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Multipoint Problem for an Extraordinary Differential Equation[1])

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Es wird ein gewisses Mehrpunktproblem für eine Differentialgleichung von nichtganzzahliger Ordnung betrachtet. Das Problem ist äquivalent zu einer nichtlinearen Integralgleichung, und auf diese Weise wird die Existenz seiner globalen Lösung bewiesen.

Рассматривается некоторая многоточечная задача для дифференциального уравнения нецелого порядка. Эта задача равносильна одному нелинейному интегральному уравнению и таким образом доказывается существование её глобального решения.

A certain multipoint problem for an extraordinary differential equation is considered. This problem is equivalent to a certain nonlinear integral equation and in this way the existence of its global solution is shown.

1. Derivative of an arbitrary order. Let $I = (0, A)$ be a finite interval of $\mathbb{R}$ and let Γ denote the Gamma function. For convenience we recall the definition and some properties of the Riemann-Liouville derivative of an arbitrary order (with respect to 0) (cf. [3: pp. 567–575]). For any integrable function $y: I \to \mathbb{R}$ we define the derivative of order $\lambda \in \mathbb{R}$

$$y^{(\lambda)}(x) = \begin{cases} \int\limits_0^x (x-t)^{-\lambda-1}\, y(t)\, dt/\Gamma(-\lambda) & \text{for } \lambda < 0, \\ y(x) & \text{for } \lambda = 0 \end{cases}$$

and $y^{(\lambda)}(x) = \left(y^{(\lambda-l)}(x)\right)^{(l)}$ for $\lambda > 0$, where $l = -[-\lambda]$ and $[\lambda]$ is the largest integer not exceeding λ provided that integrals and derivatives on the right-hand side exist. In virtue of the Fubini theorem the derivative $y^{(\lambda)}$ for $\lambda < 0$ exists almost everywhere (a.e.) on I and is integrable. Moreover, if $\mu > 0$ and $y^{(\mu)}$ exists and is integrable (which immediately yields that the functions $y^{(\mu-k)}$ for $k = 1, 2, \ldots, m$; $m = -[-\mu]$ are absolutely continuous), then for every $\lambda > 0$ the formula

$$\left(y^{(\mu)}(x)\right)^{(-\lambda)} = y^{(\mu-\lambda)}(x) - \sum_{k=1}^{m} x^{\lambda-k} y^{(\mu-k)}(0)/\Gamma(1+\lambda-k)$$

is valid a.e. on I. On the other hand we have $\left(y^{(-\lambda)}(x)\right)^{(\mu)} = y^{(\mu-\lambda)}$ $(\lambda, \mu \geqq 0)$, assuming that for $\mu \geqq \lambda$ the right-hand side exists, hence the differentiation is not commutative. In further reasoning we also need the formula

$$\left(x^{\gamma}/\Gamma(1+\gamma)\right)^{(\lambda)} = x^{\gamma-\lambda}\Gamma(1+\gamma-\lambda) \quad (\gamma > -1, \lambda \in \mathbb{R}).$$

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2. The problem. Let $n \in \mathbb{N}$, $\alpha > 0$, $\beta_l < \alpha$ $(l = 1, 2, \ldots, n)$ be fixed numbers and let $f: I \times \mathbb{R}^n \to \mathbb{R}$ be a given function. Set $p = -[-\alpha]$ and denote by $\{y^{(\beta_l)}\}$ the n-element sequence of derivatives $y^{(\beta_l)}$. We shall consider the nonlinear extraordinary differential equation

$$y^{(\alpha)}(x) = f\big(x, \{y^{(\beta_l)}(x)\}\big) \quad (x \in I) \tag{1}$$

with the multipoint conditions

$$\sum_{i=1}^{j} \left[\sum_{k=1}^{m_{ij}} z_{ijk} y^{(\alpha-i)}(x_{ijk}) + \int_0^A b_{ij}(x)\, y^{(\alpha-i)}(x)\, dx \right] = \eta_j \tag{2}$$

$(j = 1, 2, \ldots, p)$, where $0 \leqq x_{ijk} \leqq A$, $m_{ij} \in \mathbb{N}$, $z_{ijk}, \eta_j \in \mathbb{R}$ are fixed numbers, $b_{ij}: I \to \mathbb{R}$ are given functions.

The multipoint problem of similar type for a linear ordinary differential equation was first formulated by J. D. Tamarkin (cf. [9: p. 113 and references]). It can be noticed that if $x_{ijk} = 0$ $(1 \leqq i \leqq j \leqq p;\ k = 1, 2, \ldots, m_{ij})$, and m_{ij} and z_{ijk} are such that $z_{ij1} + \cdots + z_{ijm_{ij}} = \delta_{ij}$ $(1 \leqq i \leqq j \leqq p)$, where δ_{ij} is the Kronecker symbol, then the multipoint problem (1), (2) becomes the Cauchy one (cf. [1, 2, 5, 7, 8, 10]).

Let us examine the Bellman type conditions

$$\int_0^A b_j(x)\, y^{(\alpha-p)}(x)\, dx = \mathring{\eta}_j \quad (j = 1, 2, \ldots, p), \tag{3}$$

where $\mathring{\eta}_j \in \mathbb{R}$ and b_j are given functions. If the b_j have integrable derivatives of order $p - j$ $(1 \leqq j \leqq p)$, respectively, then integrating by parts the integral appearing in (3), we have the conditions (2) with $z_{ijk} = 0$ and $b_{ij} = (-1)^{p-j}\, \delta_{ij} b_j^{(p-j)}$.

By a *solution of equation* (1) *in* I we mean any integrable function $y: I \to \mathbb{R}$ such that the derivative $y^{(\alpha)}$ exists and is integrable, the derivative of order $p - 1$ of $y^{(\alpha-p)}$ is absolutely continuous and equation (1) is fulfilled a.e. on I.

We assume the following:

(I) The function f fulfils the Carathéodory conditions (cf. [4: p. 158]) and

$$|f(x, \{z_l\})| \leqq \sum_{l=1}^{n} \sum_{i=1}^{n_l} K_{il}(x)\, |z_l|^{\varkappa_{il}} \quad (z_l \in \mathbb{R};\ \text{a.e. on } I), \tag{4}$$

where $n_l \in \mathbb{N}$, $0 \leqq \varkappa_{il} \leqq \varkappa < 1$ are fixed numbers and $K_{il}: I \to \mathbb{R}$ are given functions of class $L^{1/(1-\varkappa_{il})}$, respectively.

(II) The functions b_{ij} $(1 \leqq i \leqq j \leqq p)$ are integrable and

$$\zeta_j := \sum_{k=1}^{m_{jj}} z_{jjk} + \int_0^A b_{jj}(x)\, dx \neq 0 \quad (j = 1, 2, \ldots, p). \tag{5}$$

It can be proved by classical arguments (cf. [4: pp. 161–163]) that Assumption I yields

Lemma 1: *The substitution operator* $\mathfrak{f}(s_1, \ldots, s_n) := f\big(\cdot, \{s_l(\cdot)\}\big)$ *maps* $\big(L(I)\big)^n$ *into* $L(I)$, *is continuous and bounded.*

3. Solution of the problem. If y is a solution of (1), (2), then $s := y^{(\alpha)}$ is a solution of the nonlinear integral equation

$$s(x) = f\big(x, \{w^{(\beta_l)}(x) + s^{(\beta_l-\alpha)}(x)\}\big), \tag{6}$$

where

$$w(x) = \sum_{j=1}^{p} c_j x^{\alpha-j} / \Gamma(1 + \alpha - j) \tag{7}$$

and, by setting $\sum_{r=i}^{j} a_r = 0$ if $j < i$,

$$c_j = \zeta_j^{-1} \left\{ \eta_j - \sum_{i=1}^{j} \left[\sum_{k=1}^{m_{ij}} z_{ijk} s^{(-i)}(x_{ijk}) + \int_0^A b_{ij}(x)\, s^{(-i)}(x)\, dx \right] \right.$$
$$\left. - \sum_{r=1}^{j-1} c_r \sum_{i=r}^{j} \left[\sum_{k=1}^{m_{ij}} z_{ijk} x_{ijk}^{i-r} + \int_0^A x^{i-r} b_{ij}(x)\, dx \right] \Big/ (i-r)! \right\}. \tag{8}$$

And inversely, if s is a solution of equation (6), then

$$y(x) = w(x) + s^{(-\alpha)}(x) \tag{9}$$

is a solution of the multipoint problem (1), (2).

Let us examine the transformation T defined by the right-hand side of (6). This transformation is a superposition of operators N_l, $N_l s = w^{(\beta_l)} + s^{(\beta_l - \alpha)}$, and the substitution operator $\mathfrak{f}$. In further reasoning we assume:

(III) The functions $w^{(\beta_l)}$ $(l = 1, 2, \ldots, n)$ are integrable.

Remark 1: Assumption III is always fulfilled if either $\alpha - \beta_l - p > -1$ or $\beta_l = \alpha - 1$.

Define $B_\varrho = \{s \in L(I) : \|s\| \leq \varrho\}$, where $\|\cdot\|$ denotes the norm in $L(I)$. Now we can prove

Lemma 2: *Under Assumption* III, *the operators N_l are linear, bounded and compact from $L(I)$ into $L(I)$.*

Proof: Due to relation (8) and the Fubini theorem, we have (cf. also [3: p. 569]) $\|N_l s\| \leq \text{const}\,(1 + \|s\|)$, where here and in the sequel the constant is independent of s, hence the operators N_l are bounded from $L(I)$ into itself. Let $h > 0$ and denote by $s_h = \int_{x-h}^{x+h} s(t)\, dt/2h$ the Steklov average of s. By direct calculation one can prove that the inequality $\|(N_l s)_h - N_l s\| \leq \text{const}\, \|s\|\, h^{\beta_l'}$, where $\beta_l' = \min(\alpha - \beta_l, 1)$, holds and if $s \in B_\varrho$, for fixed ϱ, then $(N_l s)_h$ tends to $N_l s$ as $h \to 0$ uniformly with respect to s. By the well-known Kolmogorov theorem on compactness in $L(I)$, the set $\{N_l s : s \in B_\varrho\}$ is compact in $L(I)$ which ends the proof ∎

Using relation (4) we can observe that the inequality

$$|Ts(x)| \leq \sum_{l=1}^{n} \sum_{i=1}^{n_l} (p+1)\, K_{il}(x) \left(\sum_{j=1}^{p} |c_j x^{\alpha-\beta_l-j} / \Gamma(1+\alpha-\beta_l-j)|^{\varkappa_{il}} + |s^{(\beta_l-\alpha)}(x)|^{\varkappa_{il}} \right)$$

holds true a.e. on I. In virtue of formula (8), the method of mathematical induction and the Jensen inequality, respectively, we obtain $|c_j| \leq \text{const}\,(1 + \|s\|)$ and $\|K_{il} |s^{(\beta_l-\alpha)}|\| \leq \text{const}\, \|s\|^{\varkappa_{il}}$. Bearing in mind these relations, we have the estimate $\|Ts\| \leq \text{const}\,(1 + \|s\|^{\varkappa})$. Hence, for ϱ so large that $\text{const}\,(1 + \varrho^{\varkappa}) \leq \varrho$, the transformation T maps B_ϱ into B_ϱ. Moreover, it is completely continuous (cf. Lemmas 1 and 2). Applying the Schauder fixed point theorem to the transformation T and the set B_ϱ we notice that the set of solutions of equation (6) is non-empty and compact in $L(I)$. The aforegoing considerations imply

Theorem 1: *If Assumptions* I—III *are fulfilled, then the multipoint problem* (1), (2) *has a solution given by* (9), *where s is a solution of* (6).

Remark 2: The operator defined by the right-hand side of (9) is linear, bounded and compact (cf. Lemma 2 with $\beta_l = 0$) and hence the set of solutions of the considered multipoint problem is compact.

4. The case $0 < \alpha - 1 \leqq \beta_l < \alpha$. In the following part of the paper we examine separately the multipoint problem (1), (2) with $0 < \alpha - 1 \leqq \beta_l < \alpha$. In this case we have to find a function y subject to the multipoint conditions (2) with $j = 2, 3, \ldots, p$ and satisfying the equation

$$y^{(\alpha-1)}(x) = u(x) \qquad (x \in I); \tag{10}$$

where u is a solution of the equation $u'(x) = f(x, \{u^{(\gamma_l)}(x)\})$ $(\gamma_l = 1 + \beta_l - \alpha)$ fulfilling the multipoint condition $\sum_{k=1}^{m_{11}} z_{11k} u(x_{11k}) + \int_0^A b_{11}(x)\, u(x)\, dx = \eta_1$. Assuming the existence of a solution of this problem and integrating equation (10), we have

$$y(x) = \sum_{r=2}^{p} c_r x^{\alpha-r}/\Gamma(1+\alpha-r) + u^{(1-\alpha)}(x) \qquad (c_r \text{ constants}).$$

Imposing on the function y the multipoint conditions (2) with $j = 2, 3, \ldots, p$, we obtain for the unknown constants $c_2, c_3, \ldots, c_p$ the system of algebraic equations

$$\sum_{r=2}^{j} c_r \sum_{i=r}^{j} \left(\sum_{k=1}^{m_{ij}} z_{ijk} x_{ijk}^{i-r} + \int_0^A x^{i-r} b_{ij}(x)\, dx \right) \Big/ (i-r)!$$
$$= \eta_j - \sum_{i=1}^{j} \left(\sum_{k=1}^{m_{ij}} z_{ijk} u^{(1-i)}(x_{ijk}) + \int_0^A b_{ij}(x)\, u^{(1-i)}(x)\, dx \right) \qquad (j = 2, 3, \ldots, p)$$

with determinant $W = \zeta_2 \cdot \ldots \cdot \zeta_p$, by relation (5), different from 0. The above results, Theorem 1 and the theory of algebraic systems imply

Theorem 2: *If Assumptions* I *and* II *are satisfied, then the multipoint problem* (1), (2) *with* $\alpha - 1 \leqq \beta_l < \alpha$ *has a solution.*

5. The linear case. Let us consider the linear extraordinary differential equation

$$y^{(\alpha)}(x) = b(x) + \sum_{l=1}^{n} a_l(x)\, y^{(\beta_l)}(x), \tag{11}$$

with multipoint conditions (2). The right-hand side of (11) does not fulfil Assumption I. We solve the linear multipoint problem (11), (2) only for $0 < \alpha \leqq 1$. All considerations for the general case are analogous. We assume the following:

(IV) The functions b and a_l $(l = 1, 2, \ldots, n)$ are integrable.

The general solution of equation (11) with $0 < \alpha \leqq 1$ has the form (9) with $p = 1$, where c (we omit the index) is an arbitrary constant and the function s satisfies the linear Volterra integral equation $s(x) = b(x) + ca(x) + \int_0^x K(x, t)\, s(t)\, dt$, where

$$K(x, t) = \sum_{l=1}^{n} a_l(x)\, (x-t)^{\alpha-\beta_l-1}/\Gamma(\alpha-\beta_l) \qquad \text{and} \qquad a(x) = K(x, 0).$$

Denoting by $\mathfrak{R}$ the resolvent kernel of K, we have the formula

$$s(x) = b(x) + ca(x) + \int_0^x \mathfrak{R}(x, t)\, [b(t) + ca(t)]\, dt.$$

Hence

$$y(x) = \left(cx^{\alpha-1} + \int_0^x (x-t)^{\alpha-1} \left[b(t) + ca(t) + \int_0^t \mathfrak{R}(t, \tau)\, [b(\tau) + ca(\tau)]\, d\tau \right] \right) \Big/ \Gamma(\alpha).$$

Imposing on this function the multipoint conditions (2) with $p = 1$ one can observe that c satisfies the algebraic equation

$$c\left(\zeta_1 + \sum_{k=1}^{m_{11}} z_{11k} \int_0^{x_{11k}} \left[a(t) + \int_0^t \Re(t, \tau)\, a(\tau)\, d\tau\right] dt + \int_0^A dx \int_0^x b_{11}(x) \left[a(t) + \int_0^t \Re(t, \tau)\, a(\tau)\, d\tau\right] dt\right)$$
$$= \eta_1 - \sum_{k=1}^{m_{11}} z_{11k} \int_0^{x_{11k}} \left[b(t) + \int_0^t \Re(t, \tau)\, b(\tau)\, d\tau\right] dt - \int_0^A dx \int_0^x b_{11}(x) \left[b(t) + \int_0^t \Re(t, \tau)\, b(\tau)\, d\tau\right] dt. \qquad (13)$$

Bearing in mind the above considerations we can establish

Theorem 3: *If Assumptions* II (*without* (5)) *and* IV *are satisfied and if* (13) *has a solution, then there exists at least one solution of the multipoint problem* (11), (2) *with* $0 < \alpha \leqq 1$. *Moreover, the solution is unique if* (13) *has a unique solution as well.*

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