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# • The Volume Problem for Pseudo-Riemannian Manifolds

R. SCHIMMING and D. MATEL-KAMINSKA

Wir stellen das Volumenproblem für pseudo-Riemannsche Mannigfaltigkeiten und legen erste 'Resultate dazu vor: Aus dem Volumen kleiner abgeschnittener Lichtkegel werden gewisse geo-The Volume Problem for Pseudo-Riemar<br>
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metrische Eigenschaften abgelesen.<br>
Мы поднимаем проб

Мы поднимаем проблему объёма псевдо-римановых многообразий и достигаем первых результатов: исследование объёма маленьких разрезанных световых конусов позволяет судить о некоторых геометрических свойствах.

We pose the volume problem for pseudo-Riemannian manifolds and present first results on it: certain geometric properties are read from the volume of small truncated light cones.

# Introduction.

The volume problem for (pseudo-) Riemannian manifolds *(M,* g) of given dimension *n* and signature ('n, "n), as we take it here, roughly reads as follows: to what extent does the volume of small naturally defined test bodies determine the geometry? Here the "test bodies" are assumed to be compact and to depend on as small a number of real parameters as possible. One should begin with test bodies of dimension *n,*  Mы поднимаем проблему объёма псевдо-римановых многообразий и достига<br>результатов: исследование объёма маленьких разрезанных световых конусов<br>судить о некоторых геометрических свойствах.<br>We pose the volume problem for pseu

The problem of the volume of small geodesic balls in properly Riemannian geometry is classical. It has historical roots in the theory of surfaces and has been studied ' in a series of papers of A. GRAY, L. VANHECKE, O. KOWALSKI and others  $[4-7,$  $10-18$ ].

F. and B. GACKSTATTER [2, 31 initiated the volume problem for Lorentzian maiifolds. They proposed truncated light cones as the test bodies.

The general idea of the definition of test bodies is the following: Choose a fixed point  $y \in M$  and define test bodies in the vector space  $T_yM$ , the tangential space at y. Map then these test bodies by means of the exponential map  $\exp_y$  with origin y into the manifold M. Effectively, the procedure is done by means of normal coordinates  $x^a$  of  $x \in M$  with respect to the origin  $y \in M$ . The ball with radius  $R > 0$  in *m* and signature  $(Tn, Tn)$ , as we take it here<br>does the volume of small naturally defin<br>Here the "test bodies" are assumed to be c<br>ber of real parameters as possible. One sho<br>but those with a positive codimension coul<br>The (*x*)<sup>2</sup> +  $(x^2)^2 + \cdots + (x^n)^2 \le R^2$ <br>(*x*) Analogously, the truncated light cone with altitude  $R > 0$  in  $T_yM$ <br>(*x*)<sup>2</sup> +  $(x^2)^2 + \cdots + (x^n)^2 \le R^2$ <br>(*x*) +  $\cdots$   $\binom{2}{2}$  +  $\cdots$   $\binom{2}{2}$  +  $\cdots$   $\binom{2}{2}$  +  $\cdots$   $\binom{2}{2$ 

$$
x^{1})^{2} + (x^{2})^{2} + \cdots + (x^{n})^{2} \leq R^{2}
$$

is mapped to the geodesic ball with radius  $R$  and centre  $y$  in a properly Riemannian manifold *(M, g)*. Analogously, the truncated light cone with altitude  $R > 0$  in  $T_{\nu}M$  $\leq R$ <br>radi<br>mca<br> $\leq$ 

$$
(x1)2 + (x2)2 + \cdots + (xn-1)2 \leq (x0)2 \leq R2
$$

is mapped to the truncated light cone with altitude  $R$  and vertex  $y$  in'a Lorentzian manifold  $(M, g)$ . For a pseudo-Riemannian manifold  $(M, g)$  of dimension *n* and signature ('n, ''n) we propose the test bodies, in  $T_{\bm{y}}\bm{M}$  to be defined by apped to the geodesic ball with radius R and centre y in a prope<br>iifold  $(M, g)$ . Analogously, the truncated light cone with altitude  $l$ <br> $(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2 \le (x^0)^2 \le R^2$ <br>apped to the truncated light cone with alti  $T_yM$ <br>  $(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 \leq R^2$ <br>
is mapped to the geodesic ball with radius R and centre y in a properly Ri<br>
manifold  $(M, g)$ . Analogously, the truncated light cone with altitude  $R > 0$ <br>  $(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2 \le$ 

$$
x'^{n+1})^2 + \cdots + (x^n)^2 \leq (x^1)^2 + \cdots + (x'^n)^2 \leq R^2,
$$

# **4** R. SCHIMMING and D. MATEL-KAMINSKA

and we call the image with respect to  $\exp_{\pmb{\nu}}$  "truncated light cone with altitude  $R$  and vertex  $y''$  in  $(M, g)$ . (We have no better name at hand.) For any signature, the volume of test bodies can be expanded into an asymptotic power, series in *1?* by means of Fubini's integral theorem and Pizzetti's expansion formula for spheres and balls in flat space. Each coefficient in the asymptotic series is a differential expression in the normal volume function  $\rho = \rho(x, y)$ . In order to extract geometrical informations from the volume of test bodies in  $(M, g)$ , it is compared with the volume of analogous test bodies in some simple "model manifold"  $(M_0, g_0)$ . The manifolds  $(M, g)$  and  $(M_0, g_0)$  are called isovolumal if *M* is covered by neighbourhoods *U* and local diffeomorphisms  $\varphi: U \to \varphi(U) \subseteq M_0$  such that  $(U, g)$  and  $(U, \varphi^* g_0)$  exhibit the same volume of test bodies. That means, the volume is calculated twice, once with respect to the proper metric g of M and once with respect to the local pull-back metric  $\varphi^*g_0$ . The parameters of the test bodies are the same in the two calculations. Clearly, the  $\tau$ simplest model manifolds  $(M_0, g_0)$  are the flat ones; for them the volume of a test body depends on R only and not on the other parameters. F. and B. GACKSTATTER [2, 31 introduced the volume defect, that is the relative deviation of the volume of a test body in  $(M, g)$  from the volume of an analogous test body, with the some  $R$ , in flat space  $\mathbb{R}^n$ . A manifold  $(M, g)$  is isovolumal to  $\mathbb{R}^n$  if and only if the volume defect vanishes for all sufficiently small  $R > 0$ . The "volume conjecture" says that a manifold with vanishing volume defect is flat. For properly Riemannian geometry, this conjecture is due to A. GRAY and L. VANHECKE [6] and it is neither confirmed nor refuted until now.

The first author of the present paper hasdecided the'volume conjecture for Lo rentzian geometry in the affirmative [20]. He has, moreover, shown that Ricciflatness is also a geometric property which can be read from the volume of small truncated light cones in a Lorentzian manifold. For  $n = 4$ , F. GACKSTATTER [3] derived the same results. It is in order to study the volume problem in general pseudo-Riemannian geometry after that in propeily Riemannian and Lorentzian geometries. This is the topic of the present paper. The programme sketched above is not realized in full generality. In this our attempt the following partial results are achieved:

1. The volume of any small truncated light cone is asymptotically expanded in powers of the altitude *R.* The first terms of the expansion are given as expressions in the curvature of  $(M, g)$ .

2. A manifold, with definite Ricci curvature or with definite four-form  $2(Riem)^2$  $= 5(Ric)^2 - 9d^2Ric$  has a non-vanishing volume defect.

3. If the pseudo-Riemannian product of two properly Riemannian manifolds ('M,'g), (''M,''g) has vanishing volume defect, then all the invariants  $(\Lambda^k_{y'}\varrho)$  ('y, 'y), ("y, "y)  $(k = 1, 2, ...)$  are constants. Here ' $\varrho = \varrho(x, 'y), ''\varrho = \varrho(x', 'y')$ 2. A manifold, with definite Ricci curvatuu<br>  $-5(Ric)^2 - 9d^2Ric$  has a non-vanishing volume 3. If the pseudo-Riemannian product of<br>  $(M, 'g), ('M, 'g)$  has vanishing volume defect<br>  $(A_{r,y}^k)'(y)'(y', 'y) (k = 1, 2, ...)$  are constants<br>
are are the normal volume functions and  $\Delta_{xy}$ ,  $\Delta_{xy}$  the so-called Euclidean-Laplace opera-  $\Delta_{xy}$ . tors of ('M, 'g), (''M, ''g), respectively. Particularly, the factor manifolds must have constant scalar curvatures 'S, "S such that  $('n + 2) S + ''n''S = 0$ , where 'n = dim 'M, "n = dim "M.  $\langle 'M, 'g \rangle$ ,  $\langle 'M, 'g \rangle$  has vani<br>  $\langle A_{r,y}^{k''} \rangle$   $\langle 'y, 'y \rangle$   $\langle k = 1, 2, 3,$ <br>
are the normal volume functions of  $\langle 'M, 'g \rangle$ ,  $\langle 'M, 'g \rangle$ <br>
constant scalar curvature<br>  $= \dim 'M, 'i\omega = \dim$ 

4. The pseudo-Riémannian product of two manifolds of constant curvature or of two two-dimensional manifolds with vanishing volume defect is flat..

5. A coordinate-independent expression for the so-called Euclidean-Laplace operator  $\Delta_y$  with respect to  $y \in M$  is presented:  $\Delta_y u = \varrho \nabla_a (\varrho^{-1} \gamma^{ab} \nabla_b u)$ . Here  $\varrho = \varrho(x, y)$  is. the normal volume function and  $y^{ab} = y^{ab}(x, y)$  is some contravariant tensor with respect to  $x$ , explicitly given in the text, which reduces to constant components  $g^{\alpha b}(y)$  in a normal coordinate system with origin y.

# **Two-point geometry**

 

Certain scalars and tensors depending on two points x, y naturally arise in (pseudo-)<br> *Riemannian geometry.* We consider smooth (i.e. of class  $C^{\infty}$ ) *n*-dimensional Rieman-<br> *nian manifolds of arbitrary signature.*<br> Riemannian geometry. We consider smooth (i.e. of class  $C^{\infty}$ ) *n*-dimensional Rieman-Two-point geometry<br>
Certain scalars and tensors depending on two po<br>
Riemannian geometry. We consider smooth (i.e.<br>
nian manifolds of arbitrary signature.<br>
Definition 1: The distance function  $\sigma = \sigma(x, g^{ab} \nabla_a \sigma \cdot \nabla_b \sigma =$ 

Definition 1: The *distance function*  $\sigma = \sigma(x, y)$  is the solution of the problem

$$
g^{ab}\nabla_a \sigma.\nabla_b \sigma = 2\sigma, \qquad (\nabla_a \sigma)(y, y) = 0, \qquad (\nabla_a \nabla_b \sigma)(y, y) = g_{ab}(y)
$$

The function  $\mu = \mu(x, y)$  is defined by  $2\mu = \Delta \sigma - n$ . The *normal volume function*  $g^{\mu\nu}V_a\sigma$ ,  $V_b\sigma = 2\sigma$ ,  $(V_a\sigma)(y, y) = 0$ ,  $(V_a V_b\sigma)(y, y) = g_{ab}(y)$ .<br>unction  $\mu = \mu(x, y)$  is defined by  $2\mu = \Delta\sigma - n$ . The normal volume function  $(x, y)$  is the solution of the problem  $g^{ab} \nabla_a \sigma \nabla_b \varrho = 2\mu \varrho$ ,  $\varrho(y, y) = 1$ the following, the differential operators  $\nabla$ ,  $\varLambda$ ,  $\tilde{d}$ , ... refer to the first argument x;  $\nabla$ dehotes the Levi-Civita derivative to *g* and  $\Delta := g^{ab} \nabla_a \nabla_b$  the usual Laplace operator acting on tensor fields.

It is known that both the two-point functions  $\sigma$  and  $\rho$  are defined in some neighbourhood of the diagonal of  $M \times M$  and are symmetric in their arguments:  $\sigma(x, y)$ .  $a = \sigma(y, x)$ ,  $\rho(x, y) = \rho(y, x)$ . For properly Riemannian manifolds  $\sigma$  equals one half of the square of the geodesic distance *s* between two sufficiently neighboured points:  $2\sigma(x, y) = s(x, y)^2$ . For pseudo-Riemannian manifolds  $\sigma$  defines the geodesic distance:  $2 |g(x,y)| =: s(x, y)^2$ . The limit for  $x \to y$ , if existing, of a two-point quantity depending on  $x, y$  is called its coincidence limit. The equality of the coincidence limits is an equivalence relation between two-point quantities and shall be denoted by  $\doteq$ . One-point quantities and constants may be looked upon as special two-point quantities. The function  $\mu = \mu(x, y)$  is defined by  $2\mu = 4\sigma - n$ . The normal volume  $g = g(x, y)$  is the solution of the problem  $g^{ab} \nabla_a \sigma \nabla_b \rho = 2\mu \rho, \rho(y, y) = 1$ . Here the following, the differential operators  $\nabla, A, d, \ldots$  refer to It is known that both in the two-point numerous o and *y* and  $\alpha$  **e** terms<br>
bourhood of the diagonal of  $M \times M$  and are symmetric in their and<br>  $= \sigma(y, x), \rho(x, y) = g(y, x)$ . For properly Riemannian manifolds  $\sigma$  ether and<br>  $2$ *roof*  $\sigma(x, y) = s(x, y)^2$ . For pseudo-Riemannian manifolds  $\sigma$  defines  $|\sigma(x, y)| = s(x, y)^2$ . For pseudo-Riemannian manifolds  $\sigma$  defines  $|\sigma(x, y)| = : s(x, y)^2$ . The limit for  $x \to y$ , if existing, of a tending on x, y is called its co ivalence relation between two-point quantities and shall be defined<br>t quantities and constants may be looked upon as special two-point qu<br>tric differential form of degree p<br> $u_p = u_{a_1a_1...a_p} dx^a_1 dx^a_2 \dots dx^{a_p}$ <br>motation for

$$
u_p = u_{a_1a_2\cdots a_p} dx^{a_1} dx^{a_2} \cdots dx^{a_p}
$$

is a new notation for a symmetric covariant tensor field of degree p: Apart from the usual tensorial notations there are specific operations for symmetric forms: usual tensorial notations there are specific operations for symmetric forms:<br>
- Symmetric product of a p-form  $u_p$  and a q-form  $v_q$ :<br>  $u_p v_q := u_{a_1 \cdots a_p} v_{b_1 \cdots b_q} dx^{a_1} \cdots dx^{a_p} dx^{b_1} \cdots dx^{b_q}$ .<br>
- Symmetric power:  $u^k := u u$ 

$$
u_{n}v_{n}:=u_{n}u_{n}v_{n}u_{n}dx^{a_{1}}\cdots dx^{a_{p}}dx^{b_{1}}\cdots dx^{b_{q}}.
$$

 $-$  Trace  $=$  tr with respect to the metric  $g$ :

\n Invariance, there are specific operations' there are specific operations. The metric product of a 
$$
p
$$
-form  $u_p$  and a  $q$ -form  $u_p v_q := u_{a_1 \cdots a_p} v_{b_1 \cdots b_q} dx^{a_1} \cdots dx^{a_p} dx^{b_1} \cdots dx^{b_q}$ .\n

\n\n Invariance, we have:\n \n- \n
$$
u_p := g^{ab} u_{ab a_1 \cdots a_p} dx^{a_1} \cdots dx^{a_p}
$$
\n for  $p \geq 3$ ,\n
	\n- \n
	$$
	u_p := g^{ab} u_{ab a_1 \cdots a_p} dx^{a_1} \cdots dx^{a_p}
	$$
	\n for  $p \geq 3$ ,\n
		\n- \n
		$$
		u_p := 0, \quad \text{tr } u_1 := 0, \quad \text{tr } u_2 := g^{ab} u_{ab}
		$$
		\n
		\n- \n
		$$
		u_p(v, v, \ldots, v) := u_{a_1 a_1 \cdots a_p} v^{a_1} v^{a_1} \cdots v^{a_p}
		$$
		\n
		\n\n

\n\n Invariance, we have:\n \n- \n
$$
u_p(v, v, \ldots, v) := u_{a_1 a_1 \cdots a_p} v^{a_1} v^{a_1} \cdots v^{a_p}
$$
\n
\n
\n

\n\n Invariance, we have:\n \n- \n
$$
u_p := \nabla_a u_{a_1 \cdots a_p} dx^a dx^{a_1} \cdots dx^{a_p}
$$
\n
\n
\n

\n\n This implies:\n \n- \n
$$
u_p(v, v, \ldots, v) := u_{a_1 a_1 \cdots a_p} v^{a_1} v^{a_1} \cdots v^{a_p}
$$
\n
\n
\n

\n\n Invariance, we have:\n \n- \n
$$
u_p(v, v, \ldots, v) := u_{a_1 a_1 \cdots a_p} v^{a_1} v^{a_1} \cdots v^{a_p}
$$
\n
\n
\n

\n\n Invariance, we have:\n \n- \n

Value of  $u_p$  on a vector or vector field  $v$ :

$$
u_p(v, v, \ldots, v) := u_{a_1 a_2 \ldots a_p} v^{a_1} v^{a_2} \ldots v^{a_p}
$$

**Symmetric differential d:** 

$$
du_n := \nabla_a u_{a_1 \ldots a_n} dx^a dx^{a_1} \cdots dx^{a_p}.
$$

*-* Powers of tr and *d:*

5 

netric differential d:<br>  $du_p := \nabla_a u_{a_1 \cdots a_p} dx^a dx^{a_1} \cdots dx^{a_p}$ .<br>
rs of tr and d:<br>  $\text{tr}^k := (\text{tr}).(\text{tr}) \cdots (\text{tr}), \qquad d^k := dd \cdots d$ .<br>
ature tensor. Ricci tensor, and scalar cur

The curvature tensor, Ricci tensor, and scàlar curvature are denoted by *Rim, Ric, 8;*  respectively. The components of  $\it{Riem}, \it{Ric}$  read  $\it{R}_{abcd}, \it{R}_{ab},$  respectively. We use the special abbreviations The<br>rest<br>spec  $\mathbf{tr}^k := (\mathbf{tr}) (\mathbf{tr}) \cdots (\mathbf{tr}), \qquad d^k := dd \cdot$ <br>
curvature tensor, Ricci tensor, and scalar<br>
sectively. The components of *Riem*, *Ric*<sup>i</sup>n<br>
cial abbreviations<br>  $\langle Riem \rangle^2 := R_{a\epsilon\beta}R_c^{e\ell}{}_{d} dx^a dx^b dx^c dx^d,$ <br>  $|Riem \rangle^2 := R_{abcd}R^{abcd}, \qquad |$ *i* the curvature tensor, Ricci tensor, and scalar curva<br>
spectively. The components of *Riem*, *Ric*<sup>1</sup>read<sup>1</sup> $R_{ab}$ <br>
is eccial abbreviations<br>  $(Riem)^2 := R_{aefb}R_c^{ef} dx^a dx^b dx^c dx^d$ ,<br>  $|Riem|^2 := R_{abcd}R^{abcd}$ ,  $|Ric|^2 := R_{ab}R^{ab}$ .

$$
(Riem)^{2} := R_{a\epsilon/b}R_{c\cdots d}^{~~\epsilon} dx^{a} dx^{b} dx^{c} dx^{d},
$$
  

$$
|Riem|^{2} := R_{abcd}R^{abcd}, \qquad |Ric|^{2} := R_{ab}R^{ab}
$$

*-:* 

## **R. SCHIMMINO and D. MATEL-KAMINSKA-**

The sign conventions for the curvature quantities are the same as in [4, 6, 20]. There holds [20]

6 R. S<sub>QHIMING</sub> and D. MATEL KAMINSKA  
\nThe sign conventions for the curvature quantities are the  
\nholds [20]  
\n
$$
d\hat{\rho} = 0
$$
,  $-3d^2\hat{\rho} = Ric$ ,  $-2d^3\hat{\rho} = d Ric$ ,  
\n $-15d^4\hat{\rho} = 2(Riem)^2 - 5(Ric)^2 + 9d^2 Ric$ .  
\nDefinition 2: The *Euclidean Laplacian*  $\Delta_{\hat{y}}$  with respe  
\nfunctions  $u = u(x)$  over  $\hat{M}$ , is given by  
\n $\Delta_y u = \hat{\rho} \nabla_a (e^{-1}y^{ab} \nabla_b u)$ ,  
\nwhere the two-point tensor field  $y^{ab} = y^{ab}(x, y)$  is defined  
\n $\sigma_{ai} = \partial/\partial x^a \partial/\partial y^i \sigma(x, y)$ ,  $\gamma_{ab} = g^{ij}(y) \sigma_{ai} \sigma_{bj}$ ,  
\nTheorem 1: In normal coordinates  $\bar{x} = (x^a) \in \mathbb{R}^n$  of  $x$ ,  
\n $y \in \hat{M}$  there holds  
\n $\sigma(x, y) = 1/2g_{ab}(y) x^a x^b$ ,  
\n $\rho(x, y) = |\det g_{ab}(x)|^{1/2} |\det g_{ai}(y)|^{1/2}$ ,

**•** *• A. Sommario and D. MATEL KAMINSKA*<br> *• a. on conventions for the curvature quantities are the same as in [4, 6, 20]. There***<br>
<b>***• do*  $\dot{=} 0$ ,  $-3d^2\varrho = Ric$ ,  $-2d^3\varrho = d Ric$ ,  $-15d^4\varrho = 2(Riem)^2 - 5(Ric)^2 + 9d^2 Ric$ . (1) <br>  $-15d^4 \varrho = 2(Riem)^2 - 5(Ric)^2 + 9d^2 Ric.$ <br>
Definition 2: The *Euclidean Laplacian*  $\Delta_{ij}$  with respect to the origin y, acting on notions  $u = u(x)$  over *M*, is given by<br>  $\Delta_{ij}u = \varrho \nabla_a(\varrho^{-1}\gamma^{ab}\nabla_b u),$  (2)<br>
nere the two-point t *CHEREAL EXECUTE DESCRIPTION*  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\gamma$ ,  $\beta$  and  $\gamma$  is defined by<br>  $A_y u = \rho \nabla_a (e^{-1} \gamma^{ab} \nabla_b u)$ ,<br>
here the two-point tensor field  $\gamma^{ab} = \gamma^{ab}(x, y)$  is defined by<br>  $\sigma'_{ai} = \partial/\partial x^a \partial/\partial y^i \sigma(x, y)$ ,  $\gamma_{ab} = g^{$ 

$$
\Delta_y u = \varrho \nabla_a (\varrho^{-1} \gamma^{ab} \nabla_b u), \tag{2}
$$

where the two-point tensor field  $\gamma^{ab} = \gamma^{ab}(x, y)$  is defined by

$$
\sigma_{ai}' = \partial/\partial x^a \partial/\partial y^i \sigma(x, y), \qquad \gamma_{ab} = g^{ij}(y) \sigma_{ai}\sigma_{bj}, (\gamma^{ab}) = (\gamma_{ab})^{-1}
$$

Theorem 1: In normal coordinates  $\bar{x} = (x^a) \in \mathbb{R}^n$  of  $x \in M$  with respect to the origin  $y \in M$  there holds *a*<sub>Q</sub> = 0,  $-3d^2Q = Ric$ ,  $-2d^3Q = dRic$ , (1)<br>  $-15d^4Q = 2(Riem)^2 - 5(Ric)^2 + 9d^2 Ric$ . (1)<br>
ition 2: The Euclidean Laplacian  $A_y$  with respect to the origin y, acting on<br>  $u = u(x)$  over M, is given by<br>  $A_y u = g \nabla_a (e^{-1}y^{ab} \nabla_b u)$ ,<br>  $\$ e( $x = 0$ ,  $x, y = 100$ ,  $x = 2$ ,  $Riem)$  =  $5$ ,  $R\varepsilon$ ) =  $5$ ,  $R\varepsilon$ ) =  $43\kappa$ ).<br>  $x = u(x)$  over  $M$ , is given by<br>  $d_y u = g \nabla_a (g^{-1}y^{ab} \nabla_b u)$ ,<br>  $e \nabla_b u = g \nabla_a (g^{-1}y^{ab} \nabla_b u)$ ,<br>  $e \nabla_b u = g \nabla_b (g^{-1}y^{ab} \nabla_b u)$ ,<br>  $e \nabla_b u = g$  $\begin{aligned} \n\phi(x,y) &= 1/2g_{ab}(y) \, x^a x^b, \\
\phi(x,y) &= | \det g_{ab}(x) |^{1/2} | \det \sigma_{ab}(x) |^{1/2} \, d\sigma_{ab} &= -g_{ab}(y), \qquad \gamma^{ab} = 0, \ \n\phi_{ab} &= \psi_{ab}(y) \, \partial/\partial x^a \, \partial/\partial x^b \, u \end{aligned}$  $\begin{align} \textit{caplacian 2}_{y} \text{ with} \ \textit{on by} \ \textit{ab} &= \gamma^{ab}(x,y) \text{ is } \textit{c} \ \textit{y}_{ab} &= g^{ij}(y) \textit{d} \ \textit{y}_{ab} &= x^{a} \in \mathbb{R}^{n} \ \textit{y}_{b} \textit{d} &= g^{a} \in \mathbb{R}^{n} \ \textit{by } g_{ij}(y)|^{1/2}, \quad \textit{by} \ \textit{g}^{ab}(y), \quad \textit{by} \ \textit{d} &= \textit{y}_{b} \textit{d} & \textit{d} \ \textit{d} &= \text$  $\sigma'_{ai} = \partial/\partial x^a \partial/\partial y^i \sigma(x, y),$   $\gamma_{ab} =$ <br>
em 1: In normal coordinates  $\bar{x} =$ <br>
gree holds<br>  $\sigma(x, y) = 1/2g_{ab}(y) x^a x^b,$ <br>  $\varrho(x, y) = |\text{det } g_{ab}(x)|^{1/2} |\text{det } g_{ij}(y)|^1$ <br>  $\sigma_{ai} = -g_{ai}(y),$   $\gamma^{ab} = g^{ab}(y),$ <br>  $\Delta_y u = g^{ab}(y) \partial/\partial x^a \partial/\partial x^b u.$ <br>

$$
\sigma(x, y) = 1/2g_{ab}(y) x^a x^b, \qquad (3)
$$

$$
\varrho(x,y) = |\det g_{ab}(x)|^{1/2} |\det g_{ij}(y)|^{1/2}, \qquad (4)
$$

$$
\sigma_{ai} = -g_{ai}(y), \qquad \gamma^{ab} = g^{ab}(y), \qquad (5)
$$

$$
\Delta_y u = g^{ab}(y) \partial/\partial x^a \partial/\partial x^b u. \tag{6}
$$

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**5**  5 ,.

Proof: The formulas  $(3)$ ,  $(4)$  and the first part of  $(5)$  are generally known  $[21, 19]$ , 1]; the second part of (5) is an immediate consequence. A well-known formula for the, divergence of a vector field implies  $\begin{aligned} \sigma(x,y) &= 1/2g_{ab}(y) x^a x^b, \\ \varrho(x,y) &= |\det g_{ab}(x)|^{1/2} |\det g_{ij}(y)|^b, \\ \sigma_{ai} &= -g_{ai}(y), \qquad \gamma^{ab} = g^{ab}(y), \\ \varDelta_y u &= g^{ab}(y) \partial/\partial x^a \partial/\partial x^b u. \\ \text{: The formulas (3), (4) and the f, \\ \text{cond part of (5) is an immediate} \\ \varphi &\text{ of a vector field implies} \\ \varDelta_y u &= \varrho |\det g_{ab}(x)|^{-1/2} \frac{\partial}{\partial x^a} \left( e^{-1} \right) \\ \text{ng that in normal coordinates} \end{aligned$  $\frac{b}{b} u$ .<br>
and the filmediate<br>
imediate<br>  $\frac{\partial}{\partial x^a} \left( e^{-1} \right)$ <br>
dinates  $y \in M$  there holds<br>  $\sigma(x, y) = 1/2g_{ab}(y) x^a x^b$ ,<br>  $\varrho(x, y) = |\det g_{ab}(x)|^{1/2} |\det g_{ij}(y)|^{1/2}$ ,  $\sigma_{ai} = -g_{ai}(y)$ ,  $\gamma^{ab} = g^{ab}(y)$ ,<br>  $A_{\mu}u = g^{ab}(y) \partial/\partial x^a \partial/\partial x^b u$ .<br>
Proof: The formulas (3), (4) and the first part of (5)<br>
1]; the seco  $\varrho(x, y) = |\det g_{ab}(x)|^{1/2} |\det g_{ij}(y)|^{1/2},$ <br>  $\sigma_{ai} = -g_{ai}(y),$   $\gamma^{ab} = g^{ab}(y),$ <br>  $\varDelta_y u = g^{ab}(y) \partial/\partial x^a \partial/\partial x^b u.$ <br>
Proof: The formulas (3), (4) and the first part of<br>
1); the second part of (5) is an inmediate consequent<br>
divergence o

\n The equation is given by:\n 
$$
\Delta_y u = \rho \left[ \det g_{ab}(x) \right]^{-1/2} \frac{\partial}{\partial x^a} \left( e^{-1} \left[ \det g_{ab}(x) \right]^{1/2} \gamma^{ab} \frac{\partial u}{\partial x^b} \right)\n \times \text{Im} \left[ \int_0^{\infty} \det g_{ab}(x) \right]^{-1/2} = \left[ \det g_{ij}(y) \right]^{-1/2}, \quad \gamma^{ab} = g^{ab}(y),
$$
\n

\n\n The result of the equation is:\n  $\rho = \frac{1}{2} \left( \int_0^{\infty} \det g_{ij}(x) \right)^{-1/2} \frac{\partial u}{\partial x^b} \frac{\partial u}{\partial x$ 

$$
\varrho |\det g_{ab}(x)|^{-1/2} = |\det g_{ij}(y)|^{-1/2}, \qquad y^{ab} = g^{ab}(y),
$$

The Euclidean Laplacian  $\Delta_{\mu}$  with respect to an origin  $y \in M$  has been explicitly introduced through .the representation (6) by A. GRAY and **T. J.-'WILLMORE** *[7,* 22]. It has also been studied by O. KOWALSKI [14, 15]. These authors consider the properly Riemannian case only and they normalize  $g^{ab}(y) = \delta^{ab}$  (diagonal matrix with entries 1 in the main'diagonal). In the pseudo-Riemannian case, "Euclidean Laplacian" is not a good name, but we have no other name to propose. In  $[7, 22]$  the coineidence limits of  $A_{\mathbf{y}}\mathbf{u}$ ,  $A_{\mathbf{y}}^3\mathbf{u}$ ,  $A_{\mathbf{y}}^3\mathbf{u}$  have been calculated. Let us reproduce the first and second by means of our covariant definition (2):  $A_{\mu}u = \rho \left[\det g_{ab}(x)\right]^{-1/2} \frac{1}{\partial x^a} \left(e^{-1} \left[\det g_{ab}(x)\right]^{1/2} \gamma^{ab} \frac{1}{\partial x^b}\right]$ .<br>Considering that in normal coordinates<br>  $\rho \left[\det g_{ab}(x)\right]^{-1/2} = |\det g_{ij}(y)|^{-1/2}, \qquad \gamma^{ab} = g^{ab}(y),$ <br>
we arrive at the result (6)  $\blacksquare$ <br>
The Euclide at in normal coordinates<br>  $\int g_{ab}(x)|^{-1/2} = |\det g_{ij}(y)|^{-1/2},$   $p^{ab} = g^{ab}(y),$ <br>
he result (6)  $\blacksquare$ <br>
and Laplacian  $\Lambda_y$  with respect to an origin  $y \in M$  has been explicition<br>
from studied by O. Kowatsky [14, 15]. These authors c we arrive at the result (6) **I**<br>
The Euclidean Laplacian  $\Delta_y$  with respect to an origin  $y \in M$  has been introduced through the representation (6) by A. GRAY and T. J. WILLM It has also been studied by O. KOWALSKI [14, 15 at the result (6)  $\blacksquare$ <br>
uclidean Laplacian  $\Delta_y$  with respect<br>
ed through the representation (6<br>
so been studied by O. Kowatsk<br>
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in the main diagonal). In the ps<br>
not a good name, but we h

 $S$ ÝNGE's book [21] provides, after simple calculations, the coincidence relations.

 $\rightarrow$ 

in the main diagonal, in the pseudorrelationian [a.s.,  
not a good name, but we have no other name to propose.]  
limits of 
$$
A_yu
$$
,  $A_y^2u$ ,  $A_y^3u$  have been calculated. Let us repro-  
by means of our covariant definition (2):  
E's book [21] provides, after simple calculations, the coincid  
 $l_a = 0$ ,  $\nabla_a l_b = -1/3R_{ab}$ ,  $Al_a = -1/2\nabla_a S$ ,  
 $\nabla_a \gamma^{ab} = 0$ ,  $\Delta \gamma^{ab} = 2/3R^{ab}$ ,  $\nabla^a \nabla_c \gamma^{bc} = 2/3R^{ab}$ ,  
 $\Delta \nabla_a \gamma^{ab} = -1/6 \nabla^b S$ ;  
 $\nabla_a \nabla_b u = -1/6 \nabla^b S$   
 $\Delta u = \gamma^{ab} \nabla_a \nabla_b u + (\nabla_a \gamma^{ab} - l_a \gamma^{ab}) \nabla_b u$   
 $= \gamma^{ab} \nabla_a \nabla_b u = g^{ab} \nabla_a \nabla_b u = \Delta u$ ,

where we abbreviate  $l_a = \nabla_a \ln \varrho$ . With this, we find

$$
A_y u = \gamma^{ab} \nabla_a \nabla_b u + (\nabla_a \gamma^{ab} - l_a \gamma^{ab}) \nabla_b u
$$
  

$$
\stackrel{d}{=} \gamma^{ab} \nabla_a \nabla_b u = g^{ab} \nabla_a \nabla_b u = \Delta u,
$$

*6* 

 

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- *I* 

The Volume Problem  
\n
$$
\Delta_y^2 u = \Delta \Delta_y u
$$
\n
$$
\Delta_y^2 u = \Delta \Delta_y u
$$
\n
$$
\Delta_z^2 u + (\Delta \gamma^{ab} + 2 \nabla^a \nabla_c \gamma^{bc} - 2 (\nabla^a l_c) \gamma^{bc}) \nabla_a \nabla_b u
$$
\n
$$
\Delta_z^2 u + (2 \nabla_a \gamma^{ab} - (\Delta l_a) \gamma^{ab}) \nabla_b u
$$
\n
$$
\Delta_z^2 u + 2 \beta R^{ab} \nabla_a \nabla_b u + 1/3 g^{ab} \nabla_a \nabla_b u.
$$

A normal coordinate system  $x \mapsto \bar{x} \in \mathbb{R}^n$  with origin y is, by definition, the inverse to the exponential map  $\exp_u$  from  $T_u M$  to M. (Both the maps  $x \mapsto \bar{x}$  and  $\exp_u$  are restricted here to appropriate domains.) Considering this, the formulas (3),  $(4)$  can be reinterpreted. Let, in the following,  $d^n\vec{x}$  denote the measure on  $T_yM$  defined by the metric  $g(y)$  with (constant) components  $g_{ij}(y)$  and let dvol denote the canonical measure on  $M$  defined by the metric  $g(x)$  with (variable) components  $g_{ab}(\dot{x})$ . Further, identify a measure with an alternating n-form and denote by  $\exp_{y}$ \* the pull-back of  $\exp_{u}$ . This pull-back transforms covariant tensors on *M* to covariant tensors on  $T_{u}M$ . With these notations, coordinate-independent expressions for  $\sigma$  and  $\rho$  can be given [19, 1]:  $\dot{x} = A^2u + 2/3R^{ab} \nabla_a \nabla_b u + 1/3g^{ab} \nabla_a S \nabla_b u$ .<br>
and coordinate system  $x \mapsto \bar{x} \in \mathbb{R}^n$  with origin y is, by definition, the ponential map  $\exp_y$  from  $T_y M$  to  $M$ . (Both the maps  $x \mapsto \bar{x}$  and here to appropria

$$
\sigma(x, y) = 1/2g(y) \left( \exp_{y}^{-1} x, \exp_{y}^{-1} x \right),
$$
  
\n
$$
\exp_{y} \exp(x) \left( \frac{\exp_{y} x}{\sqrt{x}}, y \right) d^{n} \overline{x},
$$
\n(7)

i.e. the normal volume function equals the Radon-Nikodym defivative of  $\exp_{u^*}$  dvol with respect to  $d^n\bar{x}$ .

Let us finish this section by shortly reviewing the volume problem for properly Riemannian manifolds. The following formulas are needed in the next section.

Definition 3: The numbers

\n The probability 
$$
a^2x
$$
 is a function by shortly reviewing the volume problem for properly, the probability  $a_k$  is a function. The probability  $a_k = a_k(n) = 2^{-2k}(k!)^{-2} \binom{n/2 + k - 1}{k}^{-1} = 2^{-2k}(k!)^{-1} I' \left(\frac{n}{2} + k\right)^{-1} I' \left(\frac{n}{2}\right)^{-1}$ \n

\n\n For  $k \geq 0$  are called *Pizzetti's coefficients*.\n

for integer  $k \geq 0$  are called *Pizzetti's coefficients*.

Obviously,

**'1**

 

*<sup>a</sup>0* <sup>=</sup>1, a <sup>1</sup> ' = 2n, *a2'* = *8n(n +* 2), 8 *a' = <sup>2</sup>kk!n(n +2)...(n±2k* —2) *fork ^* 1. . (S .

The name "Pizzetti's coefficients" appeared in [14]. We denote by  $B<sup>n</sup>(y, R)$  and  $S^{n-1}(y, R)$  the geodesic ball and the geodesic sphere, respectively, with centre y and radius  $R > 0$  in an *n*-dimensional properly Riemannian manifold. Further, we denote by  $B^n(R)$  and  $S^{n-1}(R)$  the ball and the sphere, respectively, with centre 0 and radius  $R > 0$  in the *n*-dimensional Euclidean space  $\mathbb{R}^n$ . The symbol Vol means the  $a_k^{-1} = 2^k k! n(n+2) \dots (n+2k-2)$  for  $k \ge 1$ .<br>
The name "Pizzetti's coefficients" appeared in [14]. We denote by  $B^n(y, R)$  and<br>  $S^{n-1}(y, R)$  the geodesic ball and the geodesic sphere, respectively, with centre y and<br>
radius  $a_k = a_k(n) = 2^{-2k}(k!)^{-2}$   $\binom{n}{k} = 2^{-2k}(k!)^{-1} \left(\frac{n}{2}\right)$ <br>
for integer  $k \ge 0$  are called *Pizzetti's coefficients*.<br>
Obviously,<br>  $a_0 = 1, \quad a_1^{-1} = 2n, \quad a_2^{-1} = 8n(n + 2),$ <br>  $a_k^{-1} = 2^k k!n(n + 2) \dots (n + 2k - 2)$  for  $k \ge 1$ .<br>
The nam volume with respect to the canonical measure.<br>Proposition 1: *There hold the asymptotic power series expansions*  $(n + 2k - 2)$  for  $k \ge 1$ .<br>
ts" appeared in [14]. We denote by  $B^n(y, R)$  and<br>
the geodesic sphere, respectively, with centre y and<br>
al properly Riemannian manifold. Further, we de-<br>
ball and the sphere, respectively, with ce

$$
B^{n}(R)
$$
 and  $S^{n-1}(R)$  the ball and the sphere, respe  
> 0 in the *n*-dimensional Euclidean space  $\mathbb{R}^{n}$ .  
11 with respect to the canonical measure.  
13.5 is given by  $\sum_{k=0}^{\infty} a_k(n) \left(\Delta_y^k e\right) (y, y) R^{2k}$ .  
 $\frac{\text{Vol } S^{n-1}(y, R)}{\text{Vol } S^{n-1}(R)} \sim \sum_{k=0}^{\infty} a_k(n) \left(\Delta_y^k e\right) (y, y) R^{2k}$ .  
 $\frac{\text{Vol } B^{n}(y, R)}{\text{Vol } B^{n}(R)} \sim \sum_{k=0}^{\infty} a_k(n+2) \left(\Delta_y^k e\right) (y, y) R^{2k}$ .

<sup>8</sup><br>R. SCHIMMING and D. MATEL-KAMINSKA<br>The proof can be read from [6, 14] and is based on the famous Pizzetti formula *-* N

8 R. SCHIMING and D. MATEL-KAMINSKA  
\nThe proof can be read from [6, 14] and is based on the famous Pizzetti formulas  
\n
$$
\frac{\int u dS}{\sqrt{0! S^{n-1}(R)}} \sim \sum_{k=0}^{\infty} a_k(n) (A_0^k u) (0) R^{2k},
$$
\n
$$
\frac{\int u d^n x}{\sqrt{0! B^n(R)}} \sim \sum_{k=0}^{\infty} a_k(n + 2) (A_0^k u) (0) R^{2k},
$$
\nwhere  $A_0$  denotes the Laplacian of  $\mathbb{R}^n$  **1**  
\nThe volume of truncated light cones  
\nIn this section we consider *n*-dimensional pseudo-Riemannian manifolds  $(M, g)$  on  
\nsignature  $(n, \text{'n})$  such that  $n \geq 1, \text{'n} \geq 1, \text{'n} + \text{'n} = n$ . The orthogonal groups to

In this section we consider *n*-dimensional pseudo-Riemannian manifolds  $(M, g)$  of signature ('n, "n) such that 'n  $\geq 1$ , "n  $\geq 1$ , "n + "n = n. The orthogonal groups to the dimensions 'n, "n are denoted by  $\overline{O}(n)$ ,  $O('n)$ , respectively, the pseudo-orthogonal group to the signature  $(n, 'n)$  is denoted by  $O((n, 'n))$ .

Definition 4: An  $O('n) \times O('n)$ -structure at the point  $y \in M$  is a representation of the metric at  $y$  as the difference of two positive semidefinite quadratic forms with for extractions in the maximal possible ranks:  $g(y) = g(y) - g(y)$ ,  $g(y) = g(y) - g(y) - g(y)$ , Vol  $B^{\pi}(R)$   $\longleftarrow_{k=0}^{n} u_k(n+2)$  ( $\Box a \rightarrow u$ ) ( $\Box f$ )  $n+7$ ,<br>
where  $\Box a_0$  denotes the Laplacian of  $\mathbb{R}^n$  **T**<br>
The volume of truncated light cones<br>
In this section we consider *n*-dimensional pseudo-Riemannian manif ich that 'n  $\geq 1$ , ''n  $\geq 1$ , ''n  $\neq$  ''n<br>
''n are denoted by  $O('n)$ ,  $O('n)$ , ro<br>
''n are denoted by  $O('n)$ ,  $O('n)$ , ro<br>
signature  $('n, 'n)$  is denoted by  $O($ <br>
in  $O('n) \times O('n)$ -structure at the positive semple ranks: ensional pseudo-Riemannian maniformers,  $m \geq 1$ ,  $n + n = n$ . The orthogotor  $O('n)$ ,  $O('n)$ , respectively, the point  $y \in M$  is a represented by  $O(n, n)$ .<br>  $\rightarrow$  structure at the point  $y \in M$  is a represented by  $O((n, n)$ .<br>  $\rightarrow$ 

Such  $O(n) \times O(m)$ -structures at a point exist. There exist normal coordinate systems  $x \mapsto \overline{x} = (x^a) \in \mathbb{R}^n$  with the origin y in which the components of 'g(y), ''g(y) are given by<br>are given by<br> $\begin{pmatrix} y_{ij}(y) \\ 0 \end{pmatrix} = \begin{pmatrix} g_{ij}(y) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} g_{ij}(y) \\ g_{ij}(y) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij}(y)$ are given by

$$
\left(\sigma_{ij}(y)\right) = \begin{pmatrix} g_{i'j}(y) & 0 \\ 0 & 0 \end{pmatrix}, \quad \left(\sigma_{ij}(y)\right) = \begin{pmatrix} 0 & 0 \\ 0 & g_{i'i'j}(y) \end{pmatrix}, \quad (11)
$$

respectively. Here we introduce and use a new index convention

$$
(90,97) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad (90,97) = \begin{pmatrix} 0 & 0 \end{pmatrix}
$$
  
respectively. Here we introduce and use a new  

$$
(a, 'b, ..., 'i, 'j, ... = 1, 2, ..., 'n;
$$

$$
(a, ''b, ..., ''i, ''j, ... = 'n + 1, ..., n).
$$

The normal coordinate system can be further specialized to<br>  $g_{ij}(y) = \delta_{ij}, \quad g_{ij}(y) = \delta_{ij}.$ 

$$
g_{i'j}(y) = \delta_{i'j}, \qquad g_{i'i'j}(\gamma) = \delta_{i'i'j}.
$$

If  $O((n) \times O((n))$ -structures are given at various points y of some domain  $U \subseteq M$  and if they depend, in a sense which can be made precise, smoothly on  $y \in M$  then we arrive at the usual notion of a local  $O('n) \times O('n)$ -structure [8, 9]. If, particularly,  $U = M$  then we have a global  $O((n) \times O((n))$ -structure. Generally, such a global structure does not exist. If it exists then it is called a reduction of the global  $O(n, 'n)$ structure defined by the metric g. A local  $O('n) \times O('n)$ -structure exists in a sufficiently small neighbourhood of each point.  $U = M$  then we have a global  $O(n) \times O(m)$ -structure. Generally, such a<br>structure does not exist. If it exists then it is called a reduction of the global  $O(n)$ <br>structure defined by the metric g. A local  $O(n) \times O(m)$ -structure  $g_{i'j}(y) = \delta_{i'j}, \quad g_{i'i'j}(y) = \delta_{i'i'j}.$ <br>  $\langle O('n)$ -structures are given at various points y of some domain U epend, in a sense which can be made precise, smoothly on  $y \in M$  the usual notion of a local  $O('n) \times O('n)$ -structu

Definition 5: The *truncated light cone*  $C(y, R, 'g, 'g)$  with vertex  $y \in M$  and altitude  $R > 0$  with respect to an  $O((n) \times O((n))$ -structure at y is described by the inequalities

$$
"g^{ij}(y)\frac{\partial\sigma}{\partial y^i}\frac{\partial\sigma}{\partial y^i}\leq "g^{ij}(y)\frac{\partial\sigma}{\partial y^i}\frac{\partial\sigma}{\partial y^j}\leq R^2.
$$

Here 'g<sup>ij</sup>(y), "g<sup>ij</sup>(y) originate from 'g<sub>ij</sub>(y), "g<sub>ij</sub>(y) by raising of the indices.

• 

• 

The first unequality of Definition 5 expresses  $\sigma(x, y) \geq 0$ , as will become clear in the following, while the second unequality is the "truncation condition".

In normal coordinates with the origin y there holds, equivalent to (5),  $\partial \sigma / \partial y^i$  $=-g_{\mu q}(y) x^a$  and as a consequence The first unequality of Definition 5 exp<br>
the following, while the second unequality<br>
In normal coordinates with the origin<br>  $=-g_{ia}(y) x^a$  and as a consequence<br>  $'g^{ij}(y) \frac{\partial \sigma}{\partial y^i} \frac{\partial \sigma}{\partial y^j} = 'g_{ab}(y) x^a x^b,$ 

The Volume Pro-  
\nstat unequality of Definition 5 expresses 
$$
\sigma(x, y) \ge 0
$$
, as will be  
\nwing, while the second unequality is the "truncation condition  
\nmal coordinates with the origin y there holds, equivalent  
\ny)  $x^a$  and as a consequence  
\n
$$
g^{ij}(y) \frac{\partial \sigma}{\partial y^i} \frac{\partial \sigma}{\partial y^j} = 'g_{ab}(y) x^a x^b, \qquad ''g^{ij}(y) \frac{\partial \sigma}{\partial y^i} \frac{\partial \sigma}{\partial y^j} = ''g_{ab}(y) x^a x^b.
$$
\nthe normal coordinates belonging to (11)  $C(y, R, \langle g, \neg g \rangle)$  is des

 $y = -g_{ia}(y) x^a$  and as a consequence<br>  $'g^{ij}(y) \frac{\partial \sigma}{\partial y^i} \frac{\partial \sigma}{\partial y^j} = 'g_{ab}(y) x^a x^b$ ,  $''g^{ij}(y) \frac{\partial \sigma}{\partial y^i} \frac{\partial \sigma}{\partial y^j} = ''g_{ab}(y) x^a x^b$ .<br>
Thus, in the hormal coordinates belonging to (11) *C(y, R, 'g, ''g*) is described inequalities *ine 1*<br>  $=$   $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br> the normal coordinates belonging to<br>
g<sub>''a''b</sub>(y)  $x''^a x''^b \leq g_{a'b}(y) x'^a x'^b \leq R^2$ .

$$
g_{\alpha\alpha\beta}(y) x^{\alpha\alpha} x^{\alpha\beta} \leq g_{\alpha\beta}(y) x^{\alpha} x^{\beta} \leq R^2.
$$

With 'r, "r defined by  $'r^2 = g_{a'b}(y) x'^a x'^b$ , " $r^2 = g_{a'b}(y) x'^a x'^b$ , the truncated light cone is also described by  $0 \leq$   $'r \leq r \leq R$ . These descriptions show that for all sufficiently small  $R > 0$  the point set  $C(y, R, 'g, 'g)$  is defined and is compact. In the normalization  $g_{a'b}(y) = \delta_{a'b}$ ,  $g_{a'b}(y) = \delta_{a'b}$  the truncated light cone is described by (*x*)  $k + \frac{1}{r}$ ,  $\frac{r}{r}$  defined by  $\frac{r^2}{2} = g_{\alpha\beta}(y) x^{\alpha} x^{\beta}$ ,  $\frac{r}{r^2} = g_{\alpha\beta}(y) x^{\beta}$ <br>
(e is also described by  $0 \leq \frac{r}{r} \leq r \leq R$ . These described<br>
ficiently small  $R > 0$  the point set  $C(y, R, 'g, 'g')$  is de

$$
(x^{n+1})^2 + \cdots + (x^n)^2 \leq (x^1)^2 + \cdots + (x^{\prime n})^2 \leq R^2.
$$

We denote the set of all points  $\bar{x} = (x^1, \ldots, x^n, x^{n+1}, \ldots, x^n)$  of the flat space  $\mathbb{R}^n$ satisfying these last inequalities by  $C(R, 'n, 'n)$ . Now we are in the position to present our main theorem.  $(x^{n+1})^2 + \cdots + (x^n)$ <br>
We denote the set of all<br>
satisfying these last inequal<br>
sent our main theorem.<br>
Theorem 2: There holds<br>
Vol  $C(y, R, 'g, ''g)$ <br>
Vol  $C(R, 'n, ''n)$ 

*T h core in 2: There holds the asymptotic power series expansion* 

We denote the set of all points 
$$
\bar{x} = (x^1, ..., x^n, x^{n+1}, ..., x^n)
$$
 of the flat space  $\mathbb{R}^n$   
satisfying these last inequalities by  $C(R, 'n, 'n)$ . Now we are in the position to pre-  
sent our main theorem.  
\nTheorem 2: There holds the asymptotic power series expansion  
\n
$$
\frac{\text{Vol } C(y, R, 'g, ''g)}{\text{Vol } C(R, 'n, ''n)}
$$
\n
$$
\frac{\sum_{k=1}^{\infty} n(n+2k)^{-1} \alpha_k('n) \alpha_k('n+2) 'tr'{}^{k} "tr"{}^{k}(d^{2k}\varrho) (y, y) R^{2k}}{\zeta_k'' k^{-\alpha}}\
$$
\nwhere 'tr denotes the contraction with 'g<sup>ij</sup>(y) of some symmetric tensor at y, 'tr denotes  
\nthe contraction with ''g<sup>ij</sup>(y), and  $k = 'k + ''k$ . The absolute term of the expansion  
\nequals 1. The coefficient of  $R^2$  is proportional to  
\n
$$
[('n + 2) 'tr + 'n "tr] Ric(y). \qquad (12)
$$
\nThe coefficient of  $R^4$  is proportional to  
\n
$$
[('n + 2) ('n + 4) 'tr^2 + 2('n + 2) ("n + 4) 'tr "tr + 'n('n + 2) "tr2]\n× [2(Riem)2 - 5(Ric)2 - 9d2 Ric] (y). \qquad (13)
$$
\n(The proportionality factors do not vanish.)  
\nProof: We apply, in a notationally simplified manner, the formula (7) and then

*where* 'tr *denotes the contraction with 'g"(y) of some symmetric tensor at y,* "tr *denotes the contraction with "g<sup>ti</sup>(y), and k = 'k + "k. The absolute term of the expansion* equals 1. The coefficient of  $R^2$  is proportional to where 'tr denotes the contraction with 'g<sup>if</sup>(y) of some symmetric<br>
the contraction with ''g<sup>if</sup>(y), and  $k = k + 2k$ . The absolute<br>
equals 1. The coefficient of  $R^2$  is proportional to<br>  $[(n + 2)$ 'tr + 'n ''tr]  $Ric(y)$ .<br>
The

$$
[(\ 'n + 2)\ 'tr + 'n\ 'tr]\ Ric(y).
$$

*The coefficient* of *R4 is proportional to* 

the contraction with "g<sup>ij</sup>(y), and 
$$
k = 'k + ''k
$$
. The absolute term of the expansion  
equals 1. The coefficient of  $R^2$  is proportional to  
\n
$$
[(\n m + 2) 'tr + 'n "tr] Ric(y).
$$
\nThe coefficient of  $R^4$  is proportional to  
\n
$$
[(\n m + 2) ('n + 4) 'tr^2 + 2(n + 2) ("n + 4) 'tr "tr + 'n("n + 2) "tr2]\n
$$
\times [2(Riem)^2 - 5(Ric)^2 - 9d^2 Ric] (y).
$$
\n(The proportionality factors do not vanish.)  
\nProof: We apply, in a notationally simplified manner, the formula (7) and then  
\nFubini's integral theorem:
$$

**•** 

Proof: We apply, in a notationally simplified manner, the formula (7) and then **a** (7) a

\n
$$
fraction\ with\ "g^{ij}(y),\ and\ k = 'k' + 'k.
$$
\nThe absolute term of the expansion.\n

\n\n
$$
The\ coefficient\ of\ R^{2}\ is\ proportional\ to
$$
\n
$$
[('n + 2) 'tr + 'n "tr] \operatorname{Ric}(y).
$$
\n

\n\n*ificient of R<sup>4</sup> is proportional to*\n
$$
[('n + 2) ('n + 4) 'tr^{2} + 2('n + 2) ('n + 4) 'tr "tr + 'n('n + 2) ''tr^{2}] \times [2(\text{Riem})^{2} - 5(\text{Ric})^{2} - 9d^{2}\text{ Ric}] (y).
$$
\n

\n\n*proportionality factors do not vanish.*\n

\n\n
$$
f: We apply, in a notationally simplified manner, the formula (7) and then
$$
\n\n*integral theorem:*\n

\n\n
$$
Vol\ C(y, R, 'g, ''g) = \int_{C(y, R', g, ''g)} dvol = \int_{C(y, R', g, ''g)} d^{\eta} \overline{x}\varrho
$$
\n
$$
= \int_{B''^{n}(R)} d^{n} \overline{x}\int_{B''^{n}(r)} d^{n} \overline{x}\varrho.
$$
\n

\n\n
$$
B'''^{n}(r)
$$
\n

# 10 R. SCHIMMING and D. MATEL-KAMINSKA

Here

R. SCHIMMING and D. MATEL-KAMINSKA  
\n
$$
d_1^{\prime n} \overline{x} = (\det g_{\alpha b}(y))^{1/2} dx^1 dx^2 \cdots dx'^n,
$$
\n
$$
d^{\prime n} \overline{x} = (\det g_{\alpha a'b}(y))^{1/2} dx^{n+1} \cdots dx^n
$$
\nole normal coordinates. The inner integral is expanded by means of the second

in suitable normal coordinates. The inner integral is expanded by means of the second *Pizzetti* formula (10),

$$
u = (\text{det } g^n a^n b(y))^{n-1} a x^{n-1} \cdots a x^n
$$
  
formula (10),  

$$
\int d^n \overline{x}_0 \sim \text{Vol } B^n(1) \sum_{k=0}^{\infty} a_{k}(n-k-2) \left( \frac{n}{2} \right)^k \left( \frac{x}{2}, 0 \right)^k r^{n+2}.
$$

by means of the first Pizzetti.formula (9):

in suitable normal coordinates. The inner integral is expanded by means of the second  
\nPizzetti formula (10),  
\n
$$
\int_{B''n(r)} d''r\overline{x}_0 \sim \text{Vol } B''n(1) \sum_{k=0}^{\infty} a_{k'}(n+2) \left( \frac{n}{4} \right)^{k} \frac{q}{2}, 0 \right) r^{n+2'k}.
$$
\nThe outer integral is decomposed according to Fubini's theorem and is expanded then  
\nby means of the first Pizzetti formula (9):  
\n
$$
\text{Vol } C(y, R, 'g, 'g) = \int_{0}^{R} d'r \int_{B''n(r)} d' s \int_{B''n(r)} d''r\overline{x}_0.
$$
\n
$$
\sim \text{Vol } B''(R) \text{ Vol } B''n(R) \sum_{k'}^{m} n(n+2k)^{-1}.
$$
\n
$$
\sim \sqrt{2} \int_{0}^{R} d''r \int_{0}^{R} d' s \int_{0}^{R} d''r\overline{x}_0.
$$
\nWe have written the integration differentials just after the integral signs in order to avoid parentheses. The formula (17) in O. Kowatski's paper [14] translates differential operators on  $T, M$  into covariant differential operators. It gives here

We have written the integration differentials just after the integral signs in order to avoid parentheses. The formula  $(17)$  in-O. KOWALSKI's paper  $[14]$  translates differential operators on  $T_yM$  into covariant differential operators. It gives here Find the country of the cou written the integration differentials just<br>rentheses. The formula (17) in O. Kowar<br>erators on  $T_yM$  into covariant differential<br> $({'A_0}^k {''A_0}^{\prime k} \varrho)$  (0) =  $'{\rm tr}'^k {\prime'}^k (d^{2k}\varrho)$  (y, y)<br>raluation of the first terms o written the integration<br>rentheses. The formula<br>erators on  $T_yM$  into economization of the first tend<br>raluation of the first tends of  $d^2\varrho$  and  $d^4\varrho$ <br>servation that the num<br>sition 2: A manifolon<br>relation that the num

$$
('A_0'^{k'''}A_0''^kq)(0) = 'tr'^{k''}tr''k(d^{2k}q)(y, y).
$$

For an evaluation of the first terms of the asymptotic expansion we have to take the coincidence limits of  $d^2\rho$  and  $d^4\rho$  from (1) and the Pizzetti coefficients from (8)  $\blacksquare$ ential operators on  $T_yM$  into covariant differential operators. It gives here<br>  $('A_0'$ <sup>*k*</sup>  $''A_0''$ <sup>*k*</sup> $\theta$ ) (0) = 'tr'<sup>k</sup> "tr"<sup>k</sup>( $d^{2k}\hat{\theta}$ ) (*y, y*).<br>
(For an evaluation of the first terms of the asymptotic expansio

The observation that the numerical coefficients in (12), (13) are positive leads to

*vanishing volume deject*

nce limits of 
$$
d^2\varrho
$$
 and  $d^4\varrho$  from (1) and the Pizz  
observation that the numerical coefficients in (1  
position 2: A manifold  $(M, g)$  with definite  
g volume defect  
Def  $C(y, R, 'g, ''g) := \frac{\text{Vol } C(y, R, 'g, ''g)}{\text{Vol } C(R, 'n, ''n)} -$   
a manifold with defining their form  $2^{(1)}(\text{Hom})^2$ .

*Likewise a manifold with definite jour form 2(Riem) — 5(Ric) 2 — 9d2 Ric has a non- -vanishing volume defect.*

Proof: The contraction of positive definite forms with the positive semidefinite matrices  $'(g^{ij}(y))$ ,  $('g^{ij}(y))$  yields positive numbers. These remain positive when multiplied with  ${}^{t}n + 2$ ,  ${}^{t}n$ , ... in (12), (13) and added up. Analogously, the contractions of negative definite forms yield negative numbers. Thus, the first terms in the connected the set of the numerical coefficients in (12), (13) are positive<br>
The observation that the numerical coefficients in (12), (13) are positive<br>
Proposition 2: A manifold  $(M, g)$  with definite Ricci curvature Ric<br>
v Def  $C(y, R, 'g, ''g) := \frac{\text{Vol } C(y, R, 'g, ''g)}{\text{Vol } C(R, 'n, ''n)} - 1$ .<br>Likewise, a manifold with definite four-form  $2(Riem)^2 - 5(Ric)^2$ <br>vanishing volume defect.<br>Proof: The contraction of positive definite forms with the<br>matrices  $({g^{ij}(y)}), ({''g^{$ Likewise, a manifold with definite four-form  $2(Riem)^2 = 5(Ric)^2 - 9d^2$  Ric has a non-<br>vanishing volume defect.<br>
Proof: The contraction of positive definite forms with the positive semidefinite<br>
multiplied with " $n + 2$ , " $n$ , . Proof: The contraction of positive definite form<br>matrices  $\binom{r}{j}(\binom{r}{j} - \binom{r}{j} - \binom{r}{j} - \binom{r}{j}$  wields positive numbe<br>multiplied with  $\binom{r}{n} + 2$ ,  $\binom{n}{n}$ ,... in (12), (13) and add<br>tions of negative definite f

Examples of manifolds with definite *Ric* or  $2(Riem)^2 - 5(Ric)^2 - 9d^2Ric$ , respec-• tively, can be constructed as the products of Einstein manifolds or of manifolds of constant curvature. Such product constructions will be considered in the next section. inite forms yield negative number of the volume defect do not<br>folds with definite Ric or 2(lucted as the products of Ein<br>Such product constructions<br>for pseudo-Riemannian products<br>for pseudo-Riemannian products<br>which we con

The class of manifolds which we consider in this section admits a more explicit treat-<br>nent of the volume problem. ets<br>
ection admits a more explicit t

•

• .

Definition 6: Let  $(M, 'g)$ ,  $('M, 'g)$  be two properly Riemannian manifolds of dimension 'n, ''n,' respectively, and  $M := M \times M$  be the product manifold. Let, further, 'p:  $M \rightarrow M$ , ''p:  $M \rightarrow$  "M denote the natural projections and 'p<sup>\*</sup>, "p<sup>\*</sup> their. pull-backs. We set  $g = 'p^{*'}g - ''p^{*''}g$  and call  $(M, g)$  the *pseudo-Riemannian prod-*. *uct* of  $('M, 'g), ('M, ''g)$ . *manifolds ('M, 'g), (''M, ''g)* be two properly R<br> *manion 'n, ''n,* respectively, and  $M := 'M \times 'M$  be the<br> *mather, 'p:*  $M \rightarrow 'M$ , 'p:  $M \rightarrow 'M$  denote the natural projectively.<br> *mather, 'p:*  $M \rightarrow 'M$ , 'p:  $M \rightarrow 'M$  denote the **Definition 6:** Let  $(M, 'g)$ ,  $('M, 'g)$  be two dimension 'n, "n, respectively, and  $M := 'M$  ;<br>further, 'p:  $M \rightarrow 'M$ , respectively, and  $M := 'M$  ;<br>further, 'p:  $M \rightarrow 'M$  denote the pull-backs. We set  $g = 'p^*g - ''p^{*'}g$  and call<br>and o

A pseudo-Riemannian product manifold carries a natural global  $O('n) \times O('n)$ structure which can be identified with the very product structure: We adopt the convention to consider truncated light cones only with respect to this natural  $O(n)$  $\times$  *O*('n)-structure! Note the change in the meaning of 'g, ''g; the formulas have to be appropriately reinterpreted. *do different* indicated in the meaning of 'g, ''g; the formulas have to be thely reinterpreted.<br>
em 3: For the pseudo-Riemannian product  $(M, g)$  of two properly Riemannian<br>  $g(M, 'g)$ ,  $('M, 'g)$  there holds<br>  $d$  Vol  $C(y; R, 'g$ *A* pseudo-Riemannian product manifold carries a natural global  $O('n)$ <br>structure which can be identified with the very product structure. We<br>convention to consider truncated light cones only with respect to this nat<br> $\times O'($ 

 $\ldots$  Theorem 3: For the pseudo-Riemannian product (M, g) of two properly Riemannian

$$
d \text{ Vol } C(y; R, 'g, ''g) / dR = \text{ Vol } S'^{n-1}('y, R) \text{ Vol } B''^{n}('y, R).
$$
 (14)

*d Vol C(y, B, 'g, "g)/dR d Vol C(R, 'n, "n)/dR - ( <sup>a</sup>k( <sup>n</sup>) (Ll) ('Y' 'Y) B2) ( a"n + 2) A) ("y, 'y) R2"k).* (15) *kO "k=O "g) = fd'r f d 'S 'o('<sup>x</sup> ) f d" 'e("x) -'* 

*Here*  $y = ('y, ''y)$ *, and '* $g = 'g('x), ''g = ''g('x)$  denote the normal volume functions of. *('M,'g), ("M,''g), respectively, and*  $A_{\lq}$ *,*  $A_{\lq}$ *, their Euclidean Laplacians.* 

Proof: The multiplicativity of the normal volume function is well known:  $\rho(x)$  $=$  ' $\rho('x)$ '' $\rho('x)$ . It implies

\n- \n
$$
(y, 'y),
$$
 and  $'g = 'o('x), ''g = ''o('x)$  denote the normic"  $(''M, ''g)$ , respectively, and  $\Delta'_{y}$ ,  $\Delta'_{y}$  their Euclidean Lap
\n- \n The multiplicative integral of the normal volume function  $'o('x)$ . It implies\n
\n- \n Vol  $C(y, R, 'g, ''g) = \int d'r \int d'S' \varrho('x) \int d'''' \bar{x}'' o('x)$ .\n
\n- \n differentiation,\n
\n

and, by differentiation,

$$
\sim \left(\sum_{k=0}^{\infty} a_k(\eta) (A_{ik}^{k} \rho) (\eta, \eta) R^{2k}\right) \left(\sum_{k=0}^{\infty} a_k(\eta + 2) A_{ik}^{k} \rho) (\eta, \eta, \eta) R^{2k}\right).
$$
  
\nHere  $y = (\eta, \eta)$ , and  $\eta = \frac{1}{2}(\eta, \eta, \eta) = \frac{1}{2}(\eta, \eta) \text{ denote the normal volume function}$   
\n $(M, g), (\eta, \eta, \eta, \eta) = \frac{1}{2}(\eta, \eta) \text{ respectively, and } A_{ij}, A_{ij} \text{ their Euclidean Laplacians.}$   
\nProof: The multiplicativeity of the normal volume function is well known:  
\n $= \frac{1}{2}(\eta, \eta, \eta, \eta, \eta, \eta) = \int_{0}^{\infty} d\eta \int_{0}^{\$ 

The asymptotic expansion follows by insertion of, Pizzetti's formulas  $\blacksquare$ 

• Proposition *3: If the pseudo-Riemannian'produci of ('M, 'g), ("M, "g) has vanish-*

$$
\frac{\mathrm{Vol} S'^{n-1}(y, R)}{\mathrm{Vol} S'^{n-1}(R)} \quad \text{and} \quad \frac{\mathrm{Vol} B'^{n}(y, R)}{\mathrm{Vol} B'^{n}(R)}
$$

*• depend only on R (i.e. do not depend on*  $y = \langle y', 'y \rangle$ *) and the product of these two quanti-*<br>ties equals 1. As a consequence, the coincidence limits of  $\Delta_{y}^{k}$  of and  $\Delta_{y}^{k}$  of  $(k = 1, 2, ...)$ The asymptotic expansion follows by insertion of Pizzetti's formulas <br> *Proposition 3: If the pseudo-Riemannian product of*  $('M, 'g)$ ,  $('M, 'g)$  has vanish-<br> *ing volume defect, then both*<br>
<u>Vol S'<sup>n-1</sup>('g, R)</u><br> *and*  $\frac{\text$ 

Proof: If Def  $C(y, R, 'y, '''y) = 0$ ; then (14) implies

Vol 
$$
S'^{n-1}(y, R)
$$
 Vol  $B''''(y, R) = Vol S'^{n-1}(R)$  Vol  $B''''(R)$ .

# **12** R. Schmming and D. Matel-Kaminska

Hereto the usual "separation of variables" argument is applied and gives the first assertion. Then the coefficients of the Pizzetti expansions in (15) have to be constants; this gives the second assertion  $\blacksquare$ 12 **R. SCHIMMING and D. MATE**<br> *Are assertion. Then the coefficients of this gives the second assertion* **F**<br> *Proposition. 4: If the pseud vanishing volume defect, then the sc*<br> *A* = -3 |'*Riem*|<sup>2</sup> + 8 |'*Ric*|<sup>2</sup> + **E.** SCHIMMING and D. MATEL-KAMINSKA<br>
<sup>1</sup><br>
(1) Then the coefficients of the Pizzetti expansic<br>
is the second assertion  $\blacksquare$ <br>
sition. 4: If the pseudo-Riemannian processition.<br>  $\blacksquare$ <br>
sition. 4: If the pseudo-Riemannian France Contains a state of this given by the state of the state of

*Proposition, 4: If the pseudo-Riemannian product- of ('M, 'g); ("M, "g) has vanishing volume deject, then. the scalar curvatures '5, "S as well as the quantities*  Proposition. 4: If the pseudo-Riemannian product of  $(M, 'g)$ ;<br>vanishing volume defect, then the scalar curvatures 'S, "S as well as the<br>' $A = -3$  |'Riem|<sup>2</sup> + 8 |'Ric|<sup>2</sup> + 5'S<sup>2</sup>, " $A = -3$  |''Riem|<sup>2</sup> + 8 |''R<br>are constants

 $\mathcal{A} = -3$  *|'Riem*|<sup>2</sup> + 8 *|'Ric*|<sup>2</sup> + 5'S<sup>2</sup>,  $\qquad$ " $A = -3$  |''Riem|<sup>2</sup> + 8 |''Ric|<sup>2</sup> + 5'S<sup>2</sup>

$$
('n + 2) 'S + 'n''S = 0,
$$

$$
('n+2) ('n+4)'A + 10('n+2) ('n+4)'S''S + 'n('n+2)''A = 0. (16)
$$

'-Proof: The constancy property follows from' Proposition 3 and the coincidence are constants such that<br>  $('n + 2)'S + 'n''S = 0,$ <br>  $('n + 2)'(n + 4)'A + 10('n + 2)('n + 4)'S''S + 'n'(n + 2)''A = 0.$  (16)<br>
Proof: The constancy property follows from Proposition 3 and the coincidence<br>
limits from [4, 6]<br>  $-3A'y'y = 'S,$ <br>  $-3A'y'y = 'S$ 

$$
('n + 2) ('n + 4)'A + 10('n + 2) ('n +
$$

• The relations (16) follow by requiring the coefficients of  $R^2$  and  $R^4$  in (15) equal to zero

**.**  $-3d'_{\mathbf{y}}$   $'e = 'S$ ,  $-3d_{\mathbf{y}}$   $''_0 = ''S$ ,<br> **45d**<sup>2</sup><sub>**y**</sub>  $''_0 = 'A$ .<br>
The relations (16) follow by requiring the coefficients of  $R^2$  and  $R^4$  in (15) equal to<br>
zero  $\blacksquare$ <br> **Example:** For manifolds ('M, 'g), (''M,  $(n + 2)$   $(n + 4)$   $A + 10(n + 2)$   $(n + 4)$   $S''S + n(n + 1)$ <br>
Proof: The constancy property follows from Proposition 3 and<br>
limits from [4, 6]<br>  $-3A'y'y'e = 'S, \t -3A'y''e = ''S,$ <br>
45 $A^2y''e = 'A, \t 45A^2y''e = ''A.$ <br>
The relations (16) follow by re vions (16) follow by requiring<br>
ple: For manifolds ('M, 'g'<br>
<sup>2</sup>, respectively, the volume of<br>
give us<br> *d* Vol (y, R, 'g, ''g)/*dR*<br>
= Vol S'<sup>n-1</sup>(1) Vol S''<sup>n-1</sup>(1)

f: The constancy property follows from Proposition 3 and the coincidence  
\nom [4, 6]  
\n
$$
-3d'_{y} / \rho = 'S, \t -3d'_{y} / \rho = ''S,
$$
\n
$$
45d_{y}^{2} / \rho = 'A, \t 45d_{z}^{2} / \rho = ''A.
$$
\ntions (16) follow by requiring the coefficients of  $R^{2}$  and  $R^{4}$  in (15) equal to  
\nuple: For manifolds (*M*, *'g*), (*'M*, *'g*) of constant curvature  $'K = '{}_{2}^{2}$ ,  
\n ${}_{2}^{2}$ , respectively, the volume of geodesic spheres and balls is known. Formulas  
\ngive us  
\n
$$
d Vol (y, R, 'g, 'g)/dR = Vol S'^{n-1}(1) \left(\frac{1}{2} \sin '2R\right)^{'n-1} \int_{0}^{R} dr \left(\frac{1}{72} \sin '2r\right)^{'n-1}.
$$
\n(17)

Herefrom Vol  $(y, R, g, g')$  follows by integration with respect to R. If  $K < 0$ , then  $d \text{Vol}(y, R, 'g, 'g)/dR$ <br>  $= \text{Vol}(S'^{n-1}(1)) \text{Vol}(S'^{n-1}(1)) \left(\frac{1}{\lambda} \sin \lambda R\right)^{n-1} \int_{0}^{R} dr \left(\frac{1}{\lambda} \sin \lambda R\right)^{n-1} \cdot \text{Vol}(M, R, 'g, 'g) \text{ follows by integration with respect to } R.$  If  $'K < 0$ , then<br>  $\frac{1}{\lambda} \sin \lambda R$  is to be replaced by  $\frac{1}{|\lambda|} \sinh |\lambda| R$ ; an analo  $\begin{align} \text{from [6]} \ \text{from [6]} \ \text{Herefront} \ \text{Herefront} \ \text{in} \ \text{$ From Vol  $(y, R, Zg, "g)$  follows by integration with respect to R. If  $K < 0$ , the  $\frac{1}{\sqrt{2}}$  is in  $|Z|$  is in  $|Z|$  is in  $|Z|$  is an analogous remark applies if  $K < 0$ .<br>
Proposition 5: *If the pseudo-Riemannian product o •*  $K = '23$ *, respectively, the volume of geodesic spheres and balls is known. Formulas<br>
from [6] give us<br>*  $d \text{Vol}(y, R, 'g, ''g)/dR$ *<br>*  $= \text{Vol}(S^{n+1}(1) \text{Vol}(S^{n-1}(1)) \left(\frac{1}{2} \sin 2R\right)^{n-1} \int_0^R dr \left(\frac{1}{72} \sin {''x}\right)^{n-1}$ *. (17)<br>
Her* 

*("111, "g)-are flat.* Proposition 5: If the pseudo-Riemannian product of two manifolds  $(M, 'g)$ ,  $'M, ''g$  of constant curvature has vanishing volume defect, then the factors  $(M, 'g)$ ,  $'M, ''g$  are flat.<br>Proof: If the volume defect vanishes, then the  $C/K < 0$ .<br>
Proposition 5: If the pseudo-Riemannian product of two manifolds  $(M, 'g)$ ,<br>  $('M, 'g)$  of constant curvature has vanishing volume defect, then the factors  $(M, 'g)$ ,<br>  $('M, 'g)$  are flat.<br>
Proof: If the volume defect v Proposition 5: If the pseudo-Riemannian product of two manifolds  $(M, 'g)$ ,<br>  $('M, 'g)$  of constant curvature has vanishing volume defect, then the factors  $(M, 'g)$ ,<br>  $('M, 'g)$  are flat.<br>
Proof: If the volume defect vanishes, t

*folds ('M,'g), ("M,''g) has vanishing volume defect, then the factors ('M,'g), ("M,''g)*  $a\ddot{r}e$  *flat.* 

Proof: Proposition 4 tells us that the scalar curvatures 'S, "S are constant. Hence the two-dimensional manifolds  $(M, 'g)$ ,  $('M, 'g)$  are of constant curvature and Proposition 5 gives the result  $\blacksquare$  $\begin{bmatrix}\n\text{arrows} & \text{arrows} \\
\text{arrows} & \text{arrows}\n\end{bmatrix}$ 

 

**•** 

*0 - -*

# The Volume Problem 13

## **Discussion**

We investigated the general pseudo-Riemannian case with signature  $(n, 'n)$ ; the Lorentzian case ' $n = 1$  (or '' $n = 1$ ) has been treated already in [2, 3, 20]. The two cases differ in the following aspects:

- Here we consider the volume- problem with respect to a fixed  $O('n) \times O('n)$ structure at a point  $y \in M$  or in a domain  $U \subseteq M$ . In [20] we considered the volume problem with respect to any  $O(1) \times O(n-1)$ -structure in a domain  $U \subseteq M$ . The ambiguity in the choice of the  $O(1) \times O(n-1)$ -structure is described by a timelike vector field  $a = a^i \partial/\partial y^i$ . Fortunately, these vectors a can be geometrically visualized as the "axes" of the truncated light cones. For general  $(n, 'n)$  the ambiguity in the choice of the  $O(n) \times O('n)$ -structure does not have such a nice description.
- Here we consider "full cones" while in [20] only the "forward half cones", characterized by non-negative time values, have been considered. It is this difference which makes here the odd powers  $R^{2k+1}$  of the altitude  $R$  cancel out from the asymptotic expansion 'and which makes in [20] both even powers *R2k* and odd powers  $R^{2k+1}$  to appear. Of course, the odd powers provide-extra informations in [20], which are not available here.
- The Lorentzian case admits geometrical' visualization as well as physical application (in the general theory of relativity). The general case admits neither the one From  $[20]$ , which and<br>  $-$  The Lorentzian<br>
tion (in the general of the other.<br>  $-$  A Lorentzian
	- nor the other.<br>A Lorentzian manifold with vanishing volume defect for each  $O(1) \times O(n 1)$ structure is shown to be flat [20]. This affirmatively answers a "volume conjecture". The answer for properly Riemannian manifolds is not known. For the remaining case ' $n \geq 2$ ,  $\sqrt[n]{n} \geq 2$  there exist non-flat pseudo-Riemannian manifolds with vanishing volume defect, namely the (non-flat) simply harmonic manifolds of signature ('n, "n). The normal volume function- of a simply harmonic manifold is constant, equal to one. Thus the "volume conjecture" in its original form should not be applied; it is to be reformulated: a pseudo-Riemannian manifold of signature  $(n, 'n)$ ,  $n \geq 2$ ,  $n \geq 2$ , with vanishing volume defect is supposed to be simply harmonic. an manifold with vanishing volume defect for each  $O(1) \times O(n - 1)$ <br>shown to be flat [20]. This affirmatively answers a "volume conject<br>answer for properly Riemannian manifolds is not known. For the<br>case ' $n \geq 2$ , " $n \geq 2$
	- The Lorentzian case is included here. We obtain, for instance the following useful formula: The Lorentzian product  $(\mathbb{R} \times M, dt^2 - g)$  of a properly Riemannian manifold  $(M, g)$  and the real number space  $(\mathbb{R}, dt^2)$  satisfies

$$
\frac{d \text{ Vol } C(t_0, y, R, dt^2, g)/dR}{d C(R, 1, n)/dR} = \frac{\text{Vol } B^n(y, R)}{\text{Vol } B^n(R)}.
$$

All  $O(n) \times O("n)$ -structures at a point  $y \in M$  of a pseudo-Riemannian manifold  $(M, g)$  are parametrized by the Grassmann space  $O((n, 'n) / (O((n) \times O('n)))$ ; its diniension equals 'n"n. All  $O('n) \times O('n)$ -structures in a domain  $U \subseteq M$  are parametrized by the sections of a Grassmann bundle over *U,* i.e. a fibre bundle with typical fibre  $O('n, 'n)/(O('n) \times O('n))$ . In order to effectively exploit the ambiguity in the  $O('n) \times O('n)$ -structure, infinitesimal Lorentz transformations should be used, i.e. elements of the vector space  $\delta(n', 'n)/(\delta(n) \times \delta('n))$ ; these can be interpreted as "infinitesimal transformations". Here  $o(...)$  denotes the Lie algebra of a Lie group *0(. ..).* Such procedures could be the topic of future work. Also, other variants of the volume problem, taken in a broad sense, for pseudo-Riemannian nianifolds could be studied, for instance the volume of tubes about curves or submanifolds.

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14 R. Schimming and D. Matel-Kaminska

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