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On the Existence of Solutions of a Semilinear Elliptic Boundary Value Problem with Superlinear Nonlinearities

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Wir beschäftigen uns mit der Lösbarkeit eines semilinearen Randwertproblems mit superlinearer Nichtlinearität in Räumen vom Besov- und Triebel-Lizorkin-Typ.

Мы исследуем разрешимость одной полулинейной эллиптической краевой задачи с суперлинейной нелинейностью в пространствах типа Бесова и Трибель-Лизоркина.

We study the solvability of a semilinear elliptic boundary value problem with superlinear nonlinearity in spaces of Besov and Triebel-Lizorkin type.

1. Introduction

The aim of this paper is the study of the following superlinear problem:

$$Lu = -g(u) + f(x) + t \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where L is a uniformly elliptic second order differential operator and $t \in \mathbf{R}_1$. Moreover, let g be a sufficiently smooth function defined on $\mathbf{R}_1$ and satisfying the following conditions:

(I) $g(x) \geqq 0$ if $x \geqq 0$ and $g(0) = 0$,
(II) The function $g(x) + \lambda_1 x$, where $\lambda_1 > 0$ is the first eigenvalue of L, is bounded from below.

We consider problems of this form in the framework of real Besov spaces $\tilde{B}^s_{p,q}(\Omega)$ and Triebel-Lizorkin spaces $\tilde{F}^s_{p,q}(\Omega)$ on bounded C^∞-domains $\Omega \subset \mathbf{R}_n$, where s is large enough. Roughly speaking, we shall prove that for each fixed f there exists a $t_0(f) \in \mathbf{R}_1$ such that

(1) has at least one solution if $t > t_0(f)$;

(1) has no solution if $t < t_0(f)$.

Furthermore, we are interested in non-negative solutions of (1). Problems of this type were considered by many authors (see e.g. H. Amann [1] and the references given there) and arise for example in the theory of nonlinear diffusion processes and in reactor theory. Questions of existence and smoothness of solutions of (1) are reduced to problems involving nonlinear mappings in the spaces considered here (§ 2.3) and a maximum principle (§ 2.2).

Nonlinear elliptic equations in the framework of Besov and Triebel-Lizorkin spaces were first considered by H. Triebel [17]. Further results in this direction may be found in D. E. Edmunds and H. Triebel [2], and J. Franke and T. Runst [5, 6].

2. Preliminaries

2.1 Spaces. Let $\mathbf{R}_n$ be the real Euclidean n-spaces. The theory of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ was developed in H. TRIEBEL [16]. We do not need the full theory, but only some properties, which we list in the sequel.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}_n$, S' the set of all tempered distributions on $\mathbf{R}_n$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse on $S'(\mathbf{R}_n)$, respectively. Now let $\varphi \in S$ be a real-valued function such that $\varphi(x) = \varphi(-x)$ if $x \in \operatorname{supp} \varphi \subset \{y \in \mathbf{R}_n \mid |y| \leqq 2\}$ and $\varphi(x) = 1$ if $|x| \leqq 1$. Then we define a sequence $\{\varphi_j\}_{j=0}^\infty$ of functions by

$$\varphi_0(x) = \varphi(x), \qquad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x) \qquad (j \geqq 1)$$

for each $x \in \mathbf{R}_n$. We have $\varphi_0(x) + \varphi_1(x) + \dots = 1$ $(x \in \mathbf{R}_n)$. If $-\infty < s < \infty$, $0 < p, q < \infty$, then by definition

$$B^s_{p,q}(\mathbf{R}_n) = \left\{f \in S' \,\middle|\, \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}\varphi_j\mathcal{F}f \mid L_p(\mathbf{R}_n)\|^q\right)^{1/q} < \infty\right\} \tag{2}$$

and

$$F^s_{p,q}(\mathbf{R}_n) = \left\{f \in S' \,\middle|\, \left\|\left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\cdot)|^q\right)^{1/q} \middle| L_p(\mathbf{R}_n)\right\| < \infty\right\} \tag{3}$$

(usual modification if $p = \infty$ and/or $q = \infty$).

It can be shown (cf. H. TRIEBEL [16]) that with the respective quasi-norms $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ are quasi-Banach spaces; moreover, they are independent of the particular choice of φ (equivalent quasi-norms).

Remark 1: By means of the fact that φ is a real-valued even function we can introduce the real part of the spaces $B^s_{p,q}(\mathbf{R}_n)$, etc., denoted by $\tilde{B}^s_{p,q}(\mathbf{R}_n)$, ... (for exact definitions see J. FRANKE and T. RUNST [5: Subsection 3.2]).

Remark 2: These two scales of function spaces include many well-known classical spaces. Equivalent quasi-norms for these spaces may be found in H. TRIEBEL [16], for instance $B^s_{\infty,\infty}(\mathbf{R}_n) = \mathscr{C}^s(\mathbf{R}_n)$ (Zygmund spaces) if $s > 0$.

Next we define the corresponding spaces on open sets. Let Ω be a bounded C^∞-domain with boundary $\partial\Omega$. Then one can introduce the spaces $B^s_{p,q}(\partial\Omega)$ and $F^s_{p,q}(\partial\Omega)$ by standard procedure via local charts, cf. H. TRIEBEL [16: Subsection 3.2.2]. The spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are defined as usual by the restriction method, cf. H. TRIEBEL [16: Subsection 3.2.2]: if $-\infty < s < \infty$ and $0 < p, q \leqq \infty$, then for instance

$$\begin{aligned} &B^s_{p,q}(\Omega) = \{f \in D'(\Omega) \mid \exists g \in B^s_{p,q}(\mathbf{R}_n) \text{ with } g \mid \Omega = f\}, \\ &\|f \mid B^s_{p,q}(\Omega)\| = \inf \|g \mid B^s_{p,q}(\mathbf{R}_n)\|, \end{aligned} \tag{4}$$

where the infimum is taken over all $g \in B^s_{p,q}(\mathbf{R}_n)$ in the sense of (4). Similarly one can define the spaces $F^s_{p,q}(\Omega)$.

2.2 Traces and linear elliptic differential operators. Let Ω be a bounded C^∞-domain in $\mathbf{R}_n$ and let f be a function defined in Ω and belonging to some function spaces of the above type. $\mathcal{R}$ denotes the restriction operator given by $\mathcal{R}f = f \mid \partial\Omega$. The following results are well-known for $0 < q \leqq \infty$ and $s > (n - 1)(1/\min(p, 1) - 1) + 1/p$ (for the proof see J. FRANKE [4] and H. TRIEBEL [16: Subsection 3.3.3]):

If $0 < p \leqq \infty$, then $\mathcal{R}$ is a linear and continuous mapping from $B^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,q}(\partial\Omega)$.

If $0 < p < \infty$, then $\mathcal{R}$, is a linear and continuous mapping from $F^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,p}(\partial\Omega)$.

As above, Ω denotes a bounded C^∞-domain in $\mathbf{R}_n$ with boundary $\partial\Omega$. Let L,

$$Lu(x) = -\sum_{|\alpha|=2} a_\alpha(x) D^\alpha u(x) + \sum_{|\alpha|=1} a_\alpha(x) D^\alpha u(x), \tag{5}$$

$x \in \Omega$, $a_\alpha \in \tilde{C}^\infty(\bar{\Omega})$ if $|\alpha| \leqq 2$ (i.e. the a_α are real C^∞-functions), be a second order uniformly elliptic operator:

$$\sum_{|\alpha|=2} a_\alpha(x) y^\alpha \geqq c\,|y|^2 > 0, \qquad y \in \mathbf{R}_n,\ y \neq 0.$$

In this paper we only consider the corresponding homogeneous Dirichlet problem.

We introduce the notations (for admissible couples (s, p))

$$B^{s+2}_{p,q,0}(\Omega) = \{f \in B^{s+2}_{p,q}(\Omega) \mid \ f|\partial\Omega = 0\}$$

and

$$F^{s+2}_{p,q,0}(\Omega) = \{f \in F^{s+2}_{p,q}(\Omega) \mid \ f|\partial\Omega = 0\}.$$

By S. Fučík [7: Theorem 34.10] we obtain the following result. If λ_1 denotes the smallest eigenvalue of $L\big|_{\bar{B}^1_{2,2,0}(\Omega)}$ with Dirichlet condition, then $\lambda_1 > 0$. Furthermore, there exists a unique normed positive eigenfunction $\varphi^* \in \tilde{C}^\infty(\bar{\Omega})$ to λ_1 with

$$\int_\Omega \varphi^*(x)^2\,dx = 1, \qquad \varphi^*(x) \geqq 0 \text{ if } x \in \bar{\Omega}, \tag{6}$$

$$\varphi^* \mid \partial\Omega = 0; \qquad L\varphi^* = \lambda_1\varphi^*$$

(cf. e.g. M. G. Krein and M. A. Rutman [9], M. A. Krasnoselski [8]). Then the following may be found in J. Franke [4] (see also H. Triebel [16: Subsection 3.3.3]) for $0 < q \leqq \infty$ and $s > (n-1)\big((1/\min(p,1) - 1)\big)_+ + 1/p - 2$:

$$\left.\begin{array}{l}\text{If } 0 < p \leqq \infty, \text{ then } L \text{ yields an isomorphic mapping}\\ \text{from } B^{s+2}_{p,q,0}(\Omega) \text{ onto } B^s_{p,q}(\Omega).\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l}\text{If } 0 < p < \infty \text{ then } L \text{ yields an isomorphic mapping}\\ \text{from } F^{s+2}_{p,q,0}(\Omega) \text{ onto } F^s_{p,q}(\Omega).\end{array}\right\} \tag{8}$$

In order to prove our main result we need some results about subsolutions and supersolutions.

Definition 1: A distribution $\psi \in \tilde{D}'(\Omega)$ is said to be *non-negative* if $\psi(\varphi) \geqq 0$ for any $\varphi \in \tilde{D}(\Omega)$ with $\varphi \geqq 0$.

Here $D(\Omega) = C_0^\infty(\Omega)$ denotes as usual the collection of all complex-valued infinitely differentiable functions f in $\mathbf{R}_n$ with supp $f \subset \Omega$, and $D'(\Omega)$ is the dual space. The set of non-negative distributions is $\sigma(D'(\Omega), D(\Omega))$-closed.

Definition 2: A function $u \in \tilde{C}(\bar{\Omega})$ is said to be a *subsolution* (*supersolution*) of (1) if $Lu \leqq -g(u) + f(x) + t$ $(Lu \geqq -g(u) + f(x) + t)$ in Ω in the above sense and $u \mid \partial\Omega = 0$.

Furthermore, $\mathring{B}^s_{p,q}(\Omega)$ denotes the completion of $D(\Omega)$ in $B^s_{p,q}(\Omega)$. In Section 3 we shall use the following

Lemma 1: *Let* $v \in \cup \{\tilde{B}^{\varepsilon}_{\infty,\infty}(\Omega): \varepsilon > 0\}$ *and* $\mu > 0$. *If* $v \mid \partial\Omega = 0$ *and* $Lv + \mu v \geqq 0$ *(in the sense of distributions), then* $v \geqq 0$.

Proof: *Step* 1. Let $w \in \tilde{B}^{\varepsilon-2}_{\infty,\infty}$, $0 < \varepsilon < 1$, be non-negative. If $\psi \in \tilde{C}^{\infty}(\bar{\Omega})$, $\psi \mid \partial\Omega = 0$, then $\psi \in \tilde{B}^{2-\varepsilon}_{1,1}(\Omega)$. We prove that if ψ is non-negative, then ψ can be approximated in $\tilde{B}^{2-\varepsilon}_{1,1}(\Omega)$ by non-negative $\tilde{C}^{\infty}_0$-functions. To do this, we apply the method used in the proof of H. Triebel [16: Subsect. 3.4.3]. Without loss of generality we may assume that $\Omega = \mathbf{R}_n^+ = \{x \in \mathbf{R}_n \mid x = (x', x_n), x_n > 0\}$. We define the extension operator $\mathcal{S}$ by

$$\mathcal{S}\psi(x', x_n) = \begin{cases} \psi(x', x_n) & \text{if } x_n > 0 \\ \psi(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

By our assumptions it holds that the characteristic function is a multiplier in $\tilde{B}^{1-\varepsilon}_{1,1}(\Omega)$ (cf. H. Triebel [16: Subsection 2.8.7]. Hence we obtain

$$\|\mathcal{S}\psi \mid \tilde{B}^{2-\varepsilon}_{1,1}(\mathbf{R}_n)\| \sim \|\mathcal{S}\psi \mid \tilde{B}^{1-\varepsilon}_{1,1}(\mathbf{R}_n)\| + \|\nabla\mathcal{S}\psi \mid \tilde{B}^{1-\varepsilon}_{1,1}(\mathbf{R}_n)\| \leq c\, \|\psi \mid \tilde{B}^{2-\varepsilon}_{1,1}(\mathbf{R}_n^+)\|.$$

Now let $\{\varphi_j\}_{j=0}^{\infty}$ be a system of non-negative C_0^{∞}-functions with $\varphi_j \to \delta$ in $D'(\mathbf{R}_n)$, where δ is the Dirac distribution. Then the system $\{\psi_j\}$, $\psi_j = \mathcal{S}\psi * \varphi_j$, is the desired approximation.

Step 2. Thus $\psi(w)$ is well-defined (for the dual space of $\tilde{B}^{2-\varepsilon}_{1,1}(\Omega)$: $\left(\tilde{B}^{2-\varepsilon}_{1,1}(\Omega)\right)' = \tilde{B}^{\varepsilon-2}_{\infty,\infty}(\Omega)$ and non-negative. The result $\left(\mathring{B}^{s}_{p,q}(\Omega)\right)' = B^{-s}_{p',q'}(\Omega)$, where $1 < p < \infty$, $1 \leqq q < \infty$, $1/p < s < \infty$, $s - 1/p \notin \mathbf{N}$, $1/p + 1/p' = 1/q + 1/q' = 1$, may be found in H. Triebel [15: Subsection 4.8.2], but it holds also if $p = 1$. Let

$$L^* = -\sum_{|\alpha|=2} D^{\alpha}a_{\alpha}(x) - \sum_{|\alpha|=1} D^{\alpha}a_{\alpha}(x)$$

(cf. (5)). If $f \in \tilde{C}^{\infty}_0(\Omega)$ is non-negative, then there exists a non-negative $g \in \tilde{C}^{\infty}(\bar{\Omega})$ with $g \mid \partial\Omega = 0$, $Lg + \mu g = f$ (cf. S. Fučík [7: Chapter 34]). According to $\int_{\Omega} \psi(x) f(x)\, dx = \int_{\Omega} \varphi(x) g(x)\, dx$ we obtain the following: If $\phi \in \tilde{C}^{\infty}_0(\Omega)$, then the unique solution ψ of $L^*\psi + \mu\psi = \varphi$, $\psi \mid \partial\Omega = 0$ is non-negative if φ is.

Step 3. Let v be the same as in the formulation of Lemma 1. Let $\varphi \in \tilde{C}^{\infty}_0(\Omega)$ be non-negative, $\varphi = L^*\psi + \mu\psi$ with $\psi \in \tilde{C}^{\infty}(\bar{\Omega})$ non-negative and $\psi \mid \partial\Omega = 0$. Then an easy limiting argument proves

$$\int_{\Omega} \varphi(x) v(x)\, dx = \int_{\Omega} \{(L^* + \mu)\, \psi(x)\}\, v(x)\, dx = \int_{\Omega} \psi(x) \left(Lv(x) + \mu v(x)\right) dx \geqq 0 \quad ∎$$

2.3 Mapping properties. In this subsection we list some results which may be found in T. Runst [14: Subsection 5.4]. In the following, let Ω be a bounded C^{∞}-domain in $\mathbf{R}_n$, while C^{ϱ} denotes as usual the classical Hölder space if $\varrho > 0$ is not an integer and the well-known Banach space of differentiable functions if $\varrho > 0$ is an integer. As mentioned above, $B^{\varrho}_{\infty,\infty}(\mathbf{R}_n) = \mathcal{C}^{\varrho}(\mathbf{R}_n)$ if $\varrho > 0$. For real s we put $s = [s]_- + \{s\}_+$, $[s]_-$ integer, $0 < \{s\}_+ \leqq 1$.

Lemma 2 (T. Runst [14: Subsection 5.4]): *Let* $0 < q \leqq \infty$, $s > n\left(1/\min(p, 1) - 1\right)$, $\varrho > \max(1, s)$ *and* $g \in \tilde{C}^{\varrho}\left(\overline{u(\Omega)}\right)$. *Then*

(i) *if* $0 < p < \infty$, *then*

$$\|g(u) \mid F^s_{p,q}\| \leqq \|g(u) \mid L_p\| + c\left(\sum_{l=1}^{[\varrho]_-} \sup_{x\in\Omega} |g^{(l)}(u(x))|\, \|u \mid F^s_{p,q}\|\, \|u \mid L_{\infty}\|^{l-1}\right.$$
$$\left. + \|g \mid C^{\varrho}(\overline{u(\Omega)})\|\, \|u \mid F^s_{p,q}\|\, \|u \mid L_{\infty}\|^{\varrho-1} + \|g \mid C^{\varrho}(\overline{u(\Omega)})\|\, \|u \mid F^{s}_{p,q}\|\right)$$

and

$$\|g(u) \mid L_p\| \leqq \|g \mid C^1(\overline{u(\Omega)})\| \, \|u \mid L_p\| \qquad \left(u \in \tilde{F}^s_{p,q}(\Omega) \cap L_\infty(\Omega)\right);$$

(ii) *if* $0 < p \leqq \infty$, *then*

$$\|g(u) \mid B^s_{p,q}\| \leqq \|g(u) \mid L_p\| + c \left(\sum_{l=1}^{[\varrho]-} \sup_{x \in \Omega} |g^{(l)}(u(x))| \, \|u \mid B^s_{p,q}\| \, \|u \mid L_\infty\|^{l-1} \right.$$

$$\left. + \|g \mid C^\varrho(\overline{u(\Omega)})\| \, \|u \mid B^s_{p,q}\| \, \|u \mid L_\infty\|^{\varrho-1} + \|g \mid C^\varrho(\overline{u(\Omega)})\| \, \|u \mid B^s_{p,q}\| \right)$$

and

$$\|g(u) \mid L_p\| \leqq \|g \mid C^1(\overline{u(\Omega)})\| \, \|u \mid L_p\| \qquad \left(u \in \tilde{B}^s_{p,q}(\Omega) \cap L_\infty(\Omega)\right).$$

The following result is a consequence of Lemma 2.

Corollary: *Let* $0 < p < \infty$ $(0 < p \leqq \infty)$, $0 < q \leqq \infty$ *and* $s > n/p$.

(i) *Let* $g \in \tilde{C}^\varrho$, $\varrho > \max(1, s)$. *Then there exists a function* γ_g, $\gamma_g\colon [0, \infty) \to [0, \infty)$, *which is independent of* u *such that*

$$\|g(u) \mid F^s_{p,q}\| \leqq \gamma_g(\|u \mid F^s_{p,q}\|) \, \|u \mid F^s_{p,q}\| \text{ for all } u \in \tilde{F}^s_{p,q}(\Omega)$$

$$\left(\|g(u) \mid B^s_{p,q}\| \leqq \gamma_g(\|u \mid B^s_{p,q}\|) \, \|u \mid B^s_{p,q}\| \text{ for all } u \in \tilde{B}^s_{p,q}(\Omega)\right).$$

(ii) *Let* $g \in \tilde{C}^{\varrho+1}$, $\varrho > \max(1, s)$. *Then* $u \to g(u)$ *is a continuous mapping from* $\tilde{F}^s_{p,q}(\Omega)$ *into* $\tilde{F}^s_{p,q}(\Omega)$ (*from* $\tilde{B}^s_{p,q}(\Omega)$ *into* $\tilde{B}^s_{p,q}(\Omega)$).

Proof: *Step* 1. We prove (i). This is a consequence of Lemma 2 and the continuous imbedding $\tilde{F}^s_{p,q}(\Omega) \hookrightarrow \tilde{L}_\infty(\Omega)$ $\left(\tilde{B}^s_{p,q}(\Omega) \hookrightarrow \tilde{L}_\infty(\Omega)\right)$ if $s > n/p$.

Step 2. We can extend Lemma 2 and consequently part (i) of the above corollary to the case $G \in \tilde{C}^\varrho$, $G\colon \mathbf{R}_m \to \mathbf{R}_1$, $m = 1, 2, \ldots$ Let $g\colon u \to g(u)$ for $u \in \tilde{F}^s_{p,q}(\Omega)$. We put $b(x, y) = \left(g(x) - g(y)\right)/(x - y)$ $(x, y \in \mathbf{R}_1)$. Then for

$$b(u, v)(\cdot) = b\left(u(\cdot), v(\cdot)\right) = \frac{g(u) - g(v)}{u - v}(\cdot)$$

we obtain that $\|b(u, v) \mid F^s_{p,q}\| \leqq \gamma(\|u \mid F^s_{p,q}\|, \|v \mid F^s_{p,q}\|)$ holds, where the mapping $\gamma\colon \{(x, y), x, y \geqq 0\} \to [0, \infty)$ is independent of $u, v \in \tilde{F}^s_{p,q}(\Omega)$. Hence we get

$$\|g(u) - g(v) \mid F^s_{p,q}\| \leqq \gamma'(\|u \mid F^s_{p,q}\|, \|v \mid F^s_{p,q}\|) \, \|u - v \mid F^s_{p,q}\|, \tag{9}$$

which yields the continuity. (9) follow from the fact that $\tilde{F}^s_{p,q}(\Omega)$ is a multiplication algebra if $s > n/p$, cf. H. Triebel [16: Subsection 3.3.2], J. Franke [3: Subsection 3.3], and T. Runst [14: Subsection 5.3]. The proofs in the case $B^s_{p,q}$ are almost the same ∎

Remark 3: Results in this direction were also proved by J. Peetre [13] in the case $\tilde{B}^s_{p,q}$, $s > n/p$, $1 \leq p, q \leq \infty$, by Y. Meyer [11, 12] in the case $\tilde{H}_p{}^s$, $1 < p < \infty$, $s > n/p$, M. Yamazaki [18] and J. Marschall [10].

3. The main result

Let Ω be a bounded C^∞-domain in $\mathbf{R}_n$ and let

$$Lu = -g(u) + f(x) + t \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{10}$$

be a second order elliptic boundary value problem, where L is a uniformly elliptic second order differential operator as described by (5) and $t \in \mathbf{R}_1$.

Theorem: *Let $0 < p, q \leq \infty$, $s > n/p$, $t \in \mathbf{R}_1$ and let $g \in \tilde{C}^{\varrho+1}(\mathbf{R}_1)$, $\varrho > \max(1, s)$, satisfy the conditions* (I) $g(x) \geq 0$ *if* $x \geq 0$ *and* $g(0) = 0$, (II) *the function* $g(x) + \lambda_1 x$, *where* $\lambda_1 > 0$ *is the first eigenvalue of* L, *is bounded from below.*

(i) *Let* $p < \infty$ *and* $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. *Then there exists a* $t_0(f) \in \mathbf{R}_1$ *such that problem* (10) *has at least one solution* $u \in \tilde{F}^s_{p,q}(\Omega)$ *if* $t > t_0$ *and has no solution if* $t < t_0$.

(ii) *Let* $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. *Then there exists a* $t_0(f) \in \mathbf{R}_1$ *such that problem* (10) *has at least one solution* $u \in \tilde{B}^s_{p,q}(\Omega)$, *if* $t > t_0$ *and has no solution if* $t < t_0$.

Proof: We prove (i). The proof of (ii) is the same.

Step 1. First of all we remark that the Dirichlet problem $Lu = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ has a unique normed positive eigenfunction φ^* corresponding to the smallest eigenvalue $\lambda_1 > 0$, see (6). It is known that $\varphi^* > 0$ in Ω and

$$\partial\varphi^*/\partial\nu < c < 0 \text{ on } \partial\Omega, \tag{11}$$

see S. FUČIK [7: Subsections 34.10—34.12].

Step 2. We have $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. Hence we can choose $t > 0$ so large that $f(x) + t > 0$ holds for $x \in \Omega$. Then $u_1 = 0$ is a subsolution of (10). Here we used the fact that $g(0) = 0$. We fix such a t.

Step 3. Because of $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$ and the properties of L (see (7), (8)) there exists a function $w \in \tilde{F}^s_{p,q}(\Omega)$ such that $Lw > f(x) + t$ in Ω and $w = 0$ on $\partial\Omega$. Notice that $\tilde{F}^s_{p,q}(\Omega) \hookrightarrow \tilde{C}(\bar{\Omega})$ if $s > n/p$. Now we choose $r > 0$ such that $w + r\varphi^* > 0$ in Ω holds. Therefore we apply (11) and the same arguments as in S. FUČIK [7: Subsections 34.11 and 34.12]. Then $u_2 = w + r\varphi^*$ is a supersolution of (10) because of $Lu_2 > -g(u_2) + f(x) + t$ in Ω, $u_2 = 0$ on $\partial\Omega$. Here we used property (I) of the function g. We have $u_2 > u_1$ in Ω.

Step 4. In what follows we show: If u_1 is a subsolution and u_2 a supersolution of (10) and $u_1(x) \leq u_2(x)$, $x \in \Omega$, then there exists a function $u \in \tilde{F}^s_{p,q}(\Omega)$ such that $u_1 \leq u \leq u_2$ in Ω and (10) holds. This result is a generalization of H. AMANN [1], see also S. FUČIK [7: Subsection 34.7]. We apply the same arguments as J. FRANKE and T. RUNST [6: Subsection 3.4]. The above conditions yield $u_{1,2} \in \tilde{C}(\bar{\Omega})$. Let $\omega > 0$ be such that

$$\omega - g'(\xi) > 0 \text{ for any } \xi \in \left[\min_{x\in\bar{\Omega}} u_1(x), \max_{x\in\bar{\Omega}} u_2(x)\right].$$

Let T be the operator which assigns to each $u \in \tilde{C}(\bar{\Omega})$ the unique solution $v \in \bigcup \{\tilde{B}^\varepsilon_{\infty,\infty}(\Omega): \varepsilon > 0\}$ of

$$Lv + \omega v = f(x) + t - g(u) + \omega u \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega. \tag{12}$$

The definition of T is correct with respect to $\omega > 0$ and the properties of L. We put $u_1^{(k)} = T^k u_1$ and $u_2^{(k)} = T^k u_2$, where as usual $T^1 = T$, $T^{k+1} = T^1 T^k$. In analogy to S. FUČIK [7: Subsection 34.7] we can show that T is monotone, i.e. if $u \leq v$ in Ω, then $Tu \leq Tv$ in Ω. For this we apply Lemma 1 (maximum principle in the sense of distributions). By induction we get a monotonically decreasing sequence

$$u_1 \leq u_1^{(1)} \leq u_1^{(2)} \leq \cdots \leq u_2^{(2)} \leq u_2^{(1)} \leq u_2. \tag{13}$$

From (8), the Corollary in Subsection 2.3 and $u_{1,2} \in L_\infty(\Omega)$ we deduce by (12) the inequality $\|u^{(k+1)}_{1,2} F^s_{p,q}\| \leq c_1 + c_2 \|u^{(k)}_{1,2} \mid F^{s-\varepsilon}_{p,q}\|$. The imbedding $\tilde{C}(\Omega) \hookrightarrow \tilde{F}^{-\varepsilon}_{p,q}(\Omega)$ and (13) yield

$$\|u^{(k)}_{1,2} \mid F^{-\varepsilon}_{p,q}\| \leq c \max(\|u_1 \mid C\|, \|u_2 \mid C\|) < c_3.$$

Applying the well-known inequality $\|w \mid F^{s-\varepsilon}_{p,q}\| \leq \delta \|w \mid F^{s}_{p,q}\| + c_\delta \|w \mid F^{-\varepsilon}_{p,q}\|$, it follows that $\|u^{(k+1)}_{1,2} \mid F^s_{p,q}\| \leq c_4 + 2^{-1} \|u^{(k)}_{1,2} \mid F^s_{p,q}\|$. By induction we get

$$\|u^{(k)}_{1,2} \mid F^s_{p,q}\| \leq C, \qquad C > \max\left(2c_4, \|u_1 \mid F^s_{p,q}\|, \|u_2 \mid F^s_{p,q}\|\right).$$

Because of $\tilde{F}^s_{p,q}(\Omega) \hookrightarrow \tilde{C}(\bar{\Omega})$, $s > n/p$, the pointwise limits

$$u_\pm(x) = \lim_{k\to\infty} u^k_{1,2}(x) \tag{14}$$

both exist. Now we apply the same arguments as J. Franke and T. Runst [6: Subsection 3.4]. Let $\mathcal{S}$ be the coretraction constructed in J. Franke [3: Subsection 4.1]. We may assume that supp $\mathcal{S}\varphi$ is uniformly bounded for all φ. The construction of $\mathcal{S}$ yields $\mathcal{S}u^{(k)}_{1,2}(x) \to \mathcal{S}u_\pm(x)$ pointwise. Lebesgue's theorem proves that this holds also for the weak $\sigma\big(S(\mathbf{R}_n), S'(\mathbf{R}_n)\big)$-topology. Applying the Fatou property (cf. J. Franke [3: Subsection 2.6]) we get $u_\pm \in \tilde{F}^s_{p,q}(\Omega)$. By means of $\tilde{F}^s_{p,q}(\Omega) \hookrightarrow \tilde{C}(\bar{\Omega})$ if $s > n/p$ it follows that $u_\pm \in \tilde{C}(\bar{\Omega})$. Finally Dini's theorem yields that (14) holds in $\tilde{C}(\bar{\Omega})$. Now it is not hard to check that $u_\pm$ are solutions of (10).

Step 5. Here we prove the following: If (10) is solvable for some $t_1 \in \mathbf{R}_1$, then it is also solvable for all $t > t_1$. Let $t_2 > t_1$. The solution of (10) with $t = t_1$ is denoted by u_1. Then $Lu_1 < -g(u_1) + f(x) + t_2$ in Ω and $u_1 = 0$ on $\partial\Omega$. Hence u_1 is a subsolution of (10) with $t = t_2$. Let us choose $v \in \tilde{F}^s_{p,q}(\Omega)$ such that $Lv > f(x) + t_2$ in Ω and $v = 0$ on $\partial\Omega$ and $\tilde{r} > 0$ such that $v + \tilde{r}\varphi^* > u_1$ in Ω. Therefore we use the same arguments as in Step 3. Then $u_2 = v + \tilde{r}\varphi^*$ is a supersolution of (10) with $t = t_2$. In analogy to Step 4 there exists a $u \in \tilde{F}^s_{p,q}(\Omega)$ satisfying $u_1 \leq u \leq u_2$ and $Lu = -g(u) + f(x) + t_2$ in Ω, $u = 0$ on $\partial\Omega$.

Step 6. We put $t_0 = \inf\{t \in \mathbf{R}_1 \mid (10) \text{ is solvable}\}$. We remark that, by Step 4, $t_0 < \infty$.

Step 7. We show that $t_0 > -\infty$ holds. According to the properties of g the function

$$s \to -g(s) - \lambda_1 s \tag{15}$$

is bounded from above. Let u be a solution of (10) for some $t \in \mathbf{R}_1$. Then we get

$$Lu - \lambda_1 u = -g(u) - \lambda_1 u + f(x) + t \text{ in } \Omega. \tag{16}$$

Note that the adjoint problem to $L\varphi - \lambda_1\varphi = 0$ in Ω, $\varphi = 0$ on $\partial\Omega$ has also only a one-dimensional space of solutions generated by a positive function $\psi^* \in \tilde{C}^\infty(\bar{\Omega})$ with $\int_\Omega \psi^*(x)^2\,dx = 1$ and $\psi^*(x) \geqq 0$ if $x \in \bar{\Omega}$ (cf. S. Fučík [7: Subsection 34.12]).

Hence it follows from (16) that

$$0 = \int_\Omega (Lu - \lambda_1 u)(x)\,\psi^*(x)\,dx$$

$$= \int_\Omega \big(-g(u) - \lambda_1 u\big)(x)\,\psi^*(x)\,dx + \int_\Omega f(x)\,\psi^*(x)\,dx + t\int_\Omega \psi^*(x)\,dx,$$

which together with the boundedness of (15) implies that $t \geq t(f)$ holds ∎

Remarks: 4. A corresponding result holds also in the case $Lu = g(u) + f(x) + t$ in Ω, $u = 0$ on $\partial\Omega$. 5. If $u_i \in \tilde{F}^s_{p,q}(\Omega)$ are solutions of $Lu = -g(u) + f_i(x)$ in Ω, $u = 0$ on $\partial\Omega$, $i = 1, 2$, then $f_1 \leq f_2$ implies $u_1 \leq u_2$. 6. Note that the above proof yields the following result: for each $f \in \tilde{F}^s_{p,q}(\Omega) \cap L_\infty(\Omega)$ there exists a $t_1(f) \geq t_0(f)$ such that (10) has a positive solution for all $t > t_1(f)$.

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