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On Function Spaces Related to Finite Element Approximation Theory

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Es werden mit Hilfe von finiten Dreieckselementen auf einem zweidimensionalen Gebiet Funktionenräume vom Besov-Typ eingeführt und untersucht. Diese Räume ermöglichen einen systematischen Zugang zu den grundlegenden Approximationsabschätzungen für finite Elemente in Besov-Sobolev-Normen, zum Problem der Approximationsverbesserung durch angepaßte Wahl der Triangulation und zu asymptotischen Fehlerschätzungen für elliptische Randwertprobleme.

Введены и исследованы некоторые функциональные пространства типа Бесова на основе треугольных конечных элементов в двухмерной области. Эти пространства позволяют систематически подходить к основным неравенствам теории аппроксимации конечными элементами в нормах классов типа Бесова-Соболева, к проблеме улучшения приближения за счёт подходящего выбора триангуляции и к оценкам ошибки для эллиптических краевых задач.

Certain function spaces of Besov type based on triangular finite elements in a two-dimensional domain are introduced and investigated. These spaces allow a systematic approach to the main estimates of finite element approximation theory in Besov-Sobolev norms, to the problem of improving the rate of approximation by choosing appropriate triangulations, and to asymptotic error estimates for elliptic boundary value problems.

The aim of this paper is to describe some results on finite element approximation in Besov-Sobolev norms. For simplicity, we concentrate on C^0-elements of type (k) over regular triangulations of a bounded plane polygonal domain. Starting from a finite element approximation scheme corresponding to a certain sequence of quasi-uniform triangulations of G, we introduce a scale of approximation spaces which allow a direct approach to questions of finite element approximation theory. It turns out that this scale is closely related to the usual Besov-Sobolev spaces.

In this way we are able to give straightforward proofs of some basic inequalities used in the finite element error estimation theory. Moreover, some new results on improved rates for nonlinear approximation by finite elements with "free" triangulation are included. We close with a short discussion of applications to error estimates for the finite element method for elliptic boundary value problems.

1. Finite element approximation schemes and function spaces

Let G be a bounded polygonal domain in $\mathbb{R}^2$ and denote by d_j and ω_j the sides and the interior angles ($0 < \omega_j \leqq 2\pi$), respectively. A finite set T of nondegenerate closed triangles K_j with pairwise disjoint interiors is called *triangulation* of G if $\bigcup_j K_j = \overline{G}$ and if the intersection of any two triangles of T is either empty or a common vertex or a common side (for the used finite element terminology we refer to Ciarlet [2]). Set $h_j = \operatorname{diam} K_j$, $h = h(T) = \max h_j$, and $r_j = \max \{\operatorname{diam} B : B \subset K_j$, B is a circle}. The triangulation T is called *γ-regular* and *γ'-quasiuniform* if $\max_j \{h_j/r_j\} \leqq \gamma$ and $\max_{j,j'} \{h_j/r_{j'}\} \leqq \gamma'$ $(2 \leqq \gamma \leqq \gamma' < \infty)$, respectively. To any triangulation T of G and any fixed $k \in \mathbb{N}$ we correspond the *finite element space* $S^{(k)}(T)$ consisting of all (continuous) functions on $\overline{G}$, whose restrictions to the triangles K_j are polynomials of total degree at most k. Let $Q_{T,j}^{(k)}$, $j = 1, \ldots, M_T^{(k)}$, denote the usual Lagrange points corresponding to T (cf. Figure 1), and introduce by the properties $B_{T,j}^{(k)} \in S^{(k)}(T)$, $B_{T,j}^{(k)}(Q_{T,j'}^{(k)}) = \delta_{j,j'}$, $j, j' = 1, \ldots, M_{T^{(k)}}$, the standard basis functions of $S^{(k)}(T)$. Clearly, $M_{T^{(k)}} = \dim S^{(k)}(T)$.

Let us mention that any triangulation T induces in a natural way a partition ∂T of the boundary ∂G of G and that $\partial S^{(k)}(T) = \operatorname{span} \{B_{T,j}|_{\partial G} : Q_{T,j}^{(k)} \in \partial G\}$ defines the corresponding finite element space on ∂G with respect to this partition.

In the following we consider finite element approximation schemes $\{S^{(k)}(T_i)\}$ generated by a sequence $\{T_i\}$ of triangulations of G satisfying the properties ($i \in \mathbb{N}_0$)

(a) $S^{(k)}(T_0) \subset \cdots \subset S^{(k)}(T_i) \subset S^{(k)}(T_{i+1}) \subset \cdots$.

(b) T_i are γ'-quasiuniform for some fixed constant γ'.

(c) $c_1 2^{2i} \leqq M_{T_i}^{(k)} \equiv M_i^{(k)} \leqq c_2 2^{2i}$ for some constants $c_1, c_2 > 0$.

For instance, if T_0 is any fixed initial triangulation of G, and the $T_i (i \in \mathbb{N})$ are obtained from T_0 by standard dyadic subdivision of the triangles (cf. Figure 2) then the corresponding finite element approximation scheme satisfies these properties. Furthermore, if (b) is fulfilled then (c) is also equivalent to

(c)' $c_1' 2^{-i} \leqq h(T_i) \leqq c_2' 2^{-i}$ for some constants $c_1', c_2' > 0$.

Now, we associate to the above introduced approximation scheme some scale of function spaces. Here, and in the following, let $0 < p, q \leqq \infty$, and $L_p(G)$, l_q denote the usual Lebesgue spaces. Let $s \geqq 0$.

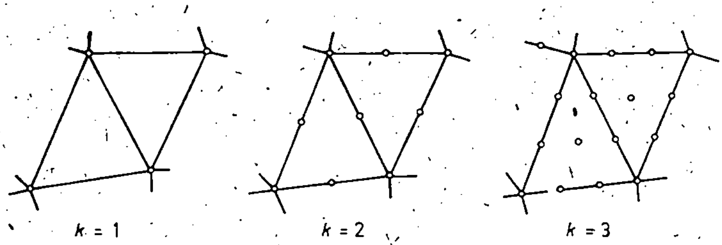

Fig. 1. The standard Lagrange points for elements of type (k)

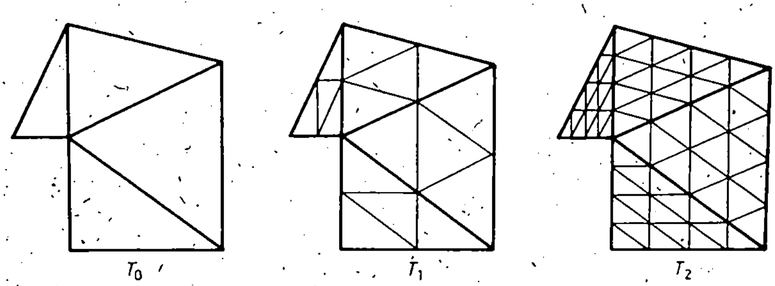

Fig. 2. The triangulations T_i generated by dyadic subdivision

Definition: A function $f \in L_p(G)$ belongs to $A^s_{p,q}(\{S^{(k)}(T_i)\}) \equiv A^{s;k}_{p,q}$ if there exists a representation

$$f = \sum_{i=0}^{\infty} g_i, \qquad g_i \in S^{(k)}(T_i) \tag{1}$$

converging in $L_p(G)$ where

$$\|\{g_i\}\|_{p,q;s} \equiv \|\{2^{is} \|g_i\|_{L_p}\}\|_{l_q} < \infty. \tag{2}$$

In this case, set

$$\|f\|_{A^{s;k}_{p,q}} \equiv \inf \left\{\|\{g_i\}\|_{p,q;s} : \{g_i\} \text{ satisfies (1), (2)}\right\}. \tag{3}$$

Remarks: 1. This definition of $A^s_{p,q}$-spaces ($s > 0$) when applied to subspaces of trigonometric polynomials of order 2^i on $\mathbb{T}^N$ or to subspaces of entire analytic functions of most exponential type 2^i on $\mathbb{R}^N$ leads to one of the classical definitions of Besov spaces on $\mathbb{T}^N$ or $\mathbb{R}^N$ (cf. NIKOLSKIJ [10: 5.6]).

2. It is easy to observe that the case $s = 0$ makes sense only for $0 < q \leqq \delta \equiv \min(p, 1)$ where $A^{0;k}_{p,q}$ isometrically coincides with $L_p(G)$ $(0 < p < \infty)$ and $C(G)$ $(p = \infty)$, respectively.

3. Introducing the best approximations $E_i^{(k)}(f)_p = \inf \{\|f - g\|_{L_p} : g \in S^{(k)}(T_i)\}$, $i \in \mathbb{N}_0$, one shows as in NIKOLSKIJ [10: 5.6] that for $s > 0$

$$\|f\|^{(1)}_{A^{s;k}_{p,q}} \equiv \|\{2^{is} E_i^{(k)}(f)_p\}\|_{l_q} + \|f\|_{L_p} \asymp \|f\|_{A^{s;k}_{p,q}} \tag{4}$$

(equivalence up to constants depending on p, q, s, only) and that $A^{s;k}_{p,q}$ are quasi-Banach spaces for all considered parameters (Banach spaces iff $1 \leqq p, q \leqq \infty$). For instance, setting for abbrevity $X = A^{s;k}_{p,q}$ and $\delta' \equiv \min(p, q, 1)$ we have $\|f + g\|_X^{\delta'} \leqq \|f\|_X^{\delta'} + \|g\|_X^{\delta'}$ $(f, g \in X)$ for any of the above considered equivalent quasinorms and arbitrary values of parameters.

4. Replacing $\{S^{(k)}(T_i)\}$ by $\{\partial S^{(k)}(T_i)\}$ we can analogously define a scale of function spaces $\partial A^{s;k}_{p,q} \equiv A^s_{p,q}(\{\partial S^{(k)}(T_i)\}) \hookrightarrow L_p(\partial G)$ on the boundary of G.

The next Lemma follows from the fact that finite element approximation schemes have, in contrast to other approximation schemes, locally supported basis functions $B^{(k)}_{i,j} \equiv B^{(k)}_{T_i,j}$ which are easy to handle with.

Lemma 1: *Let $\{S^{(k)}(T_i)\}$ be a finite element approximation scheme satisfying properties* (a)—(c). *Then, for any*

$$g_i = \sum_{j=1}^{M_i^{(k)}} a_{i,j} B^{(k)}_{i,j} \in S^{(k)}(T_i), \qquad i \in \mathbb{N}_0, \tag{5}$$

we have

$$c_3 \|g_i\|_{L_p} \leqq \|a_{i,\cdot}\|_{l_p} \leqq c_4 \|g_i\|_{L_p} \tag{6}$$

where

$$\|a_{i,\cdot}\|_{l_p} \equiv \begin{cases} \left(2^{-2i} \sum_{j=1}^{M_i^{(k)}} |a_{i,j}|^p\right)^{1/p}, & 0 < p < \infty, \\ \max\{|a_{i,j}| : j = 1, \ldots, M_i^{(k)}\}, & p = \infty \end{cases} \tag{7}$$

(here, and in the following, the positive constants c_l $(l = 3, 4, \ldots)$ depend on the approximation scheme (more precisely, on k, c_l, c_l', $(l = 1, 2)$ and γ') and on further parameters such as $p, q, s, \ldots$, but are independent of $i \in \mathbb{N}_0$ and of the functions under consideration).

Proof: In the proofs we drop the indication of k in the notations. Observe that any restriction $g_i|_K = \sum_{j: Q_{i,j} \in K} a_{i,j} B_{i,j}|K$, $K \in T_i$, coincides with some polynomial of most total degree k. Transforming K by an affine map onto a fixed reference triangle K_0 one shows in a standard way, that, according to (b), (c),

$$\|g_i\|_{L_p(K)} \asymp \Big(2^{-2i} \sum_{j: Q_{i,j} \in K} |a_{i,j}|^p\Big)^{1/p}, \quad \max_{j: Q_{i,j} \in K} \{|a_{i,j}|\}$$

(equivalence up to constants) for $0 < p < \infty$ and $p = \infty$, respectively. Summing up with respect to $K \in T_i$ we obtain (6) ∎

By Lemma 1 the following characterization of $A^{s;k}_{p,q}$ is obvious.

Corollary 1: *Let* $\{S^{(k)}(T_i)\}$ *satisfy* (a)—(c). *Then*

$$\|f\|^{(2)}_{A^{s;k}_{p,q}} \equiv \inf \big\{\|\{2^{is} \|a_{i,\cdot}\|_{l_p}\}\|_{l_q}\big\}$$

where the infimum is taken with respect to all representations (1), (2) *of the function* f (cf. (5)) *is an equivalent quasinorm on* $A^{s;k}_{p,q}$ *for all possible choices of parameters.*

Corollary 2: *Let* $0 < p < p' \leqq \infty$, $s > 0$, $0 < q \leqq \infty$, *and* $s' = s - 2(1/p - 1/p')$. *Then we have the* (*continuous*) *embeddings*

$$A^{s;k}_{p,q} \hookrightarrow A^{s';k}_{p',q} \qquad \text{if } s' > 0, \tag{8}$$

$$A^{s;k}_{p,q'} \hookrightarrow L_{p'} \quad (p' < \infty), \quad A^{s;k}_{p,1} \hookrightarrow C \quad (p' = \infty) \quad \text{if } s' = 0. \tag{9}$$

Proof: Relation (6) immediately yields a Nikolskij type inequality

$$\|g_i\|_{L_{p'}} \leqq c_5 2^{2(1/p - 1/p')i} \|g_i\|_{L_p}, \quad 0 < p < p' \leqq \infty, \tag{10}$$

for arbitrary $g_i \in S^{(k)}(T_i)$; $i \in \mathbb{N}_0$. Thus, (8) as well as (9) (for $p' \leqq 1$ or $p' = \infty$) are obvious by definition of the spaces. Finally, the proof of (9) for $1 < p' < \infty$ can be carried out by arguments as in [12: Theorem 7] ∎

Remarks: 5. The following more elementary embeddings

$$A^{s;k}_{p,q} \hookrightarrow \begin{cases} A^{s';k}_{p,q'} & \text{if } 0 < p, q, q' \leqq \infty, 0 < s' < s, \\ A^{s;k}_{p,q'} & \text{if } 0 < p \leqq \infty, 0 < q \leqq q' \leqq \infty, s > 0, \\ A^{s;k}_{p',q} & \text{if } 0 < p' \leqq p \leqq \infty, 0 < q \leqq \infty, s > 0 \end{cases} \tag{11}$$

can be combined with (8), (9) in order to obtain all possible cases of embeddings between the above defined spaces.

6. Analogous results can be stated for the spaces $\partial A^{s;k}_{p,q}$. For, we have to restrict all representations to the finite element spaces $\partial S^{(k)}(T_i)$ and their basis functions $\partial B^{(k)}_{i,j} \equiv B^{(k)}_{i,j}|_{\partial G}$ where $Q^{(k)}_{i,j} \in \partial G$ defined on the boundary and to make the changes in the exponents corresponding to the fact that $\dim \partial S^{(k)}(T_i) \asymp 2^i$.

We finish this Section by stating some results on the trace to the boundary of functions belonging to $A^{s;k}_{p,q}$. According to [10: 6.4] and [1: 20] for our situation of a plane polygonal domain the following definition will apply. Let $d = d_j$ be any side of G and fix some closed triangle $\hat{K} \subset \bar{G}$ with d as its side (without loss of generality, suppose that d lies on the x_1-axis and K in the upper half-plane as shown by Figure 3). Then, a measurable function h defined on d is called the trace of the measurable function f defined on G if, for some $0 < p < \infty$, there exists a function f_1 defined on $\hat{K}$ such that $f_1 = f$ a.e. on $\hat{K}$, $f_1 = h$ a.e. on d, the restriction of f_1 to any segment

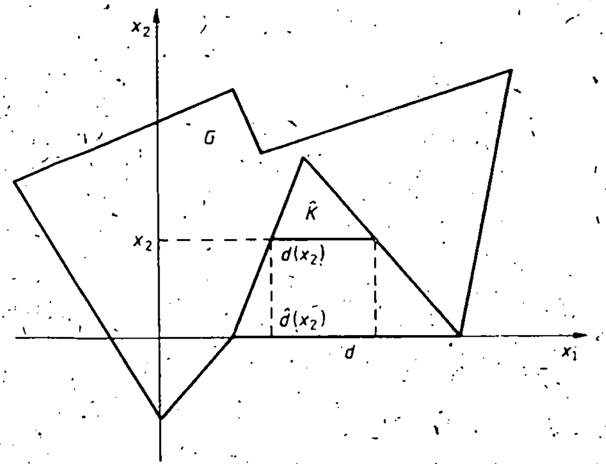

Fig. 3

$d(x_2) = \{(x_1, x_2): (x_1, x_2) \in \hat{K}\}$ is L_p-integrable ($x_2 \geqq 0$), and

$$\lim_{x_2 \to 0+} \int_{\hat{d}(x_2)} |f_1(x_1, x_2) - f_1(x_1, 0)|^p \, dx_1 = 0$$

where $\hat{d}(x_2) = \{x_1: (x_1, 0) \in d \text{ and } (x_1, x_2) \in d(x_2)\}$. Finally h defined on ∂G is the trace of f if $h|_d$ is the trace of f with respect to d for any side d of G. The so defined *trace* (if it exists) is unique and does not depend (for f belonging to some L_p space) on the particular choice of the triangles $\hat{K}$. It will be denoted by ∂f. Obviously, for continuous functions the trace exists and $\partial f = f|_{\partial G}$.

Theorem 1: *Let $0 < p, q \leqq \infty$, $s > 1/p$. Then any function $f \in A^{s;k}_{p,q}$ has a trace $\partial f \in \partial A^{s-1/p;k}_{p,q}$, and $\|\partial f\|_{\partial A^{s-1/p;k}_{p,q}} \leqq c_6 \|f\|_{A^{s;k}_{p,q}}$. Moreover, the inverse statement also holds: For arbitrary $h \in \partial A^{s-1/p;k}_{p,q}$ there exists a function $f \in A^{s;k}_{p,q}$ such that $h = \partial f$ and $\|f\|_{A^{s;k}_{p,q}} \leqq c_7 \|\partial f\|_{\partial A^{s-1/p;k}_{p,q}}$.*

Proof: The proof adapts the corresponding argument from the $\mathbb{R}^N$-case [10: 6.5–6]. Let $f \in A^{s;k}_{p,q}$, $s > 1/p$, and consider any representation (1), (2). Let d and $\hat{K}$ be as above. First we show that on the segments $d(x_2)$ the representation $\sum_{i=0}^{\infty} g_i$ converges also in $L_p(d(x_2))$ for any reasonable $x_2 \geqq 0$. For, observe that the triangulation T_i induces on the segment $d(x_2)$ a certain one-dimensional partition with maximal stepsize $c2^{-i}$. Thus, by Lemma 1 and its one-dimensional analogue, we have

$$\|g_i\|^p_{L_p(d(x_2))} \leqq c2^{-i} \sum_{K \in T_i : K \cap d(x_2) \neq \emptyset} \; \sum_{Q_{i,j} \in K} |a_{i,j}|^p \leqq c2^i \|g_i\|^p_{L_p} \tag{12}$$

and, thus,

$$\sum_{i=0}^{\infty} \|g_i\|^{\delta}_{L_p(d(x_2))} \leqq c \sum_{i=0}^{\infty} (2^{i/p} \|g_i\|_{L_p})^{\delta} \leqq c \, \|\{g_i\}\|^{\delta}_{s,p,q}$$

where in the last step we explored the elementary but useful inequality

$$\|\{2^{ib} a_i\}_{i>n}\|_{l_{q'}} \leqq c_0 2^{-n(s-b)} \|\{2^{is} a_i\}_{i>n}\|_{l_q} \tag{13}$$

which holds with a constant independent of $n \in \mathbb{N}_0$ and the sequence $\{a_i\}$ whenever $-\infty < b < s < \infty$ and $0 < q, q' \leqq \infty$. This shows the desired convergence. By (1) and the above estimates it is easy to show that the limit function $\sum_{i=0}^{\infty} g_i(x_1, x_2) = f_1(x_1, x_2)$, which is defined for any reasonable $x_2 \geqq 0$ on $d(x_2)$ and, therefore, on $\hat{K}$, does not depend (in the sense of $L_p(d(x_2))$) on the particular choice of the representation (1), (2). Furthermore, Fubini's theorem implies that f_1 considered as a function on $\hat{K}$ coincides a.e. with f. Since the functions g_i are continuous, for any $n \in \mathbb{N}$ we can find $x_2(n) > 0$ such that for sufficiently small $0 < x_2 \leqq x_2(n)$ we have the estimates

$$\begin{aligned} &\|f_1(\cdot, x_2) - f_1(\cdot, 0)\|^{\delta}_{L_p(\partial(x_2))} \\ &\leqq \sum_{i=0}^{\infty} \|g_i(\cdot, x_2) - g_i(\cdot, 0)\|^{\delta}_{L_p(\partial(x_2))} \leqq 1/n + \sum_{i=n}^{\infty} \left(\|g_i\|^{\delta}_{L_p(d(x_2))} + \|g_i\|^{\delta}_{L_p(d(0))}\right) \\ &\leqq 1/n + c \left\|\{2^{i/p} \|g_i\|_{L_p}\}_{i>n}\right\|^{\delta}_{l_\delta} \leqq 1/n + c2^{-(s-1/p)n\delta} \|\{g_i\}\|^{\delta}_{s,p,q} \end{aligned}$$

(cf. (12), (13)). This finally proves that $\partial f = \sum_{i=0}^{\infty} \partial g_i = \sum_{i=0}^{\infty} g_i|_{\partial G}$ (convergence in the sense of $L_p(\partial G)$) defines the trace of the function $f \in A^{s;k}_{p,q}$ if $s > 1/p$. Moreover, from (12) it follows that

$$\left\|\{2^{i(s-1/p)} \|\partial g_i\|_{L_p(\partial G)}\}\right\|_{l_q} \leqq c \left\|\{2^{is} \|g_i\|_{L_p}\}\right\|_{l_q}$$

and by definition of the spaces $A^{s;k}_{p,q}$ and $\partial A^{s;k}_{p,q}$, we get $\partial f \in \partial A^{s-1/p;k}_{p,q}$ as well as the continuity of the trace operator.

In order to prove the inverse statement, let $h \in \partial A^{s-1/p;k}_{p,q}$ be given and consider a corresponding representation

$$h = \sum_{i=0}^{\infty} h_i, \qquad h_i \equiv \sum_{j:Q_{i,j}\in\partial G} \bar{a}_{i,j}\, \partial B_{i,j} \in \partial S^{(k)}(T_i) \equiv S^{(k)}(\partial T_i)$$

satisfying $\left\|\{2^{i(s-1/p)} \|h_i\|_{L_p(\partial G)}\}\right\|_{l_q} \leqq c \|h\|_{\partial A^{s-1/p;k}_{p,q}}$. Extending the functions h_i to the whole domain G by defining

$$g_i = \sum_j a_{i,j} B_{i,j}, \qquad a_{i,j} = \begin{cases} \bar{a}_{i,j} & \text{if } Q_{i,j} \in \partial G, \\ 0 & \text{otherwise} \end{cases}$$

we have $\|g_i\|_{L_p} \asymp 2^{-1/p} \|h_i\|_{L_p(\partial G)}$, $\partial g_i = h_i$. Thus, $f = \sum_{i=0}^{\infty} g_i$ belongs to $A^{s;k}_{p,q}$ $(s > 1/p)$ and satisfies $\|f\|_{A^{s;k}_{p,q}} \leqq c \|h\|_{\partial A^{s-1/p;k}_{p,q}}$. Furthermore, by our construction and the above proved first part of Theorem 1 we have $\partial f = h$ ∎

Corollary 3: *Let $\Gamma \subset \partial G$ be a closed subset of the boundary which is the union of (some) segments contained in ∂T_0. Then, for $0 < p, q \leqq \infty$ and $s > 1/p$ the subspaces $A^{s;k}_{p,q;\Gamma} = \{f \in A^{s;k}_{p,q}: \partial f = 0 \text{ on } \Gamma\}$ can equivalently be defined as approximation spaces $A^s_{p,q}(\{S_\Gamma^{(k)}(T_i)\})$ (cf. the corresponding definition above) where $S_\Gamma^{(k)}(T_i) = \{g \in S^{(k)}(T_i) : g|_\Gamma = 0\}$, $i \in \mathbb{N}_0$.*

Proof: Clearly, both definitions make sense and, obviously, $A^s_{p,q}(\{S_\Gamma^{(k)}(T_i)\}) \hookrightarrow A^{s;k}_{p,q;\Gamma}$. In order to establish the inverse embedding, suppose that $s > 1/p$ and $f \in A^{s;k}_{p,q}$, $\partial f = 0$ on Γ, and consider any representation (1), (2). Let $s_n = \sum_{i=0}^{n} g_i$ and define $\bar{s}_n \in S_\Gamma^{(k)}(T_n)$ by setting

$$\bar{s}_n(Q_{i,j}) = \begin{cases} s_n(Q_{i,j}) & \text{if } Q_{i,j} \notin \Gamma, \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N}_0).$$

Since, by Lemma 1 and (13),

$$\|\bar{s}_n - s_n\|_{L_p} \leqq c2^{-n/p}\|s_n\|_{L_p(\Gamma)} \leqq c2^{-n/p}\left(\sum_{i=n+1}^{\infty}\|g_i\|^{\delta}_{L_p(\Gamma)}\right)^{1/\delta}$$
$$\leqq c2^{-n/p}\left\|\{2^{i/p}\|g_i\|_{L_p}\}_{i>n}\right\|_{l_\delta} \leqq c2^{-sn}\left\|\{2^{si}\|g_i\|_{L_p}\}_{i>n}\right\|_{l_q}$$

we get with the notation $\bar{g}_0 = \bar{s}_0$, $\bar{g}_n = \bar{s}_n - \bar{s}_{n-1}$, $n \in \mathbb{N}$, from (1) that the representation $f = \sum_{i=0}^{\infty} \bar{g}_i$, $\bar{g}_i \in S_{\Gamma}^{(k)}(T_i)$ holds in the sense of L_p. Furthermore, we can estimate (cf. (13) and set $s_{-1} = \bar{s}_{-1} = 0$) as follows:

$$\|\{\bar{g}_i\}\|^{\delta'}_{p,q,s} \leqq \|\{g_i\}\|^{\delta'}_{p,q,s} + \|\{s_i - \bar{s}_i\}\|^{\delta'}_{p,q,s} + \|\{s_{i-1} - \bar{s}_{i-1}\}\|^{\delta'}_{p,q,s}$$
$$\leqq c\left\|\left\{2^{i(s-1/p)}\|\{2^{r/p}\|g_r\|_{L_p}\}_{r>i}\|_{l_\delta}\right\}\right\|^{\delta'}_{l_q}$$
$$\leqq c\left\|\left\{\|\{2^{rs}\|g_r\|_{L_p}\}_{r>i}\|_{l_q}\right\}\right\|^{\delta'}_{l_q} \leqq c\,\|\{g_i\}\|^{\delta'}_{p,q,s}$$

Thus, $f \in A^s_{p,q}(\{S_{\Gamma}^{(k)}(T_i)\})$ with the corresponding inequality for the quasinorms ∎

Remarks: 7. Theorem 1 and Corollary 3 enable us to deal with finite element methods for boundary value problems. In particular, the spaces $A^{s;k}_{p,q;\Gamma} \cong A^s_{p,q}(\{S_{\Gamma}^{(k)}(T_i)\})$, $s > 1/p$, are adapted to second order elliptic problems with Dirichlet conditions on the part Γ of the boundary.

8. For $0 < s < 1/p$ one can prove $A^s_{p,q}(\{S_{\Gamma}^{(k)}(T_i)\}) \cong A^{s;k}_{p,q}$.

2. $A^{s;k}_{p,q}$ and Besov-Sobolev spaces

In order to translate the approximation estimates given below in terms of $A^{s;k}_{p,q}$ quasinorms into the more familiar terminology of Besov-Sobolev quasinorms we first clarify the corresponding relationships between the spaces itself.

For the Besov spaces on G we use the inner description by moduli of continuity (or, what is really equivalent, by differences) as the basic definition: Let $0 < p, q \leqq \infty$, $s > 0$, and $l \in \mathbb{N}$ be given. Then a function $f \in L_p$ belongs to the *Besov space* $B^{s;l}_{p,q}$ iff

$$\|f\|_{B^{s;l}_{p,q}} \equiv \|f\|_{L_p} + \|\{2^{is}\omega_l(2^{-i}, f)_p\}\|_{l_q} < \infty$$

where $\omega_l(t, f)_p = \sup\{\|\Delta_h{}^l f\|_{L_p(G_{l,h})}: |h| \leqq t\}$ is the total l-th order modulus of continuity of f in L_p ($h = (h_1, h_2)$, $|h|$ denotes its Euclidean norm, $G_{l,h} = \{x \equiv (x_1, x_2) \in G: [x, x + lh] \subset G\}$, and $\Delta_h{}^l f(x) = \sum_{i=0}^{l}\binom{l}{i}(-1)^{l-i} f(x + ih)$). From the well-known saturation properties of the moduli of continuity there follows that for $s \geqq l - 1 + 1/\delta$ (exept the case $s = l - 1 + 1/\delta$, $q = \infty$) this definition leads to the trivial class of polynomials of total degree smaller than l on G. Informations about other possible definitions and a lot of equivalent quasinorms can be found in, e.g., Besov, Il'in and Nikolskij [1] or Triebel [19] for $1 \leqq p, q \leqq \infty$ and in Triebel [18, 20] for the general case.

The *Sobolev-Slobodezkij spaces* $W_p{}^s$, $1 \leqq p < \infty$, $s > 0$, are defined by

$$W_p{}^m = \left\{f \in L_p(G): \|f\|_{W_p{}^m} \equiv \|f\|_{L_p} + \sum_{m_1+m_2=m}\|\partial^m f/\partial x_1{}^{m_1}\,\partial x_2{}^{m_2}\|_{L_p} < \infty\right\}$$

for integer $s = m$ and by $W_p^s = B_{p,q}^{s;l}$ for non-integer $s > 0$ (the integer $l > s$ can arbitrarily be chosen). We refer again to the above cited monographs, a detailed discussion of Sobolev spaces on plane polygonal domains can be found in Grisvard [6]. Let us mention that for $p = 2$ the Sobolev-Slobodezkij spaces can equivalently be characterized as Lebesgue spaces with a somewhat different definition (for details on these H_p^s spaces as well as on the more general scale of Lizorkin-Triebel spaces $F_{p,q}^s$ we refer to [18—20]). However, for our purposes it is sufficient to observe that the Sobolev-Slobodezkij and the Lebesgue spaces are imbedded between appropriate Besov spaces. To be precise, for $1 \leqq p < \infty$ we have

$$B_{p,\min(p,2)}^{s;l} \hookrightarrow W_p^s;\ H_p^s \hookrightarrow B_{p,\max(p,2)}^{s;l}, \qquad 0 < s < l. \tag{14}$$

Sometimes we also use the elementary imbedding relation

$$W_p^m \hookrightarrow B_{p,\infty}^{m;l} \qquad (1 \leqq p < \infty,\ m = 1, \ldots, l). \tag{14'}$$

Assertions of this kind are proved in [19: 4.6.1, 4.4.2], and [1: Theorem 18.9]. As we show below, these embeddings are sufficient for obtaining most of the Sobolev norm estimates from the corresponding ones for Besov norms. Our point of view is that Besov spaces are natural for approximation problems and more simple to deal with.

Theorem 2: *Let $0 < p, q \leqq \infty$, $k \in \mathbb{N}$, and $0 < s < k + 1/\delta$ (also $s = k + 1/\delta$ if $q = \infty$) be given. Then $B_{p,q}^{s;k+1} \hookrightarrow A_{p,q}^{s;k}$, and $A_{p,q}^{s;k} \hookrightarrow B_{p,q}^{s;k+1}$ if $s < 1 + 1/p$.*

Remarks: 9. The restriction $s < 1 + 1/p$ for the second embedding is natural because the difference properties of finite element functions are characterized by $\omega_{k+1}(t, g_i)_p \asymp t^{1+1/p}$, $t \to 0$ ($g_i \in S^{(k)}(T_i)$) if g_i does not coincide with a polynomial of total degree $\leqq k$.

10. As a corollary to Theorem 2 we have for $k \in \mathbb{N}$

$$B_{p,q}^{s;k+1} \cong A_{p,q}^{s;k}\ (0 < p, q \leqq \infty,\ 0 < s < 1 + 1/p). \tag{15}$$

The proof of Theorem 2 follows in a standard way (cf., e.g., the considerations in [12: Theorem 6]) from the following two lemmas.

Lemma 2: *For $k \in \mathbb{N}$, $0 < p \leqq \infty$, and $f \in L_p$ we have the Jackson-type inequality*

$$E_i^{(k)}(f)_p \leqq c_8 \omega_{k+1}(2^{-i}, f)_p, \qquad i \in \mathbb{N}_0. \tag{16}$$

Lemma 3: *For $k \in \mathbb{N}$, $0 < p \leqq \infty$, and $g_i \in S^{(k)}(T_i)$, $i \in \mathbb{N}_0$, we have the Bernstein-type inequality*

$$\omega_{k+1}(t, g_i)_p \leqq c_9 \min\{(t \cdot 2^i)^{1+1/p}, 1\} \|g_i\|_{L_p}, \qquad 0 < t \leqq 1 \tag{17}$$

and the inverse inequality ($i \in \mathbb{N}_0$)

$$\omega_{k+1}(2^{-i}, f)_p \leqq c_{10} 2^{-i(1+1/p)} \left(\sum_{r=0}^{i} 2^{r(1+1/p)\delta} E_r^{(k)}(f)_p^{\delta} + \|f\|_{L_p}^{\delta} \right)^{1/\delta}.$$

Proof of Lemma 3: Since the inverse inequality is a standard consequence of (17), we concentrate on the case $t \leqq 2^{-i}$ in (17) (for $t > 2^{-i}$ the assertion is trivial). First consider any basis function $B_{i,j}$. Let $0 < |h| \leqq t \leqq 2^{-i}$. According to (b) and (c)' we have

$$|\Delta_h^{k+1} B_{i,j}(x)| \begin{cases} \leqq c\, |h| \max\limits_{r=1,2} \left\{ \left\| \dfrac{\partial}{\partial x_r} B_{i,j} \right\|_{L_\infty} \right\} \leqq c\, |h|\, 2^i, & x \in E, \\ = 0, & x \in G_{k+1,h} \setminus E, \end{cases}$$

where E denotes the set of all those $x \in G_{k+1,h}$ for which the segment $[x, x + (k+1)h]$ intersects at least one side of some triangle $K \in T_i$ with $K \subset \operatorname{supp} B_{i,j}$. Since mes E

$\leqq c\,|h|\,2^{-i}$ we obtain the estimate

$$\|\Delta_h^{k+1}B_{i,j}\|_{L_p(G_{k+1,h})} = \|\Delta_h^{k+1}B_{i,j}\|_{L_p(E)} \leqq c(|h|\,2^i)^{1+1/p}\,2^{-2i/p}$$

which yields (17) for $g_i = B_{i,j}$ (cf. Lemma 1). Now, let $g_i = \sum_j a_{i,j}B_{i,j} \in S^{(k)}(T_i)$ be arbitrary. Since the sum $\Delta_h^{k+1}g_i(x) = \sum_j a_{i,j}\,\Delta_h^{k+1}B_{i,j}(x)$, for any $x \in G_{k+1,h}$, consists of at most c nonzero summands, where the constant c is independent of g_i and $|h| \leqq 2^{-i}$, we can continue by

$$\begin{aligned}\|\Delta_h^{k+1}g_i\|^p_{L_p(G_{k+1,h})} &\leqq c\sum_j |a_{i,j}|^p\,\|\Delta_h^{k+1}B_{i,j}\|^p_{L_p(G_{k+1,h})}\\ &\leqq c(|h|\,2^i)^{p+1}\,\|a_{i,\cdot}\|^p_{l_p}\\ &\leqq c(|h|\,2^i)^{p+1}\,\|g_i\|^p_{L_p},\ 0 < |h| \leqq 2^{-i}.\end{aligned}$$

This yields (17), and Lemma 3 is established ∎

Proof of Lemma 2: The idea is first to construct by Whitney type estimates a piecewise polynomial function of total degree k with respect to T_i which suitably approximates f in $L_p(G)$ but does not belong to $C(G)$. After this, it remains to smooth this piecewise polynomial in order to get a good approximant from $S^{(k)}(T_i)$. The required Whitney type estimate reads as follows. Let $0 < p \leqq \infty$, $k \in \mathbb{N}_0$. For any triangle $K \in T_i$, $i \in \mathbb{N}_0$, and $f \in L_p(K)$ there exists a polynomial $p_k = p_k(\cdot\,; f, K, p)$ of total degree k which satisfies

$$\|f - p_k\|^p_{L_p(K)} \leqq c_{11}\,2^{2i}\int_{|h|\leqq 2^{-i}} \|\Delta_h^{k+1} f\|^p_{L_p(K_{k+1,h})}\,dh$$

(with an obvious modification if $p = \infty$). Since the term at the right-hand side of this inequality is equivalent to the p-th power of the modulus of continuity of order $k+1$ of the function f in $L_p(K)$ (with $t = 2^{-i}$), this is only another form of the usual Whitney type result. For a proof, which covers the general case $0 < p \leqq \infty$, see STOROŽENKO [16, 17] and the literature cited therein (the case of triangles reduces by affine transformation and a simple extension argument to the case of squares considered in [17]). For $1 \leqq p \leqq \infty$ there is a lot of further references, cf. e.g. [4, 8].

In the following we concentrate on the case $0 < p < \infty$; the obvious modifications in the notation for $p = \infty$ are left to the reader. Defining, for a given function $f \in L_p(G)$ the piecewise polynomial function pp_i (with respect to T_i) by $pp_i(x) = p_k(x; f, K, p)$ ($x \in K$, $K \in T_i$) we obtain

$$\begin{aligned}\|f - pp_i\|^p_{L_p} &= \sum_{K\in T_i} \|f - p_k(\cdot\,; f, K, p)\|^p_{L_p(K)}\\ &\leqq c2^{2i}\int_{|h|\leqq 2^{-i}} \sum_{K\in T_i} \|\Delta_h^{k+1}f\|^p_{L_p(K_{k+1,h})}\,dh\\ &\leqq c2^{2i}\int_{|h|\leqq 2^{-i}} \|\Delta_h^{k+1}f\|^p_{L_p(G_{k+1,h})}\,dh \leqq c\omega_{k+1}(2^{-i}, f)_p{}^p, \qquad i \in \mathbb{N}_0.\end{aligned}$$

Moreover, by the definition of the differences this yields

$$\begin{aligned}&2^{2i}\int_{|h|\leqq 2^{-i}} \|\Delta_h^{k+1}pp_i\|^p_{L_p(G_{k+1,h})}\,dh\\ &\leqq c2^{2i}\int_{|h|\leqq 2^{-i}} \left(\|\Delta_h^{k+1}f\|^p_{L_p(G_{k+1,h})} + \|f - pp_i\|^p_{L_p}\right) dh\\ &\leqq c2^{2i}\int_{|h|\leqq 2^{-i}} \|\Delta_h^{k+1}f\|^p_{L_p(G_{k+1,h})}\,dh \leqq c\omega_{k+1}(2^{-i}, f)_p{}^p.\end{aligned}$$

Thus, in order to finish the argument it suffices to construct a function $g_i \in S^{(k)}(T_i)$ such that

$$\|pp_i - g_i\|_{L_p}^p \leqq c \int_{|h| \leqq 2^{-i}} \|\Delta_h^{k+1} pp_i\|_{L_p(G_{k+1,h})}^p \, dh. \tag{18}$$

For, we define g_i by

$$g_i(Q_{i,j}) = (n_{i,j})^{-1} \sum_{K \in T_i : Q_{i,j} \in K} p_k(Q_{i,j}; f, K, p), \qquad j = 1, \ldots, M_i,$$

where $n_{i,j}$ is the number of triangles $K \in T_i$ which contain the Lagrange point $Q_{i,j}$. Clearly, the $n_{i,j}$ are bounded from above by some absolute constant only depending on γ'. Thus (cf. the proof of Lemma 1),

$$\|pp_i - g_i\|_{L_p}^p \leqq c \sum_{K \in T_i} \Big(2^{-2i} \sum_{j : Q_{i,j} \in K} |p_k(Q_{i,j}; f, K, p) - g_i(Q_{i,j})|^p\Big)$$
$$\leqq c \sum_{d \in D_i} 2^{-2i} \sum_{j : Q_{i,j} \in d} |p_k(Q_{i,j}; f, K_d^+, p) - p_k(Q_{i,j}, f, K_d^-, p)|^p,$$

where D_i is the set of all sides $d \not\subset \partial G$ of the triangles $K \in T_i$, and K_d^+; K_d^- denote the two triangles in T_i for which $d \in D_i$ is the common side (cf. Figure 4). Now, for any $h \in \mathbb{R}^2$, $|h| \leqq 2^{-i}$, consider the set

$$K_{d,h}^+ = \{x \in K_d^+ : [x, x + (k+1) h] \subset K_d^+ \cup K_d^-, [x + h, x + (k+1) h] \subset K_d^-\}.$$

Observe that there exists a measurable set $E_d^+ \subset K_d^+$ (e.g. some closed triangle) such that mes $H_d \geqq c2^{-2i}$ where $H_d = \{h : |h| \leqq 2^{-i}$ and $E_d^+ \subset K_{d,h}^+\}$ and, for any polynomial $\overline{p}$ of total degree k, $\|\overline{p}\|_{L_p(E_d^+)} \geqq c \|\overline{p}\|_{L_p(K_d^+)}$. This gives, when applied to $\overline{p} = p_k(\cdot; f, K_d^+, p) - p_k(\cdot; f, K_d^-, p)$, the estimates

$$2^{-2i} \sum_{j : Q_{i,j} \in d} |p_k(Q_{i,j}; f, K_d^+, p) - p_k(Q_{i,j}; f, K_d^-, p)|^p$$
$$\leqq c \|\overline{p}\|_{L_p(K_d^+)}^p \leqq c \|\overline{p}\|_{L_p(E_d^+)}^p \leqq c\, (\text{mes } H_d)^{-1} \int_{H_d} \|\Delta_h^{k+1} pp_i\|_{L_p(E_d^+)}^p \, dh$$
$$\leqq c 2^{2i} \int_{|h| \leqq 2^{-i}} \|\Delta_h^{k+1} pp_i\|_{L_p((K_d^+ \cup K_d^-)_{k+1,h})}^p \, dh$$

since, by construction of $E_d^+ \subset K_{d,h}^+ \subset (K_d^+ \cup K_d^-)_{k+1,h}$, $\Delta_h^{k+1} pp_i(x) = \overline{p}(x)$, $x \in E_d^+$. After summing up with respect to $d \in D_i$ we get (18) ∎

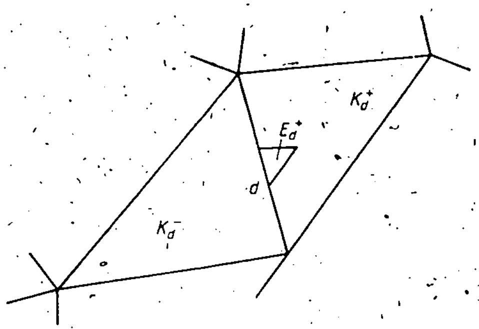

Fig. 4

Corollary 4: *Let $k \in \mathbb{N}$, $1 \leq p < \infty$. Then, for the Sobolev and Lebesgue spaces, the following embeddings hold:*

$$W_p{}^s,\ H_p{}^s \hookrightarrow A^{s;k}_{p,\max(p,2)} \text{ if } 0 < s < k + 1,$$
$$W_p{}^{k+1} \cong H_p{}^{k+1} \hookrightarrow A^{k+1;k}_{p,\infty},\quad A^{s;k}_{p,\min(p,2)} \hookrightarrow W_p{}^s, H_p{}^s \text{ if } 0 < s < 1 + 1/p.$$

Remarks: **11.** For $p = 2$ one has $A^{s;k}_{2,2} \cong B^{s;k+1}_{2,2} \cong W_2{}^s \equiv H_2{}^s \equiv H^s$; $0 < s < 3/2$. According to Corollary 3 we also have $\mathring{H}^s \cong A^{s;k}_{2,2;\partial G} \cong A^s_{2,2}(S^{(k)}_{\partial G}(T_i))$, $1/2 < s < 3/2$, where $\mathring{H}^s$ is the subspace of H^s consisting of all those functions f with $\partial f = 0$ on ∂G (or, what is equivalent for the indicated values of s, the closure of $\mathcal{D}(G)$ with respect to the H^s norm).

12. Analogous characterizations in terms of Besov-Sobolev spaces can be given for the spaces $\partial A^{s;k}_{p,q}$ defined on the boundary of G.

13. The proofs of Lemmas 2 and 3 which are of independent interest remain valid also for arbitrary γ'-quasiuniform triangulations (with $h(T)$ instead of 2^{-i}). The case of more general triangulations seems to be, in contrast to the corresponding one-dimensional results, more complicated. ◦

3. Approximation estimates

In this section, we prove in terms of the above introduced $A^{s;k}_{p,q}$ spaces some basic estimates of finite element approximation theory for quasiuniform triangulations belonging to some sequence $\{T_i\}$ satisfying the assumptions (a)—(c). Although this is not the most general case of triangulations we might be interested in the results indicate that our approach might be very useful if dealing with finite element approximation estimates in various function spaces and quasinorms.

Generally, denote by

$$E_n{}^{(k)}(f)_X = \inf \{\|f - g\|_X : g \in S^{(k)}(T_n)\}, \quad n \in \mathbb{N}_0,$$

the best approximations of $f \in X$ with respect to $S^{(k)}(T_n)$. Here, X stands for some quasinormed function space on G.

Theorem 3: *Let $X = A^{s';k}_{p',q'}$ and $Y = A^{s;k}_{p,q}$ be given such that the embedding $Y \hookrightarrow X$ holds. Then $E_n{}^{(k)}(f)_X \leq c_{12} 2^{-n(s-s'-\alpha)} \|f\|_Y$, $n \in \mathbb{N}_0$, where $\alpha \equiv \max(0, 2(1/p - 1/p'))$.*

Proof: The assumption $Y \hookrightarrow X$ yields $s - s' - \alpha \geq 0$ (cf. (8), (9), and (11)). The case $s - s' - \alpha = 0$ is obvious since $E_n{}^{(k)}(f)_X = \|f\|_X$ ($f \in X$, $n \in \mathbb{N}_0$). For $s - s' - \alpha > 0$ we introduce the intermediate space $Y' = A^{s-\alpha;k}_{p',q}$ (i.e. $Y \hookrightarrow Y' \hookrightarrow X$) and consider any representation (1), (2) of $f \in Y'$. Since, by (13),

$$\begin{aligned} E_n{}^{(k)}(f)_X &\leq \Big\| f - \sum_{i=0}^{n} g_i \Big\|_X \leq \big\| \{2^{is'} \|g_i\|_{L_{p'}}\}_{i>n} \big\|_{l_{q'}} \\ &\leq c 2^{-n(s-s'-\alpha)} \big\| \{2^{i(s-\alpha)} \|g_i\|_{L_{p'}}\}_{i>n} \big\|_{l_q} \\ &\leq c 2^{-n(s-s'-\alpha)} \|\{g_i\}\|_{p',q,s-\alpha} \end{aligned}$$

we get after taking the infimum with respect to all those representations (1), (2) $E_n{}^{(k)}(f)_X \leq c 2^{-n(s-s'-\alpha)} \|f\|_{Y'} \leq c 2^{-n(s-s'-\alpha)} \|f\|_Y$ for any $f \in Y$ ∎

Theorem 4: *Let $0 < p, p', q, q' \leq \infty$, $\alpha = \max(0, 2(1/p - 1/p'))$, and $s, s' \geq 0$. Then, for any $g \in S^{(k)}(T_n)$, we have*

$$\|g\|_{A^{s';k}_{p',q'}} \leq c_{13} \|g\|_{A^{s;k}_{p,q}} \begin{cases} 2^{n(s'+\alpha-s)}, & s' + \alpha - s > 0, \\ (n+1)^{\max(0,1/q'-1/q)}, & s' + \alpha - s = 0, \\ 1, & s' + \alpha - s < 0. \end{cases}$$

Proof: First observe that for $g \in S^{(k)}(T_n)$ we have

$$\inf\left\{\|\{g_i\}_{i\le n}\|_{p,q,s} : g = \sum_{i=0}^{n} g_i\right\} \leqq c_{14} \|g\|_{A^{s;k}_{p,q}}. \tag{19}$$

Indeed, consider any representation (1), (2) of $g \in S^{(k)}(T_n) \subset A^{s;k}_{p,q}$ and set $\bar{g}_n = \sum_{i=n+1}^{\infty} g_i$. Since $\bar{g}_n = g - \sum_{i=0}^{n} g_i \in S^{(k)}(T_n)$ and $\|\bar{g}_n\|_{L_p} \leqq \|\{\|g_i\|_{L_p}\}_{i>n}\|_{l_\delta} \leqq c2^{-ns} \|\{g_i\}\|_{p,q;s}$ we immediately get (19) (for, consider the representation $g = \sum_{i=0}^{n-1} g_i + (g_n + \bar{g}_n)$). Finally, by the Nikolskij-type inequality (10) we obtain

$$\begin{aligned}
&\|\{2^{is'} \|g_i\|_{L_{p'}}\}_{i\le n}\|_{l_{q'}} \\
&\leqq c \|\{2^{i(s'+\alpha)} \|g_i\|_{L_p}\}_{i\le n}\|_{l_{q'}} \\
&\leqq c \|\{2^{is} \|g_i\|_{L_p}\}_{i\le n}\|_{l_q} \begin{cases} 2^{n(s'+\alpha-s)}, & s'+\alpha-s>0, \\ (n+1)^{\max(0,1/q'-1/q)}, & s'+\alpha-s=0, \\ 1, & s'+\alpha-s<0. \end{cases}
\end{aligned}$$

This, together with (19), proves the theorem. ∎

Now, let $I_n^{(k)}: C(G) \to S^{(k)}(T_n)$ be the interpolation projection defined by

$$I_n^{(k)} f(Q^{(k)}_{n,j}) = f(Q^{(k)}_{n,j}), \quad j = 1, \ldots, M_n^{(k)}.$$

The approximation properties of this projection are most important for collocation methods with finite elements. Moreover, in the finite element literature, inequalities for $f - I_n^{(k)} f$ in various Sobolev norms are, as a rule, the starting point for investigations on error estimates.

Theorem 5: *Let $Y = A^{s;k}_{p,q} \hookrightarrow X = A^{s';k}_{p',q'}$ where $0 < p, q, p', q' \leqq \infty$, $s, s' \geqq 0$ $(s - s' - \alpha \geqq 0)$ be given. Suppose that $s > 2/p$. Then, for any $f \in Y$, we have $\|f - I_n^{(k)} f\|_X \leqq c_{15} 2^{-n(s-s'-\alpha)} \|f\|_Y$, $n \in \mathbb{N}_0$.*

Proof: Consider any representation (1), (2) of $f \in Y$. Since $s > 2/p$ we have $Y \hookrightarrow C(G)$, and this representation also converges uniformly. Therefore,

$$I_n^{(k)} f = \sum_{i=0}^{\infty} I_n^{(k)} g_i = \sum_{i=0}^{n} g_i + \bar{g}_n, \qquad \bar{g}_n = I_n^{(k)}\left(\sum_{i=n+1}^{\infty} g_i\right) \in S^{(k)}(T_n).$$

Thus, we can estimate

$$\|f - I_n^{(k)} f\|_X \leqq c\left(2^{ns'} \|\bar{g}_n\|_{L_{p'}} + \|\{2^{is'} \|g_i\|_{L_{p'}}\}_{i>n}\|_{l_{q'}}\right).$$

While the second term in the right-hand side can be handled as above (cf. Theorem 3), the considerations for the first term run as follows. For any Lagrange point $Q_{n,j}$, $j = 1, \ldots, M_n$, and any $i = n+1, n+2, \ldots$ we fix some triangle $K_{n,j;i} \in T_i$ containing $Q_{n,j}$. Let us observe that, for arbitrary fixed i, the maximal number of Lagrange points which corresponding triangles coincides is bounded by some absolute constant. Since, obviously,

$$|g_i(Q_{n,j})| \leqq c \|g_i\|_{L_\infty(K_{n,j;i})} \leqq c 2^{2i/p} \|g_i\|_{L_p(K_{n,j;i})}$$

we obtain

$$
\begin{aligned}
2^{ns'} \cdot \|\bar{g}_n\|_{L_{p'}} &\leqq c 2^{n(s'+\alpha)} \|\bar{g}_n\|_{L_p} \\
&\leqq c 2^{n(s'+\alpha-2/p)} \left(\sum_j \left(\sum_{i=n+1}^{\infty} |g_i(Q_{n,j})| \right)^p \right)^{1/p} \\
&\leqq c 2^{n(s'+\alpha-2/p)} \left(\sum_j \left\| \left\{ 2^{2i/p} \|g_i\|_{L_p(K_{n,j;i})} \right\}_{i>n} \right\|_{l_1}^p \right)^{1/p} \\
&\leqq c 2^{-n(\bar{s}-s'-\alpha)} \left(\sum_{i=n+1}^{\infty} 2^{i\bar{s}p} \left(\sum_j \|g_i\|_{L_p(K_{n,j;i})}^p \right) \right)^{1/p} \\
&\leqq c 2^{-n(s-s'-\alpha)} \left\| \{ 2^{is} \|g_i\|_{L_p} \}_{i>n} \right\|_{l_q}
\end{aligned}
$$

(with $2/p < \bar{s} < s$, and obvious modifications if $p = \infty$). This yields the desired estimate for the first term ∎

Clearly, by Theorem 2 and Corollary 4, the results formulated in the Theorems of this section can now be translated into estimates for Besov-Sobolev quasinorms. For instance, this yields

Corollary 5: *Let $k \in \mathbb{N}$, $1 \leqq p, p' < \infty$ and $0 \leqq s' < s - \alpha$ be given where $s \leqq k + 1$, $s' < 1 + 1/p$. Then, for any function $f \in W_p^s \hookrightarrow W_{p'}^{s'}$ we have*

$$E_n^{(k)}(f)_{W_{p'}^{s'}} \leqq c_{16} h(T_n)^{s-s'-\alpha} \|f\|_{W_p^s}, \qquad n \in \mathbb{N}_0,$$

and in the case $s > 2/p$ also

$$\|f - I_n^{(k)} f\|_{W_{p'}^{s'}} \leqq c_{17} h(T_n)^{s-s'-\alpha} \|f\|_{W_p^s}, \qquad n \in \mathbb{N}_0.$$

Proof: By Corollary 4 we have $E_n^{(k)}(f)_{W_{p'}^{s'}} \leqq c E_n^{(k)}(f)_X$, $\|f\|_Y \leqq c \|f\|_{W_p^s}$ where

$$X = \begin{cases} L_{p'} \cong A_{p',1}^{0;k}, & s' = 0, \\ A_{p',\min(p',2)}^{s';k}, & s' > 0, \end{cases} \qquad Y = \begin{cases} A_{p,\max(p,2)}^{s;k}, & s < k+1, \\ A_{p,\infty}^{k+1;k}, & s = k+1. \end{cases}$$

Since $s' < s - \alpha$ has been supposed, the assumptions of Theorem 3 and 5 are fulfilled which yields the corollary (recall that $h(T_n) \asymp 2^{-n}$) ∎

The formulation of further corollaries, e.g. for Besov quasinorms or concerning the inverse inequalities (Theorem 4), is left to the reader.

Remarks: **14.** Analogous statements can be given for the rate of approximation for functions defined on the boundary ∂G when using the spaces $\partial A_{p,q}^{s;k}$, or for functions belonging to $A_{p,q;\Gamma}^{s;k}$, $s > 1/p$ (cf. Corollary 3).

15. Corollary 5 is contained in [2: Theorems 3.1.4.6] for more general partitions but with some additional restrictions on s, s'. The results for Besov spaces (although they are closely related to those for Sobolev norms) as well as the approach via approximation spaces as introduced above seem to be new. The motivation for including the nonclassical case $p < 1$ comes from applications to the approximation of functions with singularities (see the following sections).

It should also be mentioned that the indicated ranges of parameters as well as the exponent $(s - s' - \alpha)$ occuring in the estimates cannot be improved. However, improvements of the asymptotic rate of approximation are possible if we approximate by finite element functions with variable triangulations consisting of a given number of triangles (instead of approximating with respect to a given sequence of triangulations). The use of the triangulation as additional degrees of freedom leads to a certain nonlinear approximation problem which will be discussed in the next section.

4. Nonlinear finite element approximation (variable triangulations)

One of the strategies for improving the accuracy of finite element approximations is to make a proper choice of the underlying triangulation, e.g. by adapting the triangulation to the singularities of the functions to be approximated, without essentially increasing the dimension of the corresponding finite element space. From a theoretical point of view, this question leads to a certain nonlinear approximation problem, i.e. to estimates for the nonlinear best approximations

$$e_n^{(k)}(f)_X = \inf_{|T| \leq n} \inf_{g \in S^{(k)}(T)} \|f - g\|_X, \tag{20}$$

where $|T|$ denotes the number of triangles in T. Below, these quantities will be estimated in the case that $f \in Y \hookrightarrow X$ where $X = A_{p',q'}^{s';k}$, $Y = A_{p,q}^{s;k}$, and $0 \leq s' < s - \alpha$ (as a consequence of Theorem 2 and Corollary 4, this yields the corresponding results for Besov-Sobolev spaces). Throughout this section, we suppose for simplicity that the triangulations T_i used in the definition of the approximation spaces are obtained by the standard dyadic subdivision procedure from some initial triangulation T_0.

Following the idea in [13], instead of $e_n^{(k)}(f)_X$ we first consider similar nonlinear best approximations

$$\bar{e}_n^{(k)}(f)_X = \inf \{\|f - g\|_X : g \in S_n^{(k)}\}, \qquad n \in \mathbb{N}, \tag{21}$$

where $S_n^{(k)}$ is the set of all linear combinations of at most n arbitrary basis functions $B_{i,j}^{(k)}$, i.e. $g = \sum_{i=0}^{\infty} \sum_{j=1}^{M_i^{(k)}} a_{i,j} B_{i,j}^{(k)} \in S_n^{(k)}$ if $a_{i,j} \neq 0$ for at most n pairs of indices (i, j).

Theorem 6: *Let $X = A_{p',q'}^{s';k}$, $Y = A_{p,q}^{s,k}$ where $0 < p, p', q, q' \leq \infty$, $0 \leq s' < s - \alpha$ $\left(\alpha = \max\left(0, 2(1/p - 1/p')\right)\right)$. Then, for any $f \in Y$, we have $\bar{e}_n^{(k)}(f)_X \leq c_{17} n^{-(s-s')/2} \|f\|_Y$, $n \geq n_0 \equiv M_0^{(k)}$.*

Proof: The case $\alpha = 0$, i.e. $p' \leq p$, is a trivial consequence of Theorem 3. Thus, we concentrate on $p < p'$. According to the asymptotic nature of the inequality it is sufficient to prove $\bar{e}_{n_r}^{(k)}(f)_X \leq c 2^{-r(s-s')} \|f\|_Y$ for some sequence of integers $n_r \leq c 2^{2r}$, $r \in \mathbb{N}_0$.

Let $\beta = -(2 + \varepsilon)/p$, $\beta' = -s - \beta$, where $\varepsilon > 0$ will be fixed later on. Consider any representation (1), (2), (5) of $f \in Y$, i.e.

$$f = \sum_{i=0}^{\infty} g_i = \sum_{i=0}^{\infty} \sum_{j=1}^{M_i} a_{i,j} B_{i,j}, \text{ where } \left\|\{2^{is} \|a_{i,\cdot}\|_{l_p}\}\right\|_{l_q} < \infty.$$

For given $r \in \mathbb{N}_0$ we define

$$a_{i,j}^{(r)} = \begin{cases} 0 & \text{if } i > r \text{ and } |a_{i,j}| \leq c(f)\, 2^{\beta' i + \beta r}, \\ a_{i,j} & \text{elsewhere} \end{cases}$$

(the constant $c(f) > 0$ will be specified below). First we estimate from above the number n_r of pairs of indices (i, j) such that $a_{i,j}^{(r)} \neq 0$. By the above definition of the coefficients $a_{i,j}^{(r)}$ we have

$$\begin{aligned} n_r &\leq \sum_{i=0}^{r} M_i + \sum_{i=r+1}^{\infty} \left(c(f)\, 2^{\beta' i + \beta r}\right)^{-p} \sum_{j=1}^{M_i} |a_{i,j}|^p \\ &\leq c\left(2^{2r} + c(f)^{-p}\, 2^{(2+\varepsilon)r} \left\|\{2^{i(s-\varepsilon/p)} \|g_i\|_{L_p}\}_{i>r}\right\|_{l_p}^p\right) \\ &\leq c 2^{2r}\left(1 + c(f)^{-p} \left\|\{2^{is} \|g_i\|_{L_p}\}_{i>r}\right\|_{l_q}^p\right) \leq c 2^{2r} \end{aligned}$$

by defining $c(f) \equiv \|\{2^{is} \|g_i\|_{L_p}\}_{i>r}\|_{l_q}^p$ and using once more (13).

On the other hand, if we denote

$$g_i^{(r)} = \sum_{j=1}^{M_i} a_{i,j}^{(r)} B_{i,j} \quad \text{and} \quad f^{(r)} = \sum_{i=0}^{\infty} g_i^{(r)} (\in S_{n_r}^{(k)}),$$

then, for $p' < \infty$, we can estimate as follows:

$$\begin{aligned}
\bar{e}_{n_r}^{(k)}(f)_X &\leqq \|f - f^{(r)}\|_X \leqq \left\| \left\{ 2^{is'} \|g_i - g_i^{(r)}\|_{L_{p'}} \right\} \right\|_{l_{q'}} \\
&\leqq c \left\| \left\{ 2^{i(s'-2/p')} \Big(\sum_{j: |a_{i,j}| \leqq 2^{\beta' i + \beta r} c(f)} |a_{i,j}|^{p'} \Big)^{1/p'} \right\}_{i>r} \right\|_{l_{q'}} \\
&\leqq c\, c(f)^{1-p/p'} 2^{\beta(1-p/p')r} \\
&\quad \times \left\| \left\{ 2^{i(s'-2/p'+\beta'(1-p/p'))} \Big(\sum_{j=1}^{M_i} |a_{i,j}|^p \Big)^{1/p'} \right\}_{i>r} \right\|_{l_{q'}} \\
&\leqq cc(f)^{1-p/p'} 2^{\beta(1-p/p')r} \\
&\quad \times \|\{2^{i(s'+\beta'(1-p/p'))p'/p} \|g_i\|_{L_p}\}_{i>r}\|_{l_{q'p/p'}}^{p/p'} \\
&\leqq cc(f)^{1-p/p'} 2^{\beta(1-p/p')r} \\
&\quad \times 2^{-r(s-(\beta'(1-p/p'))p'/p)p/p'} \|\{2^{is} \|g_i\|_{L_p}\}_{i>r}\|_{l_q}^{p/p'} \\
&\leqq cc(f)\, 2^{-(s-s')r} \leqq c 2^{-(s-s')r} \, \|\{g_i\}\|_{p,q;s}
\end{aligned}$$

where the use of (13) in the above estimates requires a choice of ε satisfying $0 < \varepsilon < (s - s' - \alpha)/(1/p - 1/p')$ which is possible by the assumptions. The modifications for the case $p' = \infty$ are obvious. It remains to take the infimum with respect to all considered representations of $f \in Y$ ∎

Remarks: **16.** The above construction also guarantees a certain smoothness of the approximants: $\|f^{(r)}\|_Y \leqq c_{18} \|f\|_Y$, $r \in \mathbb{N}_0$.

17. The construction of the approximants really depends on f, s, p, and ε, only. Thus, we get simultaneous approximation estimates for a lot of quasinorms $\|\cdot\|_X$, e.g. if $0 < p, p', q, q' \leqq \infty$, and $0 \leqq s' \leqq s_0'$ where $s_0' < s - \alpha$ is a fixed real number.

The above result partly holds for $s' = s - \alpha$, too.

Now, let us show how the considerations can be slightly modified in order to estimate the quantities (20) and to get some further information on the underlying triangulations. The idea is to prove that for the function $f^{(r)}$ introduced in the proof of Theorem 6 there exists a certain triangulation $T^{(r)}$ of G into at most $c2^{2r}$ triangles such that $f^{(r)} \in S^{(k)}(T^{(r)})$, $r \in \mathbb{N}_0$. Furthermore, it can be shown that the $T^{(r)}$ are γ-regular for some $\gamma \leqq c\gamma'$.

Theorem 7: *Let $0 < p, p', q, q' \leqq \infty$ and $0 \leqq s' < s - \alpha$ where $\alpha = \max\big(0, 2(1/p - 1/p')\big)$. Denote $X = A_{p',q'}^{s';k}$, $Y = A_{p,q}^{s;k}$ and consider any $f \in Y$ ($\hookrightarrow X$). Then, for $n \geqq n_0 \equiv M_0^{(k)}$, there exist triangulations T_n^* and finite element functions $f_n \in S^{(k)}(T_n^*)$ satisfying the following properties:*

(i) *T_n^* consists of at most n triangles and is γ-regular for some constant $\gamma \leqq c_{19}\gamma'$. Furthermore, $h(T_n^*) \asymp n^{-1/2}$.*

(ii) $e_n^{(k)}(f)_X \leqq \|f - f_n\|_X \leqq c_{20} n^{-(s-s')/2} \|f\|_Y$.

(iii) $\|f_n\|_Y \leqq c_{21} \|f\|_Y$.

Proof: We consider the functions $f^{(r)}$ defined in the proof Theorem 6. For the construction of the corresponding triangulations $T^{(r)}$ we introduce the following terminology: Let $K \in T_i$ for some $i \in \mathbb{N}_0$ and put $T_{-1} = T_0$. The unique triangle $K' \in T_{i-1}$ containing K we denote by $P^{-1}(K)$, and by $P(K)$ we denote the set of all triangles $K'' \in T_{i+1}$ contained in K (thus, $P^{-1}(P(K)) = K$). Let $\Theta^{(r)}$ be the set of all triangles K either belonging to T_r or satisfying, for some $i > r$, the following condition: $K \in T_i$ and there exists some j' such that $a_{i,j'}^{(r)} \neq 0$ and $K \subset \operatorname{supp} B_{i,j'}$.

Now we construct in a canonical way a new set of closed triangles $\bar{\Theta}^{(r)} \supset \Theta^{(r)}$ satisfying the following properties:

1. If $K \in \bar{\Theta}^{(r)}$, then for some $i = i(K) \geq r$ we have $K \in T_i$.

2. If $K, K' \in \bar{\Theta}^{(r)}$, and some side d of K is contained in some side d' of K' (i.e. K and K' are neighbors or coincide), then $0 \leq i(K) - i(K') \leq 1$.

3. If $K \in \bar{\Theta}^{(r)}$, then $P(P^{-1}(K)) \subset \bar{\Theta}^{(r)}$, too.

For this reason, define $i_0 \geq r$ by the requirements $T_i \cap \Theta^{(r)} = \emptyset$ for $i > i_0$, and $T_{i_0} \cap \Theta^{(r)} \neq \emptyset$. If $i_0 = r$, then $\Theta^{(r)} = \bar{\Theta}^{(r)} = T_r = T^{(r)}$. If $i_0 > r$, then put $\Theta_{i_0}^{(r)} = \Theta^{(r)}$ and construct by induction $\Theta_{i_0-1}^{(r)}, \ldots, \Theta_r^{(r)}$. The induction step consists in the following procedure: Let $\Theta_l^{(r)}$ be already defined for some $l = i_0, \ldots, r+1$. First extend $\Theta_l^{(r)}$ by adding all those triangles $K' \in T_l \setminus \Theta_l^{(r)}$ for which there exists some triangle $K \in T_l \cap \Theta_l^{(r)}$ possessing a common side with K and satisfying $P(K) \subset \Theta_l^{(r)}$. This extended set of triangles will be denoted by $\bar{\Theta}_l^{(r)}$. After this, put

$$\Theta_{l-1}^{(r)} = \bar{\Theta}_l^{(r)} \cup \Big(\cup\big((P^{-1}(K) \cup P(P^{-1}(K))\big) : K \in \Theta_l^{(r)} \cap T_l\Big)\Big).$$

It can easily be observed that $\bar{\Theta}^{(r)} \equiv \Theta_r^{(r)}$ satisfies the properties 1—3 from above (Figure 5 illustrates the construction for a fictive set $\Theta^{(r)}$ with $i_0 = 3$, $r = 0$ where G is a square and T_0 consists of two triangles as indicated).

Furthermore, if $n_i^{(r)}$ and $\bar{n}_i^{(r)}$ denotes the number of triangles belonging to $\Theta^{(r)} \cap T_i$ and $\bar{\Theta}^{(r)} \cap T_i$ $(i = r, r+1, \ldots, i_0)$, respectively, then according to the above construction and to the estimates of n_r in the proof of Theorem 7 we get for the num-

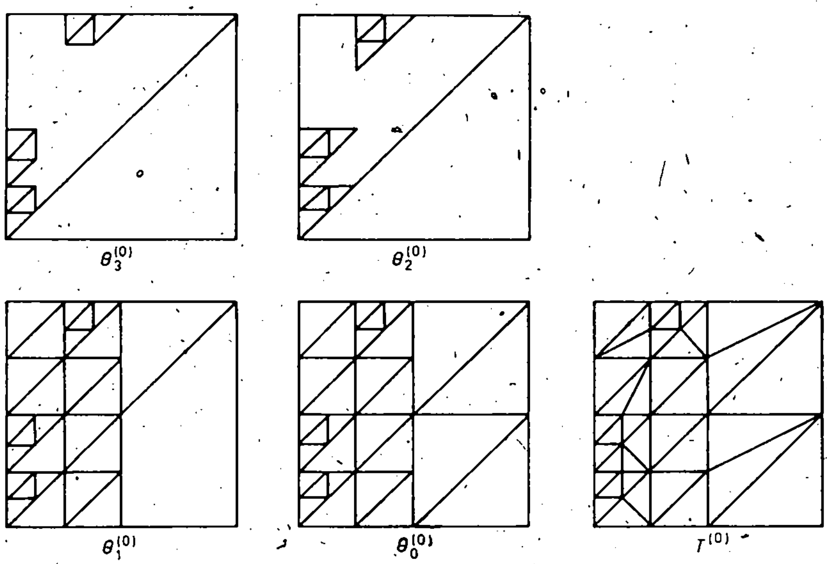

Fig. 5. The construction of $T^{(r)}$

ber $\bar{n}_r$ of triangles in $\bar{\Theta}^{(r)}$

$$\bar{n}_r = \sum_{i=r}^{i_0} \bar{n}_i^{(r)} \leq c \left(2^{(r)} + \sum_{i=r+1}^{i_0} \sum_{j=i}^{i_0} n_j^{(r)}\right)$$
$$\leq c \left(2^{2r} + \sum_{i=r+1}^{i_0} (i - r)\, c(f)^{-p}\, 2^{(2+\varepsilon)r}\, 2^{i(sp-\varepsilon)} \|g_i\|_{L_p}^p\right)$$
$$\leq c2^{2r}\left(1 + c(f)^{-p} \left\|\{2^{is} \|g_i\|_{L_p}\}_{i>r}\right\|_{l_q}^p\right) \leq c2^{2r}.$$

Now, in order to define $T^{(r)}$, we consider all those triangles $K' \in \bar{\Theta}^{(r)}$ which do not contain triangles $K'' \in \bar{\Theta}^{(r)}$ with $i(K'') > i(K')$ but possess a neighboring triangle K with $i(K) > i(K')$, i.e. there exists some $K \in \bar{\Theta}^{(r)}$ such that $K' \cap K = d$ is a side of K and $i(K) = i(K') + 1$ (cf. property 2). Such triangles K' will be subdivided into 2 or 3 or 4 new triangles (which will be added to $\bar{\Theta}^{(r)}$) in such a way that, after deleting from $\bar{\Theta}^{(r)}$ all triangles which are further subdivided, we obtain a triangulation (cf. Figure 5). This triangulation is the required $T^{(r)}$. Obviously, $f^{(r)} \in S^{(k)}(T^{(r)})$, and $T^{(r)}$ consists of at most $4\bar{n}_r$ ($\leq c2^{2r}$) triangles and is γ-regular for some $\gamma \leq c\gamma'$. This proves (i), the statements (ii) and (iii) are contained in Theorem 6 (cf. Remark 16, too). ∎

Remarks: 18. For $0 < p < p' \leq \infty$, the asymptotic estimate (ii) improves the estimate from Theorem 3. More precisely, for approximating a function $f \in Y \hookrightarrow X$ by finite element functions with respect to the quasinorm of X, a proper choice of the triangulation gives an improvement of the asymptotic rate of approximation by the factor $n^{-\alpha}$ in comparision with the corresponding approximation on a quasiuniform triangulation with the same number of triangles (as above, $\alpha = \max(0, 2(1/p - 1/p'))$). Thus, in applications to approximation estimates in $X = A_{p',q'}^{s';k}$, we should be interested in verifying $f \in Y = A_{p,q}^{s;k}$ where s is as large as possible. For this reason, p can be taken arbitrarily small but satisfying the imbedding $Y \hookrightarrow X$. Such a situation is typical for functions with singularities. For instance, let

$$f = f_0 + \sum_{j=1}^{N} \mu_j \left(\sum_{l=1}^{N_j} r_j^{\alpha_{j,l}} |\ln (r_j)|^{\beta_{j,l}} \psi_{j,l}(\varphi_j)\right) \tag{22}$$

where $f_0 \in W_{p_0}^{k+1}$; r_j, φ_j are local polar coordinates (cf. Figure 6) and $\mu_j \in C_0^\infty([0, \infty))$ cut-off functions with respect to the corner P_j, $j = 1, \ldots, N$. Furthermore, $\psi_{j,l} \in C^\infty([0, \omega_j])$, and $\alpha_{j,l}$, $\beta_{j,l} \geq 0$ are real constants. It can easily be checked that for $p \leq p_0$ satisfying the inequality $\bar{\alpha} = \min(\alpha_{j,l}) > k + 1 - 2/p$ we have $f \in B_{p,\infty}^{k+1;k+1} = Y$. Then, for a given $X = A_{p',q'}^{s';k}$, we can estimate $e_n^{(k)}(f)_X \leq cn^{-(k+1-s')/2} \|f\|_Y$, $n \geq n_0$, whenever $k + 1 - s' > 2(1/p - 1/p')$. Thus, when choosing the parameter p, we should satisfy the inequalities $k + 1 - s' + 2/p' > 2/p > k + 1 - \alpha$ and $p \leq p_0$ which is possible iff $\min(\alpha, k + 1 - 2/p_0) > s' - 2/p'$. This result covers, for instance, the particular but important case $0 < p \leq p' = q' = 2$, $s' = 1$ which will

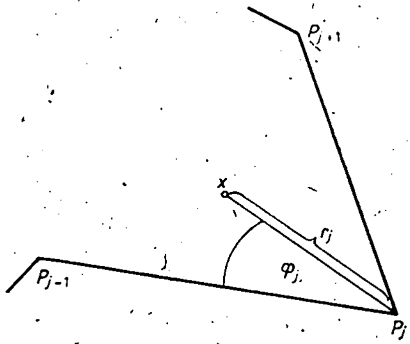

Fig. 6

be applied with some modification in Section 5 below:

$$e_n^{(k)}(f)_{H^1} \leqq cn^{-k/2} \|f\|_Y = O(n^{-k/2}), \qquad n \to \infty. \tag{23}$$

Thus, by defining appropriate triangulations, it is possible to approximate the functions (22) which are not sufficiently smooth in the L_2 sense with asymptotically high accurracy in the energy norm (as well as in other norms based on L_2 smoothness).

19. If one needs results for functions which vanish at part of the boundary (e.g. for $\mathring{H}^s$) the analogous theory for $\mathring{A}^{s;k}_{p,q;\Gamma}$ can be applied without substantial changes. To this end, use the description of these spaces given in Corollary 3, cf. also Remark 8. Some special cases have been already discussed without detailed proofs in [5].

5. Finite element error estimates for elliptic problems in polygonal domains

For simplicity, let us consider the Poisson equation

$$\left.\begin{array}{llll} -\Delta u = f & \text{in } G, & & \\ u = 0 & \text{on } \Gamma \subset \partial G; & \partial u/\partial n = 0 & \text{on } \partial G \setminus \Gamma \end{array}\right\} \tag{24}$$

with homogeneous Dirichlet and Neumann boundary conditions on Γ and $\partial G \setminus \Gamma$, respectively, where Γ is assumed to be the union of some sides d_j of the polygonal domain G (hence, the value $\omega_j = \pi$ is allowed, cf. Figure 7). Suppose that $\Gamma \neq \emptyset$. The model problem (24) possesses a unique weak solution $u = u_f \in H_\Gamma{}^1 \equiv \{v \in H^1 : \partial v = 0 \text{ on } \Gamma\}$ under rather general assumptions on f (say, $f \in L_p$ for some $p > 1$), i.e. a solution of the problem:

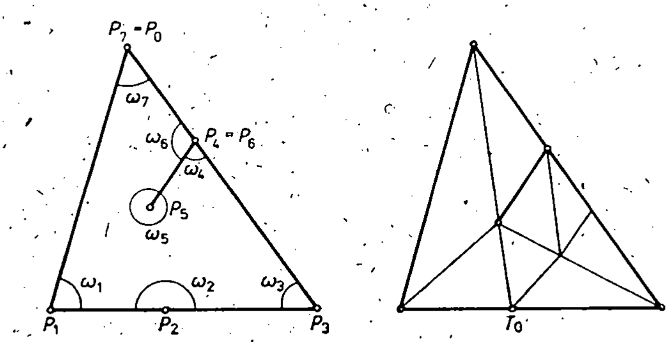

Fig. 7

Find $u \in H_\Gamma{}^1$ such that $a(u, v) = (f, v) \ \forall v \in H_\Gamma{}^1$, (25)

where

$$a(u, v) = \int_G \left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) dx, \qquad (f, v) = \int_G fv \, dx.$$

For a given triangulation T of G, denote by

$$S_\Gamma{}^{(k)}(T) \equiv S^{(k)}(T) \cap H_\Gamma{}^1 = \{v_T \in S^{(k)}(T) : v_T|_\Gamma = 0\}$$

the conforming finite element space corresponding to (25). Then, the finite element solution of the problem (25) is defined by:

Find $u_T \in S_\Gamma{}^{(k)}(T)$ such that $a(u_T, v_T) = (f, v_T) \ \forall v_T \in S_\Gamma{}^{(k)}(T)$. (26)

The well-known Lax-Milgram theorem implies the unique solvability of (26). Moreover, by the Cea lemma we have

$$\|u - u_T\|_{H^1} \leqq c(G) \inf \{\|u - v_T\|_{H^1}: v_T \in S_\Gamma^{(k)}(T)\} \tag{27}$$

and, by the Nitsche-Aubin lemma, also

$$\|u - u_T\|_{L_2} \leqq c(G) \|u - u_T\|_{H^1} \Big(\sup_{\|g\|_{L_2}=1} \inf_{v_T \in S_\Gamma^{(k)}(T)} \|u_g - v_T\|_{H^1} \Big). \tag{28}$$

where u_g denotes the solution of (25) with respect to the inhomogenity $g \in L_2$ instead of f (for these results, see [2: 2.4 and 3.2]).

Thus, the inequalities (27), (28) reduce the problem of error estimating in the energy and L_2 norm to the study of certain best approximations. First we discuss the application of the standard results of Section 3. Consider $T = T_i$, $i \in \mathbb{N}_0$ (for simplicity, suppose that each corner point P_j of G already coincides with the vertex of some triangle belonging to the initial triangulation T_0, cf. Figure 7). According to (27), (28), the relevant best approximations to be estimated are

$$E_i(w)_{A^{1:k}_{2,2:\Gamma}} = \inf \{\|w - v_i\|_{A^{1:k}_{2,2}}: v_i \in S_\Gamma^{(k)}(T_i)\}. \tag{29}$$

where $w = u = u_f$ and $w = u_0$ (for $g \in L_2$) belongs to $H_\Gamma{}^1 \cong A^{1:k}_{2,2:\Gamma}$. Thus, the results of Section 3 apply and lead to $O\big(h(T_i)^\tau\big)$ error estimates for $i \to \infty$ with some $\tau > 0$ if we can prove regularity properties of the solutions of (25). A detailed investigation of the regularity theory for elliptic problems in plane polygonal domains is given in GRISVARD [6]. For our purposes, we quote from [6: Theorem 5.1.3.5] the following assertion:

Let $f \in W_p{}^m$ for some $1 < p \leqq 2$ and $m \in \mathbb{N}_0$. Set $\Phi_j = 0$ or $\Phi_j = \pi/2$ if on the side d_j Neumann or Dirichlet boundary conditions are given, respectively, $j = 1, \ldots, N$. Furthermore, define numbers $\lambda_{j,l}$, $l \in \mathbb{N}$, by $\lambda_{j,l} = (l - 1/2)\, \pi/\omega_j$ or $\lambda_{j,l} = l\pi/\omega_j$ in dependence on whether or not the type of the boundary conditions changes at P_j. Then, if the condition

$$\big(\Phi_{j+1} - \Phi_j - m\omega_j - 2(1 - 1/p)\, \omega_j\big)/\pi \notin \mathbb{Z}, \qquad j = 1, \ldots, N \tag{30}$$

is fulfilled, the solution $u = u_f$ of (25) can be represented as

$$u = u_{\text{reg}} + \sum_{j=1}^{N} \sum_{\lambda_{j,l} < m+2-2/p} c_{j,l} S_{j,l} \tag{31}$$

$$u_{\text{reg}} \in W_p{}^{m+2}, \qquad \|u_{\text{reg}}\|_{W_p{}^{m+2}} \leqq c\, \|f\|_{W_p{}^m}, \qquad c_{j,l} \in \mathbb{R}, \qquad |c_{j,l}| \leqq c\, \|f\|_{W_p{}^m}$$

and

$$S_{j,l}(x) = \mu_j(x)\, r_j^{\lambda_{j,l}} \begin{cases} \cos t,\ t = \lambda_{j,l}\varphi_j - \Phi_{j+1} & \text{if } \lambda_{j,l} \notin \mathbb{Z}, \\ \cos t \ln r_j - \omega_j \sin t & \text{if } \lambda_{j,l} \in \mathbb{Z}, \end{cases}$$

where, as above, r_j, φ_j denote the local polar coordinates and μ_j the cut-off functions corresponding to P_j, $j = 1, \ldots, N$.

This result shows that, in general, the solution u of (25) belongs to $W_p{}^s$ only for small s. According to [6: Theorem 1.4.5.3] the precise restrictions are $s < s_0 \equiv \min_j (2/p + \lambda_{j,1})$ and $s \leqq m + 2$. Further results of this type for elliptic boundary value problems are contained in KUFNER and SÄNDIG [9].

In order to illustrate this situation, we shall consider a L-shaped domain and mixed boundary conditions as indicated in Figure 8 ($\Gamma = \partial G \setminus d_1$). For this example we have $\lambda_{1,1} = 1/3$, $\lambda_{j,1} = 2$, $j = 2, \ldots, 5$, and $\lambda_{6,1} = 1$. Checking the assumptions to be fulfilled for the representa-

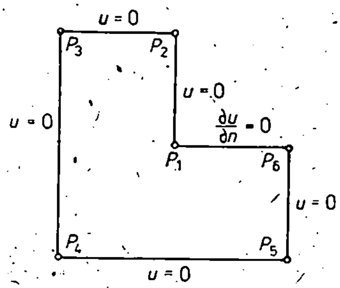

Fig. 8. Example of an L-shaped domain with boundary conditions

tion (31), in the case $m = 0$ (i.e. for $f \in L_p$, $1 < p \leqq 2$) we obtain

$$u = u_{\text{reg}} + \begin{cases} c_{1,1} S_{1,1}, & 6/5 < p < 2, \\ 0, & 1 < p < 6/5. \end{cases}$$

This immediately yields $u \in W_p^2$ for $1 < p < 6/5$ and $u \in W_p^s$ for $s < 2/p + 1/3$ and $6/5 \leqq p \leqq 2$. Now, by the results of the preceding sections we can estimate the quantities (29). Together with (27), this yields

$$\|u - u_{T_i}\|_{H^1} \leqq c E_i(u)_{A^{1;k}_{2,2;\Gamma}}$$

$$\leqq c \begin{cases} 2^{-i(2-1-2(1/p-1/2))} \|u\|_{W_p^2}, & 1 < p < 6/5, \\ 2^{-i(s-1-2(1/p-1/2))} \|u\|_{W_p^s}, & s < 2/p + 1/3,\ 6/5 \leqq p \leqq 2, \end{cases}$$

$$\leqq c\, \|f\|_{L_p} \begin{cases} 2^{-2i(1-1/p)} \\ 2^{-i(1/3-\varepsilon)} \end{cases} \leqq c\, \|f\|_{L_p} \begin{cases} h(T_i)^{2(1-1/p)}, & 1 < p < 6/5, \\ h(T_i)^{1/3-\varepsilon}, & 6/5 \leqq p \leqq 2. \end{cases}$$

Analogous considerations show that for $m \in \mathbb{N}$ we generally have

$$\|u - u_{T_i}\|_{H^1} \leqq c 2^{-i(1/3-\varepsilon)} \|f\|_{W_p^m} \leqq c h(T_i)^{1/3-\varepsilon} \|f\|_{W_p^m}$$

where $\varepsilon > 0$ is arbitrary, $i \in \mathbb{N}_0$. Thus, due to the corner singularities (here mainly to that at the corner P_1), the asymptotic error estimate in the energy norm is far from the theoretical possible $O(h(T_i)^l)$ estimate where $l = \min(k, m + 1)$. This effect is also present in practical computations based on the standard conforming finite element method with uniform or quasi-uniform triangulations (such as the above considered triangulations T_i, $i \in \mathbb{N}_0$).

There are several possibilities for avoiding this situation. For instance, special elements or special trial functions which model the singularities at the corner points can be included into the finite element space. Another strategy is based on refinements of the underlying triangulations near the corners. A brief discussion of these techniques is contained in [6: 8.4], see also [21].

We show that the results of Section 4 (especially the variant of Theorem 7 for the spaces $A^{s;k}_{p,q;\Gamma}$, and Remark 18) yield the following, in some sense optimal, asymptotic error estimate.

Theorem 8: *Let $f \in W_p^m$ for some $m \in \mathbb{N}_0$ and $1 < p \leqq 2$. Then, for $n \geqq n_0$, there exist triangulations T_n^* satisfying property* (i) *from Theorem* 8 *such that for the solutions u and $u_n \equiv u_{T_n^*}$ of* (25) *and* (26) *the estimate*

$$\|u - u_n\|_{H^1} \leqq c_{22} n^{-1/2} \|f\|_{W_p^m} \leqq c'_{22} h(T_n^*)^l \|f\|_{W_p^m} \tag{32}$$

holds where $l = \min(m + 1, k)$.

We outline the proof of this assertion. Without loss of generality, let $m + 1 \leqq k$, and $1 < p \leqq 2$ satisfy (30) (if the latter condition is not fulfilled, then consider some smaller $p - \varepsilon < \overline{p} < p$ satisfying (30) and observe that the argument given below leads to the required estimates if $\varepsilon > 0$ is sufficiently small). According to (31), we see that the solution u of (25) has exactly the form of the functions (22) considered in Remark 18, where $p_0 = p$ and $\bar{\alpha} = \min \lambda_{j,1} \geqq \min \pi/(2\omega_j) \geqq 1/4$. Now, choosing p^* such that $2/(m + 2) < p^* < 2/(m + 7/4)$ and $p^* \leqq p$ we guarantee, on the one hand, the inclusion

$$u \in B^{m+2;k+1}_{p^*,\infty;\Gamma} \equiv \{v \in B^{m+2;k+1}_{p^*,\infty}: \partial v|_\Gamma = 0\} \hookrightarrow Y = A^{m+2;k+1}_{p^*,\infty;\Gamma}$$

where $\|u\|_Y \leqq c\,\|f\|_{W_p{}^m}$ and, on the other hand, that with this Y and $X = A^{1;k+1}_{2,2;\Gamma}$ an analogon of Theorem 7 holds (observe that $(m + 2) - 1 > 2(1/p - 1/2)$ by construction). This implies the existence of triangulations $T_n{}^*$ satisfying property (i) from Theorem 8 and

$$\begin{aligned}\|u - u_n\|_{H^1} &\leqq c \inf \{\|u - v_n\|_X : v_n \in S_\Gamma{}^{(k)}(T_n{}^*)\} \\ &\leqq cn^{-((m+2)-1)/2}\,\|u\|_Y \leqq cn^{-(m+1)/2}\,\|f\|_{W_p{}^m}, \qquad n \geqq n_0.\end{aligned}$$

This actually proves (32) ∎

Remark 20. By some additional arguments one can also show that under the assumptions of Theorem 8 we have the L_2 error estimate

$$\|u - u_n\|_{L_2} \leqq c_{23} n^{-(l+1)/2}\,\|f\|_{W_p{}^m} \leqq c'_{23} h(T_n{}^*)^{l+1}\,\|f\|_{W_p{}^m}. \tag{33}$$

It should be mentioned that (32), (33) are known for $p = 2$ where the corresponding triangulations can be determined explicitly (cf. [11, 14], or [6: 8.4]).

6. Further comments

The approach presented in the preceeding sections could serve, in some sense, as the methodical basis for dealing with estimates for more general local approximation schemes. Below we briefly discuss some possible extensions and fields for further investigations.

First of all let us point out that according to (4) (cf. Remark 3) the spaces $A^{s;k}_{p,q}$ can equivalently be described by the finite element L_p best approximations. Descriptions of this kind have been used by several authors for characterizing Lipschitz and Besov spaces on special domains by spline and piecewise polynomial best approximations (see, e.g., [7, 8, 12, 15]).

The use of the representation (1), (2) which is an analogon of the Nikolskij representation in the trigonometric case seems to be a new idea for analyzing spline and finite element approximation schemes in a unified and systematic way. The most important requisites for such an approach are the existence of suitable locally supported systems of basis functions for the approximating subspaces (satisfying an analogon of Lemma 1), and, in order to relate the (Approximation) spaces by embedding theorems to the standard function spaces, e.g. to the Sobolev spaces, inequalities of Jackson and Bernstein type for the best approximations with respect to the approximating subspaces (Lemma 2 and 3). Implicitly, this approach has been already explored in [13] where nonlinear spline approximation problems (variable partitions) have been considered for certain spline schemes in one and several dimensions.

The above theory can be extended, without substantial difficulties, to finite elements of type (k) in higher dimensions. A further possible generalization which is slightly more complicated concerns the case of finite element schemes of higher smoothness order or the case of isoparametric elements (see [2] for the corresponding definitions).

Another interesting question is to relate our results to recent research on spline systems and spline representations in function spaces of Besov-Hardy-Sobolev type (for some results and references, cf. [20: 2.12.3], [3]).

For applications, it is important not only to include more general elements but to give estimates for concrete approximation processes used in numerical analysis such as Galerkin-Ritz projections or collocation resp. interpolation methods for elliptic differential and integral equations. Let us also mention the problem of obtaining realistic bounds of the constants occuring in the asymptotic estimates.

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