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# Solution in Quadratures of the Basic Problems of Thermoelasticity for a Sphere and a Spherical Cavity

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*Dedicated to S. G. Mikhlin on the occasion of his 80th birthday*

Es werden die Hauptaufgaben der Statik der klassischen Elastizitätstheorie für eine Kugel und für einen ganzen Raum mit einem kugelförmigen Hohlraum in Quadraturen gelöst. Die Lösung wird unter den allgemeinsten und natürlichsten Einschränkungen konstruiert. Viele Randwertaufgaben der mathematischen Physik werden mit Hilfe spezieller Potentiale auf singuläre Integralgleichungen zurückgeführt. Diese werden für einige Gebiete explizit gelöst, die Lösungen der entsprechenden Randwertaufgaben in Quadraturen konstruiert. Für Gebiete mit kugelförmigem Rand erweist sich eine Darstellung der Lösungen recht günstig, die manchmal nach Trefftz benannt wird (siehe z. B. [9] oder [16]), aber erstmals wahrscheinlich bereits 1904 in einer Arbeit von R. Marcolongo [11] vorkommt. Es wird gezeigt, daß diese Darstellung auch in den Aufgaben der Thermoelastizitätstheorie zum Ziel führt.

Решаются в квадратурах основные задачи статики классической теории термоупругости для шара и всего пространства с шаровой полостью. Решение строится в наиболее общих и естественных ограничениях. Многие граничные задачи математической физики с помощью специальных потенциалов сводятся к сингулярным интегральным уравнениям. Они для некоторых областей решаются явно, и решения соответствующих граничных задач строятся в крадратурах. Для областей со сферической границей оказывается весьма удобным специальное представление решений, называемое иногда представлением Треффтца (см. например, [9] или [16]), но встречающееся, по-видимому, впервые еще в 1904 году в работе Р. Марколонго [11]. Показывается, что это представление проводит к цели и в задачах термоупругости.

The basic static problems of classical thermoelasticity are solved in quadratures for a sphere and the entire space with a spherical cavity. The solution is constructed under the most general and natural restrictions. Many boundary value problems of mathematical physics are reduced by means of special potentials to singular integral equations. For some domains these equations are solved explicitly and solutions of the corresponding boundary value problems are constructed in quadratures. For domains with a spherical boundary it appears to be convenient to use a special representation of solutions sometimes called the Trefftz representation (see, for example, [9] or [16]) but evidently occuring for the first time as early as 1904 in a paper of Marcolongo [11]. It is shown that this representation is also valid for problems of thermoelasticity.

## § 1 Formulation of the problem and some auxiliary statements

Consider the system of static equations of thermoelasticity [6]

$$\begin{aligned} &\mu \Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) - \gamma \operatorname{grad} \theta(x) = 0, \\ &\Delta\theta(x) = 0, \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2, x_3)$ is a point of the three-dimensional Euclidean space $\mathbf{R}^3$, $\Delta$ is the Laplace operator, $u = (u_1, u_2, u_3)$ is the displacement vector, $\theta$ is the temperature, $\lambda$ and $\mu$ are the Lamé constants, $\gamma = \alpha(3\lambda + 2\mu)$, $\alpha$ is the coefficient of linear thermal expansion. The constants $\lambda$, $\mu$, and $\gamma$ satisfy the conditions

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \gamma \neq 0. \tag{1.2}$$

We introduce the notations $S = S(O, R) = \{y \in \mathbf{R}^3 \mid |y| = R\}$, $B^+ = B^+(O, R) = \{x \in \mathbf{R}^3 \mid |x| < R\}$, $\overline{B}^+ = B^+ \cup S$, $B^- = B^-(O, R) = \mathbf{R}^3 \setminus \overline{B}^+$, $\overline{B}^- = B^- \cup S$. Let $\varphi$ be a function defined on $B^+$ or $B^-$. Boundary values will be denoted by

$$(\varphi)^+ (y) \equiv \lim_{B^+ \ni x \to y \in S} \varphi(x), \quad (\varphi)^- (y) \equiv \lim_{B^- \ni x \to y \in S} \varphi(x).$$

The following boundary value problems will be solved: Find in $B^\pm$ a pair $(u, \theta)$ which is a solution of system (1.1) if any one of the pairs of boundary conditions given below is satisfied:

**Problem (I.I)$^\pm$:** $(u)^\pm (y) = f(y)$, $(\theta)^\pm (y) = g(y)$.

**Problem (I.II)$^\pm$:** $(u)^\pm (y) = f(y)$, $(\partial\theta/\partial n)^\pm (y) = g(y)$.

**Problem (II.I)$^\pm$:** $(\tau^{(n)} - \gamma\theta n)^\pm (y) = f(y)$, $(\theta)^\pm (y) = g(y)$.

**Problem (II.II)$^\pm$:** $(\tau^{(n)} - \gamma\theta n)^\pm (y) = f(y)$, $(\partial\theta/\partial n)^\pm (y) = g(y)$.

Here $n(y) = \big(n_1(y), n_2(y), n_3(y)\big)$ is the unit normal to $S$ at the point $y$,

$$n_k(x) = \frac{x_k}{r}, \quad x \in \mathbf{R}^3 \setminus \{O\}; \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}; \tag{1.3}$$

$\tau^{(n)} - \gamma\theta n$ is the stress vector in thermoelasticity and $\tau^{(n)}$ is the stress vector in classical elasticity: $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)})$,

$$\tau_i^{(n)} = \lambda n_i \frac{\partial u_k}{\partial x_k} + \mu n_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i = 1, 2, 3); \tag{1.4}$$

$f = (f_1, f_2, f_3)$ and $g$ are functions given on $S$. For $f = 0$ and $g = 0$ the problems will be called *homogeneous* and denoted by $(\mathrm{I.I})_0^\pm$, $(\mathrm{I.II})_0^\pm$, $(\mathrm{II.I})_0^\pm$, $(\mathrm{II.II})_0^\pm$, respectively. The solution $(u, \theta)$ of system (1.1) will be called *regular* [6] if $u, \theta \in \mathbf{C}^1(\overline{B}^\pm) \cap \mathbf{C}^2(B^\pm)$. In addition to regular solutions, we shall also be interested in classical ones. The solutions $(u, \theta)$ of Problems (I.I)$^\pm$ to (II.II)$^\pm$ will be called *classical* if $u, \theta \in \mathbf{C}(\overline{B}^\pm) \cap \mathbf{C}^2(B^\pm)$ and, additionally, if $(\partial\theta/\partial n)^\pm \in \mathbf{C}(S)$ for Problem (I.II)$^\pm$, $(\tau^{(n)})^\pm \in \mathbf{C}(S)$ for Problem (II.I)$^\pm$ and $(\tau^{(n)})^\pm$, $(\partial\theta/\partial n)^\pm \in \mathbf{C}(S)$ for Problem (II.II)$^\pm$.

For general domains the boundary value problems of classical elasticity and thermoelasticity have been investigated with sufficient completeness in [6] by the methods of a potential and integral equations. The existence and the uniqueness of regular solutions have been considered and the necessary and sufficient conditions for boundary value problems to be solvable have been established. The results of [6] readily yield the following theorems, which are helpful for our purpose.

Theorem 1.1: *If in a neighbourhood of the point* $|x| = +\infty$ *the regular solutions of Problems* $(\mathrm{I.I})_0^-$, $(\mathrm{I.II})_0^-$, $(\mathrm{II.I})_0^-$, $(\mathrm{II.II})_0^-$ *satisfy the conditions*

$$\theta(x) = O(|x|^{-1}), \quad u_i(x) = O(|x|^{-1}), \quad \partial u_i(x)/\partial x_j = o(|x|^{-1}), \tag{1.5}$$

*then these problems can have only the trivial solutions* $u(x) = 0$, $\theta(x) = 0$, $x \in B^-$.

Theorem 1.2: *Problem* $(\mathrm{I.I})_0^+$ *can have only the trivial regular solution* $u(x) = 0$, $\theta(x) = 0$, $x \in B^+$.

Theorem 1.3: *All regular solutions of Problem* $(\mathrm{I.II})_0^+$ *have the form* $u(x) = 0$, $\theta(x) = \theta_0$, $x \in B^+$, *where* $\theta_0$ *is an arbitrary constant.*

Theorem 1.4: *All regular solutions of Problem* $(\mathrm{II.I})_0^+$ *have the form* $u(x) = [a \times x] + b$, $\theta(x) = 0$, $x \in B^+$, *where* $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ *are arbitrary constant vectors and the symbol* $\times$ *denotes the vector product.*

Theorem 1.5: *All regular solutions of Problem* $(\mathrm{II.II})_0^+$ *have the form* $u(x) = \gamma\theta_0 \times \mathring{u}(x) + [a \times x] + b$, $\theta(x) = \theta_0$, $x \in B^+$, *where* $\theta_0$ *is an arbitrary constant,* $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ *are arbitrary constant vectors,* $\mathring{u}$ *is a regular solution of the boundary value problem*

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0, \qquad (\tau^{(n)})^+ (y) = n(y). \tag{1.6}$$

The latter problem is solvable and its solution will be constructed below.

Theorems 1.1 to 1.5 can be easily proved, taking into account the fact that static problems of thermoelasticity are divided into those for $\theta$ (boundary value problems for Laplace equations) and those for $u$ (boundary value problems of classical elasticity). For example, if $(u, \theta)$ is a regular solution of Problem $(\mathrm{II.II})_0^+$:

$$\Delta\theta = 0, \qquad (\partial\theta/\partial n)^+ = 0, \tag{1.7}$$

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u - \gamma \operatorname{grad} \theta = 0, \qquad (\tau^{(n)} - \gamma\theta n)^+ = 0, \tag{1.8}$$

then, as follows from [13], the Neumann problem (1.7) has the solution $\theta = \theta_0$, where $\theta_0$ is an arbitrary constant. The substitution of $\theta = \theta_0$ in (1.8) gives us the problem of classical elasticity

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0, \qquad (\tau^{(n)})^+ = \gamma\theta_0 n. \tag{1.9}$$

Since the conditions for problem (1.9) to be solvable $\int_S n(y)\, d_yS = 0$, $\int_S [n(y) \times n(y)]\, d_yS = 0$ are fulfilled and its solution is unique [6], one can easily prove Theorem 1.5.

Let us now discuss the solvability conditions of boundary value problems. Problems $(\mathrm{I.I})^\pm$, $(\mathrm{I.II})^-$, $(\mathrm{II.I})^-$, $(\mathrm{II.II})^-$ are solvable for arbitrary sufficiently smooth [6] boundary values (i.e., for arbitrary smooth $f$ and $g$). Consider Problem $(\mathrm{II.I})^+$:

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u - \gamma \operatorname{grad} \theta = 0, \qquad (\tau^{(n)} - \gamma\theta n)^+ = f, \tag{1.10}$$

$$\Delta\theta = 0, \qquad (\theta)^+ = g. \tag{1.11}$$

It is well-known that problem (1.11) is solvable for arbitrary $g$. Substituting $\theta$, determined from (1.11), in (1.10), we obtain the second boundary value problem of classical elasticity:

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = \Phi, \qquad (\tau^{(n)})^+ = F,$$

where $\Phi = \gamma \operatorname{grad} \theta$, $F = f + \gamma(\theta)^+ n$. For this problem to be solvable it is necessary and sufficient that the conditions

$$\int_{B^+} \Phi(x)\, dx - \int_S F(y)\, d_yS = 0, \qquad \int_{B^+} [x \times \Phi(x)]\, dx - \int_S [y \times F(y)]\, d_yS = 0$$

or, taking into account expressions for $\Phi$ and $F$,

$$\int_S f(y)\, d_yS = 0, \qquad \int_S [y \times f(y)]\, d_yS = 0 \tag{1.12}$$

be fulfilled [6]. These conditions will be assumed to be fulfilled for Problem (II.I)$^+$. For Problem (II.II)$^+$ the solvability conditions have the form

$$\int_S f(y)\, d_yS = 0, \qquad \int_S [y \times f(y)]\, d_yS = 0, \qquad \int_S g(y)\, d_yS = 0, \tag{1.13}$$

and for Problem (I.II)$^+$ they are written as

$$\int_S g(y)\, d_yS = 0. \tag{1.14}$$

When solving Problems (I.I)$^\pm$ to (II.II)$^\pm$, we shall make use of the well-known representations of solutions of the Neumann and Dirichlet problems for the Laplace equation. Namely, the solution of the Dirichlet problem for $B^+$:

$$\forall x \in B^+ : \Delta v(x) = 0, \qquad \forall y \in S: (v)^+ (y) = g(y),$$

is given by the Poisson formula [13]

$$v(x) = \Pi(g)(x) \equiv \frac{1}{4\pi R} \int_S \frac{R^2 - |x|^2}{|y - x|^3}\, g(y)\, d_yS, \tag{1.15}$$

and that of the Dirichlet problem for $B^-$:

$$\forall x \in B^- : \Delta v(x) = 0, \qquad \forall y \in S: (v)^- (y) = g(v),$$

is given by the Poisson formula

$$v(x) = \Pi'(g)(x) \equiv \frac{1}{4\pi R} \int_S \frac{|x|^2 - R^2}{|x - y|^3}\, g(y)\, d_yS. \tag{1.16}$$

The solution of the Neumann problem for $B^+$:

$$\forall x \in B^+ : \Delta v(x) = 0, \qquad (\partial v/\partial n)^+ (y) = g(y), \tag{1.17}$$

is given by the Neumann formula [5]

$$v(x) = N(g)(x) \equiv \frac{1}{4\pi R} \int_S \left( \frac{2R}{|x - y|} - \ln\left( (|x - y| + R)^2 - |x|^2 \right) \right) g(y)\, d_yS, \tag{1.18}$$

and that of the Neumann problem for $B^-$:

$$\forall x \in B^- : \Delta v(x) = 0, \qquad (\partial v/\partial n)^- (y) = g(y), \tag{1.19}$$

is given by the Bjerknes formula [5]:

$$v(x) = N'(g)(x) \equiv \frac{1}{4\pi R} \int_S \left( \frac{2R}{|x - y|} - \ln \frac{|x - y| + |x| + R}{|x - y| + |x| - R} \right) g(y)\, d_yS. \tag{1.20}$$

Throughout this paper we shall never use the expansion of a function in a series, but for the method to be complete, we should note that the Poisson, Neumann and Bjerknes formulas can be obtained without using the series expansion. Let us derive the Neumann and Bjerkness formulas. It is easy to verify that the Neumann problem (1.17) is equivalent to finding a function $v$ that is harmonic in $B^+$ from the equation $r(\partial v/\partial r) = R\Pi(g)$, where $g$ obeys condition (1.14). It is likewise easy to verify that the solution of this equation has the form $v(x) = R \int_0^r \Pi(g)(\eta x/r)\, \eta^{-1}\, d\eta$. Hence it follows that $v(x) = R \int_0^1 \Pi(g)(\tau x)\, \tau^{-1}\, d\tau$. Applying in the latter integral the identity

$$\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} = \tau \frac{\partial}{\partial \tau} \left( \frac{2}{|y - \tau x|} - \frac{1}{R} \ln \frac{(|y - \tau x| + R)^2 - |\tau x|^2}{2\tau} \right)$$

$(x \in B^+, y \in S, \tau \in (0, 1))$, we easily obtain formula (1.18). Similarly, problem (1.19) is equivalent to finding a function $v$ that is harmonic in $B^-$ from the equation $r(\partial v/\partial r) = R\Pi'(g)$, whose solution has the form $v(x) = -R \int_0^1 \Pi'(g)(x/\tau) \tau^{-1} d\tau$. Hence, using the identity

$$\frac{|x|^2 - \tau^2 R^2}{|x - \tau y|^3} = \frac{\partial}{\partial \tau} \left( \frac{2\tau}{|x - \tau y|} - \frac{1}{R} \ln (|x - y|^2 + 2R |x - \tau y| + 2\tau R^2 - R^2 - |x|^2) \right)$$

$(x \in B^-, y \in S, \tau \in (0, 1))$ we obtain formula (1.20).

In constructing the solutions of Problems (I.I)± to (II.II)± an essential use is made of the special representation of displacements by means of harmonic functions. We have the following theorems, which are easy to prove.

Theorem 1.6: *If*

$$u = v + 1/2(R^2 - r^2) \operatorname{grad} \psi, \tag{1.21}$$

*where* $\Delta v = 0$, $\Delta \psi = 0$,

$$r(\partial \psi/\partial r) + \alpha \psi = \beta(\operatorname{div} v - \eta \theta), \tag{1.22}$$

*$R$ is an arbitrary constant, $\theta$ is an arbitrary solution of the Laplace equation $\Delta \theta = 0$, $r = |x|$,*

$$\alpha = \mu/(\lambda + 3\mu), \quad \beta = (\lambda + \mu)/(\lambda + 3\mu), \quad \eta = \gamma/(\lambda + \mu), \tag{1.23}$$

*then $(u, \theta)$ is a solution of system* (1.1).

Theorem 1.7: *If*

$$u(x) = v(x) + x \left( \psi(x) + 2r \frac{\partial \psi(x)}{\partial r} \right) + \frac{R^2 - 3r^2}{2} \operatorname{grad} \psi(x), \tag{1.24}$$

*where* $\Delta v = 0$, $\Delta \psi = 0$,

$$2r \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + 2 \frac{2\lambda + \mu}{\lambda + \mu} r \frac{\partial \psi}{\partial r} + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi = \eta\theta - \operatorname{div} v, \quad \Delta\theta = 0, \tag{1.25}$$

*then $(u, \theta)$ is a solution of system* (1.1) *and*

$$r\left(\tau^{(n(x))}(x) - \gamma n(x)\, \theta(x)\right) = \mu h(x) + \mu(R^2 - r^2)\, r \partial \operatorname{grad} \psi(x)/\partial r. \tag{1.26}$$

*Here $n(x)$ is determined from* (1.3), $h = (h_1, h_2, h_3)$,

$$h_i(x) = x_j\left(\partial v_i(x)/\partial x_j + \partial v_j(x)/\partial x_i\right) - x_i \operatorname{div} v(x). \tag{1.27}$$

Representation (1.21) will be used for solving Problems (I.I)± and (I.II)±. Representation (1.24), which is virtually of the same form as (1.21), is convenient for solving Problems (II.I)± and (II.II)±.

## § 2 Solution of Problems (I.I)±, (I.II)±

We begin by formally solving Problem (I.I)$^+$. After the formal construction of solutions it will be proved that the formula obtained gives the classical solution for continuous boundary values.

We find the displacement vector $u$ in the form of (1.21). Now, as can be easily verified, we have the Dirichlet problem for $B^+$ relative to $\theta$ and $u$ which can be written in the form (see (1.15))

$$\theta(x) = \Pi(g)(x), \qquad v(x) = \Pi(f)(x). \tag{2.1}$$

Substituting the found values of $\theta$ and $v$ in (1.22) and treating it as a differential equation relative to $\psi$, we obtain $r(\partial\psi/\partial r) + \alpha\psi = F$, where

$$F(x) = \beta\big(\operatorname{div}\Pi(f)(x) - \eta\Pi(g)(x)\big). \tag{2.2}$$

This equation is rewritten in a more convenient form

$$r\big(\partial(\psi - c)/\partial r\big) + \alpha(\psi - c) = F - \alpha c, \tag{2.3}$$

where $c$ is a constant and $\alpha c = F(0)$. It is clear from (1.21) that the addition of the constant to the function $\psi$ does not influence $u$ and therefore to determine $\psi$ we can, instead of (2.3), consider the equation $r\big(\partial\psi(x)/\partial r\big) + \alpha\psi(x) = F(x) - F(0)$. Integrating this equation with respect to the variable $r$ and assuming that $\psi$ is a regular harmonic function, we obtain $\psi(x) = r^{-\alpha}\int_0^r [F(\eta x/r) - F(0)]\,\eta^{\alpha-1}\,d\eta$. The substitution of $\eta = \tau r$ brings this equality to the form $\psi(x) = \int_0^1 \big(F(\tau x) - F(0)\big)\,\tau^{\alpha-1}\,d\tau$. Taking into account (2.2), we have

$$\psi(x) = \frac{\beta}{4\pi R}\int_S \operatorname{div}\int_0^1 \left(\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} - \frac{3xy}{R^3}\,\tau\right)\frac{d\tau}{\tau^{2-\alpha}}\,f(y)\,d_yS$$
$$- \frac{\beta\eta}{4\pi R}\int_S\int_0^1 \left(\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R}\right)\frac{d\tau}{\tau^{1-\alpha}}\,g(y)\,d_yS, \tag{2.4}$$

where by virtue of (1.2) $0 < \alpha < 1/2$. On account of (1.21), (2.1) and (2.4) the solution of Problem (I.I)$^+$ can be represented in the form

$$u(x) = \int_S K(x, y)\,f(y)\,d_yS + \gamma\int_S \Theta(x, y)\,g(y)\,d_yS, \qquad \theta(x) = \Pi(g)(x), \tag{2.5}$$

where

$$K = \|K_{ij}\|_{3\times 3}, \tag{2.6}$$

$$K_{ij}(x, y) = \frac{1}{4\pi R}\left(\frac{R^2 - |x|^2}{|y - x|^3}\,\delta_{ij} + \frac{\beta(R^2 - |x|^2)}{2}\right.$$
$$\left.\times\frac{\partial^2}{\partial x_i\,\partial x_j}\int_0^1 \left(\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} - \frac{3xy}{R^3}\,\tau\right)\frac{d\tau}{\tau^{2-\alpha}}\right); \tag{2.7}$$

$$\Theta = (\Theta_1, \Theta_2, \Theta_3),$$

$$\Theta_i(x, y) = -\frac{\beta(R^2 - |x|^2)}{8\pi(\lambda + \mu) R} \frac{\partial}{\partial x_i} \int_0^1 \left(\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R}\right) \frac{d\tau}{\tau^{1-\alpha}}. \tag{2.7}$$

Problem (I.II)$^+$ is solved exactly in the same manner and its solution has the form

$$u(x) = \int_S K(x, y) f(y) \, d_yS + \gamma \int_S \tilde{\Theta}(x, y) \, g(y) \, d_yS,$$

$$\theta(x) = N(g)(x) + \theta_0, \tag{2.8}$$

where $K$ is determined by (2.6) and $N$ by (1.18),

$$\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3), \tag{2.9}$$

$$\tilde{\Theta}_i(x, y) = -\frac{\beta(R^2 - |x|^2)}{8\pi(\lambda + \mu) R} \frac{\partial}{\partial x_i} \int_0^1 \left(\frac{2R}{|y - \tau x|} - 2 - \ln \frac{(|y - \tau x| + R)^2 - |\tau x|^2}{4R^2}\right) \frac{d\tau}{\tau^{1-\alpha}}.$$

We have constructed the solutions of Problems (I.I)$^+$ and (I.II)$^+$. Let us now establish their differential properties. First, we shall derive some estimates to be used below. It is easy to verify that

$$I(x, y) \equiv \int_0^1 \left(\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} - \frac{3xy}{R^3} \tau\right) \frac{d\tau}{\tau^{2-\alpha}}$$

$$= \int_{0,5}^1 \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} \frac{d\tau}{\tau^{2-\alpha}} + c(x, y).$$

Here $c$ and its derivatives up to the second order inclusive are bounded functions since, for $\tau \in [0{,}5;\, 1]$,

$$|y - \tau x| \geqq R - |\tau x| \geqq R - 0{,}5R = 0{,}5R \qquad (x \in B^+, y \in S). \tag{2.10}$$

Using the identities

$$\frac{R^2 - |\tau x|^2}{|y - \tau x|^3} = \frac{1}{|y - \tau x|} + 2\tau \frac{\partial}{\partial \tau} \frac{1}{|y - \tau x|}, \tag{2.11}$$

$$\frac{1}{|y - \tau x|} = -\frac{1}{R} \tau \frac{\partial}{\partial \tau} \ln \frac{(|y - \tau x| + R)^2 - |\tau x|^2}{2\tau} \tag{2.12}$$

and the inequality

$$|y - \tau x| \geqq 1/2 \, |y - x| \qquad (\tau \in [0;\, 1], x \in B^+, y \in S), \tag{2.13}$$

we arrive at the representation

$$I(x, y) = 2/|x - y| + I_0(x, y), \qquad |\partial^2 I_0(x, y)/\partial x_i \, \partial x_j| \leqq c/|x - y|^2. \tag{2.14}$$

Therefore

$$|K(x, y)| \leqq C(R^2 - |x|^2)/|y - x|^3, \tag{2.15}$$

$$|\Theta(x, y)|, |\tilde{\Theta}(x, y)| \leqq C(R^2 - |x|^2)/|y - x|^2 \qquad (x \in B^+, y \in S), \tag{2.16}$$

and we have the representation

$$K = \overset{1}{K} + \overset{2}{K} + \overset{3}{K}, \tag{2.17}$$

where

$$\overset{1}{K}(x, y) = \left\| \frac{1}{4\pi R} \frac{R^2 - |x|^2}{|y - x|^3} \delta_{ij} \right\|_{3\times 3},$$

$$\overset{2}{K}(x, y) = \left\| \frac{\beta(R^2 - |x|^2)}{4\pi R} \frac{\partial^2 |x - y|^{-1}}{\partial x_i \, \partial x_j} \right\|_{3\times 3}, \tag{2.18}$$

$$\overset{3}{K}(x, y) = \|(R^2 - |x|^2) \chi_{ij}(x, y)\|_{3\times 3}, \qquad |\chi_{ij}(x, y)| \leqq c/|x - y|^2.$$

Moreover, by virtue of the property of the Poisson kernel

$$(1/4\pi R) \int_S (R^2 - |x|^2)/|y - x|^3 \, d_yS,$$

we easily find ($\delta_{ij}$ — Kronecker symbol)

$$\int_S K_{ij}(x, y) \, d_yS = \delta_{ij} \qquad (x \in B^+). \tag{2.19}$$

The following auxiliary theorem is valid.

Theorem 2.1: *If $f \in \mathrm{C}^{3,\gamma}(S)$, $g \in \mathrm{C}^{2,\gamma}(S)$, $0 < \gamma \leqq 1$, then the pair $(u, \theta)$ determined by formula* (2.5) *is a regular solution of Problem* (I.I)$^+$.

Proof: It is easy to verify that under the assumptions of the theorem we have

$$\Pi(f) \in \mathrm{C}^{3,\gamma}(B^+), \qquad \Pi(g) \in \mathrm{C}^{2,\gamma}(B^+). \tag{2.20}$$

Consider the expressions

$$E_i(x) \equiv \frac{1}{4\pi R} \int_0^1 \left( \frac{1}{\tau} \frac{\partial}{\partial x_i} \int_S \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} g(y) \, d_yS \right) \tau^\alpha \, d\tau$$

$$= \int_0^1 \tau^\alpha \frac{\partial}{\partial(\tau x_i)} \Pi(g)(\tau x) \, d\tau,$$

$$L_i(x) \equiv \frac{1}{4\pi R} \int_0^1 \left( \frac{1}{\tau^2} \frac{\partial^2}{\partial x_i \, \partial x_j} \int_S \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} f_j(y) \, d_yS \right) \tau^\alpha \, d\tau$$

$$= \int_0^1 \tau^\alpha \frac{\partial^2}{\partial(\tau x_i) \, \partial(\tau x_j)} \Pi(f_j)(\tau x) \, d\tau.$$

By virtue of (2.20) $E_i, L_i \in \mathrm{C}^{1,\gamma}(B^+)$. Therefore $\theta, u \in \mathrm{C}^1(\overline{B}^+)$ and, in view of the properties of the Poisson integral,

$$\lim_{B^+ \ni x \to y \in S} u(x) = f(y), \qquad \lim_{B^+ \ni x \to y \in S} \theta(x) = g(y).$$

Obviously, $\theta, u \in \mathrm{C}^2(B^+)$ and the pair $(u, \theta)$ is a solution of system (1.1) ∎

Theorem 2.2: *If $\theta, u \in C(\bar{B}^+) \cap C^2(B^+)$ and the pair $(u, \theta)$ is a solution of system* (1.1), $(u)^+(y) = 0$, $(\theta)^+(y) = 0$, *then* $u(x) = 0$, $\theta(x) = 0$, $x \in B^+$.

Proof: Since for $\theta$ we have the Dirichlet problem, under the assumptions of the theorem, as known, $\theta(x) = 0$, $x \in B^+$. Therefore for $u$ we have the first boundary value problem of classical elasticity

$$\mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0, \qquad (u)^+(y) = 0. \tag{2.21}$$

Since $u$ is continuous in $B^+$, it is uniformly continuous. Therefore by virtue of (2.21) there exists for any number $\varepsilon > 0$ a number $\delta > 0$ such that $|u(x)| < \varepsilon$ for $y \in S(O, R_0)$, where $R_0 = R - \delta_0$ and $0 < \delta_0 \leq \delta$. Let $z \in B^+$ fixed. Assuming that $\delta_0 < R - |z|$, we have $z \in B^+(O, R_0)$. From the assumptions of the theorem it follows that $u \in C^2(\bar{B}^+(O, R_0))$. Therefore [6] $u \in C^\infty(\bar{B}^+(O, R_0))$. By virtue of Theorem 2.1, we have from (2.5) the representation $u(x) = \int_{S(O,R_0)} K(z, y)\, u(y)\, d_yS$. Hence with (2.15) taken into account we obtain

$$|u(z)| \leq \frac{c}{4\pi R_0} \int_{S(O,R_0)} \frac{R_0^2 - |z|^2}{|y - z|^3} |u(y)|\, d_yS \leq \frac{c\varepsilon}{4\pi R_0} \int_{S(O,R_0)} \frac{R_0^2 - |z|^2}{|y - z|^3}\, d_yS = c\varepsilon.$$

Therefore $u(z) = 0$ and the theorem is proved ∎

Theorem 2.3: *If $f, g \in C(S)$, then the pair $(u, \theta)$ determined by formula* (2.5) *is a solution of system* (1.1): $u, \theta \in C(\bar{B}^+) \cap C^2(B^+)$ *and*

$$(u)^+(z) = f(z), \qquad (\theta)^+(z) = g(z). \tag{2.22}$$

Proof: For $\theta$ the statements of the theorem are obvious. If $f \in C(S)$, then it is clear that $u$, as determined by (2.5), is continuous and has derivatives of all orders in $B^+$. It is also easy to verify that the pair $(u, \theta)$ satisfies system (1.1). Let us verify the first of conditions (2.22). By virtue of (2.5) and (2.19) we write

$$u(x) - f(z) = \int_S K(x, y) \big(f(y) - f(z)\big)\, d_yS + \gamma \int_S \Theta(x, y)\, g(y)\, d_yS,$$

and then on account of (2.15) and (2.16) we have

$$|u(x) - f(z)| \leq c \int_S \frac{R^2 - |x|^2}{|y - x|^3} |f(y) - f(z)|\, d_yS$$

$$+ c \sqrt{R^2 - |x|^2} \max_{\xi \in S} |g(\xi)| \int_S |x - y|^{-3/2}\, d_yS. \tag{2.23}$$

But

$$\int_S |x - y|^{-3/2}\, d_yS \leq c \qquad (x \in \bar{B}^+), \tag{2.24}$$

where $c$ is a constant depending only on $R$. Passing to the limit in (2.23) and taking into account (2.24) and the properties of the Poisson integral, we obtain $\lim_{B^+ \ni x \to z \in S} u(x) = f(z)$. Hence it follows, in turn, that $u \in C(\bar{B}^+)$ ∎

Combining Theorems 2.2 and 2.3, we obtain

Theorem 2.4: *If $f, g \in C(S)$, then the pair $(u, \theta)$ determined by* (2.5) *is the unique classical solution of Problem* (I.I)$^+$.

We have a similar proof for

Theorem 2.5: *If $f, g \in C(S)$ and $g$ satisfies condition* (1.14), *then the pair* $(u, \theta)$ *determined by formula* (2.8) *is the classical solution of Problem* (I.II)$^+$. *Note that $u$ is determined uniquely and $\theta$ to within an arbitrary constant term.*

Let us prove the lemma to be used in proving Problems (II.I)$^+$, (II.II)$^+$.

Lemma 2.1: *If $f = (f_1, f_2, f_3)$, $f \in C(S)$, then*

$$\lim_{B^+\ni x\to z\in S} (R - |x|^2) \operatorname{grad} \operatorname{div} \int_S \frac{f(y)}{|x-y|} d_yS = 0. \tag{2.25}$$

Proof: It is clear from (2.23) that

$$\lim_{B^+\ni x\to z\in S} \int_S K(x, y) f(y) d_yS = f(z). \tag{2.26}$$

Consider expansion (2.17). By virtue of the property of the Poisson integral,

$$\lim_{B^+\ni x\to z\in S} \int_S \overset{1}{K}(x, y) f(y) d_yS = f(z). \tag{2.27}$$

Taking into account (2.18) and (2.24), we have

$$\lim_{B^+\ni x\to z\in S} \int_S \overset{3}{K}(x, y) f(y) d_yS = 0. \tag{2.28}$$

From (2.26) to (2.28) we obtain (2.25) ∎

Representations of solutions of Problems (I.I)$^-$ and (I.II)$^-$ are constructed similarly to (2.5) and (2.8). For Problem (I.I)$^-$ we have

$$u(x) = \int_S K'(x, y) f(y) d_yS + \gamma \int_S \Theta'(x, y) g(y) d_yS, \quad \theta(x) = \Pi'(g)(x), \tag{2.29}$$

and for Problem (I.II)$^-$,

$$u(x) = \int_S K'(x, y) f(y) d_yS + \gamma \int_S \tilde{\Theta}'(x, y) g(y) d_yS, \quad \theta(x) = N'(g)(x), \tag{2.30}$$

where

$$K' = \|K'_{ij}\|_{3\times 3},$$
$$K'_{ij}(x, y) = \frac{1}{4\pi R}\left(\frac{|x|^2 - R^2}{|x-y|^3}\delta_{ij} + \beta\frac{|x|^2 - R^2}{2}\frac{\partial^2}{\partial x_i\, \partial x_j}\int_0^1 \frac{|x|^2 - \tau^2R^2}{|x-\tau y|^3}\tau^{1-\alpha}\,d\tau\right), \tag{2.31}$$

$$\Theta' = (\Theta_1', \Theta_2', \Theta_3'),$$
$$\Theta_i'(x, y) = -\frac{\beta(|x|^2 - R^2)}{8\pi(\lambda+\mu) R}\frac{\partial}{\partial x_i}\int_0^1 \frac{|x|^2 - \tau^2R^2}{|x-\tau y|^3}\tau^{-\alpha}\,d\tau, \tag{2.32}$$

$$\tilde{\Theta}' = (\tilde{\Theta}_1', \tilde{\Theta}_2', \tilde{\Theta}_3'), \tag{2.33}$$

$$\tilde{\Theta}_i'(x, y) = \frac{\beta(|x|^2 - R^2)}{4\pi(\lambda + \mu) R} \frac{\partial}{\partial x_i} \int_0^1 \left( \frac{2R\tau}{|x - \tau y|} - \ln \frac{|x - \tau y| + |x| + \tau R}{|x - \tau y| + |x| - \tau R} \right) \frac{d\tau}{\tau^{1+\alpha}}.$$

Using the estimates

$$|K'(x, y)| \leqq c(|x|^2 - R^2)/|x - y|^3, \tag{2.34}$$

$$|\Theta'(x, y)|, |\tilde{\Theta}'(x, y)| \leqq c(|x|^2 - R^2)/|x - y|^2 \qquad (x \in B^-, y \in S), \tag{2.35}$$

$$\int_S |x - y|^{-3/2} d_yS \leqq c \qquad (x \in \bar{B}^-), \tag{2.36}$$

the following auxiliary theorem can be proved.

Theorem 2.6: *If $f \in \mathrm{C}^{3,\gamma}(S)$, $g \in \mathrm{C}^{2,\gamma}(S)$, $0 < \gamma \leqq 1$, then the pair $(u, \theta)$ determined by formula (2.29) is a regular solution of Problem (I.I)$^-$, satisfying conditions (1.5).*

Lemma 2.2: *If $x \in B^-$, then*

$$\int_S K'_{ij}(x, y)\, d_yS = \varkappa_{ij}(x), \tag{2.37}$$

*where*

$$\varkappa(x) = \|\varkappa_{ij}(x)\|_{3\times 3}, \quad \varkappa_{ij}(x) = \frac{R}{|x|} \delta_{ij} + \frac{\lambda + \mu}{2\lambda + 5\mu} \frac{|x|^2 - R^2}{2} \frac{\partial^2(R/|x|)}{\partial x_i\, \partial x_j}.$$

Proof: Consider pairs $(\overset{m}{u}, \overset{m}{\theta})$, where $\overset{m}{u}_i(x) = \varkappa_{im}(x)$, $\overset{m}{\theta}(x) = 0$ $(i, m = 1, 2, 3)$. It is easy to verify that $(\overset{m}{u}, \overset{m}{\theta})$ is a regular solution of system (1.1), satisfying conditions (1.5) and

$$\lim_{B^- \ni x \to y \in S} \overset{m}{u}_i(x) = \delta_{im}, \qquad \lim_{B^- \ni x \to y \in S} \overset{m}{\theta}(x) = 0. \tag{2.38}$$

Thus the pair $(\overset{m}{u}, \overset{m}{\theta})$ is the unique regular solution of Problem (I.I)$^-$. Therefore by Theorem 2.6 formula (2.29) holds for $(\overset{m}{u}, \overset{m}{\theta})$: $\varkappa_{im}(x) = \int_S K'_{im}(x, y)\, d_yS$ ∎

Theorem 2.7: *If $f, g \in \mathrm{C}(S)$, then the pair $(u, \theta)$ determined by formula (2.29) [(2.30)] is the unique classical solution of Problem* (I.I)$^-$ [(I.II)$^-$].

The theorem is proved just like the corresponding theorems for Problems (I.I)$^+$ and (I.II)$^+$. We shall verify only the property $\lim_{B^- \ni x \to z \in S} u(x) = f(z)$. Taking into account (2.34), (2.35) and (2.36), we have on the basis of (2.29) and (2.37)

$$\begin{aligned} &|u(x) - \varkappa(x) f(z)| \\ &= \int_S K'(x, y) \big(f(y) - f(z)\big)\, d_yS + \gamma \int_S \Theta'(x, y)\, g(y)\, d_yS \\ &\leqq c \int_S \frac{|x|^2 - R^2}{|x - y|^3} |f(y) - f(z)|\, d_yS + c \sqrt{|x|^2 - R^2} \max_{\xi \in S} |g(\xi)|: \end{aligned}$$

Hence by virtue of the properties of the Poisson integral $\lim_{B^- \ni x \to z \in S} \big(u(x) - \varkappa(x) f(z)\big) = 0$ and therefore $\lim_{B^- \ni x \to z \in S} u(x) = \lim_{B^- \ni x \to z \in S} \varkappa(x) f(z) = f(z)$ ∎

## § 3 Solution of Problems (II.I)±, (II.II)±

The solution of Problems $(\mathrm{II.I})^+$ and $(\mathrm{II.II})^+$ is sought for in the form of (1.24). It is easy to verify that $h$, as determined by (1.27), is harmonic in $B^+$; taking into account the boundary condition, we have by virtue of (1.26)

$$(h)^+(y) = (R/\mu)\, f(y). \tag{3.1}$$

Thus $h$ solves the Dirichlet problem and can therefore be represented by the Poisson integral (1.15) written in the form

$$h(x) = (1/4\pi\mu) \int_S \left(\Phi_0(x, y) + 3xy/R^3\right) d_yS, \tag{3.2}$$

where

$$\Phi_0(x, y) = (R^2 - |x|^2)/|x-y|^3 - 1/R - 3xy/R^3. \tag{3.3}$$

The latter formula takes into account the first of conditions (1.12). It is clear from relation (1.27) that

$$\operatorname{div} h = -\operatorname{div} v, \tag{3.4}$$

and therefore from the same relation we have

$$x_j\left(\partial v_i(x)/\partial x_j + \partial v_j(x)/\partial x_i\right) = p_i(x), \tag{3.5}$$

where

$$P = (P_1, P_2, P_3), \qquad P_i(x) = h_i(x) - x_i \operatorname{div} h(x). \tag{3.6}$$

Applying the operation $r\partial/\partial r$ to both parts of (3.5), we obtain

$$x_j\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i}\right) + x_k x_j \left(\frac{\partial^2 v_i(x)}{\partial x_k\, \partial x_j} + \frac{\partial^2 v_j(x)}{\partial x_k\, \partial x_i}\right) = r\, \frac{\partial P_i(x)}{\partial r}.$$

Scalar-multiplying $x$ by (3.5) and then differentiating, we have

$$x_j\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i}\right) + x_k x_j\, \frac{\partial^2 v_j(x)}{\partial x_k\, \partial x_i} = \frac{1}{2}\, \frac{\partial\left(xP(x)\right)}{\partial x_i}.$$

Since $x_k x_j \dfrac{\partial^2 v_i(x)}{\partial x_k\, \partial x_j} = r\, \dfrac{\partial}{\partial r}\, r\, \dfrac{\partial v_i(x)}{\partial r} - r\, \dfrac{\partial v_i(x)}{\partial r} = r^2\, \dfrac{\partial^2 v_i(x)}{\partial r^2}$, the latter equalities give for $v$ the equation

$$r^2\, \partial^2 v(x)/\partial r^2 = r\, \partial P(x)/\partial r - 1/2 \operatorname{grad}\left(xP(x)\right). \tag{3.7}$$

From the second of conditions (1.12) it follows that

$$\int_S y_i f_j(y)\, d_yS = \int_S y_j f_i(y)\, d_yS \qquad (i, j = 1, 2, 3). \tag{3.8}$$

By virtue of (3.2), (3.3), (3.6) and (3.8) equation (3.7) can be rewritten in the form

$$\begin{aligned} r^2\, \frac{\partial^2 v_i(x)}{\partial r^2} = \frac{1}{8\pi\mu} \int_S \Bigg( &\left(2r\, \frac{\partial \Phi_0(x, y)}{\partial r} - \Phi_0(x, y)\right) \delta_{ik} - x_k\, \frac{\partial \Phi_0(x, y)}{\partial x_i} \\ &- 2x_i\, \frac{\partial}{\partial x_k}\left(r\, \frac{\partial \Phi_0(x, y)}{\partial r} - \Phi_0(x, y)\right) + r^2\, \frac{\partial^2 \Phi_0(x, y)}{\partial x_i\, \partial x_k}\Bigg) f_k(y)\, d_yS. \end{aligned} \tag{3.9}$$

Consider a simpler condition

$$r^2\, \partial^2 w/\partial r^2 = \Phi_0. \tag{3.9'}$$

The particular solution of (3.9') is $w(x, y) = \int\limits_0^r \int\limits_0^r \Phi_0(tx/r, y)/t^2\, dt$. Using the Dirichlet formula $\int\limits_0^r \int\limits_0^r \varphi(t)\, dt = \int\limits_0^r (r - t)\, \varphi(t)\, dt$, we have

$$w = \Phi_2 - \Phi_1, \tag{3.10}$$

where

$$\Phi_1(x, y) = \int\limits_0^1 \Phi_0(\tau x, y)/\tau\, d\tau \tag{3.11}$$

$$\Phi_2(x, y) = \int\limits_0^1 \Phi_0(\tau x, y)/\tau^2\, d\tau. \tag{3.12}$$

Substituting (3.10) in (3.9'), we obtain the identity

$$\Phi_0 = r^2\, \partial^2(\Phi_2 - \Phi_1)/\partial r^2. \tag{3.13}$$

Hence, using the formulas $r\, \partial\Phi_1/\partial r = \Phi_0$, $r\, \partial\Phi_2/\partial r = \Phi_2 + \Phi_0$, we have

$$2r\, \partial\Phi_0/\partial r - \Phi_0 = r^2\, \partial^2(\Phi_2 - \Phi_1)/\partial r^2, \quad r\, \partial\Phi_0/\partial r - \Phi_0 = r^2\, \partial^2\Phi_1/\partial r^2. \tag{3.14}$$

On account of (3.13), (3.14) equation (3.9) takes the form

$$r^2 \frac{\partial^2}{\partial r^2} \left( v_i(x) - \frac{1}{8\pi\mu} \int\limits_S \left( \big(\Phi_1(x, y) + \Phi_2(x, y)\big)\, \delta_{ik} + x_k \frac{\partial\big(\Phi_1(x, y) - \Phi_2(x, y)\big)}{\partial x_i} \right.\right.$$
$$\left.\left. - 2x_i \frac{\partial\Phi_1(x, y)}{\partial x_k} + r^2 \frac{\partial^2\big(\Phi_1(x, y) - \Phi_2(x, y)\big)}{\partial x_k\, \partial x_i} \right) f_k(y)\, d_yS \right) = 0.$$

Hence

$$v_i(x) = \frac{1}{8\pi\mu} \int\limits_S \left( \big(\Phi_1(x, y) + \Phi_2(x, y)\big)\, \delta_{ik} + x_k \frac{\partial\big(\Phi_1(x, y) - \Phi_2(x, y)\big)}{\partial x_i} \right.$$
$$\left. - 2x_i \frac{\partial\Phi_1(x, y)}{\partial x_k} + r^2 \frac{\partial^2\big(\Phi_2(x, y) - \Phi_1(x, y)\big)}{\partial x_i\, \partial x_k} \right) f_k(y)\, d_yS + c_{ij}a_j + b_i, \tag{3.15}$$

where $c_{ij}$, $b_i$ are arbitrary constants.

Remark: If $\omega$ is a regular scalar function harmonic in $B^+$, then the general solution of the equation $r^2\, \partial^2\omega(x)/\partial r^2 = 0$ has the form $\omega(x) = a_k x_k + b$, where $a_1, a_2, a_3$ and $b$ are arbitrary constants. This circumstance explains why the term $c_{ij}x_j + b_i$ has been added to formula (3.15).

Choose constants $c_{ij}, b_i$ such that equality (1.27) is fulfilled. On account of (3.15) we have

$$x_j\big(\partial v_i(x)/\partial x_j + \partial v_j(x)/\partial x_i\big) - x_i \operatorname{div} v(x)$$
$$= 1/(4\pi\mu) \int\limits_S \Phi_0(x, y)\, f_i(y)\, d_yS + (c_{ij} + c_{ji} - c_{kk}\delta_{ij})\, x_j.$$

Therefore for (1.27) to be valid it is sufficient by virtue of (3.3) that $c_{ij} + c_{ji} - c_{kk}\delta_{ij} = 3/(4\pi\mu R^3) \int_S y_j f_i(y)\, d_y S$. Hence

$$c_{ij} + c_{ji} = \frac{3}{4\pi\mu R^3} \int_S \big(y_j f_i(y) - \delta_{ij} y_k f_k(y)\big)\, d_y S \qquad (i, j = 1, 2, 3).$$

Taking the latter equality into account, we have

$$\begin{aligned} c_{ij}x_j &= 1/2(c_{ij} + c_{ji})\, x_j + 1/2(c_{ij} - c_{ji})\, x_j \\ &= \frac{1}{8\pi\mu} \int_S \left(x_k \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_k}\right) \frac{3xy}{R^3} f_k(y)\, d_y S + \varepsilon_{ijk} a_j x_k, \end{aligned}$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol, $2a_1 = c_{32} - c_{23}$, $2a_2 = c_{13} - c_{31}$, $2a_3 = c_{21} - c_{12}$. Finally, for $v_i$ determined by (3.15) we have

$$\begin{aligned} v_i(x) = \frac{1}{8\pi\mu} \int_S \Big( &\big(\Phi_1(x, y) - \Phi_2(x, y)\big)\, \delta_{ik} \\ &+ x_k \frac{\partial}{\partial x_i} \left(\Phi_1(x, y) - \Phi_2(x, y) - \frac{3xy}{R^3}\right) - x_i \frac{\partial}{\partial x_k} \left(2\Phi_1(x, y) + \frac{3xy}{R^3}\right) \\ &+ r^2 \frac{\partial^2}{\partial x_i \partial x_k} \big(\Phi_2(x, y) - \Phi_1(x, y)\big)\Big) f_k(y)\, d_y S + \varepsilon_{ijk} a_j x_k + b_i. \end{aligned}$$

To determine $\psi$ we have on account of (1.25) and (3.4) the equation

$$2r \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + 2 \frac{2\lambda + \mu}{\lambda + \mu} r \frac{\partial \psi}{\partial r} + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi = \eta \operatorname{div} \theta + \operatorname{div} h,$$

which we rewrite in the equivalent form

$$2r \frac{\partial}{\partial r} r \frac{\partial(\psi - c)}{\partial r} + 2 \frac{2\lambda + \mu}{\lambda + \mu} r \frac{\partial(\psi - c)}{\partial r} + \frac{3\lambda + 2\mu}{\lambda + \mu} (\psi - c) = F, \quad (3.16)$$

where

$$c = \frac{\lambda + \mu}{3\lambda + 2\mu} \big(\eta\theta(0) + \operatorname{div} h(0)\big), \quad (3.17)$$

$$F(x) = \eta\big(\theta(x) - \theta(0) + \operatorname{div} h(x) - \operatorname{div} h(0)\big). \quad (3.18)$$

(3.16) is Euler's equation relative to $r$. Introducing the variable $t$ by the formula $t = \ln r$, $-\infty < t < \ln R$, we have

$$2 \frac{\partial^2(\psi - c)}{\partial t^2} + 2 \frac{2\lambda + \mu}{\lambda + \mu} \frac{\partial(\psi - c)}{\partial t} + \frac{3\lambda + 2\mu}{\lambda + \mu} (\psi - c) = F. \quad (3.19)$$

The characteristic equation

$$2k^2 + 2\big((2\lambda + \mu)/(\lambda + \mu)\big) k + (3\lambda + 2\mu)/(\lambda + \mu) = 0 \quad (3.20)$$

has roots $k = \left(2\lambda + \mu \pm \sqrt{-2\lambda^2 - 6\lambda\mu - 3\mu^2}\right)/2(\lambda + \mu)$. Keeping in view conditions (1.2), we shall here consider three cases

$$-\frac{2}{3}\mu < \lambda < \frac{\sqrt{3} - 3}{2}\mu, \qquad \lambda = \frac{\sqrt{3} - 3}{2}\mu, \qquad \lambda > \frac{\sqrt{3} - 3}{2}\mu$$

for which the characteristic equation has, respectively, two different real roots, a multiple real root, two complex-conjugate roots and therefore the solution of equation (3.19) differs in form. Let $\lambda > 1/2(\sqrt{3} - 3)\mu$. Then equation (3.20) has the complex-conjugate roots $-k_1 + ik_2$ and $-k_1 - ik_2$, where $k_1 = (2\lambda + \mu)/2(\lambda + \mu)$, $0{,}5 < k_1 < 1$, and $k_2 = \sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}/2(\lambda + \mu)$, $k_2 > 0$. Now it is not difficult to write the solution of the homogeneous equation in the general form $\psi(x) - c = e^{-k_1 t}(c_1 \cos k_2 t + c_2 \sin k_2 t)$. Assuming that $\psi$ is harmonic in $B^+$, the particular solution of equation (3.23) can be given in the form

$$\psi(x) - c = \frac{1}{2k_2} \int_{-\infty}^{t} e^{-k_1(t-\xi)} \sin(t - \xi)\, F\left(\frac{x}{r} e^{\xi}\right) d\xi.$$

Taking into account that in this formula $t = \ln r$ and introducing the variable $\tau = r^{-1} e^{\xi}$, we finally obtain

$$\psi(x) = -\frac{1}{2k_2} \int_0^1 F(\tau x) \sin(k_2 \ln \tau)\, \tau^{k_1 - 1}\, d\tau + c.$$

The cases $-2/3\mu < \lambda < 1/2(\sqrt{3} - 3)\mu$ and $\lambda = 1/2(\sqrt{3} - 3)\mu$ are treated similarly. Note that $F$ is determined by (3.18) and therefore

$$\overset{1}{\psi}(x) = \frac{\gamma}{8\pi(\lambda + \mu) R} \int_S P_1(x, y)\, g(y)\, d_y S + \frac{1}{8\pi\mu} \operatorname{div} \int_S \Phi_3(x, y)\, f(y)\, d_y S + c_1$$

in the case of Problem (II.I)$^+$,

$$\overset{2}{\psi}(x) = \frac{\gamma}{8\pi(\lambda + \mu) R} \int_S P_2(x, y)\, g(y)\, d_y S + \frac{1}{8\pi\mu} \operatorname{div} \int_S \Phi_3(x, y)\, f(y)\, d_y S + c_2$$

in the case of Problem (II.II)$^+$. Here

$$\Phi_3(x, y) = -\frac{1}{k_2} \int_0^1 \left( \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} - \frac{3xy}{R^3} \tau \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{2-k_1}}, \tag{3.21}$$

$$P_1(x, y) = -\frac{1}{k_2} \int_0^1 \left( \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1-k_1}}, \tag{3.22}$$

$$P_2(x, y) = -\frac{1}{k_2} \int_0^1 \left( \frac{2R}{|y - \tau x|} - 2 - \ln \frac{(|\tau x - y| + R)^2 - |\tau x|^2}{4R^2} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1-k_1}}, \tag{3.23}$$

$$c_1 = \frac{\gamma}{4\pi(3\lambda + 2\mu) R^2} \int\limits_S g(y) \, d_yS + \frac{3(\lambda + \mu)}{4\pi\mu(3\lambda + 2\mu) R^3} \int\limits_S yf(y) \, d_yS,$$

$$c_2 = \frac{3(\lambda + \mu)}{4\pi\mu(3\lambda + 2\mu) R^3} \int\limits_S yf(y) \, d_yS.$$

On account of (1.24) the solution of Problem (II.I)$^+$ takes the form

$$\theta(x) = \Pi(g)(x),$$

$$\begin{aligned} u_i(x) &= \frac{1}{8\pi\mu} \int\limits_S \Bigg( \big(\Phi_1(x, y) + \Phi_2(x, y)\big) \delta_{ik} + x_k \frac{\partial}{\partial x_i} \left( \Phi_1(x, y) - \Phi_2(x, y) + \frac{3xy}{R^3} \right) \\ &\quad + x_i \frac{\partial}{\partial x_k} \left( 2r \frac{\partial}{\partial r} \Phi_3(x, y) - \Phi_3(x, y) - 2\Phi_1(x, y) - \frac{\lambda}{3\lambda + 2\mu} \frac{3xy}{R^3} \right) \\ &\quad + \frac{R^2 - r^2}{2} \frac{\partial^2 \Phi_3(x, y)}{\partial x_i \, \partial x_k} + r^2 \frac{\partial^2 (\Phi_2(x, y) - \Phi_1(x, y) - \Phi_3(x, y))}{\partial x_i \, \partial x_k} \Bigg) f_k(y) \, d_yS \\ &\quad + \frac{\gamma}{8\pi(\lambda + \mu) R} \int\limits_S \Bigg( x_i \left( \frac{2(\lambda + \mu)}{(3\lambda + 2\mu) R} + P_1(x, y) + 2r \frac{\partial P_1(x, y)}{\partial r} \right) \\ &\quad + \frac{R^2 - 3r^2}{2} \frac{\partial P_1(x, y)}{\partial x_i} \Bigg) g(y) \, d_yS + \varepsilon_{ijk} a_j x_k + b_i, \end{aligned} \tag{3.24}$$

and the stress vector has by virtue of (1.26) the form

$$\begin{aligned} \tau^{n(x)}(x) - \gamma\theta(x) \, n(x) &= \frac{1}{4\pi r} \int\limits_S \frac{R^2 - r^2}{|x - y|^3} f(y) \, d_yS \\ &\quad + \frac{R^2 - r^2}{8\pi r} \int\limits_S \operatorname{grad} \operatorname{div} \left( r \frac{\partial}{\partial r} \Phi_3(x, y) - 2\Phi_3(x, y) \right) f(y) \, d_yS \\ &\quad + \frac{\gamma\mu(R^2 - r^2)}{8\pi(\lambda + \mu) Rr} \int\limits_S \operatorname{grad} \left( r \frac{\partial}{\partial r} P_1(x, y) - P_1(x, y) \right) g(y) \, d_yS. \end{aligned} \tag{3.25}$$

The solution of Problem (II.II)$^+$ can be written as follows:

$$\theta(x) = N(g)(x) + \theta_0,$$

$$\begin{aligned} u_i(x) &= \frac{1}{8\pi\mu} \int\limits_S \Bigg( \big(\Phi_1(x, y) + \Phi_2(x, y)\big) \delta_{ik} + x_k \frac{\partial}{\partial x_i} \left( \Phi_1(x, y) - \Phi_2(x, y) + \frac{3xy}{R^3} \right) \\ &\quad + x_i \frac{\partial}{\partial x_k} \left( 2r \frac{\partial}{\partial r} \Phi_3(x, y) - \Phi_3(x, y) - 2\Phi_1(x, y) - \frac{\lambda}{3\lambda + 2\mu} \frac{3xy}{R^3} \right) \\ &\quad + \frac{R^2 - r^2}{2} \frac{\partial^2}{\partial x_i \, \partial x_k} \Phi_3(x, y) \end{aligned}$$

$$+ r^2 \frac{\partial^2}{\partial x_j \, \partial x_k} \left( \Phi_2(x, y) - \Phi_1(x, y) - \Phi_3(x, y) \right) \Big) f_k(y) \, d_y S$$

$$+ \frac{\gamma}{8\pi(\lambda + \mu) R} \int_S \left( P_2(x,y) + 2r \frac{\partial}{\partial r} P_2(x, y) + \frac{R^2 - 3r^2}{2} \frac{\partial}{\partial x_i} P_2(x,y) \right) g(y) \, d_y S$$

$$+ \gamma \theta_0 \mathring{u}_i(x) + \varepsilon_{ijk} a_j x_k + b_i . \tag{3.26}$$

The stress vector has the form

$$\begin{aligned} &\tau^{(n(x))}(x) - \gamma\theta(x) n(x) \\ &= \frac{1}{4\pi r} \int_S \frac{R^2 - r^2}{|x - y|^3} f(y) \, d_y S \\ &\quad + \frac{R^2 - r^2}{8\pi r} \int_S \operatorname{grad} \operatorname{div} \left( r \frac{\partial}{\partial r} \Phi_3(x, y) - 2\Phi_3(x, y) \right) f(y) \, d_y S \\ &\quad + \frac{\mu\gamma(R^2 - r^2)}{8\pi(\lambda + \mu) Rr} \int_S \operatorname{grad} \left( r \frac{\partial}{\partial r} P_2(x, y) - P_2(x, y) \right) g(y) \, d_y S \\ &\quad + \mathring{\tau}^{(n(x))}(x) - \gamma\theta_0 n(x), \end{aligned} \tag{3.27}$$

$$\mathring{\tau}_i^{(n(x))}(x) = \lambda n_i(x) \frac{\partial \mathring{u}_k(x)}{\partial x_k} + \mu n_j(x) \left( \frac{\partial \mathring{u}_i(x)}{\partial x_j} + \frac{\partial \mathring{u}_j(x)}{\partial x_i} \right).$$

Classical solutions of Problems $(\mathrm{II.I})^+$ and $(\mathrm{II.II})^+$ can be obtained, using the uniqueness theorems similar to Theorems 1.1, 1.4 and 1.5. They result from the validity of Green's formula for classical solutions [6].

Let us investigate solutions of Problems $(\mathrm{II.I})^+$ and $(\mathrm{II.II})^+$ when the boundary values $f$ and $g$ are continuous on $S$. We have to prove that the inclusion $u \in \mathbf{C}(\overline{B}^+)$ is valid for $u$ determined by (3.24) or (3.26) and the inclusion $(\tau^{(n)} - \gamma\theta n)^+ \in \mathbf{C}(S)$ is valid for $\tau^{(n)} - \gamma\theta n$ determined by (3.25) or (3.27). We begin by establishing some properties of the functions $\Phi_1, \Phi_2, \Phi_3, P_1, P_2$ that are determined by equalities (3.11), (3.12), (3.21)–(3.23), respectively. It is easy to verify that $r \, \partial\Phi_3/\partial r = (1 - k_1) \Phi_3 + \Phi_4$, where

$$\Phi_4(x, y) = \int_0^1 \left( \frac{R^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{R} - \frac{3xy}{R^3} \tau \right) \cos(k_2 \ln \tau) \frac{d\tau}{\tau^{2-k_1}} .$$

On account of (2.10) to (2.12) we have

$$\begin{aligned} &\Phi_1(x, y) = \frac{2}{|x - y|} - \frac{1}{R} \ln \left( (|x - y| + R)^2 - |x|^2 \right) \frac{3xy}{R^3} - \frac{1}{R} \ln (4R^2 - 2), \\ &|P_k(x, y)|, \left| \frac{\partial P_k(x, y)}{\partial x_i} \right| \leqq \frac{c}{|x - y|}, \left| \frac{\partial^2 P_k(x, y)}{\partial x_i \, \partial x_j} \right| \leqq \frac{c}{|x - y|^2} \\ &(k = 1, 2; i, j = 1, 2, 3), \end{aligned} \tag{3.28}$$

$$\Phi_3(x, y) = -\frac{2}{R} \ln \left((|x - y| + R^2) - |x|^2\right)$$

$$+ \int_{0.5}^{1} \ln \left((|\tau x - y| + R)^2 - |\tau x|^2\right) \varphi_1(\tau)\, d\tau + \Phi_3'(x, y), \quad (3.29)$$

$$\Phi_4(x, y) = \frac{2}{|x - y|} - \frac{\lambda + 2\mu}{(\lambda + \mu) R} \ln \left((|x - y| + R)^2 - |x|^2\right)$$

$$+ \int_{0.5}^{1} \ln \left((|\tau x - y| + R)^2 - |\tau x|^2\right) \varphi_2(\tau)\, d\tau + \Phi_4'(x, y), \quad (3.30)$$

where $\Phi_3'$ and $\Phi_4'$ and their derivatives up to the second order inclusive, as well as $\varphi_1$ and $\varphi_2$, are bounded functions. Using the formula

$$\int \left(\frac{t^2 - 1}{t^2(t^2 - 3at + 1)^{3/2}} + \frac{1}{t^2} + \frac{3a}{t}\right) dt = -\frac{2}{t\sqrt{t^2 - 2at + 1}}$$

$$+ \frac{3\sqrt{t^2 - 2at + 1}}{t} + 3a \ln \left(\sqrt{t^2 - 2at + 1} + 1 - at\right), \qquad |a| \leqq 1,$$

we obtain

$$\Phi_2(x, y) = \frac{2}{|x - y|} - \frac{3\,|x - y|}{R^2} - \frac{3xy}{R^3} \ln \left((|x - y| + R^2) - |x|^2\right)$$

$$+ (3 \ln 4R^2 - 5)\, \frac{xy}{R^3} + \frac{1}{R}. \quad (3.31)$$

By virtue of (3.28) and (3.31), a vector $u$ determined by (3.24) or (3.26) can be represented as $u = \overset{1}{u} + \overset{2}{u} + \overset{3}{u}$, where

$$\overset{1}{u}(x) = \int_S L^1(x, y)\, f(y)\, d_yS,$$

$$\overset{2}{u}(x) = (R^2 - |x|^2) \int_S L^2(x, y)\, f(y)\, d_yS,$$

$$\overset{3}{u}(x) = \frac{|x|^2}{8\pi\mu} \int_S L^3(x, y)\, f(y)\, d_yS,$$

$$L^m = \|L_{ij}^m\|_{3\times 3} \quad (m = 1, 2, 3),$$

$$|L^1(x, y)| \leqq \frac{c}{|x - y|}, \quad |L^2(x, y)| \leqq \frac{c}{|x - y|^2},$$

$$L_{i,j}^3(x, y) = \partial^2 \left(\Phi_2(x, y) - \Phi_1(x, y) - \Phi_3(x, y)\right) / \partial x_i\, \partial x_j.$$

Therefore $\overset{1}{u}, \overset{2}{u} \in C(\bar{B}^+)$. Now we shall prove that $\overset{3}{u} \in C(\bar{B}^+)$. By virtue of equalities (3.28), (3.29), (3.31) we write

$$\Phi_2(x, y) - \Phi_1(x, y) - \Phi_3(x, y) = \Phi(x, y)$$

$$+ \int_{0.5}^{1} \ln \left((|\tau x - y| + R)^2 - |\tau x|^2\right) \varphi(\tau)\, d\tau,$$

where $\varphi$, as well as $\Phi$ and its derivatives of the first order, are bounded functions and $|\partial^2\Phi(x, y)/\partial x_i\, \partial x_j| \leqq c/|x - y|$ $(i, j = 1, 2, 3)$. Thus to prove the inclusion $\overset{3}{u} \in \mathbf{C}(\bar{B}^+)$ it is sufficient to show that $\chi \in \mathbf{C}(\bar{B}^+)$, where

$$\chi(x) = \int_S \left( \int_0^1 \operatorname{grad} \operatorname{div} \ln \left( (|\tau x - y| + R)^2 - |\tau x|^2 \right) \varphi(\tau)\, d\tau \right) f(y)\, d_y S.$$

Obviously, $\chi \in \mathbf{C}(B^+)$ and it can therefore be assumed that $0{,}5R \leqq |x| \leqq R$. Since $(|z - y| + R)^2 - |z|^2 \geqq 2R\, |z - y|$, $|z_i - y_i|/|z - y| \leqq 1$, $|y_i| \leqq R$, for $z \in B^+$, $y \in S$, we easily obtain

$$\left|\partial \ln \left( (|\tau x - y| + R)^2 - |\tau x|^2 \right) / \partial x_i\, \partial x_j \right| \leqq 4/|\tau x - y|.$$

On account of this estimate, for expressions

$$\Psi_{ij}(x, y) \equiv \int_{0,5}^1 \frac{\partial^2}{\partial x_i\, \partial x_j} \ln \left( (|\tau x - y| + R)^2 - |\tau x|^2 \right) \varphi(\tau)\, d\tau$$

we have

$$|\Psi_{ij}(x, y)| \leqq c \int_{0,5}^1 \frac{d\tau}{|\tau x - y|^2} = \frac{c}{R\, |x| \sqrt{1 - a^2}} \operatorname{arctg} \frac{t - a}{\sqrt{1 - a^2}} \Bigg|_{t = |x|/2R}^{t = |x|/R},$$

where $a = xy/R\, |x| \equiv \cos \gamma$. Therefore $|\Psi_{ij}(x, y)| \leqq c/\sqrt{1 - a^2}$. Let $x \in B^+$ and let $x_0$ be a point of the sphere $S$ such that $|x - x_0| = \min \{|x - y| : y \in S\}$. Then $|y - x_0| = 2R\, |\sin (\gamma/2)|$, $|y + x_0| = 2R\, |\cos (\gamma/2)|$, and therefore $\sqrt{1 - a^2} = \sqrt{1 - \cos^2 \gamma} = |\sin \gamma| = |y - x_0|\, |y + x_0|/2R$. Thus

$$|\Psi_{ij}(x, y)| \leqq \frac{c}{|y - x_0|\, |y + x_0|}, \qquad |\chi(x)| \leqq c \int_S \frac{f(y)\, d_y S}{|y - x_0|\, |y + x_0|}.$$

Keeping in mind that $|y - x_0| > R$ if $|y + x_0| \leqq R$ and, conversely, $|y + x_0| > R$ if $|y - x_0| \leqq R$ and repeating the reasoning used in proving the continuity of the harmonic single-layer potential [13], we can easily establish the inclusion $\chi \in \mathbf{C}(\bar{B}^+)$. Thus, $u \in \mathbf{C}(\bar{B}^+)$.

We next prove the inclusion $(\tau^{(n)} - \gamma\theta n)^+ \in \mathbf{C}(S)$. On the basis of representations (3.29) and (3.30) for $\tau^{(n)} - \gamma\theta n$ determined by (3.25) or (3.27) we have

$$\tau^{(n(x))}(x) - \gamma\theta(x)\, n(x) = \frac{R}{|x|}\, \Pi(f)\,(x) + (R^2 - |x|^2) \int_S Q(x, y)\, d_y S$$

$$+ \frac{R^2 - |x|^2}{4\pi\, |x|} \int_S \operatorname{grad} \operatorname{div} \frac{1}{|x - y|}\, f(y)\, d_y S + \eta(x),$$

where $|Q(x, y)| \leqq c/|x - y|^2$, $\lim\limits_{B^+ \ni x \to y \in S} \eta(x) = 0$. Now by virtue of Lemma 2.1, estimate (2.24) and the property of the Poisson integral it can be easily verified that

$$\lim_{B^+ \ni x \to y \in S} \left( \tau^{(n(x))}(x) - \gamma\theta(x)\, n(x) \right) = f(y)$$

and therefore the inclusion $(\tau^{(n)} - \gamma\theta n)^+ \in C(S)$ is valid. Thus we have

Theorem 3.1: *If $f, g \in C(S)$ and conditions* (1.12) [(1.13)] *are satisfied, then the pair $(u, \theta)$ determined by formula* (3.24) [(3.26)] *is the classical solution of Problem* (II.I)$^+$ [(II.II)$^+$].

Problems (II.I)$^-$ and (II.II)$^-$ are solved in a similar manner. No new difficulty arises in constructing and investigating the differential properties of their solutions.

## § 4 Concluding remarks. Historical background

Static equations of thermoelasticity in terms of displacement and temperature components give rise to system (1.1), which consists of four equations for four unknown values $u_1, u_2, u_3, \theta$. The temperature $\theta$ satisfies the Laplace equation. If $\theta$ is known, then the displacement $u = (u_1, u_2, u_3)$ is determined by the system of classical elasticity. At the same time, in the case of Problems (I.I)$^\pm$ to (II.II)$^\pm$, to find $\theta$ we have the Dirichlet or the Neumann problem for the Laplace equation. $\theta$ can therefore be determined in quadratures by formulas (1.15), (1.16), (1.18) or (1.20). Substituting the obtained value of $\theta$ in system (1.1) and in the boundary conditions, we determine $u$ by solving the following problems:

Find in $B^\pm$ a solution of the system

$$\mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = F \tag{4.1}$$

satisfying the condition $(u)^\pm = \psi$ in case of Problem (I)$^\pm$, or the condition $(\tau^{(n)})^\pm = \psi$ in case of Problem (II)$^\pm$, as well as the condition of damping at infinity (1.5) in the case of the domain $B^-$.

Here $F$ and $\psi$ are expressed explicitly (effectively) by means of quadratures in terms of boundary values $f$ and $g$.

One particular solution of system (4.1) is given by the volume potential $V(F)$ with the density $F$ [6]. It is expressed by means of quadratures in terms of $F$. In this case, if $u$ is represented in the form $u = V(F) - u_0$, then $u_0$ must be the solution of the same Problems (I)$^\pm$ or (II)$^\pm$ in which $F = 0$ and $\psi$ is expressed by means of quadratures in terms of $F$. We denote these problems by (I)$_0^\pm$ and (II)$_0^\pm$. Their solutions are constructed in quadratures [15, 20].

Though the presented algorithm for the solution of Problems (I.I)$^\pm$ to (II.II)$^\pm$ leads to the purpose, the solution representations obtained in this manner are rather cumbersome (they contain both surface and volume integrals) as compared with representations (2.5), (2.8), (2.29), (2.30), (3.24), (3.26).

Quite a few mathematicians have tried to solve Problems (I)$_0^\pm$ and (II)$_0^+$. The first work in this direction belongs to G. Lamé [7], who obtained the representation in terms of series of spherical coordinates of the displacement vector. W. Thomson [18] obtained the representation in terms of series of Cartesian coordinates of the displacement vector. The results of G. Lamé and W. Thomson were afterwards repeated and used in the specific situations. Some authors represented them in a different form. The first paper in which solutions of the above problems are constructed in quadratures belongs to C. W. Borchardt [1]. Simpler representations of solutions in quadratures are also obtained in Cerruti [3], O. Tedone and C. Somigliana [17], and in others. Special mention should be made of R. Marcolongo's paper [11: pp 279—299], in which a simple procedure of deriving the solution by means of representation (1.21) was evidently given for the first time. This representation was later used by E. Trefftz [19], who is often believed to be its author [9, 16]. Problem (II)$_0^\pm$ was solved in quadratures for the first time in [20]. The same problem is solved by the expansion in a series in LURYE [10]. Detailed bibliographical references concerning all these problems can be found in [4, 8–11, 16].

Note that we can also solve in quadratures problems with concentrated singularities for the sphere. They are formulated as follows: Let $x_1, \ldots, x_r$ be interior points of the sphere $B^+$, and let $p_1, \ldots, p_r$ be some non-negative numbers.

**Problem $(\mathrm{I.I})_p^+$:** Find in $B^+$ a pair $(u, \theta)$ which is regular in $B^+ \setminus \{x_1, \ldots, x_r\} \equiv B_r^+$ and satisfies system (1.1) in $B_r^+$, the boundary conditions $(u)^+ = f$, $(\theta)^+ = g$ on $S$, and the condition $|u(x)| \leq c/|x - x_i|^{p_i}$ $(i = 1, \ldots, r)$.

If all $p_i < 1$, then Problems $(\mathrm{I.I})^+$ and $(\mathrm{I.I})_p^+$ are equivalent, i.e., they have the same solution. If at least one $p_i \geq 1$, then Problem $(\mathrm{I.I})_p^+$ will have — in addition to the solution of Problem $(\mathrm{I.I})^+$ — other singular solutions which can be constructed in quadratures, using the formulas given in the present paper and the algorithm from [2]. Problems $(\mathrm{I.I})^-$, $(\mathrm{I.II})^\pm$, $(\mathrm{II.I})^\pm$, $(\mathrm{II.II})^\pm$ are formulated and solved in a similar manner.

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