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A Nehari Type Problem for Matricial Schur Functions

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Unter Verwendung von Methoden der Schur-Analysis wird ein spezielles Erweiterungsproblem für matrizielle Schur-Funktionen behandelt, welches als ein Analogon zu einem jüngst von den Autoren untersuchten verallgemeinerten Nehari-Problem für matrizielle Carathéodory-Funktionen angesehen werden kann.

Используя методы анализа Шура изучается некоторая проблема дополнения для матричных функций Шура, которая может быть рассмотрена как аналог обобщённой проблемы Нехари для матричных функций Каратеодори, изученной авторами недавно.

Using Schur analysis methods a particular completion problem for matricial Schur functions will be treated which can be conceived as an analogue of a generalized Nehari problem for matricial Carathéodory functions recently studied by the authors.

1. Introduction

This paper is strongly related to the author's investigations [11] on a generalized Nehari problem raised by Kaznelson [14]. He named that problem after Nehari because it contains as a special case the matricial Nehari problem studied in the fundamental paper of Adamjan, Arov and Krein [1] (compare also Dym and Gohberg [8] and Gohberg, Kaashoek and van Schagen [13]), which in turn can be considered as a matricial generalization of the classical work of Nehari [15]. It should be mentioned that Kaznelson stated several equivalent versions of the problem in question (compare also [11]). In [11] we worked mainly in that version which can be interpreted as a certain completion problem for matricial Carathéodory functions. The method used there can be classified as a certain form of what is called today Schur analysis. Schur analysis has its origin in the famous paper of Schur [16] where bounded holomorphic functions in the unit disc were parametrized via a sequence of contractive parameters. In the meantime several matricial and operatorial versions of that method were developed (compare e.g. the survey paper of Arsene, Ceausescu and Constantinescu [4]). In this paper, using Schur analysis methods a special completion problem for matricial Schur functions will be studied, which can be conceived as an analogue of the afore-mentioned generalized Nehari problem for matricial Carathéodory functions. In order to formulate the main problem in this paper we give some notations.

Throughout this paper, let $p, q, r, s \in \mathbf{N}$ and let $\mathfrak{M}_{p\times q}$ be the set of all complex $p \times q$ matrices. Denote I_q the unit matrix of order q. Let $p \times q$-$\mathfrak{S}$ be the $p \times q$ Schur class in the open unit disc $\mathbf{D} = \{w \in \mathbf{C}: |w| < 1\}$, i.e. the set of all matrix functions $f: \mathbf{D} \to \mathfrak{M}_{p\times q}$ which are holomorphic in $\mathbf{D}$ and satisfy $I_p - f(w)[f(w)]^* \geqq 0$ for each $w \in \mathbf{D}$. We shall consider the following *completion problem (S) for matricial Schur functions*:

Problem (S): Let $\alpha: \mathbf{D} \to \mathfrak{M}_{p\times r}$, $\beta: \mathbf{D} \to \mathfrak{M}_{p\times s}$ and $\delta: \mathbf{D} \to \mathfrak{M}_{q\times s}$ be matrix functions which are holomorphic in $\mathbf{D}$.

(S1) Give necessary and sufficient conditions for the existence of a matrix function $\xi: \mathbf{D} \to \mathfrak{M}_{q\times r}$ such that

$$f = \begin{pmatrix} \alpha & \beta \\ \xi & \delta \end{pmatrix}$$

belongs to the Schur class $(p + q) \times (r + s)$-$\mathfrak{S}$. Describe the set of all matrix functions ξ for which $f \in (p + q) \times (r + s)$-$\mathfrak{S}$.

(S2) Let $n \in \mathbf{N}_0$ and $x_0, x_1, \ldots, x_n \in \mathfrak{M}_{q\times r}$. Give necessary and sufficient conditions for the existence of a matrix function $\xi: \mathbf{D} \to \mathfrak{M}_{q\times r}$ such that $f \in (p + q) \times (r + s)$-$\mathfrak{S}$ and that $x_0, x_1, \ldots, x_n$ are just the first $n + 1$ Taylor coefficients of ξ. Describe the set of all matrix functions ξ satisfying these conditions.

In the case $\alpha = 0$, $\beta = 0$ and $\delta = 0$ (S2) coincides with the matricial version of the classical Schur problem which was studied indepedently by different methods in AROV and KREIN [3], DUBOVOJ [5], DYM [7] and in the author's papers [10]. The scalar version of which was treated in SCHUR [16]. It should be remarked that DUBOVOJ [5, Part VI] considered several other types of completion problems for matricial Schur functions.

This paper is organized as follows: In Section 2, we shall recall some basic facts on matrix balls due to SMULJAN [17]. Furthermore, we shall state two algebraic lemmas. In Section 3, we shall work out a Schur analysis approach to problem (S), which enables us to give a description of all solutions in terms of Taylor coefficients. Moreover, what concerns part (S1) of (S) we formulate a necessary and sufficient condition in terms of so-called kernels of mixed Toeplitz-Hankel type.

2. Preliminaries

First, we give some results on matrix balls due to SMULJAN [17]. Let $\mathbf{K}_{\mathfrak{M}_{p\times q}}$ be the set of all contractive $p \times q$ matrices, i.e. the set of all $K \in \mathfrak{M}_{p\times q}$ satisfying $I_p - KK^* \geqq 0$. For $M \in \mathfrak{M}_{p\times q}$, $A \in \mathfrak{M}_{p\times p}$ and $B \in \mathfrak{M}_{q\times q}$ denote $\mathfrak{K}(M; A, B) = \{M + AKB: K \in \mathbf{K}_{\mathfrak{M}_{p\times q}}\}$ the matrix ball with center M, left semi-radius A and right semi-radius B. Then $\mathfrak{K}(M; A, B) = \{M\}$ iff $A = 0$ or $B = 0$. Assuming $A \neq 0$ and $B \neq 0$, $\mathfrak{K}(M; A, B) = \mathfrak{K}(\tilde{M}; \tilde{A}, \tilde{B})$ iff $\tilde{M} = M$ and there exists $\varrho \in \mathbf{R}$, $\varrho > 0$, such that $\tilde{A}\tilde{A}^* = \varrho AA^*$ and $\tilde{B}^*\tilde{B} = 1/\varrho B^*B$. Hence, $\mathfrak{K}(M; A, B) = \mathfrak{K}\left(M; \sqrt{AA^*}, \sqrt{B^*B}\right)$. The following result is basic for our investigations.

Lemma 1 (SMULJAN [17]): *Let $\left(\mathfrak{K}(M_n; A_n, B_n)\right)_{n\in N}$ be a nested sequence of matrix balls. Then:*

a) *The sequence (M_n) is convergent. If M denotes the limit, then $\bigcap_n \mathfrak{K}(M_n; A_n, B_n)$ is a matrix ball $\mathfrak{K}$ with center M.*

b) *There exist sequences $(\tilde{A}_n) \subset \mathfrak{M}_{p\times p}$, $(\tilde{B}_n) \subset \mathfrak{M}_{q\times q}$ such that*

$$\mathfrak{K}(M_n; \tilde{A}_n, \tilde{B}_n) = \mathfrak{K}(M_n; A_n, B_n), \qquad n \in \mathbf{N}, \tag{1}$$

and

$$\tilde{A}_n\tilde{A}_n^* \geqq \tilde{A}_{n+1}\tilde{A}_{n+1}^*, \qquad \tilde{B}_n^*\tilde{B}_n \geqq \tilde{B}_{n+1}^*\tilde{B}_{n+1}, \qquad n \in \mathbf{N}. \tag{2}$$

If $(\tilde{A}_n) \subset \mathfrak{M}_{p\times p}$, $(\tilde{B}_n) \subset \mathfrak{M}_{q\times q}$ are arbitrary sequences satisfying (1) and (2), then $\mathfrak{K} = \mathfrak{K}\left(M; \sqrt{C}, \sqrt{D}\right)$ where $C \doteq \lim \tilde{A}_n\tilde{A}_n^*$ and $D \doteq \lim \tilde{B}_n^*\tilde{B}_n$.

For $A \in \mathfrak{M}_{p\times q}$ denote A^+ the *Moore-Penrose inverse* of A.

Lemma 2: *For $j \in \{1, 2, 3\}$ and $k \in \{1, 2\}$ let $m_j, n_k \in \mathbf{N}$ and $C_{jk} \in \mathfrak{M}_{m_j\times n_k}$. Suppose that $C = (C_{jk})$ satisfies*

$$CC^* \leqq I_{m_1+m_2+m_3}. \tag{3}$$

Then $O \leqq \underline{L} \leqq L$ where $L = I_{m_3} - C_{31}(I_{n_1} - C_{21}^*C_{21})^+ C_{31}^*$,

$$\underline{C} := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \qquad c := (C_{31}, C_{32}) \tag{4}$$

and $\underline{L} = I_{m_3} - c(I_{n_1+n_2} - \underline{C}^*\underline{C})^+ c^*$.

Proof: Clearly, the condition (3) is fulfilled if and only if

$$G := \begin{pmatrix} I_{n_1+n_2} & C^* \\ C & I_{m_1+m_2+m_3} \end{pmatrix} \geqq O \tag{5}$$

holds (compare e.g. Efimov and Potapov [9: p. 88]). We choose the block partition

$$G = \begin{pmatrix} I_{n_1} & (0, \; C_{11}^*) & C_{21}^* & C_{31}^* \\ \begin{pmatrix} 0 \\ C_{11} \end{pmatrix} & \begin{pmatrix} I_{n_2}, & C_{12}^* \\ C_{12}, & I_{m_1} \end{pmatrix} & \begin{pmatrix} C_{22}^* \\ 0 \end{pmatrix} & \begin{pmatrix} C_{32}^* \\ 0 \end{pmatrix} \\ C_{21} & (C_{22}, 0) & I_{m_2} & 0 \\ C_{31} & (C_{32}, 0) & 0 & I_{m_3} \end{pmatrix}$$

and apply Lemma 3 in [11]. Then we obtain $0 \leqq \underline{\mathscr{L}} \leqq \mathscr{L}$ where

$$\underline{\mathscr{L}} = I_{m_3} - (c, 0) \begin{pmatrix} I_{n_1+n_2} & \underline{C}^* \\ \underline{C} & I_{m_1+m_2} \end{pmatrix}^+ \begin{pmatrix} c^* \\ 0 \end{pmatrix},$$

$$\mathscr{L} = I_{m_3} - (C_{31}, 0) \begin{pmatrix} I_{n_1} & C_{21}^* \\ C_{21} & I_{m_2} \end{pmatrix}^+ \begin{pmatrix} C_{31}^* \\ 0 \end{pmatrix}.$$

Using Lemma 4 from [12] and (5) we get $\underline{\mathscr{L}} = \underline{L}$ and $\mathscr{L} = L$ ∎

Let us present a further result which can be proved similarly.

Lemma 3: *For $j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$ let $m_j, n_k \in \mathbf{N}$ and $C_{jk} \in \mathfrak{M}_{m_j\times n_k}$. Suppose that $C = (C_{jk})$ satisfies $CC^* \leqq I_{m_1+m_2}$. Then $O \leqq \underline{R} \leqq R$ where $R = I_{n_1} - C_{21}^*(I_{m_2} - C_{22}C_{22}^*)^+ C_{21}$ and*

$$\underline{R} = I_{n_1} - (C_{11}^*, C_{21}^*) \left[I_{m_1+m_2} - \begin{pmatrix} C_{12} & C_{13} \\ C_{22} & C_{23} \end{pmatrix} \begin{pmatrix} C_{12} & C_{13} \\ C_{22} & C_{23} \end{pmatrix}^* \right]^+ \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}.$$

3. Solution of Problem (S)

Assume that $\alpha : \mathbf{D} \to \mathfrak{M}_{p\times r}$, $\beta : \mathbf{D} \to \mathfrak{M}_{p\times s}$ and $\delta : \mathbf{D} \to \mathfrak{M}_{q\times s}$ are given matrix functions holomorphic in $\mathbf{D}$. Denote by

$$\alpha(w) = \sum_{k=0}^{\infty} a_k w^k, \qquad w \in \mathbf{D}, \tag{6}$$

$$\beta(w) = \sum_{k=0}^{\infty} b_k w^k, \qquad w \in \mathbf{D}, \tag{7}$$

$$\delta(w) = \sum_{k=0}^{\infty} d_k w^k, \qquad w \in \mathbf{D}, \tag{8}$$

their Taylor series. Obviously in order to study problem (S) it is sufficient to consider holomorphic matrix functions ξ. Therefore, we can write the Taylor series

$$\xi(w) = \sum_{k=0}^{\infty} x_k w^k, \qquad w \in \mathbf{D}. \tag{9}$$

For $k \in \mathbf{N}_0$ and $c_0, c_1, \ldots, c_k \in \mathfrak{M}_{p\times q}$ we define

$$\mathcal{T}_k\langle c_0, c_1, \ldots, c_k\rangle = \begin{pmatrix} c_0 & 0 & \ldots & 0 \\ c_1 & c_0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ c_k & c_{k-1} & \ldots & c_0 \end{pmatrix}. \tag{10}$$

Setting

$$E_k = \begin{pmatrix} a_k & b_k \\ x_k & d_k \end{pmatrix}, \qquad k \in \mathbf{N}_0,$$

from DUBOVOJ [5: Theorem 1.2] (compare also [10: Part II, Theorem 3]) we know that $f \in (p+q) \times (r+s)$-$\mathfrak{S}$ iff for each $k \in \mathbf{N}_0$ the matrix $S_k = \mathcal{T}_k\langle E_0, E_1, \ldots, E_k\rangle$ satisfies

$$S_k S_k^* \leqq I_{(k+1)(p+q)}. \tag{11}$$

Clearly, for $k \in \mathbf{N}_0$ there exist unitary matrices $U_k \in \mathfrak{M}_{(k+1)(p+q)\times(k+1)(p+q)}$ and $V_k \in \mathfrak{M}_{(k+1)(r+s)\times(k+1)(r+s)}$ such that $U_k S_k V_k = H_k$ where

$$H_k = \begin{pmatrix} A_k & B_k \\ X_k & D_k \end{pmatrix} \tag{12}$$

and

$$A_k = \mathcal{T}_k\langle a_0, a_1, \ldots, a_k\rangle, \qquad B_k = \mathcal{T}_k\langle b_0, b_1, \ldots, b_k\rangle, \tag{13}$$

$$X_k = \mathcal{T}_k\langle x_0, x_1, \ldots, x_k\rangle, \qquad D_k = \mathcal{T}_k\langle d_0, d_1, \ldots, d_k\rangle \tag{14}$$

(the matrices U_k and V_k only change rows and columns, respectively). Hence, for $k \in \mathbf{N}_0$ the inequality (11) is equivalent to

$$H_k H_k^* \leqq I_{(k+1)(p+q)}. \tag{15}$$

Consequently, $f \in (p+q) \times (r+s)$-$\mathfrak{S}$ iff (15) is true for $k \in \mathbf{N}_0$. Now we shall study certain submatrices of the matrices H_k.

Lemma 4: *Let $j, m, n \in \mathbf{N}_0$, and let $a_0, a_1, \ldots, a_{j+m} \in \mathfrak{M}_{p\times r}$, $b_0, b_1, \ldots, b_{j+m+n} \in \mathfrak{M}_{p\times s}$, $x_0, x_1, \ldots, x_j \in \mathfrak{M}_{q\times r}$ and $d_0, d_1, \ldots, d_{j+n} \in \mathfrak{M}_{q\times s}$. Further, let the matrices $A_0, A_1, \ldots, A_{j+m}, B_0, B_1, \ldots, B_{j+m+n}, X_0, X_1, \ldots, X_j$ and $D_0, D_1, \ldots, D_{j+n}$ be defined by (13) and (14). Denote $G_{l,0,0} = H_l$ and*

$$G_{-1,\mu,\nu} = \begin{cases} \begin{pmatrix} B_{\nu-1} \\ D_{\nu-1} \end{pmatrix}, \\[2ex] (A_{\mu-1}, B_{\mu-1}), \\[2ex] \begin{pmatrix} \begin{pmatrix} 0 \\ A_{\mu-1} \end{pmatrix} & B_{\mu+\nu-1} \\ 0 & (D_{\nu-1}, 0) \end{pmatrix}, \end{cases} \qquad G_{l,\mu,\nu} = \begin{cases} \begin{pmatrix} \begin{pmatrix} 0 \\ A_l \end{pmatrix} & B_{l+\nu} \\ \begin{pmatrix} 0 \\ X_l \end{pmatrix} & D_{l+\nu} \end{pmatrix}, & \mu = 0, \ \nu \in \{1, \ldots, n\}; \\[2ex] \begin{pmatrix} A_{l+\mu} & B_{l+\mu} \\ (X_l, 0) & (D_l, 0) \end{pmatrix}, & \mu \in \{1, \ldots, n\}, \ \nu = 0, \\[2ex] \begin{pmatrix} \begin{pmatrix} 0 \\ A_{l+\mu} \end{pmatrix} & B_{l+\mu+\nu} \\ \begin{pmatrix} 0 & 0 \\ X_l & 0 \end{pmatrix} & (D_{l+\nu}, 0) \end{pmatrix}. & \mu \in \{1, \ldots, m\}, \ \nu \in \{1, \ldots, n\}, \end{cases} \tag{16}$$

for $l \in \{0, 1, \ldots, j\}$. Suppose

$$G_{jmn} G_{jmn}^* \leqq J_{(j+n+1)(p+q)+mp}. \tag{17}$$

Then for $l \in \{-1, 0, \ldots, j\}$, $\mu \in \{0, 1, \ldots, m\}$ and $\nu \in \{\max(0, -1-\mu), \ldots, n\}$ there follows $G_{l\mu\nu} G_{l\mu\nu}^* \leqq I_{(l+\nu+1)(p+q)+\mu p}$.

Proof: Recall the well-known fact that each submatrix of a contractive matrix is contractive, too. Therefore it is sufficient to show that $G_{l\mu\nu}$ is a submatrix of G_{jmn}. First we consider the case $l > -1$ and $\mu, \nu > 0$. Then

$$A_{j+m} = \begin{pmatrix} A_{l+\mu} & 0 \\ \underbrace{*}_{(l+\mu+1)r} & \underbrace{*}_{(j+m-l-\mu)r} \end{pmatrix} \begin{matrix} \}\,(l+\mu+1)p \\ \}\,(j+m-l-\mu)p \end{matrix}, \qquad X_j = \begin{pmatrix} X_l & 0 \\ \underbrace{*}_{(l+1)r} & \underbrace{*}_{(j-l)r} \end{pmatrix} \begin{matrix} \}\,(l+1)q \\ \}\,(j-l)q \end{matrix}$$

$$B_{j+m+n} = \begin{pmatrix} * & 0 & 0 \\ * & B_{l+\mu+\nu} & 0 \\ \underbrace{*}_{(n-\nu)s} & \underbrace{*}_{(l+\mu+\nu+1)s} & \underbrace{0}_{(j+m-l-\mu)s} \end{pmatrix} \begin{matrix} \}\,(n-\nu)p \\ \}\,(l+\mu+\nu+1)p \\ \}\,(j+m-l-\mu)p \end{matrix},$$

$$D_{j+n} = \begin{pmatrix} * & 0 & 0 \\ * & D_{l+\nu} & 0 \\ \underbrace{*}_{(n-\nu)s} & \underbrace{*}_{(l+\nu+1)s} & \underbrace{*}_{(j-l)s} \end{pmatrix} \begin{matrix} \}\,(n-\nu)q \\ \}\,(l+\nu+1)q \\ \}\,(j-l)q \end{matrix}$$

and, consequently,

$$G_{jmn} = \begin{pmatrix} \begin{pmatrix} 0 \\ A_{j+m} \end{pmatrix} & B_{j+m+n} \\ \begin{pmatrix} 0 & 0 \\ X_j & 0 \end{pmatrix} & (D_{j+n}, 0) \end{pmatrix}$$

$$= \begin{array}{c@{}l} \left[\begin{array}{ccccc} * & * & * & * & * \\ \begin{pmatrix} 0 \\ A_{l+\mu} \end{pmatrix} & * & * & B_{l+\mu+\nu} & * \\ * & * & * & * & * \\ * & * & * & * & * \\ \begin{pmatrix} 0 & 0 \\ X_l & 0 \end{pmatrix} & * & * & (D_{l+\nu}, 0) & * \\ * & * & * & * & * \end{array}\right] & \begin{array}{l} \} \, (n-\nu)p \\ \} \, (l+\mu+\nu+1)p \\ \} \, (j+m-l-\mu)p \\ \} \, (n-\nu)q \\ \} \, (l+\nu+1)q \\ \} \, (j-l)q \end{array} \\ \begin{array}{ccccc} \underbrace{\qquad}_{(l+\mu+1)r} & \underbrace{\quad}_{(j+m-l-\mu)r} & \underbrace{\quad}_{(n-\nu)s} & \underbrace{\qquad}_{(l+\mu+\nu+1)s} & \underbrace{\quad}_{(j+m-l-\mu)s} \end{array} & \end{array}$$

Analogously, in the other cases one can also show that $G_{l\mu\nu}$ is a submatrix of G_{jmn} ∎

Lemma 5: *Let $k \in \mathbf{N}_0$, and let $a_j \in \mathfrak{M}_{p\times r}$, $b_j \in \mathfrak{M}_{p\times s}$, $x_j \in \mathfrak{M}_{q\times r}$, $d_j \in \mathfrak{M}_{q\times s}$, $j \in \{0, 1, \ldots, k\}$. Further, let H_k be defined by* (12). *Then* (15) *is fulfilled iff* (17) *holds for* $j \in \{-1, 0, \ldots, k\}$, $m \in \{0, 1, \ldots, k-j\}$ *and* $n \in \{\max(0, -j-m), \ldots, k-j-m\}$.

Proof: First suppose that (15) is satisfied. From $H_k = G_{k00}$, $0 \leqq j+m+n \leqq k$, and Lemma 4 we see that (15) implies

$$G_{j+m+n,0,0} G^*_{j+m+n,0,0} \leqq I_{(j+m+n+1)(p+q)}. \tag{18}$$

Now consider the case $k \geqq 2$, $j > -1$ and $m > 0$. Then

$$G_{j+m+n,0,0} = \begin{pmatrix} A_{j+m+n} & B_{j+m+n} \\ X_{j+m+n} & D_{j+m+n} \end{pmatrix}$$

$$= \begin{array}{c@{}l} \left(\begin{array}{cc} \begin{pmatrix} * & 0 \\ * & A_{j+m} \end{pmatrix} & B_{j+m+n} \\ \begin{pmatrix} * & \begin{pmatrix} 0 & 0 \\ X_j & 0 \end{pmatrix} \\ * & * \end{pmatrix} & \begin{pmatrix} (D_{j+n} & 0) \\ & * \end{pmatrix} \end{array}\right) & \}\, mq \\ \underbrace{\;\;}_{nr} \qquad\qquad\qquad\qquad & \end{array},$$

i.e.,

$$G_{j+m+n,0,0} = \begin{array}{c@{}l} \begin{pmatrix} * & G_{jmn} \\ * & * \end{pmatrix} & \}\, mq \\ \underbrace{\;}_{nr} \qquad & \end{array} \tag{19}$$

and thus, in view of (18), the inequality (17) is satisfied. Analogously, in all other cases one can obtain the block partition (19) and, consequently, (17). Conversely, for $j = k$, $m = n = 0$ the condition (17) coincides with (15) ∎

Remark 1: Let $j, m \in \mathbf{N}_0$, and let $b_0, b_1, \ldots, b_{j+2m} \in \mathfrak{M}_{p \times s}$, $d_0, d_1, \ldots, d_{j+m} \in \mathfrak{M}_{q \times s}$. If $j + m > 0$, then let $a_0, a_1, \ldots, a_{j+m-1} \in \mathfrak{M}_{p \times r}$. If $j > 0$, then let $x_0, x_1, \ldots, x_{j-1} \in \mathfrak{M}_{q \times r}$. Then

$$G_{j-1,m,m+1} = \begin{pmatrix} \mathcal{S}_{j-2,m+1} \\ e_{j-1,m} \end{pmatrix} \tag{20}$$

where

$$\mathcal{S}_{j-2,m+1} = \begin{cases} B_0, & j = m = 0, \\ \begin{pmatrix} \begin{pmatrix} 0 \\ A_{m-1} \end{pmatrix} & B_{2m} \\ 0 & (D_{m-1}, 0) \end{pmatrix}, & j = 0,\ m > 0, \\ G_{j-2,m+1,m+1}, & j > 0,\ m \geqq 0; \end{cases} \tag{21}$$

$$e_{j-1,m} = \begin{cases} D_0, & j = m = 0, \\ (\Xi_{j-1,m}, \Delta_{j-1,m}), & j + m > 0; \end{cases} \tag{22}$$

$$\Xi_{j-1,m} = \begin{cases} (x_{j-1}, x_{j-2}, \ldots, x_0), & j > 0,\ m = 0, \\ 0 \in \mathfrak{M}_{q \times mr}, & j = 0,\ m > 0, \\ (x_{j-1}, x_{j-2}, \ldots, x_0, 0) \in \mathfrak{M}_{q \times (j+m)r}, & j > 0,\ m > 0; \end{cases} \tag{23}$$

$$\Delta_{j-1,m} = \begin{cases} (d_j, d_{j-1}, \ldots, d_0), & m = 0, \\ (d_{j+m}, d_{j+m-1}, \ldots, d_0, 0) \in \mathfrak{M}_{q \times (j+2m+1)s}, & m > 0. \end{cases} \tag{24}$$

If

$$G_{j-1,m,m+1} G^*_{j-1,m,m+1} \leqq I_{(j+m+1)(p+q)+mp}, \tag{25}$$

then, in view of [6: Theorem 5], and (20) the matrix

$$L_{jm} = I_q - e_{j-1,m}(I_{(j+m)(r+s)+(m+1)s} - \mathcal{S}^*_{j-2,m+1} \mathcal{S}^*_{j-2,m+1})^+ e^*_{j-1,m} \tag{26}$$

is non-negative Hermitian.

Remark 2: Let $j, m \in \mathbf{N}_0$, and let $a_0, a_1, \ldots, a_{j+m} \in \mathfrak{M}_{p \times r}$, $b_0, b_1, \ldots, b_{j+2m} \in \mathfrak{M}_{p \times s}$. If $j + m > 0$, then let $d_0, d_1, \ldots, d_{j+m-1} \in \mathfrak{M}_{q \times s}$. If $j > 0$, then let $x_0, x_1, \ldots, x_{j-1} \in \mathfrak{M}_{q \times r}$. Then we have the block partition

$$G_{j-1,m+1,m} = (f_{j-1,m}, \mathcal{S}_{j-2,m+1}) \tag{27}$$

where $\mathcal{S}_{j-2,m+1}$ is defined by (21) and

$$f_{j-1,m} = \begin{cases} a_0, & j = m = 0, \\ \begin{pmatrix} \varphi_{j-1,m} \\ \psi_{j-1,m} \end{pmatrix} & j + m > 0, \end{cases} \qquad \varphi_{j-1,0} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \end{pmatrix}, \quad \psi_{j-1,0} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{j-1} \end{pmatrix}, \quad j > 0,$$

and in the case $m > 0$.

$$\varphi_{j-1,m} = \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ \vdots \\ a_{j+m} \end{pmatrix} \in \mathfrak{M}_{(j+2m+1)p \times r}, \quad \psi_{j-1,m} = \begin{cases} 0 \quad \in \mathfrak{M}_{mq \times r}, & j = 0, \\ \begin{pmatrix} 0 \\ x_0 \\ x_1 \\ \vdots \\ x_{j-1} \end{pmatrix} \in \mathfrak{M}_{(j+m)q \times r}, & j > 0. \end{cases}$$

If

$$G_{j-1,m+1,m} G^*_{j-1,m+1,m} \leqq I_{(j+m)(p+q)+(m+1)p}, \tag{28}$$

then, in view of [6: Theorem 5], the matrix

$$R_{jm} = I_r - f^*_{j-1,m}(I_{(j+m)(p+q)+(m+1)p} - \mathcal{S}_{j-2,m+1} \mathcal{S}^*_{j-2,m+1})^+ f_{j-1,m} \tag{29}$$

is non-negative Hermitian.

Lemma 6: *Let $j, m \in \mathbf{N}_0$, and let $a_0, a_1, \ldots, a_{j+m} \in \mathfrak{M}_{p\times r}$, $b_0, b_1, \ldots, b_{j+2m} \in \mathfrak{M}_{p\times s}$, $x_0, x_1, \ldots, x_j \in \mathfrak{M}_{q\times s}$ and $d_0, d_1, \ldots, d_{j+m} \in \mathfrak{M}_{q\times s}$. Let $\mathscr{G}_{jm}$ be defined by* (21) *and*

$$\mathscr{G}_{j0} := H_j, \tag{30}$$

respectively, where H_j is given by (12). *Then*

$$\mathscr{G}_{jm}\mathscr{G}_{jm}^* \leqq I_{(j+m+1)(p+q)+mp} \tag{31}$$

if and only if (25), (28) *and*

$$x_j \in \mathfrak{K}_{jm} := \mathfrak{K}(M_{jm}; \sqrt{L_{jm}}, \sqrt{R_{jm}}) \tag{32}$$

are satisfied, where

$$M_{jm} = -e_{j-1,m}(I_{(j+m)(r+s)+(m+1)s} - \mathscr{G}_{j-2,m+1}^*\mathscr{G}_{j-2,m+1})^+ \mathscr{G}_{j-2,m+1}^* f_{j-1,m}. \tag{33}$$

Proof: For $j, m \in \mathbf{N}_0$ we have

$$\mathscr{G}_{jm} = \begin{pmatrix} f_{j-1,m} & \mathscr{G}_{j-2,m+1} \\ x_j & e_{j-1,m} \end{pmatrix}.$$

In view of Remarks 1, 2 the application of [6: Theorem 5] completes the proof ∎

Lemma 7: *Let $j \in \mathbf{N}_0$ and $m \in \mathbf{N}$. Assume $a_0, a_1, \ldots, a_{j+m} \in \mathfrak{M}_{p\times r}$, $b_0, b_1, \ldots, b_{j+2m} \in \mathfrak{M}_{p\times s}$ and $d_0, d_1, \ldots, d_{j+m} \in \mathfrak{M}_{q\times s}$. Furthermore, if $j > 0$, then let $x_0, x_1, \ldots, x_{j-1} \in \mathfrak{M}_{q\times r}$. Suppose that* (25) *and* (28) *are fulfilled. Then $\mathfrak{K}_{jm} \subseteqq \mathfrak{K}_{j,m-1}$.*

Proof: Assume $x \in \mathfrak{K}_{jm}$. Setting $x_j := x$ from (25), (28) and Lemma 6 we obtain (31). Because of $\mathscr{G}_{jm} = G_{jmm}$ then Lemma 4 provides $\mathscr{G}_{j,m-1}\mathscr{G}_{j,m-1}^* = G_{j,m-1,m-1}G_{j,m-1,m-1}^* \leqq I_{(j+m)(p+q)+(m-1)p}$. In view of Lemma 6 thus we have $x = x_j \in \mathfrak{K}_{j,m-1}$ ∎

Lemma 8: *Let $j \in \mathbf{N}_0$ and $m \in \mathbf{N}$. Assume $a_0, a_1, \ldots, a_{j+m} \in \mathfrak{M}_{p\times r}$, $b_0, b_1, \ldots, b_{j+2m} \in \mathfrak{M}_{p\times s}$ and $d_0, d_1, \ldots, d_{j+m} \in \mathfrak{M}_{q\times s}$. Furthermore, if $j > 0$, then let $x_0, x_1, \ldots, x_{j-1} \in \mathfrak{M}_{q\times s}$. Then*

a) (25) *implies* $0 \leqq L_{jm} \leqq L_{j,m-1}$, b) (28) *implies* $0 \leqq R_{jm} \leqq R_{j,m-1}$.

Proof: First assume $j \geqq 2$. From (21) then we have

$$\mathscr{G}_{j-2,m+1} = \left(\begin{array}{cccc} * & * & * & * \\ \begin{pmatrix}0\\ A_{j+m-1}\end{pmatrix} & * & B_{j+2m-2} & * \\ * & * & * & * \\ \underbrace{\begin{pmatrix}0 & 0\\ X_{j-2} & 0\end{pmatrix}}_{(j+m-1)r} & \underbrace{*}_{r+s} & \underbrace{(D_{j+m-2}, 0)}_{(j+2m-1)s} & \underbrace{*}_{s} \end{array}\right) \begin{array}{l} \}\, p \\ \}\, (j+2m-1)p \\ \}\, p+q \\ \}\, (j+m-1)q \end{array} \tag{34}$$

and

$$\mathscr{G}_{j-2,m} = \begin{pmatrix} \begin{pmatrix}0\\ A_{j+m-1}\end{pmatrix} & B_{j+2m-2} \\ \begin{pmatrix}0 & 0\\ X_{j-2} & 0\end{pmatrix} & (D_{j+m-2}, 0) \end{pmatrix}. \tag{35}$$

In view of (22)—(24),

$$e_{j-1,m} = (\underbrace{\Xi_{j-1,m-1}}_{(j+m-1)r}, \underbrace{*}_{r+s}, \underbrace{\Delta_{j-1,m-1}}_{(j+2m-1)s}, \underbrace{*}_{s}), \quad e_{j-1,m-1} = (\Xi_{j-1,m-1}, \Delta_{j-1,m-1}). \tag{36}$$

Defining the unitary matrices

$$U_{jm} = \begin{pmatrix} 0 & I_{p+q} \\ I_{(j+2m-1)p} & 0 \end{pmatrix}, \qquad V_{jm} = \begin{pmatrix} 0 & I_{r+s} \\ I_{(j+2m-1)s} & 0 \end{pmatrix}, \tag{37}$$

$$\tilde{U}_{jm} = \operatorname{diag}(I_p, U_{jm}, I_{(j+m-1)q}), \qquad \mathcal{V}_{jm} = \operatorname{diag}(I_{(j+m-1)r}, V'_{jm}, I_s) \tag{38}$$

and

$$\mathcal{U}_{jm} = \operatorname{diag}(\tilde{U}_{jm}, I_q) \tag{39}$$

from (20), (34)—(36) we see that the matrix

$$K_{jm} = \mathcal{U}_{jm} G_{j-1,m,m+1} \mathcal{V}_{jm} \tag{40}$$

has the block partition

$$K_{jm} = (C_{kl})_{\substack{k=1,2,3 \\ l=1,2}} \tag{41}$$

with

$$C_{21} = \mathcal{S}_{j-2,m}, \qquad C_{31} = e_{j-1,m-1}. \tag{42}$$

Moreover, we have

$$\mathcal{S}_{-2,m-1} = \tilde{U}^*_{jm} \underline{C} \mathcal{V}^*_{jm}, \qquad e_{j-1,m} = c\mathcal{V}^*_{jm} \tag{43}$$

where $\underline{C}$ and $\underline{c}$ are defined in (4). Analogously, in the cases $j = 0, 1$ one can prove that the matrix K_{jm} defined by (40) fulfils (41)—(43). (In the case $j = 0$ and $m = 1$, (38) must be replaced by $\tilde{U}_{01} = \operatorname{diag}(I_p, U_{01})$, $\mathcal{V}_{01} = \operatorname{diag}(V_{01}, I_s)$.) From (25) and (40) we get $K_{jm}K^*_{jm} \leqq I_{(j+m+1)(p+q)+mp}$. Applying Lemma 2 with $C = K_{jm}$ from (42) and (4) we obtain

$$0 \leqq \underline{L} := I_q - c(I_{(j+m)(r+s)+(m+1)s} - \underline{C}^*\underline{C})^+ c^* \leqq L_{j,m-1}. \tag{44}$$

In view of (43) and the fact that $(UEU^*)^+ = UE^+U^*$ holds for each $E \in \mathfrak{M}_{p\times p}$ and each unitary $p \times p$ matrix U we can conclude

$$\begin{aligned} \underline{L} &= I_q - c[\mathcal{V}^*_{jm}(I_{(j+m)(r+s)+(m+1)s} - \mathcal{V}_{jm}\underline{C}^*\tilde{U}_{jm}\tilde{U}^*_{jm}\underline{C}\mathcal{V}^*_{jm})\,\mathcal{V}_{jm}]^+ c^* \\ &= I_q - c\mathcal{V}^*_{jm}[I_{(j+m)(r+s)+(m+1)s} - (\tilde{U}^*_{jm}\underline{C}\mathcal{V}^*_{jm})^*\,(\tilde{U}^*_{jm}\underline{C}\mathcal{V}^*_{jm})]^+\,\mathcal{V}_{jm}c^* \\ &= I_q - e_{j-1,m}(I_{(j+m)(r+s)+(m+1)s} - \mathcal{S}^*_{j-2,m+1}\mathcal{S}_{j-2,m+1})^+\,e^*_{j-1,m} \\ &= L_{jm}. \end{aligned}$$

Thus, because of (44) part a) is proved.

To verify part b) first one shows that

$$\tilde{U}_{jm} G_{j-1,m+1,m} \tilde{V}_{jm} = \begin{pmatrix} * & * & * \\ f_{j-1,m-1} & \mathcal{S}_{j-2,m} & * \end{pmatrix}$$

holds where $\tilde{V}'_{jm} = \operatorname{diag}(I_r, \mathcal{V}_{jm})$. Applying Lemma 3 then one can obtain $0 \leqq R_{jm} \leqq R_{j,m-1}$ analogously to part a). We omit the details ∎

Now we are able to formulate an answer to part (S1) of Problem (S).

Theorem 1: *Let* $\alpha\colon \mathbf{D} \to \mathfrak{M}_{p\times r}$, $\beta\colon \mathbf{D} \to \mathfrak{M}_{p\times s}$ *and* $\delta\colon \mathbf{D} \to \mathfrak{M}_{q\times s}$ *be matrix functions holomorphic in* $\mathbf{D}$. *Denote* (6)—(8) *the Taylor series of* α, β *and* δ, *respectively. Then:*

a) *There exists a matrix function* $\xi\colon \mathbf{D} \to \mathfrak{M}_{q\times r}$ *such that*

$$f = \begin{pmatrix} \alpha & \beta \\ \xi & \delta \end{pmatrix} \tag{45}$$

belongs to the Schur class $(p+q) \times (r+s)$-$\mathfrak{S}$ *if and only if*

$$\mathcal{S}_{-1,m}\mathcal{S}^*_{-1,m} \leqq I_{m(2p+q)}, \quad m \in \mathbf{N}, \tag{46}$$

where $\mathcal{S}_{-1,m}$ *is defined by* (16) *and* (21).

b) *Let* $\xi\colon \mathbf{D} \to \mathfrak{M}_{q\times r}$ *be a matrix function such that* $f \in (p+q) \times (r+s)$-$\mathfrak{S}$. *Then* ξ *is holomorphic in* $\mathbf{D}$. *Denote* (9) *the Taylor series of* ξ. *Then for* $j \in \mathbf{N}_0$ *there exist the limits*

$$M_j = \lim_{m\to\infty} M_{jm}, \qquad L_j = \lim_{m\to\infty} L_{jm}, \qquad R_j = \lim_{m\to\infty} R_{jm} \tag{47}$$

where M_{jm}, L_{jm} *and* R_{jm} *are defined by* (33), (26) *and* (29), *respectively. For* $j \in \mathbf{N}_0$ *the matrices* L_j *and* R_j *are non-negative Hermitian and, moreover,* $x_j \in \mathfrak{R}(M_j; \sqrt{L_j}, \sqrt{R_j})$.

c) *Assuming* (46) *the following procedure for the successive determination of the coefficients* x_j *in the Taylor series* (9) *yields all matrix functions* $\xi\colon \mathbf{D} \to \mathfrak{M}_{q\times r}$ *such that* $f \in (p+q) \times (r+s)$-$\mathfrak{S}$:

Step j: *If* $j > 0$, *then assume that* $x_0, x_1, \ldots, x_{j-1}$ *are already determined. Then there exist the limits* (47), *where* L_j *and* R_j *are non-negative Hermitian. Choose* $x_j \in \mathfrak{R}(M_j; \sqrt{L_j}, \sqrt{R_j})$.

Proof: First suppose that there exists a matrix function ξ such that $f \in (p+q) \times (r+s)$-$\mathfrak{S}$. Then ξ is holomorphic in $\mathbf{D}$. Denote (9) the Taylor series of ξ. As we showed above, then (15) is fulfilled for $k \in \mathbf{N}_0$, where H_k is defined by (12). Therefore, according to Lemma 5 we get

$$\mathcal{S}_{j-1,m}\mathcal{S}^*_{j-1,m} \leqq I_{(j+m)(p+q)+mp} \tag{48}$$

for all $j, m \in \mathbf{N}_0$ with $j + m > 0$. In particular, we have (46). Furthermore, from Lemma 6 we see that (25), (28) and (32) hold for $j, m \in \mathbf{N}_0$. Hence,

$$x_j \in \bigcap_{m=0}^{\infty} \mathfrak{R}_{jm}, \quad j \in \mathbf{N}_0. \tag{49}$$

Let $j \in \mathbf{N}_0$. Then Lemma 7 shows that

$$\mathfrak{R}_{j,m+1} \subseteqq \mathfrak{R}_{jm} \tag{50}$$

holds. Moreover, Lemma 8 provides

$$0 \leqq L_{j,m+1} \leqq L_{jm}, \qquad 0 \leqq R_{j,m+1} \leqq R_{jm}, \qquad m \in \mathbf{N}_0.$$

Consequently, in view of Lemma 1, there exist the limits M_j, L_j and R_j and

$$\bigcap_{m=0}^{\infty} \mathfrak{R}_{jm} = \mathfrak{R}(M_j; \sqrt{L_j}, \sqrt{R_j}). \tag{51}$$

Thus, part b) is proved. Now we assume (46) and show that the procedure given in c) yields a matrix function ξ with $f \in (p+q) \times (r+s)$-$\mathfrak{S}$. Let $j \in \mathbf{N}_0$. If $j > 0$, we suppose that $x_0, x_1, \ldots, x_{j-1}$ are already chosen in the prescribed way. Additionally, we assume (48) for $m \in \mathbf{N}$. (In the case $j = 0$ these conditions coincide with (46).) Thus, Lemma 4 provides (25) and (28) for $m \in \mathbf{N}_0$. Hence, from Lemma 7 we have (50). From Lemma 8 we can conclude that $(L_{jm})_{m\in\mathbf{N}_0}$ and $(R_{jm})_{m\in\mathbf{N}_0}$ are non-increasing sequences of non-negative Hermitian $q \times q$ and $r \times r$ matrices, respectively. Therefore, according to Lemma 1 there exist the limits (47) where L_j and R_j are non-negative Hermitian and, moreover, (51) follows. Consequently, we can choose $x_j \in \mathfrak{R}(M_j; \sqrt{L_j}, \sqrt{R_j})$. Clearly, (32) holds for $m \in \mathbf{N}_0$. Using Lemma 6 from (25), (28) and (32) we obtain (31) for $m \in \mathbf{N}_0$. Therefore, the procedure given in c) can always be carried out and (31) is ensured for $j, m \in \mathbf{N}_0$. In particular, in view of (30), the inequality (15) is fulfilled for $k \in \mathbf{N}_0$. Hence, in order to prove that $f \in (p+q) \times (r+s)$-$\mathfrak{S}$ it remains to show that the series (9) converges for $w \in \mathbf{D}$. For $j \in \mathbf{N}_0$ the matrix x_j is a submatrix of H_j and, consequently, satisfies $x_j x_j^* \leqq I_q$. Thus, the sequence $(x_j)_{j \in \mathbf{N}_0}$ is bounded and (9) defines a matrix function ξ which is holomorphic in $\mathbf{D}$. Therefore, $f \in (p+q) \times (r+s)$-$\mathfrak{S}$ ∎

A slight modification of the last proof shows that the following theorem holds which gives an answer to part (S2) of Problem (S).

Theorem 2: *Let $\alpha: \mathbf{D} \to \mathfrak{M}_{p \times r}$, $\beta: \mathbf{D} \to \mathfrak{M}_{p \times s}$ and $\delta: \mathbf{D} \to \mathfrak{M}_{q \times s}$ be matrix functions holomorphic in $\mathbf{D}$. Denote* (6)—(8) *the Taylor series of α, β and δ, respectively. Furthermore, let $k \in \mathbf{N}_0$, and let $x_0, x_1, \ldots, x_k \in \mathfrak{M}_{q \times r}$. Then:*

a) *There exists a (holomorphic) matrix function $\xi: \mathbf{D} \to \mathfrak{M}_{q \times r}$ such that f defined by* (45) *belongs to $(p+q) \times (r+s)$-$\mathfrak{S}$ and that $x_0, x_1, \ldots, x_k$ are just the first $k+1$ Taylor coefficients of ξ iff*

$$\mathscr{S}_{km} \mathscr{S}_{km}^* \leqq I_{(k+m+1)(p+q)+mp}, \qquad m \in \mathbf{N}_0. \tag{52}$$

b) *Suppose* (52). *Then for $j \in \{k+1, k+2, \ldots\}$ the successive determination of the Taylor coefficients x_j in* (9) *according to step j described in Theorem* 1 *yields all matrix functions $\xi: \mathbf{D} \to \mathfrak{M}_{q \times r}$ such that $f \in (p+q) \times (r+s)$-$\mathfrak{S}$ and that $x_0, x_1, \ldots, x_k$ are the first $k+1$ Taylor coefficients of ξ.*

Now we reformulate part a) of Theorem 1 in terms of a kernel of mixed Toeplitz-Hankel type (compare [10: Theorem 2]).

Theorem 3: *Let $\alpha: \mathbf{D} \to \mathfrak{M}_{p \times r}$, $\beta: \mathbf{D} \to \mathfrak{M}_{p \times s}$ and $\delta: \mathbf{D} \to \mathfrak{M}_{q \times s}$ be matrix functions holomorphic in $\mathbf{D}$. Denote* (6)—(8) *the Taylor series of α, β and δ, respectively. Define $P: \mathbf{N}_0 \times \mathbf{N}_0 \to \mathfrak{M}_{(2p+q+r+2s) \times (2p+q+r+2s)}$ by*

$$P(m, n) = \begin{bmatrix} \begin{pmatrix} I_r & 0 & a_{m-n}^* \\ 0 & I_s & b_{m-n}^* \\ a_{m-n} & b_{m-n} & I_p \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{m+n+1} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & b_{m+n+1}^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} I_s & b_{n-m}^* & d_{n-m}^* \\ b_{n-m} & I_p & 0 \\ d_{n-m} & 0 & I_q \end{pmatrix} \end{bmatrix},$$

where $a_{-j} = 0 \in \mathfrak{M}_{p \times r}$, $b_{-j} = 0 \in \mathfrak{M}_{p \times s}$, $d_{-j} = 0 \in \mathfrak{M}_{q \times s}$, $j \in \mathbf{N}$. Then the following statements are equivalent:

(i) *There exists a matrix function $\xi: \mathbf{D} \to \mathfrak{M}_{q \times r}$ such that f defined by* (45) *belongs to $(p+q) \times (r+s)$-$\mathfrak{S}$.*

(ii) *The kernel P is positive definite.*

Proof: Let $l \in \mathbf{N}_0$. Then the block matrix $\big(P(m, n)\big)_{m,n=0,\ldots,l}$ arises by simultaneous interchanging of rows and columns from

$$\begin{pmatrix} (I_{(l+1)(r+2s)} & \mathscr{S}_{-1,l+1}^* \\ \mathscr{S}_{-1,l+1} & I_{(l+1)(2p+q)} \end{pmatrix}.$$

The application of part a) of Theorem 1 completes the proof ∎

Finally, it should be mentioned that some kernels of mixed Toeplitz-Hankel type play a key role in Arocena and Cotlar [2] (compare also the references therein).

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