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A Class of Nonlinear Singular Integro-Differential Equations

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Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Mit Hilfe des Hauptsatzes der Theorie pseudo-monotoner Operatoren wird ein Existenzsatz für eine Klasse von nichtlinearen singulären Integrodifferentialgleichungen vom Cauchyschen Typ und zwei zugehörige Klassen von nichtlinearen Integralgleichungen bewiesen.

С помощью основной теоремы теории псевдо-монотонных операторов доказывается теорема существования для одного класса нелинейных сингулярных интегро-дифференциальных уравнений типа Коши и двух связанных с ним классов нелинейных интегральных уравнений.

By means of the main theorem of the theory of pseudo-monotone operators, an existence theorem is proved for a class of nonlinear singular integro-differential equations of Cauchy type and two related classes of nonlinear integral equations.

Introduction

In recent papers (cf. [3] and [5] for an overview) methods of monotone operator theory were applied for proving the existence of a solution to various classes of nonlinear singular integral and integro-differential equations of Cauchy type. In particular, in [4] by means of the theory of pseudo-monotone operators, the author proves an existence theorem for a class of singular integro-differential equations of second order with linear main part. In the present paper we extend this approach to a corresponding class of second-order equations with nonlinear main part. Moreover, a class of nonlinear integral equations occuring in contact problems of elasticity theory [2] are reduced to special singular integro-differential equations of this type.

1. Formulation of problem

We deal with the *nonlinear singular integro-differential equation*

$$\begin{aligned} &-(\gamma P(u'))' + \delta Q(u') + \alpha S[u] - \beta S[u'] \\ &+\varphi(u)\, u' + \lambda \psi(u) = f \qquad \text{on } [-a, a] \end{aligned} \tag{1}$$

under the boundary conditions $u(-a) = u(a) = 0$, where S denotes the Cauchy operator

$$S[u]\,(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{u(y)}{y - x}\, dy.$$

The data fulfil the basic Assumptions I:

(i) P, Q are continuous functions on $\mathbb{R}$ satisfying the growth conditions

$$|P(U)| \leqq c_1 |U|^{p-1} + d_1, \qquad |Q(U)| \leqq c_2 |U|^p + d_2 \tag{2}$$

for any $U \in \mathbb{R}$ with positive constants $c_i, d_i, i = 1, 2,$ and some $p \in [2, \infty)$.

(ii) $\gamma, \delta \in L_\infty(-a, a)$.

(iii) $f, \lambda \in L_1(-a, a)$.

(iv) $\alpha, \beta \in \mathbb{R}$.

(v) $\varphi, \psi \in C(\mathbb{R})$.

In the sequel we are looking for *generalized solutions* $u \in \mathring{W}_p{}^1(-a, a)$ of (1) which are defined by the integral identity

$$a_0(u, v) + a_1(u, v) + a_2(u, v) = b(v) \tag{3}$$

for any $v \in \mathring{W}_p{}^1(-a, a)$, where

$$a_0(u, v) := \int_{-a}^{a} \gamma P(u') v' \, dx + \int_{-a}^{a} \delta Q(u') v \, dx,$$

$$a_1(u, v) := \alpha \int_{-a}^{a} S[u] v \, dx - \beta \int_{-a}^{a} S[u'] v \, dx,$$

$$a_2(u, v) := \int_{-a}^{a} \varphi(u) u'v \, dx + \int_{-a}^{a} \lambda\psi(u) v \, dx$$

$$= - \int_{-a}^{a} \Phi(u) v' \, dx + \int_{-a}^{a} \lambda\psi(u) v \, dx$$

with Φ a primitive of φ and

$$b(v) := \int_{-a}^{a} fv \, dx. \tag{4}$$

The problem (3) is equivalent to the *operator equation*

$$Au = b \qquad \text{for} \quad u \in X := \mathring{W}_p{}^1(-a, a), \tag{5}$$

where $A := A_0 + A_1 + A_2$, the operators $A_k: X \to X^*$, $k = 0, 1, 2$, are defined by $\langle A_k u, v\rangle_X := a_k(u, v)$ for $u, v \in X$ and $b \in X^*$ is defined by (4). Namely, since $f \in L_1(-a, a)$ and the Sobolev space $W_p{}^1(-a, a)$ is continuously imbedded in the space $C[-a, a]$ of continuous functions on $[-a, a]$, we have $|b(v)| \leqq \|f\|_{L_1} \|v\|_C \leqq C_p \|f\|_{L_1} \|v\|$, where $\|\cdot\|$ denotes the norm in X defined by $\|u\| := \|u\|_{W_p{}^1} = \left(\int_{-a}^{a} [|u'|^p + |u|^p] \, dx \right)^{1/p}$ and C_p is the imbedding constant of $W_p{}^1(-a, a)$ in $C[-a, a]$. Analogously one proves that under Assumptions I for any fixed $u \in X$ the expressions $a_k(u, \cdot)$, $k = 0, 1, 2$, represent bounded linear functionals on X. On account of (i), (ii) we have

$$\begin{aligned} |a_0(u, v)| &\leqq \|\gamma\|_{L_\infty} \int_{-a}^{a} [c_1 |u'|^{p-1} + d_1] |v'| \, dx + \|\delta\|_{L_\infty} \int_{-a}^{a} [c_2 |u'|^p + d_2] |v| \, dx \\ &\leqq \|\gamma\|_{L_\infty} [c_1 \|u'\|_{L_p}^{p/q} + d_1(2a)^{1/q}] \|v'\|_{L_p} \\ &\quad + \|\delta\|_{L_\infty} [c_2 \|u'\|_{L_p}^p + d_2 2a] \|v\|_C \leqq h_0(\|u\|) \|v\| \end{aligned} \tag{6}$$

with

$$h_0(\|u\|) := \|\gamma\|_{L_\infty} [c_1 \|u\|^{p/q} + (2a)^{1/q} d_1] + \|\delta\|_{L_\infty} C_p[c_2 \|u\|^p + 2ad_2],$$

where $q = p/(p-1)$ is the exponent conjugate to p. By (iv) and the boundedness of the Cauchy operator S in $L_p(-a, a)$ there holds

$$\begin{aligned} |a_1(u, v)| &\leq [|\alpha| \, \|S[u]\|_{L_p} + |\beta| \, \|S[u']\|_{L_p}] \, \|v\|_{L_q} \\ &\leq [|\alpha| + |\beta|] \, B_p D_q \, \|u\| \, \|v\|, \end{aligned} \tag{7}$$

where B_p is the norm of S in $L_p(-a, a)$ and D_q the imbedding constant of $W_p{}^1(-a, a)$ in $L_q(-a, a)$. Finally, in view of (v), $u \in C[-a, a]$, and $\lambda \in L_1(-a, a)$ we have

$$\begin{aligned} |a_2(u, v)| &\leq \|\Phi(u)\|_{L_q} \|v'\|_{L_p} + \|\lambda\|_{L_1} \|\psi(u)\|_C \|v\|_C \\ &\leq [\|\Phi(u)\|_{L_q} + C_p \|\lambda\|_{L_1} \|\psi(u)\|_C] \, \|v\|. \end{aligned}$$

2. Existence theorem

At first we state the needed *boundedness* and *continuity properties* of the operators A_k, $k = 0, 1, 2$.

The operator A_0 is bounded since by (6) we have $\|A_0 u\| \leq h_0(\|u\|)$. Further A_0 is continuous as follows from the estimations

$$\begin{aligned} |a_0(u, v) - a_0(u_n, v)| \leq{}& \|\gamma\|_{L_\infty} \|P(u') - P(u_n')\|_{L_q} \|v'\|_{L_p} \\ &+ \|\delta\|_{L_\infty} \|Q(u') - Q(u_n')\|_{L_1} \|v\|_C \end{aligned}$$

and

$$\begin{aligned} \|A_0 u - A_0 u_n\| &= \sup \{|a_0(u, v) - a_0(u_n, v)| : \|v\| \leq 1\} \\ &\leq \|\gamma\|_{L_\infty} \|P(u') - P(u_n')\|_{L_q} + C_p \|\delta\|_{L_\infty} \|Q(u') - Q(u_n')\|_{L_1}. \end{aligned}$$

Under assumptions (i) the Nemitskyi operators of P and Q are continuous from $L_p(-a, a)$ to $L_q(-a, a)$ and $L_1(-a, a)$, respectively. Since $u_n' \to u'$ in $L_p(-a, a)$ if $u_n \to u$ in $X = W_p{}^1(-a, a)$, the assertion follows.

In view of (7) the linear operator A_1 is bounded and continuous.

Finally, the operator A_2 is *completely continuous* in the sense that it maps weakly convergent sequences (towards $u \in X$) into strongly convergent ones (towards $A_2 u \in X^*$). Namely, let $u_n \rightharpoonup u$ in X. Then $\|u_n\| \leq$ Const and, due to the compact imbedding of $X = \mathring{W}_p{}^1(-a, a)$ in $C[-a, a]$, we have $u_n \to u$ in $C[-a, a]$. By assumption (v) then also $\psi(u_n) \to \psi(u)$ and $\Phi(u_n) \to \Phi(u)$ in $C[-a, a]$. Therefore,

$$\begin{aligned} \|A_2 u - A_2 u_n\| &= \sup \{|a_2(u, v) - a_2(u_n, v)| : \|v\| \leq 1\} \\ &\leq \sup_{\|v\| \leq 1} \{\|\Phi(u) - \Phi(u_n)\|_{L_q} \|v'\|_{L_p} + \|\lambda\|_{L_1} \|\psi(u) - \psi(u_n)\|_C \|v\|_C\} \\ &\leq \|\Phi(u) - \Phi(u_n)\|_{L_q} + C_p \|\lambda\|_{L_1} \|\psi(u) - \psi(u_n)\|_C \end{aligned}$$

tends to zero as $n \to \infty$. As a completely continuous operator, A_2 is bounded, too. Hence also $A = A_0 + A_1 + A_2$ is a *bounded operator*.

For proving the *monotonicity* of the operator $B := A_0 + A_1$ we make the additional Assumptions II:

(i) There exist constants $c_0 > 0$ and $d_0 \geq 0$ such that

$$\begin{aligned} &[P(U_1) - P(U_2)] \, [U_1 - U_2] \geq c_0 (U_1 - U_2)^2, \\ &|Q(U_1) - Q(U_2)| \leq d_0 \, |U_1 - U_2| \qquad \text{for } U_1, U_2 \in \mathbb{R}. \end{aligned} \tag{8}$$

(ii) $\gamma(x) \geqq \gamma_0 > 0$, $\beta \geqq 0$.

(iii) There holds the inequality, with $\Delta := \|\delta\|_{L_\infty}$,

$$\gamma_0 c_0 \geqq \begin{cases} \Delta^2 d_0^2 a/4\beta & \text{if } \beta \geqq (\pi/4)\, \Delta d_0, \\ 2\Delta d_0 a/\pi + 4a\beta/\pi^2 & \text{if } \beta < (\pi/4)\, \Delta d_0. \end{cases}$$

Then we have

$$\begin{aligned} D &:= \langle Bu_1 - Bu_2, u_1 - u_2 \rangle_X \\ &= [a_0(u_1, u_1 - u_2) - a_0(u_2, u_1 - u_2)] + a_1(u_1 - u_2, u_1 - u_2) \\ &= \int_{-a}^{a} \gamma[P(u_1') - P(u_2')]\,(u_1' - u_2')\,dx + \int_{-a}^{a} \delta[Q(u_1') - Q(u_2')]\,(u_1 - u_2)\,dx \\ &\quad + \int_{-a}^{a} (\alpha S[u_1 - u_2] - \beta S[u_1' - u_2'])\,(u_1 - u_2)\,dx \\ &\geqq \gamma_0 c_0 \int_{-a}^{a} (u_1' - u_2')^2\,dx - \Delta d_0 \int_{-a}^{a} |u_1' - u_2'|\,|u_1 - u_2|\,dx + \frac{\beta}{a} \int_{-a}^{a} (u_1 - u_2)^2\,dx \end{aligned}$$

since (cf. [3])

$$\int_{-a}^{a} S[u]\,u\,dx = 0, \qquad -\int_{-a}^{a} S[u']\,u\,dx \geqq \frac{1}{a} \int_{-a}^{a} u^2\,dx \quad \text{for } u \in \mathring{W}_p^1(-a, a).$$

By means of the elementary inequality $2wz \leqq \mu w^2 + z^2/\mu$ there follows

$$D \geqq \left(\gamma_0 c_0 - \frac{\Delta d_0 \mu}{2}\right) \int_{-a}^{a} (u_1' - u_2')^2\,dx + \left(\frac{\beta}{a} - \frac{\Delta d_0}{2\mu}\right) \int_{-a}^{a} (u_1 - u_2)^2\,dx$$

for any $\mu > 0$. In the sequel we choose

$$\mu = \begin{cases} (\Delta d_0/2\beta)\, a & \text{if } \beta \geqq (\pi/4)\, \Delta d_0, \\ (2/\pi)\, a & \text{if } \beta < (\pi/4)\, \Delta d_0 \end{cases}$$

and obtain

$$D \geqq \left(\gamma_0 c_0 - \frac{\Delta^2 d_0^2 a}{4\beta}\right) \int_{-a}^{a} (u_1' - u_2')^2\,dx \qquad \text{if } \beta \geqq \frac{\pi}{4}\, \Delta d_0,$$

$$\begin{aligned} D &\geqq \left(\gamma_0 c_0 - \frac{\Delta d_0 a}{\pi}\right) \int_{-a}^{a} (u_1' - u_2')^2\,dx - \left(\frac{\pi}{4}\, \Delta d_0 - \beta\right) \frac{1}{a} \int_{-a}^{a} (u_1 - u_2)^2\,dx \\ &\geqq \left(\gamma_0 c_0 - \frac{2\Delta d_0 a}{\pi} + \frac{4a\beta}{\pi^2}\right) \int_{-a}^{a} (u_1' - u_2')^2\,dx \qquad \text{if } \beta < \frac{\pi}{4}\, \Delta d_0 \end{aligned}$$

in virtue of Wirtinger's inequality

$$\int_{-a}^{a} u^2\,dx \leqq (4a^2/\pi^2) \int_{-a}^{a} u'^2\,dx \qquad \text{for } u \in \mathring{W}_p^1(-a,a).$$

This yields $D \geqq 0$ if the inequality (9) is fulfilled. Therefore, $B = A_0 + A_1$ is a continuous monotone operator and since A_2 is a completely continuous operator, the operator $A = B + A_2$ is *pseudo-monotone*.

Finally we show the *coercivity* of the operator A under the following additional Assumptions III:

(i) There exist constants $c_3 > 0$, $d_3 \geqq 0$ and $c_4 \geqq 0$, $d_4 \geqq 0$ such that

$$P(U)\,U \geqq c_3\,|U|^p - d_3, \quad |Q(U)| \leqq c_4\,|U|^r + d_4 \quad \text{for } U \in \mathbb{R}, \tag{10}$$

where $0 < r \leqq p - 1$ and in case of $r = p - 1$ there holds the inequality

$$\gamma_0 c_3 > \alpha_p a \Delta c_4, \tag{11}$$

where α_p with $\alpha_2 = 2/\pi$ is the constant in the generalized Wirtinger inequality below.

(ii) $\lambda(x) \geqq 0$ and there exists a constant $\nu \geqq 0$ such that

$$u\psi(u) \geqq -\nu \quad \text{for } u \in \mathbb{R}. \tag{12}$$

Remark: Obviously, the conditions (2) and (10) for P are fulfilled for functions of type $P(U) = |U|^{p-2}\,U$ being moreover monotonically increasing. Further the condition (8) for P is satisfied if P possesses a derivative greater than a positive constant. Therefore the conditions (2), (8), and (10) for P (in case of $p > 2$) are especially fulfilled for functions of the form $P(U) = |U|^{p-2}\,U + c_0U$, $c_0 > 0$.

The condition (10) for Q implies the condition (2) and the Lipschitz condition (8) for Q is satisfied if Q possesses a bounded derivative. Therefore the conditions (2), (8), and (10) for Q are fulfilled if Q possesses a bounded derivative and grows at most as the power $|U|^{p-1}$ at infinity, for instance for the functions $Q(U) = \arctan U + cU$, $c \in \mathbb{R}$, to mention a concrete example.

Under the additional assumptions (10) and (12) with $\lambda \geqq 0$ besides $\gamma \geqq \gamma_0 > 0$, $\beta \geqq 0$ we have

$$\begin{aligned} D_0 &:= \langle Au, u\rangle_X \\ &= \int_{-a}^{a} \gamma P(u')\,u'\,dx + \int_{-a}^{a} \delta Q(u')\,u\,dx \\ &\quad + \alpha \int_{-a}^{a} S[u]\,u\,dx - \beta \int_{-a}^{a} S[u']\,u\,dx - \int_{-a}^{a} \Phi(u)\,u'\,dx + \int_{-a}^{a} \lambda\psi(u)\,u\,dx \\ &\geqq \gamma_0 \int_{-a}^{a} [c_3|u'|^p - d_3]\,dx - \Delta \int_{-a}^{a} [c_4\,|u'|^r + d_4]\,|u|\,dx - \nu_0, \end{aligned}$$

where $\nu_0 := \nu \int_{-a}^{a} \lambda\,dx$ and we further used that $\int_{-a}^{a} \Phi(u)\,u'\,dx = 0$ for $u \in \mathring{W}_p^1(-a,a)$.

In the following we introduce the equivalent norm $\|\cdot\|_0$ in $\mathring{W}_p^1(-a,a)$ defined by $\|u_0\| := \|u'\|_{L_p}$. If $r < p - 1$, we put $s = p/r$ and t the exponent conjugate to s and obtain

$$\int_{-a}^{a} |u'|^r\,|u|\,dx \leqq \|\,|u'|^r\|_{L_s} \|u\|_{L_t} = \|u'\|_{L_p}^r\,\|u\|_{L_t} \leqq E_t\,\|u\|_0^{r+1},$$

where E_t, $t \geqq 1$, is the imbedding constant of $\mathring{W}_p^1(-a, a)$ in $L_t(-a, a)$. Therefore,

$$D_0 \geqq \gamma_0 c_3 \|u\|_0^p - \gamma_0 d_3 2a - \Delta c_4 E_t \|u\|_0^{r+1} - \Delta d_4 E_1 \|u\|_0 - \nu_0$$

from which in view of $r + 1 < p$ the coercivity of A follows. In case $r = p - 1$ we have

$$\int_{-a}^{a} |u'|^{p-1} |u| \, dx \geqq \|u'\|_{L_p}^{p-1} \|u\|_{L_p} \leqq \alpha_p a \|u\|_0^p$$

in virtue of the *generalized* Wirtinger inequality (cf. [1])

$$\int_{-a}^{a} |u|^p \, dx \leqq \alpha_p{}^p a^p \int_{-a}^{a} |u'|^p \, dx \qquad \text{for } u \in \mathring{W}_p^1(-a, a).$$

Under the assumption (11) again the operator A is coercive.

The main theorem of the theory of pseudo-monotone operators by Brézis (cf. [6: Theorem 27.2]) now yields the existence of a solution u to the operator equation (5).

Theorem 1: *Under Assumptions* I—III *the integro-differential equation* (1) *possesses a generalized solution* $u \in \mathring{W}_p^1(-a, a)$, $2 \leqq p < \infty$.

Remark 2: If $\varphi = \psi = 0$ and (9) is fulfilled as true inequality, the operator $A = B$ is strictly monotone and the solution $u \in \mathring{W}_p^1(-a, a)$ of (1) is unique.

Remark 3: Theorem 1 also holds if in the left-hand side of (1) an additional term of the form $K_1[u] + K_2[u']$ is present, where K_1 is a positive linear bounded operator in $L_p(-a, a)$ (or, more generally, from $L_\infty(-a, a)$ into $L_1(a, -a)$) and K_2 is a linear bounded operator in $L_p(-a, a)$ satisfying the condition $\int_{-a}^{a} K_2[u'] \, u \, dx \geqq 0$ for $u \in \mathring{W}_p^1(-a, a)$.

3. Application to integral equations

We consider the integral equation

$$-\gamma P(u') - \alpha N[u] - \beta S[u] + \Phi(u) = F + c \tag{13}$$

with a free constant $c \in \mathbb{R}$, where N denotes the operator

$$N[u](x) = \frac{1}{\pi} \int_{-a}^{a} u(y) \ln |y - x| \, dy,$$

Φ is a continuously differentiable function on $\mathbb{R}$ and F an absolutely continuous function on $[-a, a]$. This equation is equivalent to the equation (1) with $Q \equiv 0$, $\varphi = \Phi'$, $\psi \equiv 0$, $f = F'$ obtained by differentiating (13). As a corollary to Theorem 1 we therefore get

Theorem 2: *Let there be* $\gamma \in L_\infty(-a, a)$ *with* $\gamma(x) \geqq \gamma_0 > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$, *and* P *a continuous function on* $\mathbb{R}$ *satisfying the conditions* (2), (8), *and* (10). *Then the equation* (13) *possesses a solution* $u \in \mathring{W}_p^1(-a, a)$, $2 \leqq p < \infty$, *for some* $c \in \mathbb{R}$.

Further we deal with the *integral equation*

$$-\gamma P(v) + \int_{-a}^{a} \delta Q(v) \, d\xi + \beta N[v] = F + c \tag{14}$$

with a free constant $c \in \mathbb{R}$ and the additional condition

$$\int_{-a}^{a} v(x)\,dx = d \tag{15}$$

with a given constant $d \in \mathbb{R}$. Again F is a given absolutely continuous function on $[-a, a]$.

Equation (14) (with constant coefficients δ, γ) occurs in a plane contact problem of denting a stamp into a plate with rough surface, where v is the sought function of the contact pressure, F the function describing the form of the basis of the stamp, P the tangential pressure on the surface of the plate as function of v and Q a characteristic quantity for the roughness of the surface of the plate ("dislocation of micro-roughness") also as a function of v (cf. [2]).

We again differentiate the equation (14) and introduce the new unknown function $u(x) = \int_{-a}^{x} v(\xi)\,d\xi - d(x + a)/2a$ satisfying the conditions $u(-a) = u(a) = 0$ in view of (15). This function u is a solution of the equation

$$-\big(\gamma P(u' + D)\big)' + \delta Q(u' + D) - \beta S[u'] = f \tag{16}$$

with $D := d/2a$ and $f := F' + \beta D S[1] = F' + \beta D \ln [(a - x)/(a + x)]$. This is a special case of the equation (1). If u is a solution of (16), $v(x) = u'(x) + d/2a$ is a solution of the equation (14) with (15). Theorem 1 and Remark 2 imply the following

Theorem 3: *Let there be $\gamma \in L_\infty(-a, a)$ with $\gamma(x) \geqq \gamma_0 > 0$, $\delta \in L_\infty(-a, a)$, $\beta \in \mathbb{R}_+$, P and Q continuous functions on $\mathbb{R}$ satisfying the conditions* (2), (8) *with* (9), *and* (10) *with* (11). *Then the equation* (14) *with* (15) *possesses a solution $v \in L_p(-a, a)$, $2 \leqq p < \infty$, which is uniquely determined if in* (9) *the strict inequality sign holds.*

Remark 4: If Q is a linear function and δ is a constant, the term with Q in (16) can be viewed as a term of the form $\varphi(u)\,u'$ in (1) (with an additional given function). Then the existence of a solution $v \in L_p(-a, a)$ to (14) with (15) follows without assuming the inequality (9). Also the solution is unique since $\int_{-a}^{a} u'u\,dx = 1/2[u^2]_{-a}^{a} = 0$ so that the corresponding operator A is strictly monotone.

Remark 5: In [2], without proof, Kudisch states an existence and uniqueness theorem for nonnegative solutions of the equation (14) with (15) under certain monotonicity assumptions on the functions P and Q without assuming an inequality of the form (9). In particular, he assumes that, if δ is a negative constant, Q is a nonnegative increasing continuous function on $\mathbb{R}_+$ with $Q(0) = 0$ and the Nemitskyi operator of the integral of Q in (14) is a monotone operator on a certain class of nonnegative functions.

The corresponding assumption in our treatment of (solutions with arbitrary sign of) the equation (14) in case of a positive constant δ would be

$$\int_{-a}^{a} [Q(u_1') - Q(u_2')]\,(u_1 - u_2)\,dx \geqq 0 \quad \text{for} \quad u_1, u_2 \in \mathring{W}_p{}^1(-a, a).$$

But as is easily seen this requirement on Q is only fulfilled if Q is a linear function so that the foregoing Remark 4 applies.

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