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Removable Singularities of the Functional Wave Equation

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Dedicated to Prof. Dr. S. G. Mikhlin on the occasion of his 80th birthday

Hebbare Singularitäten für Lösungen der Funktional-Wellengleichung, die einer Verschärfung der Lipschitz-Bedingung genügen, werden unter Verwendung des eindimensionalen Lebesgueschen Maßes charakterisiert.

Устранимые особенности решений функционального волнового уравнения удовлетворяющих усиленному условию Липшица характеризуются при помощи одномерной меры Лебега.

Removable singularities for solutions of the functional wave equation satisfying a strengthened Lipschitz condition are characterized with help of one-dimensional Lebesgue measure.

Let $G \subset \mathbb{R}^2$ be an open set. (This assumption will be kept throughout the paper.) We shall denote by $\mathcal{W}(G)$ the set of functions $u: G \to \mathbb{R}^2$ for which for every point $z_0 = (x_0, y_0) \in G$ there is a square

$$Q(z_0, r) = [x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r] \tag{1}$$

of side length $2r > 0$ (depending on z_0) such that $Q(z_0, r) \subset G$ and the equation

$$u(x + h, y + h) + u(x, y) = u(x + h, y) + u(x, y + h) \tag{2}$$

holds whenever $(x, y), (x + h, y + h) \in Q(z_0, r)$.

The functional equation (2) and its modifications arise in connection with the wave equation. Indeed, a continuous function u belongs to $\mathcal{W}(G)$ if and only if it satisfies in G the equation $\partial^2 u/\partial x\, \partial y = 0$ in the sense of distribution theory (cf. [1–3]); note that, introducing the variables $\xi = x + y$, $\eta = x - y$ one may transform $\partial^2 u/\partial x\, \partial y = 0$ into the usual form of the wave equation $\partial^2 v/\partial \xi^2 - \partial^2 v/\partial \eta^2 = 0$ satisfied by $v(\xi, \eta) = u(1/2(\xi + \eta), 1/2(\xi - \eta))$.

We shall be engaged with investigation of removable singularities of solutions of the equation (2). For this purpose we adopt the following terminology. For a set $M \subset \mathbb{R}^2$, $u|_M$ denotes the restriction of u to M. Let $\mathcal{K}(G)$ be a set of functions defined in G. A set $F \subset G$, which is relatively closed in G, will be termed *removable* for $\mathcal{K}(G)$, if, for any $u \in \mathcal{K}(G)$ with $u|_{G \setminus F} \in \mathcal{W}(G \setminus F)$, $u \in \mathcal{W}(G)$. As a sample result we quote a theorem of M. Dont [2] who proved, using a result of B.-W. Schulze [4], that a relatively closed set $F \subset G$ is removable for the set $\mathcal{C}(G)$ of all continuous functions in G if and only if F can be covered by countably many straight lines parallel to coordinate axes.

We shall deal with a subspace $\text{V-Lip}_{\text{loc}}(G) \subset \mathcal{C}(G)$ to be defined below, for which removable singularities can be characterized with help of linear measure. For this, for $M \subset \mathbb{R}$ we denote by $|M|$ the outer Lebesque measure of M. If $I \subset \mathbb{R}$ is an interval then total variation of a function $f: I \to \mathbb{R}$ is defined as usual and will be denoted by $\text{var}(f; I)$. Let now $K = [a, b] \times [c, d]$ be a rectangle. Then V-Lip (K)

will stand for the set of all functions $u: K \to \mathbb{R}$ for which there exists a constant $\varkappa \geqq 0$ (depending on u) such that, for any couple of points $x_1, x_2 \in [a, b]$, the function

$$u(x_1, \cdot) - u(x_2, \cdot): y \mapsto u(x_1, y) - u(x_2, y)$$

has bounded variation on $[c, d]$ satisfying the estimate

$$\operatorname{var}\big(u(x_1, \cdot) - u(x_2, \cdot);\ [c, d]\big) \leqq \varkappa\, |x_1 - x_2| \tag{3}$$

and, for any couple of points $y_1, y_2 \in [c, d]$, the function

$$u(\cdot, y_1) - u(\cdot, y_2): x \mapsto u(x, y_1) - u(x, y_2)$$

has bounded variation on $[a, b]$ satisfying the estimate

$$\operatorname{var}\big(u(\cdot, y_1) - u(\cdot, y_2); [a, b]\big) \leqq \varkappa\, |y_1 - y_2|. \tag{4}$$

Finally, let V-Lip$_{\text{loc}}$ (G) be the set of all functions $u: G \to \mathbb{R}$ such that, for any closed rectangle $K \subset G$, $u|_K \in$ V-Lip (K).

Denoting by $\pi_1, \pi_2: \mathbb{R}^2 \to \mathbb{R}$ the projections $\pi_1: (x, y) \mapsto x$, $\pi_2: (x, y) \mapsto y$, we are now in position to formulate our main result.

Theorem: *Let $F \subset G$ be relatively closed in G. Then F is removable for* V-Lip$_{\text{loc}}$ (G) *if and only if there are sets $F_1, F_2 \subset G$ such that*

$$F = F_1 \cup F_2 \tag{5}$$

and

$$|\pi_j(F_j)| = 0 \qquad (j = 1, 2). \tag{6}$$

For the proof of this theorem we shall need a series of auxiliary results.

Lemma 1: *Suppose that $F \subset \mathbb{R}^2$ cannot be decomposed into subsets F_1, F_2 satisfying* (5), (6). *Then there is an $\varepsilon > 0$ such that, for each couple of sets $L_1, L_2 \subset \mathbb{R}^2$ fulfilling $F \subset L_1 \cup L_2$, the estimate $|\pi_1(L_1)| + |\pi_2(L_2)| \geqq \varepsilon$ is valid.*

Proof is easy.

Lemma 2: *Suppose that $F \subset \mathbb{R}^2$ is a compact set which cannot be decomposed into subsets F_1, F_2 satisfying* (5), (6). *Then there exists a finite non-trivial Borel measure μ with support contained in F such that the function*

$$V\mu: (x, y) \mapsto \mu\big((-\infty, x) \times (-\infty, y)\big) \tag{7}$$

satisfies the Lipschitz condition on $\mathbb{R}^2$.

Proof: We shall assume, for the sake of simplicity, that $F \subset [0, 1] \times [0, 1]$. Let $\mathbb{N}$ be the set of all positive integers. For each $n \in \mathbb{N}$ we subdivide $[0, 1]$ into the intervals $I_i{}^n = [(i-1)/2^n, i/2^n]$ and put $K_{ij}^n = I_i{}^n \times J_j{}^n$ $(1 \leqq i, j \leqq 2^n)$, $\mathcal{M}^n = \{K_{ij}^n: F \cap K_{ij}^n \neq \emptyset\}$. We are going to construct a non-trivial Borel measure μ_n with support in $F^n = \bigcup \mathcal{M}^n$ such that $V\mu_n$ satisfies the Lipschitz condition with coefficient $\tilde{\varkappa} \leqq 1$ on $\mathbb{R}^2$ and $\mu_n(\mathbb{R}^2) \leqq 1$.

Since $\mathcal{M}^n \neq \emptyset$, we can choose a $K_{i_1 j_1}^n \in \mathcal{M}^n$ and define first the Borel measure ν_1 whose density with respect to the Lebesgue measure in $\mathbb{R}^2$ vanishes on $\mathbb{R}^2 \setminus K_{i_1 j_1}^n$ and equals 2^n on $K_{i_1 j_1}^n$. Consequently, we have $\nu_1(K_{i_1 j_1}^n) = 1/2^n$, $\nu_1(K_{ij}^n) = 0$ whenever $K_{ij}^n \neq K_{i_1 j_1}^n$. Put $\mathcal{M}_1{}^n = \{K_{ij}^n: i = i_1 \text{ or } j = j_1\}$. Suppose now that, for given $k \in \mathbb{N}$, we have already constructed the system $\mathcal{M}_k{}^n \subset \{K_{ij}^n\}$ and the Borel measure ν_k whose density with respect to the Lebesgue measure in $\mathbb{R}^2$ vanishes outside F^n

and remains constant almost everywhere on each square in $\mathcal{M}^n$ in such a way that $\nu_k(\mathbb{R}^2) > 0$ and the following properties are satisfied:

(i) For each K_{ij}^n, $\nu_k(K_{ij}^n) \in \{0, 1/2^n\}$.

(ii) If $\nu_k(K_{i_0j_0}^n) \neq 0$ for some (i_0, j_0), then $K_{i_0j_0}^n \in \mathcal{M}^n$, $K_{i_0j}^n \in \mathcal{M}_k{}^n$ for $1 \leqq j \leqq 2^n$ and $\nu_k(K_{i_0j}^n) = 0$ whenever $j \neq j_0$, and also $K_{ij_0}^n \in \mathcal{M}_k{}^n$ for $1 \leqq i \leqq 2^n$ and $\nu_k(K_{ij_0}^n) = 0$ whenever $i \neq i_0$. Note that these properties guarantee that the function $V\nu_k$ (defined by (7) with μ replaced by ν_k) satisfies the Lipschitz condition with coefficient $\varkappa \leqq 1$. (For $k = 1$ such $\mathcal{M}_k{}^n$, ν_k have, indeed, been constructed above.) If $\mathcal{M}_k{}^n \supset \mathcal{M}^n$, then the process stops and we put $\mu_n = \nu_k$. If $\mathcal{M}^n \setminus \mathcal{M}_k{}^n \neq \emptyset$, then we choose a $K_{i_{k+1}j_{k+1}}^n \in \mathcal{M}^n \setminus \mathcal{M}_k{}^n$ and proceed to define the Borel measure ν_{k+1}, whose density with respect to the Lebesgue measure in $\mathbb{R}^2$ vanishes on $\mathbb{R}^2 \setminus F^n$ and remains constant almost everywhere on each square in $\mathcal{M}^n$, in such a way that $\nu_{k+1}(K_{i_{k+1}j_{k+1}}^n) = 1/2^n$, $\nu_{k+1}(K_{pq}^n) = \nu_k(K_{pq}^n)$ whenever $K_{pq}^n \in \mathcal{M}_k{}^n$, $\nu_{k+1}(K_{ij}^n) = 0$ whenever $K_{ij}^n \notin \{K_{i_{k+1}j_{k+1}}^n\} \cup \mathcal{M}_k{}^n$. Further put

$$\mathcal{M}_{k+1}^n = \{K_{ij}^n : i = i_{k+1} \text{ or } j = j_{k+1}\} \cup \mathcal{M}_k{}^n. \tag{8}$$

Clearly, $\mathcal{M}_{k+1}^n$ and ν_{k+1} enjoy all the properties listed above for $\mathcal{M}_k{}^n$ and ν_k with k replaced by $k+1$. After a finite number of steps we inevitably arrive at $\mathcal{M}_k{}^n \supset \mathcal{M}^n$ and thus obtain the non-trivial Borel measure $\mu_n = \nu_k$ with support in $F^n \subset [0, 1] \times [0, 1]$ for which $V\mu_n$ satisfies the Lipschitz condition with coefficient $\varkappa \leqq 1$. Since for each i there is at most one j with $\mu_n(K_{ij}^n) \neq 0$, we have also $\mu_n(\mathbb{R}^2) \leqq 1$.

Passing to a subsequence, if necessary, we may achieve that the sequence $\{\mu_n\}$ converges vaguely to a measure μ with support contained in $\bigcap F^n = F$. It is easy to see that $V\mu$ satisfies the Lipschitz condition with coefficient $\varkappa \leqq 1$ and $\mu(\mathbb{R}^2) \leqq 1$. It remains to prove that $\mu(\mathbb{R}^2) > 0$.

Let $\mathcal{M}^{n1}$ be the system of all the squares K_{ij}^n for which there exists a K_{iq}^n with $\mu_n(K_{iq}^n) \neq 0$, and let $\mathcal{M}^{n2}$ be the system of all the remaining squares K_{ij}^n. We assert that, for each $K_{ij}^n \in \mathcal{M}^{n2} \cap \mathcal{M}^n$, there exists a p such that $\mu_n(K_{pj}^n) \neq 0$. This is obvious if $K_{ij}^n \in \mathcal{M}_1{}^n$, which was the first system of squares defined in connection with the process of constructing the measure ν_1. Assuming $K_{ij}^n \in (\mathcal{M}^n \cap \mathcal{M}^{n2}) \setminus \mathcal{M}_1{}^n$ we can choose $k \in \mathbb{N}$ such that $K_{ij}^n \in \mathcal{M}_{k+1}^n \setminus \mathcal{M}_k{}^n$. Taking into account the definition (8) and the fact that $K_{ij}^n \notin \mathcal{M}^{n1}$ we conclude that necessarily $j = j_{k+1}$ whence we get for $p = i_{k+1}$ the relation $\mu_n(K_{pj}^n) = \mu_n(K_{i_{k+1}j_{k+1}}^n) \neq 0$ as claimed. Set now $F_1{}^n = \bigcup (\mathcal{M}^n \cap \mathcal{M}^{n1})$, $F_2{}^n = \cup (\mathcal{M}^n \cap \mathcal{M}^{n2})$, so that $F_1{}^n \cup F_2{}^n = F^n \supset F$. We have seen that, for each $K_{ij}^n \in \mathcal{M}^n \cap \mathcal{M}^{n2}$, there is a K_{pj}^n with $\mu_n(K_{pj}^n) = 1/2^n$. Consequently, $|\pi_2(F_2{}^n)| \leqq \mu_n(F_2{}^n)$. It follows similarly from the definition or $\mathcal{M}^{n1}$ that $|\pi_1(F_1{}^n)| \leqq \mu_n(F_1{}^n)$, whence

$$|\pi_1(F_1{}^n)| + |\pi_2(F_2{}^n)| \leqq \mu_n(F_1{}^n) + \mu_n(F_2{}^n) = \mu_n([0, 1] \times [0, 1]).$$

Applying Lemma 1 we get an $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, $\mu_n([0, 1] \times [0, 1]) \geqq |\pi_1(F_1{}^n)| + |\pi_2(F_2{}^n)| \geqq \varepsilon$, whence it follows that $\mu(\mathbb{R}^2) \geqq \varepsilon$, which completes the proof ∎.

Lemma 3: *Let μ be a finite Borel measure in $\mathbb{R}^2$ such that the function $V\mu$ defined by* (7) *satisfies locally the Lipschitz condition. Then $V\mu \in$ V-Lip$_{\text{loc}}(\mathbb{R}^2)$.*

Proof: Let $K = [a, b] \times [c, d]$ be arbitrary and choose $\varkappa \in [0, \infty)$ such that $V\mu$ satisfies the Lipschitz condition with coefficient $\varkappa$ in K. Let $a \leqq x_1 \leqq x_2 \leqq b$. We have for $y \in \mathbb{R}$

$$V\mu(x_2, y) - V\mu(x_1, y) = \mu\big([x_1, x_2) \times [-\infty, y)\big),$$

which is a non-negative and non-decreasing function of the variable $y \in \mathbb{R}$ bounded on $[c, d]$ by its value in d. Using the Lipschitz condition we get $|V\mu(x_2, d) - V\mu(x_1, d)| \leqq \varkappa |x_2 - x_1|$, so that $\operatorname{var}\left(V\mu(x_2, \cdot) - V\mu(x_1, \cdot); [c, d]\right) \leqq \varkappa |x_2 - x_1|$. Similar reasoning yields for $c \leqq y_1 \leqq y_2 \leqq d$ the inequality $\operatorname{var}\left(V\mu(\cdot, y_2) - V\mu(\cdot, y_1); [a, b]\right) \leqq \varkappa |y_2 - y_1|$, so that $V\mu|_K \in$ V-Lip (K) ∎

Lemma 4: *Let $F \subset G$ be relatively closed in G and suppose that $u: G \to \mathbb{R}$ is a function such that $u|_{G \setminus F} \in \mathscr{W}(G \setminus F)$ and $u|_K \in$ V-Lip (K), $K = [a, b] \times [c, d] \subset G$. If $F \cap K$ can be decomposed into subsets F_1, F_2 satisfying* (6), *then*

$$u(a, c) + u(b, d) = u(b, c) + u(a, d). \tag{9}$$

Proof: If $I = [\alpha, \beta] \times [\gamma, \delta] \subset K$, we define $w(I) = u(\beta, \delta) - u(\alpha, \delta) - u(\beta, \gamma) + u(\alpha, \gamma)$. Then w is an additive interval function. Let us fix $\varkappa \in [0, \infty)$ such that (3) holds for all $x_1, x_2 \in [a, b]$ and (4) holds for all $y_1, y_2 \in [c, d]$. Then

$$w([\alpha, \beta] \times [\gamma, \delta]) \leqq \varkappa \min\{\beta - \alpha, \delta - \gamma\}. \tag{10}$$

Since the relation (9) is obvious if $a = b$ or $c = d$, we shall assume that $a < b$ and $c < d$. Noting that K can be decomposed into a rectangle $[a, b] \times [c, d_0]$ (where $c < d_0 < d$) with commensurable side lengths $b - a, d_0 - c$ and another rectangle $[a, b] \times [d_0, d]$ with arbitrarily small side length $d - d_0$, we conclude easily from (10) that it is sufficient to verify (9) for squares contained in K; so we shall suppose, without any loss of generality, that $b - a = d - c$. (Note that, in case $u \in C(G)$, such reduction is also possible without reference to (10).)

Fix an arbitrary $\varepsilon > 0$. Then there are open sets $U_i \subset \mathbb{R}$ $(i = 1, 2)$ such that $\pi_i(K \cap F) \subset U_i$, $|U_i| < \varepsilon$. In view of $K \cap F \subset (U_1 \times \mathbb{R}) \cup (\mathbb{R} \times U_2)$ we can choose $r > 0$ small enough to guarantee that from $Q(z_0, r) \subset K$, $Q(z_0, r) \cap F \neq \emptyset$ follows $Q(z_0, r) \subset (U_1 \times \mathbb{R}) \cup (\mathbb{R} \times U_2)$, where $Q(z_0, r)$ is defined by (1). Next fix $n \in \mathbb{N}$ large enough to have $(b - a)/2^n = (d - c)/2^n < r$ and put

$$\left.\begin{aligned} I_j &= [a + (j - 1)(b - a)/2^n, a + j(b - a)/2^n] \\ J_k &= [c + (k - 1)(d - c)/2^n, c + k(b - a)/2^n] \end{aligned}\right\} \quad (1 \leqq j, k \leqq 2^n).$$

Let

$$A = \{(j, k): (I_j \times J_k) \cap F = \emptyset\};$$

$$B = \{(j, k): (I_j \times J_k) \cap F \neq \emptyset, I_j \subset U_1\}$$

and denote by C the set of all the remaining couples (j, k), so that

$$C = \{(j, k): (I_j \times J_k) \cap F \neq \emptyset, J_k \subset U_2\} \setminus B.$$

We claim that, for any $(j, k) \in A$, the corresponding square $L_0 = I_j \times J_k$ satisfies $w(L_0) = 0$. Indeed, in the opposite case one of the squares obtained by quartering L_0, to be denoted by L_1, would also satisfy $w(L_1) \neq 0$. Proceeding in this way we should obtain, by the method of consecutive quartering, a sequence of squares $\{L_p\}_{p \geq 0}$, $L_0 \supset L_1 \supset \cdots$ such that, for each p, $w(L_p) \neq 0$ and the side length of L_p equals $(b - a)/2^{n+p}$. Then $\bigcap L_p = \{z\}$, where $z \in G \setminus F$; this would contradict $u \in \mathscr{W}(G \setminus F)$. Writing $\sum_A$ for the sum extended over all $(j, k) \in A$ we have thus

$$\sum_A w(I_j \times J_k) = 0. \tag{11}$$

Consider now $(j, k) \in B$ and put $I_j = [\alpha, \beta]$; j being fixed, we arrange all the intervals J_k with $(j, k) \in B$ into a finite sequence $\{[\gamma_s^j, \delta_s^j]\}_{s=1}^{p_j}$ such that $\gamma_1^j < \delta_1^j \leqq \gamma_2^j$

$< \delta_2^j \leqq \cdots$. Using (3) we get

$$\left|\sum_{s=1}^{p_j} w(I_j \times [\gamma_s^j, \delta_s^j])\right|$$
$$= \left|\sum_{s=1}^{p_j} \left(u(\beta, \delta_s^j) - u(\alpha, \delta_s^j) - u(\beta, \gamma_s^j) + u(\alpha, \gamma_s^j)\right)\right|$$
$$\leqq \operatorname{var}\left(u(\beta, \cdot) - u(\alpha, \cdot); [c, d]\right) \leqq \varkappa\, |\beta - \alpha| = \varkappa\, |I_j|.$$

Hence we get for the sum $\sum\limits_B$ extended over all $(j, k) \in B$ the estimate

$$\left|\sum_B w(I_j \times J_k)\right| \leqq \varkappa\, |U_1| \leqq \varkappa\varepsilon. \tag{12}$$

Similar reasoning based on (4) yields the inequality

$$\left|\sum_C w(I_j \times J_k)\right| \leqq \varkappa\, |U_2| \leqq \varkappa\varepsilon. \tag{13}$$

Summarizing (11)—(13) we arrive at

$$|w(K)| \leqq \left|\sum_A w(I_j \times J_k)\right| + \left|\sum_B w(I_j \times J_k)\right| + \left|\sum_C w(I_j \times J_k)\right| \leqq 2\varkappa\varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small independently of $\varkappa$, we have $w(K) = 0$, which is the relation (9) ∎

Remark: Applying (9) to the rectangles $[a, x] \times [c, y] \subset K$ one gets $u(x, y) = -u(a, c) + u(x, c) + u(a, y)$ which shows that there are functions $f(= u(\cdot, c) - u(a, c))$ and $g(= u(a, \cdot))$ such that u can be represented in the form

$$u(x, y) = f(x) + g(y), \qquad (x, y) \in K. \tag{14}$$

The reasoning described in the course of the above proof of Lemma 4 shows that, if $u \in \mathscr{C}(G) \cap \mathscr{W}(G)$, then $w(K) = 0$ for each rectangle $K = I \times J \subset G$, which is equivalent to saying that there exist continuous functions $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$ such that u can be represented in the form (14) (cf. [3]).

Now we are in position to complete the proof of our main result.

Proof of the Theorem: Suppose that $u \in$ V-Lip$_{\mathrm{loc}}(G)$, $F \subset G$ is relatively closed in G and $u|_{G \setminus F} \in \mathscr{W}(G \setminus F)$. If there exist sets F_1, F_2 satisfying (5), (6) then, according to Lemma 4, for each rectangle $[a, b] \times [c, d] \subset G$ the relation (9) is valid, which means that $u \in \mathscr{W}(G)$ and F is removable for V-Lip$_{\mathrm{loc}}(G)$.

Conversely, suppose that $F \subset G$ is a relatively closed subset in G which cannot be written as a union (5) of subsets satisfying (6). Then there is a closed rectangle $K \subset G$ such that $\hat{F} = K \cap F$ also cannot be decomposed into subsets $\hat{F}_1, \hat{F}_2$ satisfying $|\pi_i(\hat{F}_i)| = 0$, $i = 1, 2$. Applying Lemma 2 we get a finite non-trivial Borel measure μ with support contained in $\hat{F}$ such that the function $V\mu$ defined by (7) satisfies the Lipschitz condition on $\mathbb{R}^2$. Hence it follows by Lemma 3 that $V\mu \in$ V-Lip$_{\mathrm{loc}}(\mathbb{R}^2)$. If $[\alpha, \beta] \times [\gamma, \delta]$ is any square disjoint with $\hat{F}$, then $V\mu(\beta, \delta) - V\mu(\alpha, \delta) - V\mu(\beta, \gamma) + V\mu(\alpha, \gamma) = \mu\left([\alpha, \beta) \times [\gamma, \delta)\right) = 0$, so that $V\mu|_{\mathbb{R}^2 \setminus \hat{F}} \in \mathscr{W}(\mathbb{R}^2 \setminus \hat{F})$. Since μ is non-trivial, we can choose $z_0 = (x_0, y_0) \in \hat{F} \subset G$ such that, for each $r > 0$, $\mu\left((x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r)\right) > 0$ and, consequently,

$$V\mu(x_0 + r, y_0 + r) + V\mu(x_0 - r, y_0 - r) - V\mu(x_0 - r, y_0 + r)$$
$$- V\mu(x_0 + r, y_0 - r) = \mu\left([x_0 - r, x_0 + r) \times [y_0 - r, y_0 + r)\right) > 0.$$

Denoting $u = V\mu|_G$ we see that $u \notin \mathscr{W}(G)$, although $u \in$ V-Lip$_{\mathrm{loc}}(G)$ and $u|_{G \setminus F} \in \mathscr{W}(G \setminus F)$. Thus F is not removable for V-Lip$_{\mathrm{loc}}(G)$ ∎

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