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The Bergman and Szegö Kernels for Separately Monogenic Functions

D. CONSTALES

Es werden die Existenz des Bergman- und Szegö-Kernes für separiert monogene Funktionen in der Einheitskugel bewiesen und explizite Reihenentwicklungen für diese angegeben.

Доказывается существование ядер Бергмана и Сеге для отдельно-моногенных функций в единичном шаре и приводится их явное разложение в ряд.

The existence of the Bergman and Szegö kernels for separately monogenic functions in the unit ball is proved and explicit series expansions are obtained for them.

1. Preliminaries. Let $n \in \mathbf{N}$ and let

$$\mathbf{R}^n = \mathbf{R}^{d_1} \oplus \cdots \oplus \mathbf{R}^{d_p} \tag{1.1}$$

be an orthogonal decomposition, where the d_i are integers larger than 2. If $\mathbf{A}$ is the Clifford algebra constructed over a real quadratic n-dimensional vector space, we can generate it from an orthonormal basis $(e_1{}^1, \ldots, e^1_{d_1}, e_1{}^2, \ldots, e^2_{d_2}, \ldots, e_1{}^p, \ldots, e^p_{d_p})$ of $\mathbf{R}^n$, where each set $(e_1{}^i, \ldots, e^i_{d_i})$ is an orthonormal basis of a term in (1.1). To do so, we define the geometric product by the relation $e_j{}^i e_m{}^k + e_m{}^k e_j{}^i = -2\delta_{ik}\delta_{jm}$. Then $\mathbf{A}$ is a 2^n-dimensional real associative algebra with a basis consisting of all products of different basis vectors $e_j{}^i$, taken in some fixed order. A trace $\mathbf{A} \to \mathbf{R}: \alpha \to \langle\alpha\rangle_0$ is defined by taking the coefficient of the empty product in the linear combination of basis elements that represents α. An involution $\alpha \to \bar{\alpha}$ is defined on $\mathbf{A}$ by the axioms $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$, $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$, $\overline{e_j{}^i} = -e_j{}^i$, $\bar{\lambda} = \lambda$ if $\lambda \in \mathbf{R}$. A norm in $\mathbf{A}$ is given by $\|\alpha\|_0{}^2 = 2^n\langle\alpha\bar{\alpha}\rangle_0$; we will often write $|\alpha|$ for $\sqrt{\langle\alpha\bar{\alpha}\rangle_0}$. The point $x = (x_1{}^1, \ldots, x^1_{d_1}, x_1{}^2, \ldots, x^p_{d_p}) \in \mathbf{R}^n$ is identified with $\sum_{i=1}^{p} x^i$, where $x^i = \sum_{j=1}^{d_i} x_j{}^i e_j{}^i$, so $\mathbf{R}^n$ can be viewed as a subspace of $\mathbf{A}$.

The classical Cauchy-Riemann operator $\partial/\partial\bar{z}_i$ is generalized to $D^i = \sum_{j=1}^{d_i} e_j{}^i \partial/\partial x_j{}^i$ and we also write D for $\sum_{i=1}^{p} D^i$. A function $f \in C^1(\Omega, \mathbf{A})$, $\Omega \subseteq \mathbf{R}^n$ being open, is called *monogenic* if $Df = 0$ holds in Ω and *separately monogenic* if $D^i f = 0$ in Ω for $i = 1, \ldots, p$. Monogenic functions are a generalization of holomorphic functions of one complex variable; separately monogenic functions generalize holomorphic functions of several complex variables. We refer to [2] for monogenic and to [3] for separately monogenic functions.

Function spaces having a reproducing kernel are of great relevance to function theory. Certainly one of the most important examples of them is given by the space of square integrable holomorphic functions in the unit ball of $\mathbf{C}^n$ (see [7]). We will

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examine the case of separately monogenic functions in the unit ball of $\mathbf{R}^n$. For the case of monogenic functions the reader is referred to [2]. Throughout this article we shall use the notations and the results of [3] for separately monogenic functions.

2. Existence of reproducing kernels. $ML_2(B_n)$ is the right Hilbert $\mathbf{A}$-module of all monogenic functions f defined in the unit ball B_n of $\mathbf{R}^n$ such that

$$\|f\|^2_{ML_2(B_n)} = \frac{1}{V_n} \int_{B_n} \|f\|_0^2 \, dV < \infty,$$

given the inner product $\langle f, g\rangle = (1/V_n) \int_{B_n} \bar{f} g \, dV$, where V_n is the Lebesgue measure of B_n. Similarly, $ML_2(\partial B_n)$ is the right Hilbert-$\mathbf{A}$-module of all monogenic functions defined in B_n such that

$$\|f\|^2_{ML_2(\partial B_n)} = \lim_{r\to 1} \frac{1}{\omega_n} \int_{S^{n-1}} \|f(ru)\|_0^2 \, dS_u < \infty,$$

with inner product $\langle f, g\rangle = \lim_{r\to 1} \frac{1}{\omega_n} \int_{S^{n-1}} (\bar{f} g)(ru) \, dS_u$, ω_n being the area of S^{n-1}.

Assuming a given decomposition (1.1) such that $d_i > 2$ for $i = 1, \ldots, p$, we may consider the space $SM(B_n)$ of separately monogenic functions defined in the unit ball.

Definition 1: The L_2 spaces associated to the separately monogenic functions in the unit ball B_n of $\mathbf{R}^n$ are $SML_2(B_n) = ML_2(B_n) \cap SM(B_n)$ and $SML_2(\partial B_n) = ML_2(\partial B_n) \cap SM(B_n)$.

The Szegö kernel for monogenic functions, i.e. the reproducing kernel for $ML_2(\partial B_n)$, is explicitly known to equal (see [2]) $S_{ML_2(\partial B_n)}(u, t) = (1 + ut)/|1 + ut|^n$. We can perform the canonical expansion

$$S_{ML_2(\partial B_n)}(u, t) = \sum_{k=0}^{\infty} P_{k,ML_2(\partial B_n)}(u, t), \tag{2.1}$$

where, for all k, $P_{k,ML_2(\partial B_n)}$ is a monogenic polynomial of degree k in u. By the orthogonality properties of spherical monogenics (see [2]) the polynomial $P_{k,ML_2(\partial B_n)}$ must be the reproducing kernel for the subspace $ML_{2,k}(\partial B_n)$ of homogeneous monogenic polynomials of degree k in $ML_2(B_n)$. In [4] $ML_2(B_n)$ and $ML_2(\partial B_n)$ were shown to be right Hilbert $\mathbf{A}$-modules with reproducing kernel, by establishing, for all $t \in B_n$, the existence of positive constants C_t, C_t' such that the estimates

$$\|f(t)\|_0 \leq C_t \, \|f\|_{ML_2(B_n)} \qquad \text{and} \qquad \|f(t)\|_0 \leq C_t' \, \|f\|_{ML_2(\partial B_n)} \tag{2.2}$$

hold for all monogenic functions defined in B_n. Furthermore, these constants can be chosen to depend smoothly on t, so $\sup_{t\in K} C_t$ and $\sup_{t\in K} C_t'$ are finite for all compact subsets K of B_n. By the very definition of $SML_2(B_n)$ and $SML_2(\partial B_n)$, similar inequalities hold for them; invoking [2] and Weierstrass' Theorem (see [3]) we get

Theorem 1: *The spaces $SML_2(B_n)$ and $SML_2(\partial B_n)$ are right Hilbert* $\mathbf{A}$*-modules with reproducing kernel.*

3. Series expansion of the Bergman and Szegö kernels. We now try to obtain a series expansion for the reproducing kernels. This leads us to define on $\mathbf{R}^n$ a coordinate system associated to the chosen decomposition, as follows. Take the usual polar coordinates on each $\mathbf{R}^{d_i}$ and call them r_i, u^i where u^i stands for the $(d_i - 1)$-dimensional spherical part. Then clearly the set $(r_1, u^1, \ldots, r_p, u^p)$ is a coordinate system on $\mathbf{R}^n$. The following lemmas group some computations required later on.

Lemma 2: *The surface average of* $\prod_{i=1}^{p} r_i^{2k_i}$ *is given by*

$$\frac{1}{\omega_n}\int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i}\, dS = \left(\prod_{i=1}^{p} (d_i/2)_{k_i}\right) \Big/ (n/2)_{k_1+\cdots+k_p},$$

where for $a \in \mathbf{R}$ *and* $k \in \mathbf{N}$, $(a)_k$ *stands for* $\prod_{j=1}^{k} (a + j - 1)$.

Proof: Consider $I = \int_{\mathbf{R}^n} \left(\prod_{i=1}^{p} r_i^{2k_i}\right) \exp(-r_1^2 - \cdots - r_p^2)\, dV$. As

$$I = \left(\int_0^\infty r^{n-1+2k_1+\cdots+2k_p} \exp(-r^2)\, dr\right) \left(\int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i}\, dS\right),$$

where $r = \sqrt{r_1^2 + \cdots + r_p^2}$ is the polar distance on $\mathbf{R}^n$, we find

$$I = 1/2\Gamma\left(n/2 + \sum_{i=1}^{p} k_i\right) \int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i}\, dS.$$

On the other hand,

$$I = \prod_{i=1}^{p} \left(\int_{\mathbf{R}^{d_i}} r_i^{2k_i} \exp(-r_i^2)\, dV_i\right)$$

$$= \prod_{i=1}^{p} \left(\int_0^\infty r_i^{2k_i+d_i-1} \exp(-r_i^2)\, dr_i \int_{S^{d_i-1}} dS_i\right) = \prod_{i=1}^{p} \frac{1}{2}\, \omega_{d_i} \Gamma\left(\frac{d_i}{2} + k_i\right)$$

whence

$$\frac{1}{\omega_n}\int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i} = \left(\prod_{i=1}^{p} \frac{\Gamma(d_i/2 + k_i)}{\Gamma(d_i/2)}\right) \Big/ \left(\frac{\Gamma(n/2 + \sum k_i)}{\Gamma(n/2)}\right) \blacksquare$$

In the sequel we will put

$$\gamma_{k_1,\ldots,k_p} = \frac{1}{\omega_n}\int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i}\, dS. \tag{3.1}$$

Lemma 3: *The volume average of* $\prod_{i=1}^{p} r_i^{2k_i}$ *is given by*

$$\frac{1}{V_n}\int_{B_n} \prod_{i=1}^{p} r_i^{2k_i}\, dV = \gamma_{k_1,\ldots,k_p} \Big/ \left(1 + \frac{2}{n} \sum k_i\right).$$

Proof: Using polar coordinates in $\mathbf{R}^n$, the integral equals

$$\left(\int_0^1 r^{n-1+2\Sigma k_i}\, dr\right) \left(\int_{S^{n-1}} \prod_{i=1}^{p} r_i^{2k_i}\, dS\right)$$

and the result follows from the previous lemma $\blacksquare$

Let us now consider the subspace $SML_{2,(k_1,\ldots,k_p)}(\partial B_n) \subseteq SML_2(\partial B_n)$ of separately monogenic polynomials homogeneous of degree k_i in the variables $x_1^i, \ldots, x_{d_i}^i$ for every $i = 1, \ldots, p$. This homogeneity and the orthogonality properties of spherical monogenics ensure that these spaces are pairwise orthogonal. By the Taylor expansion for separately monogenic functions (see [3]) the span of all the $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$ is the whole of $SML_2(\partial B_n)$. Hence $SML_2(\partial B_n)$ has an orthogonal decomposition as direct sum of the $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$. But on each $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$ the reproducing kernel can be computed as follows. Putting $P_{k_i}^i$ for the polynomials of the expansion (2.1) applied to $\mathbf{R}^{d_i}$, we obtain

Lemma 4: *The reproducing kernel of $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$ is*

$$P^S_{(k_1,\ldots,k_p)}(u,t) = \frac{1}{\gamma_{k_1,\ldots,k_p}} \prod_{i=1}^{p} r_i^{k_i} p_{k_i}^i(u^i, t^i).$$

Proof: Clearly $P^S_{(k_1,\ldots,k_p)} \in SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$. Any $f \in SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$ is the sum of terms $\left(\prod_{i=1}^{p} V^i(r_i u^i)\right)\lambda$, where the V_i are Fueter polynomials in $ML_{2,k_i}(\partial B_{d_i})$ (see [3]) and λ is a constant Clifford number. For such a term,

$$\left\langle \overline{P}_{(k_1,\ldots,k_p)}, \prod_{i=1}^{p} V_i(r_i u^i)\lambda \right\rangle$$

$$= \frac{1}{\gamma_{k_1,\ldots,k_p}\omega_n} \int_{S^{n-1}} \prod_{i=1}^{p} \left(r_i^{2k_i} \overline{P}_{k_i}^i(u^i, t^i)\, V^i(u^i)\right) \lambda\, dS$$

$$= \frac{1}{\gamma_{k_1,\ldots,k_p}\omega_n} \int_{S^{n-1}} \prod_{i=1}^{p} \left(r_i^{2k_i} \frac{1}{\omega_{d_i}} \int_{S^{d_i-1}} \overline{P}_{k_i}^i(u^i, t^i)\, V^i(u^i)\, dS_{u^i}\right) \lambda\, dS$$

$$= \frac{1}{\gamma_{k_1,\ldots,k_p}\omega_n} \int_{S^{n-1}} \left(\prod_{i=1}^{p} r_i^{2k_i} V^i(t^i)\right) \lambda\, dS = \left(\prod_{i=1}^{p} V^i(t^i)\right)\lambda,$$

where we have exploited the commutativity of P^i and V^j when $i \neq j$ and averaged over each of the u^i. This proves the reproducing property of $P^S_{(k_1,\ldots,k_p)}$ ∎

We now only have to sum these kernels to obtain the Szegö kernel for $SML_2(\partial B_n)$.

Theorem 5: *The Szegö kernel of $SML_2(\partial B_n)$ is given by*

$$S_{SML_2(\partial B_n)} = \sum_{k_1,\ldots,k_p=0}^{\infty} P^S_{(k_1,\ldots,k_p)}(u,t),$$

where the multiple series converges both in $SML_2(\partial B_n)$ and uniformly on the compact subsets of B_n.

Proof: We know $S_{SML_2(\partial B_n)}$ to exist; if we decompose it with respect to the $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$ we get a series converging in $SML_2(\partial B_n)$, for fixed t:

$$S_{SML_2(\partial B_n)}(u,t) = \sum_{k_1,\ldots,k_p=0}^{\infty} Q_{(k_1,\ldots,k_p)}(u,t)$$

in which $Q_{(k_1,\ldots,k_p)}$ must be the reproducing kernel for $SML_{2,(k_1,\ldots,k_p)}(\partial B_n)$. But this kernel is unique, so $Q_{(k_1,\ldots,k_p)}(u,t) = P^S_{(k_1,\ldots,k_p)}(u,t)$ almost everywhere, proving the series expansion in $SML_2(\partial B_n)$. Uniform convergence on compact subsets follows at once from the estimate (2.2) ∎

Merely replacing the constants $\gamma_{k_1,\ldots,k_p}$ by $\gamma_{k_1,\ldots,k_p}\Big/\left(1+\frac{2}{n}\sum_{i=1}^{p}k_i\right)$ in this section, we can prove a similar result for the Bergman kernel $B_{SML_2(B_n)}$, according to Lemma 3.

Theorem 6: *The Bergman kernel is given by*

$$B_{SML_2(B_n)}(u,t)=\sum_{k_1,\ldots,k_p=0}^{\infty}\left(1+\frac{2}{n}\sum_{i=1}^{p}k_i\right)P^{S}_{(k_1,\ldots,k_p)}(u,t),$$

the series converging both in $SML_2(B_n)$ and uniformly on all compact subsets of B_n.

These convergence properties allow us to prove the following result, where Γ_u is as in [2].

Corollary 7: *The Bergman and Szegö kernels for separately monogenic functions in the unit ball of $\mathbf{R}^n$ are related by $B_{SML_2(B_n)}(u,t)\equiv(1-2n^{-1}\Gamma_n)\,S_{SML_2(\partial B_n)}(u,t)$.*

4. Explicit formulas for the reproducing kernels. The Szegö kernel for monogenic functions in B_d is given by

$$S_{ML_2(\partial B_d)}(u,t)=(1+ut)/|1+ut|^d,\qquad u,t\in B_d.$$

Henceforth we shall write $v=ut$ and $C(v)=S_{ML_2(\partial B_d)}(u,t)$. Notice that $v=u\cdot t+u\wedge t$, where $u\wedge t$ is a bivector and $u\cdot t$ is real. (We refer to [6] for the definition of dot and wedge products in a Clifford algebra.) Furthermore, $v\bar v=\bar v v=|u|^2\,|t|^2$ and $|1+v|^2=(1+v)(1+\bar v)$. Let u and t be fixed. The subalgebra of $\mathbf{A}$ generated by $\mathbf{R}$ and $u\wedge t$ will be written $\mathbf{R}_{u\wedge t}$; from $(u\wedge t)^2\in\mathbf{R}$ it follows that $\mathbf{R}_{u\wedge t}=\mathbf{R}+(u\wedge t)\,\mathbf{R}$. Also, $v,\bar v\in\mathbf{R}_{u\wedge t}$. We define the mapping $\varphi\colon\mathbf{R}_{u\wedge t}\to\mathbf{C}\colon a+(u\wedge t)\,b\to a+i\,|u\wedge t|\,b$. (This makes sense even if $u\wedge t=0$.)

Lemma 8: *φ has the following properties: for all $v_1,v_2\in\mathbf{R}_{u\wedge t}$,*

$$\begin{aligned}&\varphi(v_1+v_2)=\varphi(v_1)+\varphi(v_2),\qquad \varphi(v_1v_2)=\varphi(v_1)\,\varphi(v_2),\\&\varphi(\lambda)=\lambda\text{ if }\lambda\in\mathbf{R},\qquad \varphi(\bar v_1)=\overline{\varphi(v_1)},\qquad |\varphi(v_1)|=|v_1|,\\&\varphi(v_1)=0\Leftrightarrow v_1=0.\end{aligned}\tag{4.1}$$

This φ will now be used to relate the reproducing kernels with their values for complex arguments.

Theorem 9: *Let $v=ut$, $u,t\in B_d$, then*

$$C(v)=\sum_{l,m=0}^{\infty}(d/2-1)_l\,(d/2)_m\,(-1)^{l+m}\,v^l\bar v^m/(l!m!),$$

the series converging absolutely.

Proof: Relying on the properties of φ obtained in Lemma 8, we have

$$\varphi\big(C(v)\big)=\frac{1+\varphi(v)}{|1+v|^d}=\frac{1+\varphi(v)}{|\varphi(1+v)|^d}=\frac{1+\varphi(v)}{|1+\varphi(v)|^d}.$$

From now on, let us write $z=\varphi(v)$, then $|z|<1$ and

$$\begin{aligned}\frac{1+z}{|1+z|^d}&=(1+z)^{1-d/2}\,(1+\bar z)^{-d/2}=\sum_{l,m=0}^{\infty}\left(\frac{d}{2}-1\right)_l\left(\frac{d}{2}\right)_m(-1)^{l+m}\frac{z^l\bar z^m}{l!m!}\\&=\sum_{l,m=0}^{\infty}\varphi\left(\left(\frac{d}{2}-1\right)_l\left(\frac{d}{2}\right)_m(-1)^{l+m}\frac{v^l\bar v^m}{l!m!}\right)\\&=\lim_{k\to\infty}\varphi\left(\sum_{l+m\le k}\left(\frac{d}{2}-1\right)_l\left(\frac{d}{2}\right)_m(-1)^{l+m}\frac{v^l\bar v^m}{l!m!}\right)\end{aligned}$$

$$= \varphi \left(\lim_{k\to\infty} \sum_{l+m\leq k} \left(\frac{d}{2} - 1\right)_l \left(\frac{d}{2}\right)_m (-1)^{l+m} \frac{v^l\bar{v}^m}{l!m!} \right)$$
$$= \varphi \left(\sum_{l,m=0}^{\infty} \left(\frac{d}{2} - 1\right)_l \left(\frac{d}{2}\right)_m (-1)^{l+m} \frac{v^l\bar{v}^m}{l!m!} \right),$$

where φ and the limit can be exchanged because of (4.1); this also ensures absolute convergence. The theorem then follows from the injectivity of φ ∎

Corollary 10: *The Szegö kernel can be expressed as a series:*

$$P_{k,ML_2(\partial B_n)}(u, t) = \sum_{l+m=k} (d/2 - 1)_l (d/2)_m (-1)^{l+m} \frac{v^l\bar{v}^m}{l!m!}.$$

Theorem 11: *Let* $F(v, \bar{v}) = \sum_{l,m=0}^{\infty} c_{l,m} v^l \bar{v}^m$, *where* $c_{l,m} \in \mathbf{R}$, *be absolutely convergent and let* $z = \varphi(v)$. *Then*

$$F(v, \bar{v}) = \begin{cases} F(z, \bar{z}), & \text{if } u \wedge t = 0, \\ \frac{1}{2}\left(1 - \frac{u\wedge t}{|u \wedge t|}\,\mathrm{i}\right) F(z, \bar{z}) + \frac{1}{2}\left(1 + \frac{u \wedge t}{|u \wedge t|}\,\mathrm{i}\right) F(\bar{z}, z) & \text{otherwise}. \end{cases}$$

Proof: If $u \wedge t = 0$, $v = \bar{v} = z = \bar{z}$ and the result is obvious. In the general case, notice that for all $v \in \mathbf{R}_{u\wedge t}$,

$$v = \left(\left(\varphi(v) + \overline{\varphi(v)}\right) - (u \wedge t/|u \wedge t|)\,\mathrm{i}\left(\varphi(v) - \overline{\varphi(v)}\right)\right)/2$$

and that $\varphi\left(F(v, \bar{v})\right) = F(z, \bar{z})$; $\overline{F(z, \bar{z})} = F(\bar{z}, z)$ ∎

We apply these results to the computation of the reproducing kernels. To start, we add indices corresponding to the $\mathbf{R}^{d_i}$, giving the notations $v_i = u_i t_i$, $u_i, v_i \in B_{d_i}$, $\varphi_i\colon \mathbf{R}_{u_i\wedge t_i} \to \mathbf{C}$, $z_i = \varphi_i(v_i)$; writing $\Phi = \varphi_1\varphi_2 \cdots \varphi_p$ and $S(v) = S_{SML_2(\partial B_n)}\left((u_1, \ldots, u_p), (t_1, \ldots, t_p)\right)$, we see that through repeated application of Theorem 11, explicit knowledge of $\Phi\left(S(v)\right)$ extends to $S(v)$.

Relying on Theorem 5, we see that

$$\Phi\left(S(v)\right) = \sum_{l_1,\ldots,l_p,m_1,\ldots,m_p=0}^{\infty} \left(\frac{n}{2}\right)_{\Sigma(l_i+m_i)} \prod_{i=1}^{p} \frac{(d_i/2 - 1)_{l_i} (d_i/2)_{m_i}}{(d_i/2)_{l_i+m_i}} (-1)^{l_i+m_i} \frac{z_i^{l_i}}{l_i!} \frac{\bar{z}_i^{m_i}}{m_i!}$$

converging absolutely in the set $|z_1| + \cdots + |z_p| < 1$ (cf. [5] for a general method to obtain the convergence domain of such a series). In the special case where all z_i are real, we obtain (through an elementary identity involving binomial coefficients)

$$\Phi\left(S(v)\right) = \sum_{k_1,\ldots,k_p=0}^{\infty} \left(\frac{n}{2}\right)_{k_1+\cdots+k_p} \prod_{i=1}^{p} \frac{(d_i - 1)_{k_i} (-z_i)^{k_i}}{(d_i/2)_{k_i} k_i!}$$
$$= F_A\left(n/2; (d_i - 1)_{i=0}^p; (d_i/2)_{i=0}^p; (-z_i)_{i=1}^p\right),$$

where F_A stands for the generalized hypergeometric function of Lauricella type

$$F_A\left(a; (b_i)_{i=1}^p; (c_i)_{i=1}^p; (z_i)_{i=1}^p\right) = \sum_{k_i=0}^{\infty} (a)_{k_1+\cdots+k_p} \prod_{i=1}^{p} \frac{(b_i)_{k_i} z_i^{k_i}}{(c_i)_{k_i} k_i!}.$$

This function has the interesting property

$$F_A\left(a; (b_i)_{i=1}^p; (c_i)_{i=1}^p; (z_i)_{i=1}^p\right)$$
$$= \left(1 - \sum_{j=1}^{p} z_j\right)^{-a} F\left(a; (c_i - b_i)_{i=1}^p; (c_i)_{i=1}^p; \left(-z_i \middle/ \left(1 - \sum_{j=1}^{p} z_j\right)\right)_{i=1}^{p}\right), \qquad (4.2)$$

so

$$\Phi(S(v)) = \left(1 + \sum_{j=1}^{p} z_j\right)^{-n/2} F_A\left(n/2; (1 - d_i/2)_{i=1}^p; (d_i/2)_{i=1}^p; \left(z_i \Big/ \left(1 + \sum_{j=1}^{p} z_j\right)\right)_{i=1}^p\right)$$

which, if all dimensions d_i are even, is a *rational* function of the z_i. We refer to [5] for a full treatment of these generalized hypergeometric functions. Relying on the properties of the beta function, we can now link the Szegö kernel for general arguments to this functions.

Theorem 12: *The Szegö kernel for $SML_2(\partial B_n)$ is given by*

$$\Phi(S(v)) = \frac{1}{\prod_{i=1}^{p} B(d_i/2, d_i/2 - 1)} \int_{t_1=0}^{1} \cdots \int_{t_p=0}^{1} \prod_{i=1}^{p} t_i^{d_i/2-2}(1 - t_i)^{d_i/2-1}$$

$$\times F_A\left(n/2; (d_i - 1)_{i=1}^p; (d_i/2)_{i=1}^p; \left(-t_1 z_i - (1 - t_1)\,\bar{z}_i^1\right)_{i=1}^p\right) dt_1 \cdots dt_p.$$

If all d_i are even, one can apply the identity (4.2) under the integral sign; the integral can then be computed explicitly, yielding an expression in terms of rational functions and logarithms. As a consequence of Theorem 6 and because

$$\left(1 + (2/n) \sum_{i=1}^{p} (l_i + m_i)\right) (n/2)_{\Sigma(l_i + m_i)} = (n/2 + 1)_{\Sigma(l_i + m_i)}$$

similar results for the Bergman kernel may be obtained by substituting $n + 2$ for n in the formulas of this section.

REFERENCES

[1] Aronszajn, N.: Theory of reproducing kernels. Trans. Amer. Math. Soc. 68 (1950), 337 to 404.

[2] Brackx, F., Delanghe, R., and F. Sommen: Clifford Analysis (Research Notes in Math. 76). London: Pitman 1982.

[3] Constales, D.: On separately monogenic functions. To appear.

[4] Delanghe, R., and F. Brackx: Hypercomplex function theory and Hilbert modules with reproducing kernel. Proc. London Math. Soc. 37 (1978), 545–576.

[5] Exton, H.: Multiple hypergeometric functions and applications. Chichester: Ellis Horwood 1976.

[6] Hestenes, D., and G. Sobczyk: Clifford Algebra to Geometric Calculus. A unified language for mathematics and physics. Dordrecht—Boston—Lancester: D. Reidel Publ. Comp. 1984.

[7] Rudin, W.: Function theory in the unit ball of $\mathbb{C}^n$. Berlin—Heidelberg—New York: Springer-Verlag 1980.

[8] Sommen, F.: Planes waves, biregular functions and hypercomplex Fourier analysis. Suppl. Rend. Circ. Math. Palermo (II) 9 (1985), 205–219.

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