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Overdetermined Systems of Nonlinear Partial Differential Equations

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Dedicated to S. G. Mikhlin on the occasion of his 80th birthday

Für die in der Theorie der Lie-Reihen auftretenden partiellen Differentialgleichungssysteme wird die allgemeine Struktur der Lösungen angegeben, ohne die Holomorphievoraussetzungen der Lie-Theorie zu verwenden.

Не используя предположения голоморфности мы изучаем общую структуру решений систем дифференциальных уравнений в частных производных возникающих в теории рядов Ли.

For the systems of partial differential equations appearing in the theory of Lie series we deduce the general structure of the solutions without assuming the data to be holomorphic ones.

This paper is concerned with initial value problems of the kind

$$Z_{\underline{t}}(\underline{t}) = \underline{A}(Z), \qquad Z(0) = z \tag{1}$$

with $Z_{\underline{t}}(\underline{t}) = (\partial/\partial\underline{t})\, Z(\underline{t})$, the vectors

$$Z = (Z_1, \ldots, Z_n), \qquad z = (z_1, \ldots, z_n),$$

$$\underline{t} = (t_1, \ldots, t_m), \qquad \partial/\partial\underline{t} = (\partial/\partial t_1, \ldots, \partial/\partial t_m)^{\mathsf{T}},$$

the zero vector 0 and an $m \times n$-matrix $\underline{A}$. Such problems for nonlinear partial differential equations appear in the framework of holomorphic functions in the theory of Lie series, cf. W. Gröbner [3]. Here we only assume that the matrix $\underline{A}$ possesses continuous partial derivatives of first order, so that we can use the real theory as in E. Kamke [4].

In case of $m > 1$ the system (1) is overdetermined, and the theorem of Schwarz $Z_{t_it_j} = Z_{t_jt_i}$ implies the necessary compatibility conditions

$$\underline{A}_{j\cdot}\underline{A}'_{i\cdot} = \underline{A}_{i\cdot}\underline{A}'_{j\cdot} \tag{2}$$

for $i, j = 1, \ldots, m$, where $\underline{A}_{i\cdot}$ denotes the i-th row vector of $\underline{A}$, and a prime denotes the derivative with respect to the argument, i.e. in this case the derivative $\underline{A}'_{i\cdot} = (\partial/\partial Z)\, \underline{A}_{i\cdot}(Z)$ with

$$\partial/\partial Z = (\partial/\partial Z_1, \ldots, \partial/\partial Z_n)^{\mathsf{T}}. \tag{3}$$

The trivial case $\underline{A}(z) = 0$ with the solution $Z = z$ of (1) shall be excluded, i.e. we always assume the initial value z to be chosen in such a way that $\underline{A}(z) \neq 0$. Here, of course, 0 denotes a zero matrix, and later on I a unit matrix of suitable size.

Theorem 1: *If the rank of $\underline{A}(Z)$ is equal to m and if the conditions* (2) *are satisfied, then there exist two mutually inverse n-dimensional vector functions*

$$w = f(z), \qquad z = F(w) \tag{4}$$

with $w = (w_1, \ldots, w_n)$, so that, for suitable small t, the solution of (1) *possesses the structure*

$$Z = F\big(\underline{t}C + f(z)\big), \tag{5}$$

where C is an $m \times n$-matrix with the block representation $C = (I, 0)$.

In view of the hypothesis on the rank of $\underline{A}$, Theorem 1 is concerned with the case $m \leq n$. If $\underline{A}$ possesses a rank $r < m$, then after suitable permutations we can make the splitting

$$\underline{t} = (t, s), \quad t = (t_1, \ldots, t_r), \quad s = (t_{r+1}, \ldots, t_m),$$

$$\underline{A} = \begin{pmatrix} I \\ B \end{pmatrix} A = \begin{pmatrix} A \\ BA \end{pmatrix},$$

where $A = A(Z)$ is the $r \times n$-matrix of the first rows of $\underline{A}$, and where $B = B(Z)$ is a suitable $(m - r) \times r$-matrix. With these notations and an analogous interpretation of the partial derivatives as above the problem (1) turns over into

$$Z_t(t, s) = A(Z), \quad Z_s(t, s) = B(Z)\, A(Z), \quad Z(0, 0) = z. \tag{6}$$

Theorem 2: *If the system* (1) *possesses the form* (6) *with* rank $A = r$ *and if the conditions* (2) *are satisfied, then for suitable small t, s the solution of* (6) *possesses the structure*

$$Z = F\big((t + sB(z))\, C + f(z)\big) \tag{7}$$

with f, F as in (4) *and an $r \times n$-matrix $C = (I, 0)$.*

In both theorems the solutions (5) and (7), respectively, exist also for large arguments, so far as the premisses are satisfied. The case $m = 1$ of ordinary differential equations was already treated in [1], the cases $m = 2$, n arbitrary, and $n \leq 2$, m arbitrary, are accomplished in [2]. In these two papers you also find some examples. In what follows we consider the case of Theorem 1 as a special case of Theorem 2 with $r = m$, where the second equation in (6) must be ignored. This means that in case of Theorem 1 we write in (1) simply $\underline{A} = A$.

Proof of Theorem 1: An unessential change in the formulations of [1] and [2] shows that Theorem 1 is valid in the cases $m = 1$ and $m = 2$. Hence we can prove it by induction. For this reason we assume that Theorem 1 is valid for a fixed $m - 1$ instead of m, and we shall show the validity for m. Let

$$U = g(Z), \quad G(U) = Z \tag{8}$$

be mutually inverse vector functions, such that

$$Z = G\big(t_1\underline{1} + g(Z_0)\big), \quad g(Z) = t_1\underline{1} + g(Z_0) \tag{9}$$

with $\underline{1} = (1, \ldots, 1)$ is the explicit and implicit form, respectively, of the solution of $Z_{t_1} = A_1.(Z)$; $Z|_{t_1=0} = Z_0$, where Z_0 depends only on $t_2, \ldots, t_m$. The functions (8) exist according to [1]. We consider (8) as a substitution and go over from Z to U. From (8) and (1) we find

$$U_t = Z_t g'(Z) = A(Z)\, g'(Z) = A(Z)\, G'^{-1}(U), \quad U(0) = g(z). \tag{10}$$

The matrix AG'^{-1} possesses the same rank $r = m$ as A.

Now we show that the compatibility conditions (2) corresponding to (10) are satisfied, i.e. that $U_{t_i t_j} = (A_i.G'^{-1})_{t_j}$ is symmetrical in i and j. According to the product rule and

$$(A_{i\cdot})_{t_j} = Z_{t_j} A'_{i\cdot} = A_{j\cdot} A'_{i\cdot}, \qquad (G'^{-1})_{t_j} = -G'^{-1} G'_{t_j} G'^{-1}$$

we find

$$U_{t_i t_j} = A_{j\cdot} A'_{i\cdot} G'^{-1} - A_{i\cdot} G'^{-1} G'_{t_j} G'^{-1}.$$

The first term at the right-hand side is symmetrical in view of (2). According to $A_{i\cdot} G'^{-1} = U_{t_i}$ and

$$(G'_{t_j})_{k\cdot} = \left(\frac{\partial G}{\partial U_k}\right)_{t_j} = \sum_{\nu=1}^{n} (U_\nu)_{t_j} \frac{\partial^2 G}{\partial U_k \, \partial U_\nu}$$

we have further

$$U_{t_i} G'_{t_j} = \sum_{\mu=1}^{n} (U_\mu)_{t_i} (G'_{t_j})_{\mu\cdot} = \sum_{\mu,\nu=1}^{n} (U_\mu)_{t_i} (U_\nu)_{t_j} \frac{\partial^2 G}{\partial U_\mu \, \partial U_\nu},$$

so that also the second term is symmetrical and the compatibility conditions are satisfied. In view of (8) and (9) the first component of the matrix equation in (10) reads

$$U_{t_1} = \underline{1}. \tag{11}$$

According to the compatibility conditions this implies as in [1] that (10) possesses the structure

$$U_t = a(U_* - U_1 e), \qquad U(0) = g(z), \tag{12}$$

where we have used the splitting

$$U = (U_1, U_*), \qquad U_* = (U_2, \ldots, U_n), \qquad \underline{1} = (1, e) \tag{13}$$

with an $(n-1)$-dimensional vector e all components of which are equal to 1. The compatibility conditions now read

$$(a_{i\cdot} - a_{i1} e)\, a'_{j\cdot} = (a_{j\cdot} - a_{j1} e)\, a'_{i\cdot} \tag{14}$$

for $i, j = 2, \ldots, m$. With an analogous splitting $g = (g_1, g_*)$, $a = (a_{\cdot 1}, a_*)$, where $a_{\cdot 1}$ denotes the vector of the first column of a, and the substitutions $V = U_* - U_1 e$, $b = a_* - a_{\cdot 1} e$ we obtain from (12)

$$V_t = b(V), \qquad V(0) = g_*(z) - g_1(z)\, e, \tag{15}$$

and the compatibility conditions (14) transfer to $b_{i\cdot} b'_{j\cdot} = b_{j\cdot} b'_{i\cdot}$. Equation (11) implies that $V_{t_1} = 0$, so that V is independent of t_1, and b is a matrix of rank $m-1$. Hence (15) considered as a system with respect to the $(m-1)$-dimensional vector $t_* = (t_2, \ldots, t_m)$ possesses by hypothesis a solution of the form

$$V = F\big(t_* C_* + f(V(0))\big) \tag{16}$$

with $C_* = (I, 0)$, where here I is the $(m-1)$-dimensional unit matrix. Equation (16) means that

$$U_* - U_1 e = F(t_* C_* + p) \tag{17}$$

with $p = f\big(g_*(z) - g_1(z)\, e\big)$. Now we obtain from (12)

$$U_{1t} = a_{\cdot 1}\big(F(t_* C_* + p)\big), \qquad U_1(0) = g_1(z). \tag{18}$$

The corresponding compatibility conditions

$$F_{t_i} a'_{j1} = F_{t_j} a'_{i1} \tag{19}$$

with $i, j = 2, \ldots, m$ are satisfied in view of (14) and $F_{t_i} = b_i = a_{i\cdot} - a_{i1} e$, where the last equations follow from (15) and (16). From (11) and (12) we find $a_{11} \equiv 1$, so that we can make the splitting $a_{\cdot 1} = \begin{pmatrix} 1 \\ a_{*1} \end{pmatrix}$, $a_{*1} = (a_{21}, \ldots, a_{m1})^{\mathsf{T}}$. According to (19) the contour integral

$$\int_0^{t_*} ds\, a_{*1}\big(F(sC_* + p)\big) = K(t_* C_* + p) - K(p) \tag{20}$$

with an $(m-1)$-dimensional vector $s = (s_2, \ldots, s_m)$ is independent of the path of integration. Hence (18) possesses the solution $U_1 = t_1 + K(t_* C_* + p) - K(p) + g_1(z)$, and this together with (17) in the form $t_* C_* + p = f(U_* - U_1 e)$ with $p = f\big(g_*(z) - g_1(z)\, e\big)$ implies

$$U_1 - K\big(f(U_* - U_1 e)\big) = t_1 - K\big(f(g_*(z) - g_1(z)\, e)\big) + g_1(z),$$
$$f(U_* - U_1 e) = t_* C_* + f\big(g_*(z) - g_1(z)\, e\big).$$

This means the intermediate result

$$h(U) = tC + h\big(g(z)\big) \tag{21}$$

with $h(U) = \big(U_1 - K(f(U_* - U_1 e)),\ f(U_* - U_1 e)\big)$ and C as in the theorem. The determinant

$$\det h'(U) = \det \begin{pmatrix} 1 + ef'K' & -ef' \\ -f'K' & f' \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ -f'K' & f' \end{pmatrix} = \det f'$$

is different from zero, so that h is invertible, and according to (8) we obtain from (21) the wanted result (5) only with other notations ∎

Proof of Theorem 2: According to Theorem 1 the first equation of (6) possesses the solution

$$Z(t, s) = F\big(tC + f(Z(0, s))\big). \tag{22}$$

The differential equations in (6) read in components $Z_{t_i} = A_{i\cdot}$, $Z_{s_j} = B_{j\cdot}A$, so that the compatibility conditions $(A_{i\cdot})_{s_j} = (B_{j\cdot}A)_{t_i} = (B_{j\cdot})_{t_i} A + B_{j\cdot}A_{t_i}$ imply

$$B_{j\cdot}AA'_{i\cdot} = (B_{j\cdot})_{t_i} A + B_{j\cdot}A_{t_i}. \tag{23}$$

Writing (2) in the form $A_{k\cdot}A'_{i\cdot} = A_{i\cdot}A'_{k\cdot} = (A_{k\cdot})_{t_i}$, we see from (23) that $B_{t_i}A = 0$ and, since AA^{T} is a regular matrix, $B_{t_i}(Z) = Z_{t_i}B' = A_{i\cdot}B' = 0$ for $i = 1, \ldots, m$, i.e. $AB' = 0$. This implies also

$$B_{s_i}(Z) = Z_{s_i}B' = B_{i\cdot}AB' = 0, \tag{24}$$

so that for a solution of (6) the matrix $B(Z)$ is independent of t and s, i.e.

$$B(Z) = B(z). \tag{25}$$

But $Z_s(0, s) = B(z)\, A(Z)$, $Z(0, 0) = z$ possesses the solution $Z(0, s) = F\big(sB(z)\, C + f(z)\big)$ with the same vector functions F, f as in (22), so that together with (22) we obtain (7). On the other side, we can easily check that (7) satisfies both (25) and (6) ∎

Let us mention that we did not use explicitly the compatibility conditions arising from the symmetry of $Z_{s_i s_j} = (B_i.A)_{s_j} = (B_i.)_{s_j} A + B_i.A_{s_j}$. But

$$B_i.A_{s_j} = \sum_{k=1}^{r} B_{ik}(A_k.)_{s_j} = \sum_{k=1}^{r} \sum_{l=1}^{r} B_{ik}B_{jl}A_l.A_k'.$$

is symmetrical in view of (2), and the other term on the right-hand side equals to zero in view of (24), so that the compatibility conditions with respect to the vector s are a consequence of the other ones.

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