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The Volume Problem for Pseudo-Riemannian Manifolds

R. SCHIMMING and D. MATEL-KAMINSKA

Wir stellen das Volumenproblem für pseudo-Riemannsche Mannigfaltigkeiten und legen erste Resultate dazu vor: Aus dem Volumen kleiner abgeschnittener Lichtkegel werden gewisse geometrische Eigenschaften abgelesen.

Мы поднимаем проблему объёма псевдо-римановых многообразий и достигаем первых результатов: исследование объёма маленьких разрезанных световых конусов позволяет судить о некоторых геометрических свойствах.

We pose the volume problem for pseudo-Riemannian manifolds and present first results on it: certain geometric properties are read from the volume of small truncated light cones.

Introduction

The volume problem for (pseudo-) Riemannian manifolds (M, g) of given dimension n and signature $('n, ''n)$, as we take it here, roughly reads as follows: to what extent does the volume of small naturally defined test bodies determine the geometry? Here the "test bodies" are assumed to be compact and to depend on as small a number of real parameters as possible. One should begin with test bodies of dimension n, but those with a positive codimension could be studied too.

The problem of the volume of small geodesic balls in properly Riemannian geometry is classical. It has historical roots in the theory of surfaces and has been studied in a series of papers of A. GRAY, L. VANHECKE, O. KOWALSKI and others [4–7, 10–18].

F. and B. GACKSTATTER [2, 3] initiated the volume problem for Lorentzian manifolds. They proposed truncated light cones as the test bodies.

The general idea of the definition of test bodies is the following: Choose a fixed point $y \in M$ and define test bodies in the vector space T_yM, the tangential space at y. Map then these test bodies by means of the exponential map $\exp_y$ with origin y into the manifold M. Effectively, the procedure is done by means of normal coordinates x^a of $x \in M$ with respect to the origin $y \in M$. The ball with radius $R > 0$ in T_yM

$$(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 \leq R^2$$

is mapped to the geodesic ball with radius R and centre y in a properly Riemannian manifold (M, g). Analogously, the truncated light cone with altitude $R > 0$ in T_yM

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2 \leq (x^0)^2 \leq R^2$$

is mapped to the truncated light cone with altitude R and vertex y in a Lorentzian manifold (M, g). For a pseudo-Riemannian manifold (M, g) of dimension n and signature $('n, ''n)$ we propose the test bodies in T_yM to be defined by

$$(x^{'n+1})^2 + \cdots + (x^n)^2 \leqq (x^1)^2 + \cdots + (x^{'n})^2 \leqq R^2,$$

and we call the image with respect to $\exp_y$ "truncated light cone with altitude R and vertex y" in (M, g). (We have no better name at hand.) For any signature, the volume of test bodies can be expanded into an asymptotic power series in R by means of Fubini's integral theorem and Pizzetti's expansion formula for spheres and balls in flat space. Each coefficient in the asymptotic series is a differential expression in the normal volume function $\varrho = \varrho(x, y)$. In order to extract geometrical informations from the volume of test bodies in (M, g), it is compared with the volume of analogous test bodies in some simple "model manifold" (M_0, g_0). The manifolds (M, g) and (M_0, g_0) are called isovolumal if M is covered by neighbourhoods U and local diffeomorphisms $\varphi\colon U \to \varphi(U) \subseteq M_0$ such that (U, g) and $(U, \varphi^* g_0)$ exhibit the same volume of test bodies. That means, the volume is calculated twice, once with respect to the proper metric g of M and once with respect to the local pull-back metric $\varphi^* g_0$. The parameters of the test bodies are the same in the two calculations. Clearly, the simplest model manifolds (M_0, g_0) are the flat ones; for them the volume of a test body depends on R only and not on the other parameters. F. and B. Gackstatter [2, 3] introduced the volume defect, that is the relative deviation of the volume of a test body in (M, g) from the volume of an analogous test body, with the some R, in flat space $\mathbb{R}^n$. A manifold (M, g) is isovolumal to $\mathbb{R}^n$ if and only if the volume defect vanishes for all sufficiently small $R > 0$. The "volume conjecture" says that a manifold with vanishing volume defect is flat. For properly Riemannian geometry, this conjecture is due to A. Gray and L. Vanhecke [6] and it is neither confirmed nor refuted until now.

The first author of the present paper has decided the volume conjecture for Lorentzian geometry in the affirmative [20]. He has, moreover, shown that Ricci-flatness is also a geometric property which can be read from the volume of small truncated light cones in a Lorentzian manifold. For $n = 4$, F. Gackstatter [3] derived the same results. It is in order to study the volume problem in general pseudo-Riemannian geometry after that in properly Riemannian and Lorentzian geometries. This is the topic of the present paper. The programme sketched above is not realized in full generality. In this our attempt the following partial results are achieved:

1. The volume of any small truncated light cone is asymptotically expanded in powers of the altitude R. The first terms of the expansion are given as expressions in the curvature of (M, g).

2. A manifold with definite Ricci curvature or with definite four-form $2(Riem)^2 - 5(Ric)^2 - 9d^2Ric$ has a non-vanishing volume defect.

3. If the pseudo-Riemannian product of two properly Riemannian manifolds $('M, 'g)$, $(''M, ''g)$ has vanishing volume defect, then all the invariants $(\Delta^k_{'y}\,'\varrho)\,('y, 'y)$, $(\Delta^k_{''y}\,''\varrho)\,(''y, ''y)$ $(k = 1, 2, \ldots)$ are constants. Here $'\varrho = '\varrho('x, 'y)$, $''\varrho = ''\varrho(''x, ''y)$ are the normal volume functions and $\Delta_{'y}$, $\Delta_{''y}$ the so-called Euclidean-Laplace operators of $('M, 'g)$, $(''M, ''g)$, respectively. Particularly, the factor manifolds must have constant scalar curvatures $'S$, $''S$ such that $(''n + 2)\,'S + ''n''S = 0$, where $'n = \dim 'M$, $''n = \dim ''M$.

4. The pseudo-Riemannian product of two manifolds of constant curvature or of two two-dimensional manifolds with vanishing volume defect is flat.

5. A coordinate-independent expression for the so-called Euclidean-Laplace operator Δ_y with respect to $y \in M$ is presented: $\Delta_y u = \varrho\, \nabla_a(\varrho^{-1}\gamma^{ab}\, \nabla_b u)$. Here $\varrho = \varrho(x, y)$ is the normal volume function and $\gamma^{ab} = \gamma^{ab}(x, y)$ is some contravariant tensor with respect to x, explicitly given in the text, which reduces to constant components $g^{ab}(y)$ in a normal coordinate system with origin y.

Two-point geometry

Certain scalars and tensors depending on two points x, y naturally arise in (pseudo-) Riemannian geometry. We consider smooth (i.e. of class C^∞) n-dimensional Riemannian manifolds of arbitrary signature.

Definition 1: The *distance function* $\sigma = \sigma(x, y)$ is the solution of the problem

$$g^{ab}\,\nabla_a\sigma . \nabla_b\sigma = 2\sigma, \qquad (\nabla_a\sigma)\,(y, y) = 0, \qquad (\nabla_a\nabla_b\sigma)\,(y, y) = g_{ab}(y).$$

The function $\mu = \mu(x, y)$ is defined by $2\mu = \Delta\sigma - n$. The *normal volume function* $\varrho = \varrho(x, y)$ is the solution of the problem $g^{ab}\,\nabla_a\sigma\,\nabla_b\varrho = 2\mu\varrho$, $\varrho(y, y) \doteq 1$. Here and in the following, the differential operators ∇, Δ, d, ... refer to the first argument x; ∇ denotes the Levi-Civita derivative to g and $\Delta := g^{ab}\,\nabla_a\,\nabla_b$ the usual Laplace operator acting on tensor fields.

It is known that both the two-point functions σ and ϱ are defined in some neighbourhood of the diagonal of $M \times M$ and are symmetric in their arguments: $\sigma(x, y) = \sigma(y, x)$, $\varrho(x, y) = \varrho(y, x)$. For properly Riemannian manifolds σ equals one half of the square of the geodesic distance s between two sufficiently neighboured points: $2\sigma(x, y) = s(x, y)^2$. For pseudo-Riemannian manifolds σ defines the geodesic distance: $2\,|\sigma(x, y)| =: s(x, y)^2$. The limit for $x \to y$, if existing, of a two-point quantity depending on x, y is called its coincidence limit. The equality of the coincidence limits is an equivalence relation between two-point quantities and shall be denoted by $\doteq$. One-point quantities and constants may be looked upon as special two-point quantities. A symmetric differential form of degree p

$$u_p = u_{a_1a_2\cdots a_p}\,dx^{a_1}\,dx^{a_2}\ldots dx^{a_p}$$

is a new notation for a symmetric covariant tensor field of degree p. Apart from the usual tensorial notations there are specific operations for symmetric forms:

— Symmetric product of a p-form u_p and a q-form v_q:

$$u_pv_q := u_{a_1\cdots a_p}v_{b_1\cdots b_q}\,dx^{a_1}\cdots dx^{a_p}\,dx^{b_1}\cdots dx^{b_q}.$$

— Symmetric power: $u^k := u\,u \ldots u$ (k times).

— Trace $=$ tr with respect to the metric g:

$$\operatorname{tr} u_p := g^{ab}u_{aba_3\cdots a_p}\,dx^{a_3}\cdots dx^{a_p} \quad \text{for } p \geqq 3,$$

$$\operatorname{tr} u_0 := 0, \qquad \operatorname{tr} u_1 := 0, \qquad \operatorname{tr} u_2 := g^{ab}u_{ab}.$$

— Value of u_p on a vector or vector field v:

$$u_p(v, v, \ldots, v) := u_{a_1a_2\cdots a_p}v^{a_1}v^{a_2}\cdots v^{a_p}.$$

— Symmetric differential d:

$$du_p := \nabla_a u_{a_1\cdots a_p}\,dx^a\,dx^{a_1}\cdots dx^{a_p}.$$

— Powers of tr and d:

$$\operatorname{tr}^k := (\operatorname{tr})\,(\operatorname{tr})\cdots(\operatorname{tr}), \qquad d^k := dd\cdots d.$$

The curvature tensor, Ricci tensor, and scalar curvature are denoted by *Riem*, *Ric*, S, respectively. The components of *Riem*, *Ric* read R_{abcd}, R_{ab}, respectively. We use the special abbreviations

$$(Riem)^2 := R_{aefb}R_{c\cdot\cdot d}^{\;\;ef}\,dx^a\,dx^b\,dx^c\,dx^d,$$

$$|Riem|^2 := R_{abcd}R^{abcd}, \qquad |Ric|^2 := R_{ab}R^{ab}.$$

The sign conventions for the curvature quantities are the same as in [4, 6, 20]. There holds [20]

$$d\varrho \doteq 0, \quad -3d^2\varrho \doteq Ric, \quad -2d^3\varrho \doteq d\,Ric,$$

$$-15d^4\varrho \doteq 2(Riem)^2 - 5(Ric)^2 + 9d^2\,Ric. \tag{1}$$

Definition 2: The *Euclidean Laplacian* $\overset{\circ}{\Delta}_y$ with respect to the origin y, acting on functions $u = u(x)$ over M, is given by

$$\Delta_y u = \varrho\, \nabla_a(\varrho^{-1}\gamma^{ab}\,\nabla_b u), \tag{2}$$

where the two-point tensor field $\gamma^{ab} = \gamma^{ab}(x, y)$ is defined by

$$\sigma_{ai} = \partial/\partial x^a\, \partial/\partial y^i\, \sigma(x, y), \quad \gamma_{ab} = g^{ij}(y)\,\sigma_{ai}\sigma_{bj},\ (\gamma^{ab}) = (\gamma_{ab})^{-1}.$$

Theorem 1: *In normal coordinates* $\bar{x} = (x^a) \in \mathbb{R}^n$ *of* $x \in M$ *with respect to the origin* $y \in M$ *there holds*

$$\sigma(x, y) = 1/2 g_{ab}(y)\, x^a x^b, \tag{3}$$

$$\varrho(x, y) = |\det g_{ab}(x)|^{1/2}\, |\det g_{ij}(y)|^{1/2}, \tag{4}$$

$$\sigma_{ai} = -g_{ai}(y), \quad \gamma^{ab} = g^{ab}(y), \tag{5}$$

$$\Delta_y u = g^{ab}(y)\, \partial/\partial x^a\, \partial/\partial x^b\, u. \tag{6}$$

Proof: The formulas (3), (4) and the first part of (5) are generally known [21, 19, 1]; the second part of (5) is an immediate consequence. A well-known formula for the divergence of a vector field implies

$$\Delta_y u = \varrho\, |\det g_{ab}(x)|^{-1/2} \frac{\partial}{\partial x^a}\left(\varrho^{-1}\, |\det g_{ab}(x)|^{1/2}\, \gamma^{ab} \frac{\partial u}{\partial x^b}\right).$$

Considering that in normal coordinates

$$\varrho\, |\det g_{ab}(x)|^{-1/2} = |\det g_{ij}(y)|^{-1/2}, \quad \gamma^{ab} = g^{ab}(y),$$

we arrive at the result (6) ▮

The Euclidean Laplacian Δ_y with respect to an origin $y \in M$ has been explicitly introduced through the representation (6) by A. Gray and T. J. Willmore [7, 22]. It has also been studied by O. Kowalski [14, 15]. These authors consider the properly Riemannian case only and they normalize $g^{ab}(y) = \delta^{ab}$ (diagonal matrix with entries 1 in the main diagonal). In the pseudo-Riemannian case, "Euclidean Laplacian" is not a good name, but we have no other name to propose. In [7, 22] the coincidence limits of $\Delta_y u$, $\Delta_y{}^2 u$, $\Delta_y{}^3 u$ have been calculated. Let us reproduce the first and second by means of our covariant definition (2):

Synge's book [21] provides, after simple calculations, the coincidence relations

$$l_a \doteq 0, \quad \nabla_a l_b \doteq -1/3 R_{ab}, \quad \Delta l_a \doteq -1/2 \nabla_a S,$$

$$\nabla_a \gamma^{ab} \doteq 0, \quad \Delta\gamma^{ab} \doteq 2/3 R^{ab}, \quad \nabla^a \nabla_c \gamma^{bc} \doteq 2/3 R^{ab},$$

$$\Delta\, \nabla_a \gamma^{ab} \doteq -1/6\, \nabla^b S;$$

where we abbreviate $l_a = \nabla_a \ln \varrho$. With this, we find

$$\begin{aligned}\Delta_y u &= \gamma^{ab}\,\nabla_a\,\nabla_b u + (\nabla_a\,\gamma^{ab} - l_a\gamma^{ab})\,\nabla_b u \\ &\doteq \gamma^{ab}\,\nabla_a\,\nabla_b u \doteq g^{ab}\,\nabla_a\,\nabla_b u = \Delta u,\end{aligned}$$

$$\begin{aligned}\Delta_y{}^2 u &\doteq \Delta\Delta_y u \\ &\doteq \Delta^2 u + \left(\Delta\gamma^{ab} + 2\nabla^a \nabla_c\gamma^{bc} - 2(\nabla^a l_c)\,\gamma^{bc}\right)\nabla_a\nabla_b u \\ &\quad + \left(\Delta\,\nabla_a\gamma^{ab} - (\Delta l_a)\,\gamma^{ab}\right)\nabla_b u \\ &\doteq \Delta^2 u + 2/3R^{ab}\,\nabla_a\nabla_b u + 1/3g^{ab}\,\nabla_a S\,\nabla_b u.\end{aligned}$$

A normal coordinate system $x \mapsto \bar{x} \in \mathbb{R}^n$ with origin y is, by definition, the inverse to the exponential map $\exp_y$ from T_yM to M. (Both the maps $x \mapsto \bar{x}$ and $\exp_y$ are restricted here to appropriate domains.) Considering this, the formulas (3), (4) can be reinterpreted. Let, in the following, $d^n\bar{x}$ denote the measure on T_yM defined by the metric $g(y)$ with (constant) components $g_{ij}(y)$ and let dvol denote the canonical measure on M defined by the metric $g(x)$ with (variable) components $g_{ab}(\dot{x})$. Further, identify a measure with an alternating n-form and denote by $\exp_y{}^*$ the pull-back of $\exp_y$. This pull-back transforms covariant tensors on M to covariant tensors on T_yM. With these notations, coordinate-independent expressions for σ and ϱ can be given [19, 1]:

$$\begin{aligned}&\sigma(x, y) = 1/2g(y)\,(\exp_y{}^{-1} x, \exp_y{}^{-1} x),\\ &\exp_y{}^*\,\mathrm{dvol} = \varrho(\exp_y \bar{x}, y)\,d^n\bar{x},\end{aligned} \tag{7}$$

i.e. the normal volume function equals the Radon-Nikodym derivative of $\exp_y{}^*$ dvol with respect to $d^n\bar{x}$.

Let us finish this section by shortly reviewing the volume problem for properly Riemannian manifolds. The following formulas are needed in the next section.

Definition 3: The numbers

$$a_k = a_k(n) = 2^{-2k}(k!)^{-2}\binom{n/2+k-1}{k}^{-1} = 2^{-2k}(k!)^{-1}\,\Gamma\left(\frac{n}{2}+k\right)^{-1}\Gamma\left(\frac{n}{2}\right)$$

for integer $k \geqq 0$ are called *Pizzetti's coefficients.*

Obviously,

$$\begin{aligned}&a_0 = 1, \qquad a_1{}^{-1} = 2n, \qquad a_2{}^{-1} = 8n(n+2),\\ &a_k{}^{-1} = 2^k k!\,n(n+2)\dots(n+2k-2) \quad \text{for } k \geqq 1.\end{aligned} \tag{8}$$

The name "Pizzetti's coefficients" appeared in [14]. We denote by $B^n(y, R)$ and $S^{n-1}(y, R)$ the geodesic ball and the geodesic sphere, respectively, with centre y and radius $R > 0$ in an n-dimensional properly Riemannian manifold. Further, we denote by $B^n(R)$ and $S^{n-1}(R)$ the ball and the sphere, respectively, with centre 0 and radius $R > 0$ in the n-dimensional Euclidean space $\mathbb{R}^n$. The symbol Vol means the volume with respect to the canonical measure.

Proposition 1: *There hold the asymptotic power series expansions*

$$\frac{\operatorname{Vol} S^{n-1}(y, R)}{\operatorname{Vol} S^{n-1}(R)} \sim \sum_{k=0}^{\infty} a_k(n)\,(\Delta_y{}^k\varrho)\,(y, y)\,R^{2k}, \tag{9}$$

$$\frac{\operatorname{Vol} B^n(y, R)}{\operatorname{Vol} B^n(R)} \sim \sum_{k=0}^{\infty} a_k(n+2)\,(\Delta_y{}^k\varrho)\,(y, y)\,R^{2k}. \tag{10}$$

The proof can be read from [6, 14] and is based on the famous Pizzetti formulas

$$\frac{\int\limits_{S^{n-1}(R)} u\, dS}{\operatorname{Vol} S^{n-1}(R)} \sim \sum_{k=0}^{\infty} a_k(n)\, (\Delta_0{}^k u)\,(0)\, R^{2k},$$

$$\frac{\int\limits_{B^n(R)} u\, d^n x}{\operatorname{Vol} B^n(R)} \sim \sum_{k=0}^{\infty} a_k(n+2)\, (\Delta_0{}^k u)\,(0)\, R^{2k},$$

where Δ_0 denotes the Laplacian of $\mathbb{R}^n$ ∎

The volume of truncated light cones

In this section we consider n-dimensional pseudo-Riemannian manifolds (M, g) of signature $('n, ''n)$ such that $'n \geqq 1$, $''n \geqq 1$, $'n + ''n = n$. The orthogonal groups to the dimensions $'n$, $''n$ are denoted by $O('n)$, $O(''n)$, respectively, the pseudo-orthogonal group to the signature $('n, ''n)$ is denoted by $O('n, ''n)$.

Definition 4: An $O('n) \times O(''n)$-*structure* at the point $y \in M$ is a representation of the metric at y as the difference of two positive semidefinite quadratic forms with the maximal possible ranks: $g(y) = 'g(y) - ''g(y)$, $'g(y)\,(v, v) \geqq 0$, $''g(y)\,(v, v) \geqq 0$ for every $v \in T_y M$, rank $'g(y) = 'n$, rank $''g(y) = ''n$.

Such $O('n) \times O(''n)$-structures at a point exist. There exist normal coordinate systems $x \mapsto \bar{x} = (x^a) \in \mathbb{R}^n$ with the origin y in which the components of $'g(y)$, $''g(y)$ are given by

$$\left('g_{ij}(y)\right) = \begin{pmatrix} g_{'i'j}(y) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \left(''g_{ij}(y)\right) = \begin{pmatrix} 0 & 0 \\ 0 & g_{''i''j}(y) \end{pmatrix}, \tag{11}$$

respectively. Here we introduce and use a new index convention

$$'a, 'b, \ldots, 'i, 'j, \ldots = 1, 2, \ldots, 'n;$$

$$''a, ''b, \ldots, ''i, ''j, \ldots = 'n + 1, \ldots, n.$$

The normal coordinate system can be further specialized to

$$g_{'i'j}(y) = \delta_{'i'j}, \qquad g_{''i''j}(y) = \delta_{''i''j}.$$

If $O('n) \times O(''n)$-structures are given at various points y of some domain $U \subseteqq M$ and if they depend, in a sense which can be made precise, smoothly on $y \in M$ then we arrive at the usual notion of a local $O('n) \times O(''n)$-structure [8, 9]. If, particularly, $U = M$ then we have a global $O('n) \times O(''n)$-structure. Generally, such a global structure does not exist. If it exists then it is called a reduction of the global $O('n, ''n)$-structure defined by the metric g. A local $O('n) \times O(''n)$-structure exists in a sufficiently small neighbourhood of each point.

Definition 5: The *truncated light cone* $C(y, R; 'g, ''g)$ with vertex $y \in M$ and altitude $R > 0$ with respect to an $O('n) \times O(''n)$-structure at y is described by the inequalities

$$''g^{ij}(y)\, \frac{\partial \sigma}{\partial y^i}\, \frac{\partial \sigma}{\partial y^j} \leqq 'g^{ij}(y)\, \frac{\partial \sigma}{\partial y^i}\, \frac{\partial \sigma}{\partial y^j} \leqq R^2.$$

Here $'g^{ij}(y)$, $''g^{ij}(y)$ originate from $'g_{ij}(y)$, $''g_{ij}(y)$ by raising of the indices.

The first unequality of Definition 5 expresses $\sigma(x, y) \geqq 0$, as will become clear in the following, while the second unequality is the "truncation condition".

In normal coordinates with the origin y there holds, equivalent to (5), $\partial\sigma/\partial y^i = -g_{ia}(y)\, x^a$ and as a consequence

$$'g^{ij}(y)\,\frac{\partial\sigma}{\partial y^i}\,\frac{\partial\sigma}{\partial y^j} = 'g_{ab}(y)\, x^a x^b, \qquad ''g^{ij}(y)\,\frac{\partial\sigma}{\partial y^i}\,\frac{\partial\sigma}{\partial y^j} = ''g_{ab}(y)\, x^a x^b.$$

Thus, in the normal coordinates belonging to (11) $C(y, R, 'g, ''g)$ is described by the inequalities

$$g_{''a''b}(y)\, x''^a x''^b \leqq g_{'a'b}(y)\, x'^a x'^b \leqq R^2.$$

With $'r$, $''r$ defined by $'r^2 = g_{'a'b}(y)\, x'^a x'^b$, $''r^2 = g_{''a''b}(y)\, x''^a x''^b$, the truncated light cone is also described by $0 \leqq ''r \leqq 'r \leqq R$. These descriptions show that for all sufficiently small $R > 0$ the point set $C(y, R, 'g, ''g)$ is defined and is compact. In the normalization $g_{'a'b}(y) = \delta_{'a'b}$, $g_{''a''b}(y) = \delta_{''a''b}$ the truncated light cone is described by

$$(x^{'n+1})^2 + \cdots + (x^n)^2 \leqq (x^1)^2 + \cdots + (x^{'n})^2 \leqq R^2.$$

We denote the set of all points $\bar{x} = (x^1, \ldots, x^{'n}, x^{'n+1}, \ldots, x^n)$ of the flat space $\mathbb{R}^n$ satisfying these last inequalities by $C(R, 'n, ''n)$. Now we are in the position to present our main theorem.

Theorem 2: *There holds the asymptotic power series expansion*

$$\frac{\operatorname{Vol} C(y, R, 'g, ''g)}{\operatorname{Vol} C(R, 'n, ''n)}$$

$$\sim \sum_{'k, ''k=0}^{\infty} n(n + 2k)^{-1}\, a_{'k}('n)\, a_{''k}(''n + 2)\, '\mathrm{tr}^{'k}\, ''\mathrm{tr}^{''k}(d^{2k}\varrho)\,(y, y)\, R^{2k}$$

where $'\mathrm{tr}$ *denotes the contraction with* $'g^{ij}(y)$ *of some symmetric tensor at* y, $''\mathrm{tr}$ *denotes the contraction with* $''g^{ij}(y)$, *and* $k = 'k + ''k$. *The absolute term of the expansion equals* 1. *The coefficient of* R^2 *is proportional to*

$$[(''n + 2)\, '\mathrm{tr} + 'n\, ''\mathrm{tr}]\, Ric(y). \tag{12}$$

The coefficient of R^4 *is proportional to*

$$[(''n + 2)(''n + 4)\, '\mathrm{tr}^2 + 2('n + 2)(''n + 4)\, '\mathrm{tr}\, ''\mathrm{tr} + 'n('n + 2)\, ''\mathrm{tr}^2]$$
$$\times [2(Riem)^2 - 5(Ric)^2 - 9d^2\, Ric]\,(y). \tag{13}$$

(The proportionality factors do not vanish.)

Proof: We apply, in a notationally simplified manner, the formula (7) and then Fubini's integral theorem:

$$\operatorname{Vol} C(y, R, 'g, ''g) = \int_{C(y,R,'g,''g)} d\,vol = \int_{C(R,'n,''n)} d^n\bar{x}\,\varrho$$
$$= \int_{B'^n(R)} d^{'n}\bar{x} \int_{B''^n('r)} d^{''n}\bar{x}\,\varrho.$$

Here

$$d'^n\bar{x} = \left(\det g_{'a'b}(y)\right)^{1/2} dx^1\, dx^2 \cdots dx'^n,$$
$$d''^n\bar{x} = \left(\det g_{''a''b}(y)\right)^{1/2} dx'^{n+1} \cdots dx^n$$

in suitable normal coordinates. The inner integral is expanded by means of the second Pizzetti formula (10),

$$\int\limits_{B''^n('r)} d''^n\bar{x}\varrho \sim \operatorname{Vol} B''^n(1) \sum_{''k=0}^{\infty} a_{''k}(''n + 2)\, (''\Delta_0{}''^k\varrho)\, ('\bar{x}, 0)'\, r''^{n+2''k}.$$

The outer integral is decomposed according to Fubini's theorem and is expanded then by means of the first Pizzetti formula (9):

$$\operatorname{Vol} C(y, R, 'g, ''g) = \int\limits_0^R d'r \int\limits_{S^{n-1}('r)} d'S \int\limits_{B''^n('r)} d''^n\bar{x}\varrho$$

$$\sim \operatorname{Vol} B'^n(R) \operatorname{Vol} B''^n(R) \sum_{'k,''k=0}^{\infty} n(n + 2k)^{-1}$$
$$\times a'^k('n)\, a_{''k}(''n + 2)\, ('\Delta_0{}'^k\, ''\Delta_0{}''^k\varrho)\, (0)\, R^{2k}.$$

We have written the integration differentials just after the integral signs in order to avoid parentheses. The formula (17) in O. KOWALSKI's paper [14] translates differential operators on T_yM into covariant differential operators. It gives here

$$('\Delta_0{}'^k\, ''\Delta_0{}''^k\varrho)\, (0) = '\mathrm{tr}'^k\, ''\mathrm{tr}''^k(d^{2k}\varrho)\, (y, y).$$

For an evaluation of the first terms of the asymptotic expansion we have to take the coincidence limits of $d^2\varrho$ and $d^4\varrho$ from (1) and the Pizzetti coefficients from (8) ∎

The observation that the numerical coefficients in (12), (13) are positive leads to

Proposition 2: *A manifold (M, g) with definite Ricci curvature Ric has a non-vanishing volume defect*

$$\operatorname{Def} C(y, R, 'g, ''g) := \frac{\operatorname{Vol} C(y, R, 'g, ''g)}{\operatorname{Vol} C(R, 'n, ''n)} - 1.$$

Likewise, a manifold with definite four-form $2(Riem)^2 - 5(Ric)^2 - 9d^2\,Ric$ has a non-vanishing volume defect.

Proof: The contraction of positive definite forms with the positive semidefinite matrices $\left('g^{ij}(y)\right)$, $\left(''g^{ij}(y)\right)$ yields positive numbers. These remain positive when multiplied with $''n + 2$, $'n$, ... in (12), (13) and added up. Analogously, the contractions of negative definite forms yield negative numbers. Thus, the first terms in the asymptotic expansion of the volume defect do not vanish ∎

Examples of manifolds with definite Ric or $2(Riem)^2 - 5(Ric)^2 - 9d^2Ric$, respectively, can be constructed as the products of Einstein manifolds or of manifolds of constant curvature. Such product constructions will be considered in the next section.

The volume problem for pseudo-Riemannian products

The class of manifolds which we consider in this section admits a more explicit treatment of the volume problem.

Definition 6: Let $('M, 'g)$, $(''M, ''g)$ be two properly Riemannian manifolds of dimension $'n, ''n$, respectively, and $M := 'M \times ''M$ be the product manifold. Let, further, $'p\colon M \to 'M$, $''p\colon M \to ''M$ denote the natural projections and $'p^*$, $''p^*$ their pull-backs. We set $g = 'p^*'g - ''p^*''g$ and call (M, g) the *pseudo-Riemannian product* of $('M, 'g)$, $(''M, ''g)$.

A pseudo-Riemannian product manifold carries a natural global $O('n) \times O(''n)$-structure which can be identified with the very product structure. We adopt the convention to consider truncated light cones only with respect to this natural $O('n) \times O(''n)$-structure! Note the change in the meaning of $'g$, $''g$; the formulas have to be appropriately reinterpreted.

Theorem 3: *For the pseudo-Riemannian product (M, g) of two properly Riemannian manifolds $('M, 'g)$, $(''M, ''g)$ there holds*

$$d \operatorname{Vol} C(y; R, 'g, ''g)/dR = \operatorname{Vol} S'^{n-1}('y, R) \operatorname{Vol} B''^{n}(''y, R). \tag{14}$$

As a consequence, there holds the asymptotic power series expansion

$$\frac{d \operatorname{Vol} C(y, R, 'g, ''g)/dR}{d \operatorname{Vol} C(R, 'n, ''n)/dR} \sim \left(\sum_{'k=0}^{\infty} a_k('n)\,(\Delta'^{k}_{y}\varrho)\,('y, 'y)\, R^{2'k}\right)\left(\sum_{''k=0}^{\infty} a_k(''n+2)\,\Delta''^{k}_{y}\varrho)\,(''y, ''y)\, R^{2''k}\right). \tag{15}$$

Here $y = ('y, ''y)$, and $'\varrho = '\varrho('x)$, $''\varrho = ''\varrho(''x)$ denote the normal volume functions of $('M, 'g)$, $(''M, ''g)$, respectively, and $\Delta_{'y}$, $\Delta_{''y}$ their Euclidean Laplacians.

Proof: The multiplicativity of the normal volume function is well known: $\varrho(x) = '\varrho('x)\, ''\varrho(''x)$. It implies

$$\operatorname{Vol} C(y, R, 'g, ''g) = \int_0^R d'r \int_{S'^{n-1}('r)} d'S\, '\varrho('x) \int_{B''^{n}('r)} d''^{n}\bar{x}\, ''\varrho(''x)$$

and, by differentiation,

$$\frac{d}{dR} \operatorname{Vol} C(y, R, 'g, ''g) = \int_{S'^{n-1}(R)} d'S\, '\varrho('x) \int_{B''^{n}(R)} d''^{n}\bar{x}\, ''\varrho(''x)$$
$$= \operatorname{Vol} S'^{n-1}('y, R) \operatorname{Vol} B''^{n}(''y, R).$$

The asymptotic expansion follows by insertion of Pizzetti's formulas ∎

Proposition 3: *If the pseudo-Riemannian product of $('M, 'g)$, $(''M, ''g)$ has vanishing volume defect, then both*

$$\frac{\operatorname{Vol} S'^{n-1}('y, R)}{\operatorname{Vol} S'^{n-1}(R)} \quad \text{and} \quad \frac{\operatorname{Vol} B''^{n}(''y, R)}{\operatorname{Vol} B''^{n}(R)}$$

depend only on R (i.e. do not depend on $y = (y', ''y)$) and the product of these two quantities equals 1. As a consequence, the coincidence limits of $\Delta'^{k}_{y}\, '\varrho$ and $\Delta''^{k}_{y}\, ''\varrho$ $(k = 1, 2, \ldots)$ are constants.

Proof: If $\operatorname{Def} C(y, R, 'g, ''g) = 0$, then (14) implies

$$\operatorname{Vol} S'^{n-1}('y, R) \operatorname{Vol} B''^{n}(''y, R) = \operatorname{Vol} S'^{n-1}(R) \operatorname{Vol} B''^{n}(R).$$

Hereto the usual "separation of variables" argument is applied and gives the first assertion. Then the coefficients of the Pizzetti expansions in (15) have to be constants; this gives the second assertion ∎

Proposition 4: *If the pseudo-Riemannian product of $('M, 'g)$, $(''M, ''g)$ has vanishing volume defect, then the scalar curvatures $'S$, $''S$ as well as the quantities*

$$'A = -3\,|'Riem|^2 + 8\,|'Ric|^2 + 5'S^2, \qquad ''A = -3\,|''Riem|^2 + 8\,|''Ric|^2 + 5''S^2$$

are constants such that

$$(''n + 2)\,'S + 'n''S = 0,$$

$$(''n + 2)\,(''n + 4)\,'A + 10('n + 2)\,(''n + 4)\,'S''S + 'n('n + 2)\,''A = 0. \qquad (16)$$

Proof: The constancy property follows from Proposition 3 and the coincidence limits from [4, 6]

$$-3\Delta_{'y}\,'\varrho \doteq 'S, \qquad -3\Delta_{''y}\,''\varrho \doteq ''S,$$

$$45\Delta^2_{'y}\,''\varrho \doteq 'A, \qquad 45\Delta^2_{''y}\,''\varrho \doteq ''A.$$

The relations (16) follow by requiring the coefficients of R^2 and R^4 in (15) equal to zero ∎

Example: For manifolds $('M, 'g)$, $(''M, ''g)$ of constant curvature $'K = '\lambda^2$, $''K = ''\lambda^2$, respectively, the volume of geodesic spheres and balls is known. Formulas from [6] give us

$$d \operatorname{Vol}(y, R, 'g, ''g)/dR$$
$$= \operatorname{Vol} S^{'n-1}(1) \operatorname{Vol} S^{''n-1}(1) \left(\frac{1}{'\lambda} \sin '\lambda R\right)^{'n-1} \int_0^R dr \left(\frac{1}{''\lambda} \sin ''\lambda r\right)^{''n-1}. \qquad (17)$$

Herefrom $\operatorname{Vol}(y, R, 'g, ''g)$ follows by integration with respect to R. If $'K < 0$, then $\frac{1}{'\lambda} \sin '\lambda R$ is to be replaced by $\frac{1}{|'\lambda|} \sinh |'\lambda|\, R$; an analogous remark applies if $''K < 0$.

Proposition 5: *If the pseudo-Riemannian product of two manifolds $('M, 'g)$, $(''M, ''g)$ of constant curvature has vanishing volume defect, then the factors $('M, 'g)$, $(''M, ''g)$ are flat.*

Proof: If the volume defect vanishes, then the quantity (17) is proportional to R^{n-1}. This is possible only in the limit $'\lambda \to 0$, $''\lambda \to 0$ ∎

Proposition 6: *If the pseudo-Riemannian product of two two-dimensional manifolds $('M, 'g)$, $(''M, ''g)$ has vanishing volume defect, then the factors $('M, 'g)$, $(''M, ''g)$ are flat.*

Proof: Proposition 4 tells us that the scalar curvatures $'S$, $''S$ are constant. Hence the two-dimensional manifolds $('M, 'g)$, $(''M, ''g)$ are of constant curvature and Proposition 5 gives the result ∎

Discussion

We investigated the general pseudo-Riemannian case with signature $('n, ''n)$; the Lorentzian case $'n = 1$ (or $''n = 1$) has been treated already in [2, 3, 20]. The two cases differ in the following aspects:

- Here we consider the volume problem with respect to a fixed $O('n) \times O(''n)$-structure at a point $y \in M$ or in a domain $U \subseteqq M$. In [20] we considered the volume problem with respect to any $O(1) \times O(n-1)$-structure in a domain $U \subseteqq M$. The ambiguity in the choice of the $O(1) \times O(n-1)$-structure is described by a timelike vector field $a = a^i\partial/\partial y^i$. Fortunately, these vectors a can be geometrically visualized as the "axes" of the truncated light cones. For general $('n, ''n)$ the ambiguity in the choice of the $O('n) \times O(''n)$-structure does not have such a nice description.
- Here we consider "full cones" while in [20] only the "forward half cones", characterized by non-negative time values, have been considered. It is this difference which makes here the odd powers R^{2k+1} of the altitude R cancel out from the asymptotic expansion and which makes in [20] both even powers R^{2k} and odd powers R^{2k+1} to appear. Of course, the odd powers provide extra informations in [20], which are not available here.
- The Lorentzian case admits geometrical visualization as well as physical application (in the general theory of relativity). The general case admits neither the one nor the other.
- A Lorentzian manifold with vanishing volume defect for each $O(1) \times O(n-1)$-structure is shown to be flat [20]. This affirmatively answers a "volume conjecture". The answer for properly Riemannian manifolds is not known. For the remaining case $'n \geqq 2$, $''n \geqq 2$ there exist non-flat pseudo-Riemannian manifolds with vanishing volume defect, namely the (non-flat) simply harmonic manifolds of signature $('n, ''n)$. The normal volume function of a simply harmonic manifold is constant, equal to one. Thus the "volume conjecture" in its original form should not be applied; it is to be reformulated: a pseudo-Riemannian manifold of signature $('n, ''n)$, $'n \geqq 2$, $''n \geqq 2$, with vanishing volume defect is supposed to be simply harmonic.
- The Lorentzian case is included here. We obtain, for instance the following useful formula: The Lorentzian product $(\mathbb{R} \times M, dt^2 - g)$ of a properly Riemannian manifold (M, g) and the real number space $(\mathbb{R}, dt^2)$ satisfies

$$\frac{d \operatorname{Vol} C(t_0, y, R, dt^2, g)/dR}{dC(R, 1, n)/dR} = \frac{\operatorname{Vol} B^n(y, R)}{\operatorname{Vol} B^n(R)}.$$

All $O('n) \times O(''n)$-structures at a point $y \in M$ of a pseudo-Riemannian manifold (M, g) are parametrized by the Grassmann space $O('n, ''n)/(O('n) \times O(''n))$; its dimension equals $'n''n$. All $O('n) \times O(''n)$-structures in a domain $U \subseteqq M$ are parametrized by the sections of a Grassmann bundle over U, i.e. a fibre bundle with typical fibre $O('n, ''n)/(O('n) \times O(''n))$. In order to effectively exploit the ambiguity in the $O('n) \times O(''n)$-structure, infinitesimal Lorentz transformations should be used, i.e. elements of the vector space $o('n, ''n)/(o('n) \times o(''n))$; these can be interpreted as "infinitesimal transformations". Here $o(\ldots)$ denotes the Lie algebra of a Lie group $O(\ldots)$. Such procedures could be the topic of future work. Also, other variants of the volume problem, taken in a broad sense, for pseudo-Riemannian manifolds could be studied, for instance the volume of tubes about curves or submanifolds.

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VERFASSER:

Doz. Dr. sc. RAINER SCHIMMING
Sektion Mathematik der Ernst-Moritz-Arndt-Universität
Friedrich-Ludwig-Jahn-Str. 15a
DDR-2200 Greifswald

Dr. DONATA MATEL-KAMINSKA
Instytut Matematyki, Politechnika Szczecińska
al. Piastów 50
P-70-310 Szczecin