mapping $t \mapsto H_{\mathfrak{w}}(t)$ is bounded and measurable. In the same way $G_w(\cdot) \in L_\infty(\mathcal{L}(L_\alpha(\Gamma)))$ is obtained. The derivative of Φ **V V** at $w^{0}(T)$ is written $\Phi'(w^{0}(T)) = \nabla \Phi$. Note that in general $\nabla \Phi \in L_q(\Omega)$, $q = p/(p - 1)$. **Pefore stating the next result, which is basic for all that follows, we introduce a more general notation, which will be frequently used in the next sections. We define** $\times w(x)$, and h_w is bounded and measurable with respect to t and x. Hence $H_w(t)$
 $\in \mathcal{L}(L_{\alpha}(\Omega))$ for all $1 \leq \alpha \leq \infty$ (t fixed), and the mapping $t \mapsto H_w(t)$ is bounded and

measurable. In the same way $G_w(\cdot) \in L_{\infty}$ for $1 < r < \infty$ operators $A_r: L_r(\Omega) \supset D(A_r) \to L_r(\Omega)$ by be that in the set $\Phi'(w^0(T)) = \nabla \Phi$. Note that in general $\nabla \Phi \in L_q(\Omega)$, $q = p/(p$ stating the next result, which is basic for all that follows, we introdered in the next result, which will be frequently used in the next $L_{\infty}(\Omega), L_{\infty}(\Omega), L_{\infty}(\Gamma), L_{\infty}(\Gamma), \mathcal{L}\{L_p(\Omega)\}, \mathcal{L}\{L_p(\Omega)\}$
 i and measurable with respect to the and x ;
 1. ∞ , *(t* fixed), and the mapping $t \mapsto H_w(t)$ is
 ∞ , *(t* fixed), and the mapping $t \mapsto H_w(t)$ is

$$
D(A_r)=\bigg\{w\in W_r^2(\Omega)\bigg|\frac{\partial w}{\partial n}=0\bigg\},\qquad A_r w=-\Delta w+b w,\quad w\in D(A_r).
$$

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 $\frac{1}{2}$

 $\int_{-\infty}^{\infty}$ These operators are linear, closed and densely defined in $L_r(\Omega)$ and generate analytic semigroups in $L_r(Q)$; which we denote by $\{S_r(t)\}_{t\geq 0}$. Moreover, $N_r: L_r(I) \to W_r^{1+1/r}(Q)$ is defined according to (2.5) for $g \in L_r(\Gamma)$. Note that we have $A = A_p$, $N = N_p$, $S(t) = S_n(t)$. **436 • F. TRÖLTZSCH**
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 are given in L_t(Ω *)***; which we denote by** $\{S_r(t)\}_t$ **

is defined according to (2.5) for** $g \in L_r(\Gamma)$ **. N
** $S(t) = S_p(t)$ **.

Lemma 1:** Assume that operato

Lemma 1: *Assume that operator-valued abstract functions* $H \in L_{\infty}(0, T; \mathcal{L}(L_r(\Omega)))$ *,* $G \in L_{\infty}(0, T; \mathcal{L}(L_r(\Gamma))),$ and an abstract function $c: [0, T] \to W_r^{\sigma}(\Omega), 1/r < \sigma < 1 + 1/r$, are given. Assume further that (i) $S = S_p(t)$.

Lemma 1: Assume that operator-valued abstract functions
 $\in L_\infty(0, T; \mathcal{L}(L_r(T)))$, and an abstract function $c : [0, T] \to W$
 $\in g$ iven. Assume further that

(i) $c \in L_r(0, T; W_r(\Omega))$ or (ii) $c \in C[0, T; W_r(\Omega)]$.

Wen

Then the abstract integral qua'tion

$$
x(t) = c(t) + \int_{0}^{t} S_r(t-s) H(s) x(s) ds + \int_{0}^{t} A_r S_r(t-s) N_r G(s) \tau x(s) ds \quad (3.4)
$$

has a unique solution in $L_r(0, T; W_r(\Omega))$, which is continuous on $[0, T]$ in the case (ii).

'Proof: We formally define the operator *L* to be the integral operator standing on the right-hand side of (3.4), i.e.

$$
(Lx(\cdot)) (t) = \int_{0}^{t} k(t,s) x(s) ds,
$$

where $k(t, s)$ $x = S_r(t - s)H(s)$ $x + A_rS_r(t - s)N_rG(s)$ *cx* is linear and continuous from $W_r^{\sigma}(\Omega)$ to W_r^{σ} for $t > s$ and $\sigma > 1/r$. At $t = s$ this operator has a "weak singularity", as (2.8), (2.9) imply $||k(t, s)|| \leq c(t - s)^{-\lambda}$, where $\lambda = \max (c/2, 1 + (c - \varepsilon)/2)$
 $\in (0, 1)$ (cf. (2.10)). We compare L with an operator L acting in spaces of real funcon the right-hand side of (3.4), i.e.
 $(Lx(\cdot)) (t) = \int_{0}^{t} k(t, s) x(s) ds$,

where $k(t, s) x = S_r(t - s) H(s) x + A_r S_r(t - s) N_r G(s) rx$ is linear and continuous

from $W_r^s(\Omega)$ to W_r^s for $t > s$ and $\sigma > 1/r$. At $t = s$ this operator has a "weak tions defined by (Lz) $(t) = \int c(t-s)^{-\lambda} z(s) ds$. It is known (cf. KRASNOSELSKII a.o. (17) that \tilde{L} is continuous in each space $L_r(0, T)$, $1 \leq r \leq \infty$, and that $\tilde{L}: L_r(0, T)$
 $\to C(0, T]$ for $r > 1/(1 - \lambda)$. In particular, $\tilde{L}: L_\infty(0, T) \to C[0, T]$. Therefore it can where $k(t, s)$ $x = S_r(t - s) H(s)$
from $W_r^q(\Omega)$ to W_r^s for $t > s$
larity", as $(2.8), (2.9)$ imply $||k|| \in (0, 1)$ (cf. (2.10)). We compo
tions defined by $(\tilde{L}z)(t) = \int_0^t$
[7]) that \tilde{L} is continuous in e
 $\rightarrow C(0, T]$ fo $(Lx(\cdot)) (t) = \int_{0}^{t-\epsilon} k(t,s) x(s) ds,$

where $k(t,s) x = S_{\tau}(t-s) H(s) x + A_{\tau}S_{\tau}(t-s) N_{\tau}G(s) \tau x$ is linear and co

from $W_{\tau}^{*}(2)$ to W_{τ}^{*} for $t > s$ and $\sigma > 1/\tau$. At $t = s$ this operator has a 'we

larity'', as (2.8) , (2.9) [7]) that \tilde{L} is continuous in each space $L_r(0, T)$, $1 \le r \le \infty$, and that $\tilde{L}: L_r(0, T)$
 $\to C(0, T]$ for $r > 1/(1 - \lambda)$. In particular, $\tilde{L}: L_\infty(0, T) \to C[0, T]$. Therefore it can

be shown that $\{w_t(t)\}_t$,
 $w_t(t) = \begin$ *f* $f_r(t-s) H(s) x + A_r S_r(t-s) N_r G(s) rx$ is linear and continue
 f, ℓ for $t > s$ and $\sigma > 1/r$. At $t = s$ this operator has a "weak sing

9) imply $||k(t, s)|| \le c(t - s)^{-\lambda}$, where $\lambda = \max (\sigma/2, 1 + (\sigma - \varepsilon))$.

We compare *L* with an operato

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if $0 \le t < \varepsilon$,
if $\varepsilon \le t \le T$,

 $\begin{aligned}\n\mathcal{L}(0, T) \text{ for } r > 1/(1 - \lambda). \text{ In particular, } \hat{L}: L_{\infty}(0, T) \to C[0, T]. \text{ Therefore, it can be shown that } \{w_{\epsilon}(t)\}, \\
\mathcal{L}(0, T) \text{ for } r > 1/(1 - \lambda). \text{ In particular, } \hat{L}: L_{\infty}(0, T) \to C[0, T]. \text{ Therefore, it can be shown that } \{w_{\epsilon}(t)\}, \\
\mathcal{L}(t) &= \begin{cases}\n0 & \text{if } 0 \leq t < \varepsilon, \\
\int_0^t k(t, s) x(s)$ $\varepsilon \to +0$. In this way the continuity of *L* in *L*, or C , respectively, is shown¹. Furthermore it is easy to show by induction that $||L^n|| \le ||L^n||$, $n \in \mathbb{N}$. L^n is known to be a contraction in $L_r(0, T)$ for $n \in \mathbb{N}$ sufficiently large (cf. KRASNOSELSKI a.o. [7]). Hence $Lⁿ$ is in this case contractive, too. Now the statement of the lemma follows from the Banach fixed point theorem \blacksquare is a Cauchy sequence in $L_r(0, T; W_r(Q))$ (case (i)) or $C[0, T; W_r(Q)]$ (case (ii)) f
 $\epsilon \to +0$. In this way the continuity of L in L_r or C , respectively, is shown to be

more it is easy to show by induction that $|L_r|| \leq |$

• For convenience we introduce the non-linear operator $K = K(w, u_1, u_2)$ which assigns to $(w, u_1, u_2) \in C[0, T; W] \times U_1 \times U_2$ the right-hand side of (3.2). *K* is continuous from $C[0, T; W] \times U_1 \times U_2$ to $C[0, T; W]$. The continuity of $S(t) w_0$

I). It should be remarked that more general results can be proved using methods from singular integral theory, we refer to F_{ATTOBINI} [5] and L_{ASIECKA} [9]. **-** e remarked that more
eory, we refer to Farr
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follows from $w_0 \in W_n^{\delta}(\Omega)$, as

$$
||S(t) w_0 - S(t') w_0||_{p,\sigma} \leq c ||A^{\sigma/2}(S(t) - S(t')) w_0||_p
$$

$$
\leq c ||(S(t) - S(t')) A^{\sigma/2}w_0||_p
$$

and $S(t)$ is strongly continuous in $L_p(\Omega)$. Moreover, $H(s, w(s), u_1(s)), G(s, w(s), u_2(s))$ belong to $L_{\infty}(0, T; L_{\infty}(\Omega))$ and $L_{\infty}(0, T; L_{\infty}(T))$, respectively, and depend continuously on (w, u_1, u_2) (Carathéodory type conditions). In the proof of the preceding lemma the continuity of L was shown. Altogether this implies that K is continuous. It can further be proved that J and K are continuously differentiable on $C[0, T; W] \times U_1 \times U_2$. This follows from the considerations on differentiability above (where $t \in [0, T]$ was fixed) along the lines of [14, Thm. 2.2.2]. In accordance with our previous notation we write K_w , K_{u_1} , K_{u_1} , J_w , J_u , J_u , for the corresponding F -derivatives at the locally optimal triple (w^0, u_1^0, u_2^0) . The operator K_w admits the form

$$
K_w w (t) = \int_{0}^{t} S(t-s) H_w(s) w(s) ds + \int_{0}^{t} AS(t-s) N G_w(s) \tau w(s) ds
$$

It is clear from the proof of Lemma 1 that K_w is continuous in $C[0, T; W]$, and Lemma 1 applied for $r = p$, $H(t) = H_w(t)$, $G(t) = G_w(t)$ directly implies the following

Corollary:
$$
(I - K_w)^{-1}
$$
 exists in $\mathcal{L}(C[0, T; W])$.

This result can also be derived after having endowed $C[0, T, W]$ with the equivalent norm $|w(\cdot)|_{\beta} = \max \{ \exp(-\beta t) ||w(t)||_{p,\sigma} | t \in [0,T] \}.$ Then K_w is contractive for $\beta > 0$ sufficiently large.

Theorem 1: The locally optimal triple (w^0, u_1^0, u_2^0) satisfies the variational inequality

$$
\nabla \Phi_{,w}(T) - w^{0}(T) \big) + \int_{0}^{T} \{ \big(F_{w}^{-1}(s), w(s) - w^{0}(s) \big) + \big(F_{w}^{2}(s), w(s) - w^{0}(s) \big) \big| \} ds
$$

$$
+ \int_{0}^{1} \left\{ \left(F_{u}^{1}(s), u_{1}(s) - u_{1}^{0}(s) \right) + \left(F_{u}^{2}(s), u_{2}(s) - u_{2}^{0}(s) \right) \right\} ds \geq 0 \tag{3.4}
$$

for all $(w, u_1, u_2) \in C[0, T; W] \times U_1^{\text{ad}} \times U_2^{\text{ad}}$ which solve the linearized equation

$$
\ddot{w}(t) - w^{0}(t) = \int_{0}^{t} S(t-s) \left[H_{w}(s) (w(s) - w^{0}(s)) + H_{u}(s) (u_{1}(s) - u_{1}^{0}(s)) \right] ds
$$

+
$$
\int_{0}^{t} AS(t-s) N \left[G_{w}(s) (rw(s) - rw^{0}(s)) + G_{u}(s) (u_{2}(s) - u_{2}^{0}(s)) \right] ds.
$$
 (3.5)

Proof: Linearization results of the form (3.4-5) hold true if a certain regularity condition is fulfilled. In our case this is the assumption that $(I - K_w)$ is surjective. Then (w^0, u_1^0, u_2^0) satisfies

$$
\langle J_w, w - w^0 \rangle + \langle J_{u_1}, u_1 - u_1^0 \rangle + \langle J_{u_1}, u_2 - u_2^0 \rangle \geq 0 \tag{3.6}
$$

for all (w, u_1, u_2) with $u_i \in U_i^{\text{ad}}$ and

$$
v - w0 = K_w(w - w0) + K_{u_1}(u_1 - u_10) + K_{u_2}(u_2 - u_20), \qquad (3.7)
$$

cf. TRÖLTZSCH [14, Thms 1.2.2 and 1.3.1]. This is obviously equivalent to $(3.4-5)$ 29 Analysis Bd. 8, Heft 5 (1989)

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ing problem in a Banach space 438 F. TRÖLTZSCH

Remark: It follows from $(3.6-7)$ that (w^0, u)

ing problem in a Banach space
 $\langle J_w, w \rangle + \langle J_{u_1}, u_1 \rangle + \langle J_{u_1}, u_2 \rangle = \min \{$

subject to (3.7) and $u_i \in U_i^{\text{ad}}$. Problems of this

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(*x*) \cdot **k**: It follows from (3.6-7) that (w^0, u)

(*j_w*, w) + $\langle J_u, u_1 \rangle + \langle J_{u_1}, u_2 \rangle = \min !$

(3.7) and $u_i \in U_i$ ^{ad}. Problems of this

of feasible directions in order to find a Remark: It follows from $(3.6-7)$ that (w^0, u_1^0, u_2^0) is the solution of the linearing problem in a Banach space
 $\langle J_w, w \rangle + \langle J_{u_1}, u_1 \rangle + \langle J_{u_1}, u_2 \rangle = \min!$

subject to (3.7) and $u_i \in U_i^{ad}$. Problems of this type are

$$
\langle J_w, w \rangle + \langle J_{u_1}, u_1 \rangle + \langle J_{u_2}, u_2 \rangle = \min!
$$

subject to (3.7) and $u_i \in U_i^{ad}$. Problems of this type are of particular interest for numerical methods of feasible directions in order to find a new direction of descent.

4. **Adjoint operators**

I . - For the necessary optimality conditions we need some adjoint operators, which will

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Remark: It follows from (3.6-7) that (w^0, u_1^0, u_2^0) is the solution

ing problem in a Banach space
 $\langle J_w, w \rangle + \langle J_u, u_1 \rangle + \langle J_u, u_2 \rangle = \min!$

subject to (3.7) and $u_i \in U_i^{ad}$. Problems of this type are o Lemma 2: The adjoint operator A^* to A is given by $A^* = A_o$ $(q = p/(p - 1)),$ and $S(t)^* = S_q(t)$ holds true.

Proof: It is known that $-A^*$ is the generator of a C_0 -semigroup in $L_q(\Omega)$ and that $\exp(-A^*t) = S(t)^*$, as L_q is reflexive, see PAZY [11]. Therefore it remains to show $A^* = A_q$. We define $D' = \{y \in W_q^2(\Omega) \mid \partial y/\partial n = 0\}$. For $y \in D', w \in D(A)$, *I.emma 2: The adjoint operator* A^* *to* A *is given by* $A^* = A_q$ $(q =$
 Proof: It is known that $-A^*$ is the generator of a C_0 -semigroup in

that $\exp(-A^*t) = S(t)^*,$ as L_q is reflexive, see PAZY [11]. Therefore it **•** *Proof:* **It is known that** $-A^*$ **is the generator of a** C_0 **-semigroup in 1
that** $\exp(-A^*t) = S(t)^*$ **, as** L_0 **is reflexive, see PAzy [11]. Therefore it re
show** $A^* = A_q$ **. We define** $D' = \{y \in W_q^2(2) \mid \partial y/\partial n = 0\}$ **. For y **

$$
(y, Aw) = -\left(y, \frac{\partial w}{\partial n}\right)_{\Gamma} + \left(\frac{\partial y}{\partial n}, w\right)_{\Gamma} - (Ay, w) + b(y, w) = (A_{q}y, w)
$$

by Green's formula (see MIXHAILOV [10]). Hence $D' \subset D(A^*)$. The opposite inclusion can now be proved completely analogous to the proof of Lemma 3.4 in Pazy [11, $p. 213$] $-$ • at $\exp(-A^*t) = S(t)$

ow $A^* = A_q$. We defin
 $(y, Aw) = -\left(\frac{1}{2}\right)$
 y Green's formula (see

n now be proved com

213] \blacksquare

Lemma 3: Assume

ken $(A_r/y, N_r x) = (\tau y)$

Proof: We put $w = (A_r y, N_r x) = ($
 $= (\tau y \in D(A_r))$ and the

ow
$$
A^* = A_q
$$
. We define $D' = \{y \in W_q^2(\Omega) \mid \text{oy}(on = 0)\}$. For $y \in D$, $w \in D(A)$,
\n
$$
(y, Aw) = -\left(y, \frac{\partial w}{\partial n}\right)_\Gamma + \left(\frac{\partial y}{\partial n}, w\right)_\Gamma - (Ay, w) + b(y, w) = (A_q y, w)
$$
\nGreen's formula (see MIKHALLOV [10]). Hence $D' \subseteq D(A^*)$. The opposite inclusion
\nn now be proved completely analogous to the proof of Lemma 3.4 in Pazz [11,
\n213] \blacksquare
\n**Lemma 3:** Assume $1 < r < \infty$, $x \in L_r(\Gamma)$ and $y \in D(A_r)$, where $r' = r/(r - 1)$.
\nthen $(A_r/y, N_r x) = (ry, x)f$.
\nProof: We put $w = N_r x$, where $x \in W_r^{1-1/r}(\Gamma)$. Then $w \in W_r^2(\Omega)$ and
\n
$$
(A_r y, N_r x) = (-\Delta y + by, w)
$$
\n
$$
= (y, -\Delta w + bw) - \left(\frac{\partial y}{\partial n}, w\right)_\Gamma + \left(y, \frac{\partial w}{\partial n}\right)_\Gamma = (y, x)_\Gamma
$$
\n
$$
y \in D(A_r)
$$
 and the definition of w. The statement follows from the density of
\n $r^{1-1/r}$ in L_r

by $y \in D(A_r)$ and the definition of *w*. The statement follows from the density of

by Green's formula (see MIKHAILOV [10]). Hence $D' \subseteq D(A^*)$. The opposite inclusion
can now be proved completely analogous to the proof of Lemma 3.4 in PAZY [11,
p. 213] **I**
Lemma 3: $Assume 1 < r < \infty$, $x \in L_r(\Gamma)$ and $y \in D(A_r)$ Now it is easy to derive the form of several adjoint operators. We know for $t > 0$ that $\tau AS(t) N \in \mathcal{L}(L_p(\Gamma))$, $AS(t) N \in \mathcal{L}(L_p(\Gamma), L_p(\Omega))$, $\tau S(t) \in \mathcal{L}(L_p(\Omega), L_p(\Gamma))$. In what follows, we shall regard these operators in these $\psi = (y, -2iw + bw) - \left(\frac{1}{\partial n}, w\right)_f + \left(y, \frac{1}{\partial n}\right)_f = (y, x)_f$

by $y \in D(A_r)$ and the definition of w. The statement follows from the density of
 $W_r^{1-1/r}$ in L_r $(\tau AS(t) N)^* \in \mathcal{L}(L_q(\Gamma)), (AS(t) N)^* \in \mathcal{L}(L_q(\Omega), L_q(\Gamma)),$ and $(\tau S(\overline{t}))^* \in \mathcal{L}(L_q(\Gamma), L_q(\Omega))$.

In this sense we can prove

Lemma 4: *For* $t > 0$,

(i) $(\tau AS(t) N)^* = \tau A_q S_q(t) N_q$,

(ii) $(A S(t) N)^* = -S(t)$ *- -* **In** this sense we can prove by $y \in D(A_r)$ and
 $W_r^{1-1/r}$ in L_r **ii**

Now it is easy t

that $rAS(t)$ $N \in \mathcal{L}$

what follows, we
 $(rAS(t) N)^* \in \mathcal{L}$ [*L*

In this sense we c

Lemma 4: Fo

(i) $(rAS(t) N)^*$

(ii) $(AS(t) N)^*$

(iii) $(AS(t) N)^*$

(iii) $(rS(\mathring$ what follows, we shall regard these operators in these L_p -spaces. Thus
 $(\tau AS(t) N)^* \in \mathcal{L}(L_q(T)), (AS(t) N)^* \in \mathcal{L}(L_q(\Omega), L_q(T)),$ and $(\tau S(t))^* \in \mathcal{L}(L_q(\Omega))$.

In this sense we can prove

Lemma 4: For $t > 0$,

(i) $(\tau AS(t) N)^* = \tau A_q S_q(t)$

Lemma 4: For
$$
t > 0
$$
,
\n(i) $(\tau AS(t) N)^* = \tau A_q S_q(t) N_q$,
\n(ii) $(AS(t) N)^* = \tau S_q(t)$,
\n(iii) $(\tau S(\tilde{t}))^* = A_q S_q(t) N_q$ $(q = p/(p-1))$.

Proof: To show (i) we take $y \in L_q(\Gamma)$, $x \in L_p(\Gamma)$ fixed and find for $t > 0$

$$
(y,\tau AS(t) Nx)_\Gamma = -\left(y,\tau \frac{d}{dt} S(t)Nx\right)_\Gamma = -\frac{d}{dt} (y,\tau S(t) Nx)_\Gamma
$$

(the operator τ and the pairing are continuous)

On the Semigr
\n
$$
= -\frac{d}{dt} (N_q y, AS(t) Nx) = \frac{d^2}{dt^2} (N_q y, S(t) Nx)
$$
\nby Lemma 3 with $r = q$)\n
$$
d^2
$$

On the Semigroup A
\n
$$
= -\frac{d}{dt} \left(N_q y, AS(t) Nx \right) = \frac{d^2}{dt^2} \left(N_q y, S(t) Nx \right)
$$
\n(by Lemma 3 with $r = q$)
\n
$$
= \frac{d^2}{dt^2} \left(S(t)^* N_q y, Nx \right) = -\frac{d}{dt} \left(A^* S(t)^* N_q y, Nx \right)
$$
\n
$$
= -\frac{d}{dt} \left(A_q S_q(t) N_q y, Nx \right) = -\frac{d}{dt} \left(r S_q(t) N_q y, x \right),
$$
\n(by Lemma 2, 3)
\n
$$
= \left(r A_q S_q(t) N_q y, x \right),
$$
\nhence (i) is shown. Similarly, for $y \in L_q(\Omega)$, $x \in L_p(\Gamma)$

$$
(y, AS(t) Nx) = -\frac{d}{dt} (y, S(t) Nx) = -\frac{d}{dt} (S(t)^* y, Nx)
$$

= $(A^*S(t)^* y, Nx) = (A_qS_q(t) y, Nx) = (rS(t) y, x)_r,$

$$
(y, AS(t) Nx) = -\frac{d}{dt} (y, S(t) Nx) = -\frac{d}{dt} (S(t)^* y, Nx)
$$

\n
$$
= (A^*S(t)^* y, Nx) = (A_qS_q(t) y, Nx) = (rS(t) y, x)_r,
$$

\ni.e. (ii). Finally, for $y \in L_q(\Gamma)$ and $x \in L_p(\Omega)$ by Lemma 3
\n
$$
(y, rS(t) x)_r = (N_qy, AS(t) x) = -\frac{d}{dt} (N_qy, S(t) x)
$$

\n
$$
= -\frac{d}{dt} (S(t)^* N_qy, x) = (A_qS_q(t) N_qy, x)
$$

5. Necessary optimality condition \leq minimum principle

After introducing a suitable adjoint state the linearization Theorem 1 can be expressed by a minimum principle We define the *adjoint state* y as the solution of the equation

$$
dt \t\t(1) \t\t(2) \t
$$

where $q = p/(p - 1)$. In this equation we regard $H_w(t)$ and $G_w(t)$ as operators in $L_p(Q)$ and $\hat{L}_p(\Gamma)$, respectively. Hence their adjoints are operators in $L_q(Q)$ and $L_q(\Gamma)$.
Actually, we have even $H_w^* \in L_\infty(0, T; L_\alpha(Q))$ and $G_w^* \in L_\infty(0, T; L_\alpha(Q))$ for all $\begin{aligned}\ng(e) &= \omega_q(1-t)/v + \int \omega_q(s-t) \, F_w(s) \, ds + \int A_q S_q(s-t) \, N_q F_w(s) \, ds \\
&\quad + \int_1 S_q(s-t) \, H_w(s)^* \, y(s) \, ds + \int A_q S_q(s-t) \, N_q G_w(s)^* \, \tau y(s) \, ds, \\
\text{where } q &= p/(p-1). \text{ In this equation we regard } H_w(t) \text{ and } G_w(t) \text{ as operators in } L_q(\Omega) \text{ and } L_p(\Gamma), \\
\text{Re}(\Omega) \text{ and } L_p(\Gamma), \text{ respectively. Hence their adjoints are operators in } L_q(\Omega) \text{ and$ $1 \leq \alpha \leq \infty$, as H_w and G_w are formally self-adjoint. Now it follows from Lemma 1 after the change of variables $t' = T - \tilde{t}$, which transforms the "backward" equation (5.1) into a "forward" one, that (5.1) has a unique solution in $L_q(0, T; W_{q}(Q))$, provided that $1/q < \sigma' < 2/q$. $=p/(p-1)$. In this equation we regard $H_w(t)$ and $G_w(t)$ as operators in $L_p(\Gamma)$, respectively. Hence their adjoints are operators in $L_q(\Omega)$ and $L_q(\Gamma)$, we have even $H_w^* \in L_\infty(0, T; L_a(\Omega))$ and $G_w^* \in L_\infty(0, T; L_a(\Omega))$ for a

Theorem 2: The locally optimal triple (w^0, u_1^0, u_2^0) must satisfy

-:

$$
\int_{0}^{1} \left\{ \left(H_{u}(t)^{*} y(t) + F_{u}^{1}(s), u_{1}(t) - u_{1}^{0}(t) \right) + \left(G_{u}(t)^{*} y(t) + F_{u}^{2}(s), u_{2}(t) - u_{2}^{0}(t) \right) r \right\} dt \geq 0 \ \forall u_{i} \in U_{i}^{ad} \qquad (i = 1, 2).
$$
\n
$$
29^{*} \qquad (5.2)
$$

(5.2)

I)

F. Tronduce for short $z = w - w^0$, $v_i = u_i - u_i^0$ and start from the variational inequality (3.4). The term containing *w* will be transformed. The equation Proof: We introduce for short $z = w - w^0$, $v_i = u_i - u_i^0$ and start from the 440 F. TRÖLTZSCH

Proof: We introduce for s

variational inequality (3.4). The

(3.5) reads now
 $z(t) = \int S(t-s) [H_m(t)]$

F. TRötrzisch
\n: We introduce for short
$$
z = w - w^0
$$
, $v_i = u_i - u_i^0$ and start from the
\nal inequality (3.4). The term containing w will be transformed. The equation
\n
$$
z(t) = \int_{0}^{t} S(t-s) [H_w(s) z(s) + H_u(s) v_1(s)] ds
$$
\n
$$
+ \int_{0}^{t} AS(t-s) N[G_w(s) z(s) + G_u(s) v_2(s)] ds.
$$
\n
$$
\varphi_z(t) = \int_{0}^{t} (S(t-s) H_w(s) z(s) + AS(t-s) N[G_w(s) z(s)] ds.
$$
\n(5.3)\n
$$
\varphi_u(t) = \int_{0}^{t} (S(t-s) H_u(s) v_1(s) + AS(t-s) N G_u(s) v_2(s)) ds.
$$
\n(5.4)

Setting

$$
\varphi_{z}(t) = \int_{0}^{t} \left(S(t-s) H_{w}(s) z(s) + AS(t-s) NG_{w}(s) z(s) \right) ds,
$$

$$
\varphi_{u}(t) = \int_{0}^{t} \left(S(t-s) H_{u}(s) v_{1}(s) + AS(t-s) NG_{u}(s) v_{2}(s) \right) ds
$$

we find for all *z* satisfying (5.3) Setting
 $\frac{1}{\sqrt{1-\frac{1}{2}}}$

we find for the set of the

$$
I = (V\Phi, z(T)) + \int_{0}^{T} (F_w^{-1}(s), z(s)) ds + \int_{0}^{T} (F_w^{-2}(s), \tau z(s)) r ds
$$

$$
= (V\Phi, \varphi_z(T)) + \int_{0}^{T} (F_w^{-1}(s), z(s)) ds + \int_{0}^{T} (F_w^{-2}(s), \tau z(s)) r ds
$$

$$
= (V\Phi, \varphi_z(T)) + \varphi_u(T)) + \int_{0}^{T} (F_w^{-1}(t), \varphi_z(t)) + \varphi_u(t)) dt
$$

$$
= (\nabla \Phi, \varphi_z(T) + \varphi_u(T)) + \int_{0}^{T} (F_w^{-1}(t), \varphi_z(t) + \varphi_u(t)) dt
$$

\n
$$
\times \left(+ \int_{0}^{T} (F_w^{-2}(t), \varphi_z(t) + \varphi_u(t)) \cdot dt \right)
$$

\nand
\n
$$
= \int_{0}^{T} (H_w(t)^* \psi(t), z(t)) dt + \int_{0}^{T} (G_w(t)^* \cdot \tau \psi(t), \tau z(t)) \cdot dt
$$

\n
$$
\psi(t) = S_q(T - t) \cdot \nabla \Phi + \int_{t}^{T} S_q(s - t) \cdot F_w^{-1}(s) ds + \int_{t}^{T} A_q S
$$

by (5.3) , and

5,

$$
I = \int_{0}^{T} \left(F_w^2(t), \varphi_z(t) + \varphi_u(t)\right) \varphi_t dt
$$

and

$$
I = \int_{0}^{T} \left(H_w(t)^* \psi(t), z(t)\right) dt + \int_{0}^{T} \left(G_w(t)^* \varphi(v), \tau z(t)\right) \varphi_t dt + R,
$$

$$
\psi(t) = S_q(T-t) \nabla \Phi + \int_{t}^{T} S_q(s-t) F_w^{-1}(s) ds + \int_{t}^{T} A_q S_q(s-t)
$$

$$
R = \left(\nabla \Phi, \varphi_u(T)\right) + \int_{0}^{T} \left(F_w^{-1}(t), \varphi_u(t)\right) dt + \int_{0}^{T} \left(F_w^{-2}(t), \tau \varphi_u(t)\right) \varphi_t dt
$$

joining the last expression by Lemma 4. According to the d

where

$$
I = \int_{0}^{T} \left(H_{\omega}(t)^{*} \psi(t), z(t) \right) dt + \int_{0}^{T} \left(G_{\omega}(t)^{*} \tau \psi(t), \tau z(t) \right) r dt + R,
$$

re

$$
\psi(t) = S_{q}(T-t) \nabla \Phi + \int_{t}^{T} S_{q}(s-t) F_{\omega}^{1}(s) ds + \int_{t}^{T} A_{q} S_{q}(s-t) N_{q} F_{\omega}^{2}(s) ds
$$

and

by (5.3), and
\n
$$
I = \int_{0}^{T} (F_w^{2}(t), \varphi_z(t) + \varphi_u(t)) \, dt + \int_{0}^{T} (G_w(t)^* \tau \psi(t), \tau z(t)) \, dt + R,
$$
\nwhere
\n
$$
\psi(t) = S_q(T - t) \, \nabla \Phi + \int_{t}^{T} S_q(s - t) \, F_w^{1}(s) \, ds + \int_{t}^{T} A_q S_q(s - t) \, N_q F_w^{2}(s) \, ds
$$
\nand
\n
$$
R = (\nabla \Phi, \varphi_u(T)) + \int_{0}^{T} (F_w^{1}(t), \varphi_u(t)) \, dt + \int_{0}^{T} (F_w^{2}(t), \tau \varphi_u(t)) \, dt + \int_{0}^{T} (F_w^{2}(t), \tau \varphi_u(t)) \, dt \qquad (5.5)
$$
\nafter adjoining the last expression by Lemma 4. According to the definition of \dot{y} we have
\n
$$
\psi(t) = \dot{y}(t) - \int S_q(s - t) \, H_w(s)^* \, y(s) \, ds - \int_{0}^{T} A_q S_q(s - t) \, N_q G_w(s)^* \, \tau y(s) \, ds.
$$

have
have
have

$$
\psi(t) = y(t) - \int\limits_t^T S_q(s-t) \, H_w(s)^* \, y(s) \, ds - \int\limits_t^t A_q S_q(s-t) \, N_q G_w(s)^* \, \tau y(s) \, ds.
$$

Hence, inserting this term into (5.5) and "adjoining back" we continue

$$
\psi(t) = y(t) - \int_{t}^{T} S_q(s-t) H_w(s)^* y(s) ds - \int_{t}^{T} A_q S_q(s-t) N_q G_w(s)^* \tau y(s) ds.
$$

nce, inserting this term into (5.5) and "adjoining back" we continue

$$
I = \int_{0}^{T} (H_w(t)^* y(t), z(t) - \varphi_1(t)) dt + \int_{0}^{T} (G_w(t)^* \tau y(t), \tau z(t) - \tau \varphi_2(t)) \tau dt + R
$$

$$
= \int_{0}^{T} (H_w(t)^* y(t), \varphi_u(t)) dt + \int_{0}^{T} (\tilde{G}_w(t)^* \tau y(t), \tau \varphi_u(t)) \tau dt + R
$$

by 5.3) From (5.4) and Lemma 4, **/** *^I*= J (H(t)* *Sq(S -* t) *H(s)* y(s) ds [±]f AqSq (S - 1) NqGw(s)* Ty(s) ds} (t)) dl ⁺f (Gt* t{ }, *Vi(t))F di + R ! (H(t)* y(l), vj(t)) di+ f (G(** ry(t), *vi(l))r dt (H^u t* [S^q (T -) V. + f S(s t) F(s) ds* **0** *⁺f ^AqS^q (S 1) Nq^F ² (s) ds]* vl(t)) *di* **^I ' S ::•S** *f (G(t)* t[*I, oⁱ(1))rdt + R* **^I** *=f (H(i)t y(t), vi(t)) di + f* (G(t)* Ty(t) *vi(t))r dl, (5.6)*

as a simple calculation yields the equivalence of R with the minus part in the expression above. Now (5.2) follows immediately from $(3.4-5)$

After returning to the original quantities introduced in $(1.1-3)$ the minimum
inciple (5.2) admits the form
 $\int_{0}^{T} \int_{0}^{T} (h_u^0(t, x) y(t, x) + f_{1_u}^0(t, x)) (u_1(t, x) - u_1^0(t, x)) dx dt$ principle (6.2) admits the form

is a simple calculation yields the equivalence of R with the minus part in the
\nsion above. Now (5.2) follows immediately from
$$
(3.4-5)
$$
 ••
\nAfter returning to the original quantities introduced in $(1.1-3)$ the mir-
\nprimeible (5.2) admits the form
\n
$$
\int_{0}^{T} \int_{0}^{R} (h_u^{0}(t, x) y(t, x) + f_{1_u}^{0}(t, x)) (u_1(t, x) - u_1^{0}(t, x)) dx dt
$$
\n
$$
\int_{0}^{T} \int_{0}^{R} (g_u^{0}(t, x) y(t, x) + f_{2_u}^{0}(t, x)) (u_2(t, x) - u_2^{0}(t, x)) dS_x dt \ge 0.
$$
\nfor all $u_i \in U_i^{nd}$ $(i = 1, 2)$, where $h_u^{0}(t, x) = h_u(t, x, w^{0}(t, x), u_1^{0}(t, x))$ and g_u^{0} ,
\ndefined analogously. Finally, this amounts to pointwise minimum principle
\nknown arguments. For instance, min $\{[h_u^{0}(t, x) y(t, x) + f_{1_u}^{0}(t, x)] u | u \in [u_1,$
\nattained almost everywhere on $[0, T] \times \Omega$ by $u_1^{0}(t, x)$.
\nThe optimality conditions in the paper are obtained by means of lineari-
\nthey are so-called *local minimum principles*. An entirely different approach
\ndiscussed by FATTORINT [4]. He derived a sequence maximum principle by me
\nthe Ekeland variational principle.
\nWe shall finish the paper with an interpretation of y as the solution of an a
\npartial differential equation. It is quite clear from (5.1) that y should, in an
\npriate sense, solve the adjoint system

for all $u_i \in U_i^{ad}$ $(i = 1, 2)$, where $h_u^0(t, x) = h_u(t, x, w^0(t, x), u_1^0(t, x))$ and g_u^0, f_u^0 are defined' analogously. Finally, this amounts to. *poiniwise minimum principles* 'by known arguments. For instance; min $\{[h_u^0(t, x), y(t, x) + f^0_{1u}(t, x)]u \mid u \in [u_1, \overline{u}_1]\}$ is attained almost everywhere on $[0, T] \times \Omega$ by $u_1^0(t, x)$. $\begin{aligned}\n\delta \stackrel{\delta}{=} \delta \end{aligned}$
 $+ \int_0^T \int \left(g_u^0(t, x) y(t, x) + \int_{x_u}^2 (t, x) \right) \left(u_2(t, x) - u_2^0(t, x) \right) dS_x dt \geq 0$.

for all $u_i \in U_i^{ad} \ (i = 1, 2)$, where $h_u^0(t, x) = h_u(t, x, w^0(t, x), u_1^0(t, x))$ and g_u^0, f_u^0 are defined analogously. Fina

The optimality conditions in the paper are obtained by means of linearization,. they are so-called *local minimum* principles. An entirely different approach was discussed by FATTORTNI [4]. He derived a sequence maximum principle by means of Fracturity The optimal
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We shape partial diversity
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We shall finish the paper with an interpretation of y as the solution of an adjoint
rtial differential equation. It is quite clear from (5.1) that y should, in an appro-
iate sense, solve the adjoint system
 $-y'(t) = \Delta y(t) - by(t$ partial differential equation. It is quite clear from (5.1) that y should, in an appro-

partial differential equation. It is quite clear from (3.1) that
\nprivate sense, solve the adjoint system
\n
$$
-y'(t) = \Delta y(t) - by(t) + H_w(t)^* y(t) + F_w^1(t),
$$
\n
$$
y(T) = \nabla \Phi,
$$
\n
$$
\partial y/\partial n = G_w(t)^* y(t) + F_w^2(t),
$$

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\nwhich admits with the original quantities the form
\n
$$
-y_i(t, x) = \Delta y(t, x) + h_w(0, x) y(t, x) + f_{1_w}^0(t, x) \text{ in } \Omega,
$$
\n
$$
y(T, x) = (\Phi'(w^0(T, \cdot))) (x) \text{ in } \Omega,
$$
\n
$$
\partial y/\partial n(t, x) = g_w(0, x) y(t, x) + f_{2_w}^0(t, x) \text{ on } \Gamma,
$$
\n
$$
0 \le t < T. \text{ We shall not thoroughly discuss the question in which sense } y \text{ solves (5.7)}.
$$
\nIn our important particular case, however, *u* is seen to be a mild solution of (5.7).

In our important particular case, however, y is seen to be a mild solution of (5.7) .

Theorem :3: Suppose that $\nabla \Phi \in W_q^{\sigma'}(\Omega)$. Then y is a mild solution of (5.7) in the use that $v, v(t) = y(T - t)$, is a mild solution of $v_t(t, x) = Av(t, x) + h_w^0(T - t, x) v(t, x) + f_{1w}^0(T - t, x)$, $v(0, x) = (\Phi'(w^0(T - t)))$ (x) *sense that v, v(t)* = $y(T - t)$, *is a mild solution of*

$$
v_t(t, x) = \Delta v(t, x) + h_w^0(T - t, x) v(t, x) + f_{1_w}^0(T - t, x),
$$

\n
$$
v(0, x) = (\Phi'(w^0(T, \cdot))) (x),
$$

\n
$$
\frac{\partial v}{\partial n}(t, x) = g_w^0(T - t, x) v(t, x) + f_{2_w}^0(T - t, x).
$$
\n(5.8)

 $\texttt{Proof}\colon A$ mild solution v of (5.8) is defined as continuous solution of

$$
\frac{\partial v}{\partial n}(t, x) = g_w^{0}(T - t, x) v(t, x) + f_{2_w}^{0}(T - t, x).
$$

:. A mild solution v of (5.8) is defined as continuous

$$
v(t) = c(t) + \int_{0}^{t} (S_q(t - s) H_w(T - s)^* v(s) + A_q S_q(t - s) N_q G_w(T - s)^* v(s)) ds,
$$

where

$$
c(\tilde{t}) = \int_{0}^{t} \left(S_q(t-s) F_w^{-1}(T-s) + A_q S_q(t-s) N_q F_w^{-2}(T-s) \right) ds
$$

+
$$
S_q(t) \nabla \Phi
$$

 $F_w^1(t)$, $F_w^2(t)$ are bounded and measurable with values in $L_q(Q)$ and $L_q(\Gamma)$, respectively, and $S_q(T-t) \nabla \Phi$ is continuous according to the assumption of the theorem. Hence $c(\cdot) \in C[0, T; W_{q}(Q)]$. Moreover, $H_w(t)^*$ and $G_w(t)^*$ are bounded and measurable with respect to *t*. Now Lemma 1/(ii), applied for $r = q$ and $\sigma := \sigma'$ yields the existence of $v(\cdot) \in C[0, T; W_q^{\sigma'}(Q)]$. It is easy to see that $y(t) = v(T - t)$ solves (5.7) in the mild sense (substitute $t' = T$ \overline{a} \overline{b} \overline{c} \overline{d} \overline{c} \overline{d} $\overline{$ **I**

Remark: The assumption $\nabla \Phi \in W_q^{\sigma'}(\Omega)$ is satisfied in the following example: We take $p > \max (n - 1, 2)$, σ according to (2.10) (this is possible due to $n - 1 < p$), $1/q < \sigma' <$ $1 + 1/q$ and assume $\sigma' \leq \sigma$ (take σ close to $1 + 1/p$ and σ' close to $1/q = 1 - 1/p$). The function-Remark: The assumption $\nabla \Phi \in W_q^{\sigma'}(\Omega)$ is satisfied in the following example: We take
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 $1 + 1/q$ and assume $\sigma' \leq \sigma$ (take σ close to $1 + 1/p$ and σ' close to $1/q = 1 - 1/p$). The fu
al Φ is defined by $\Phi(w(\cdot)) = \int_{\Omega} (w(x) - z(x$

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