On the Existence of Solutions of a Semilinear Elliptic Boundary Value Problem with Superlinear Nonlinearities

P. DRÁBEK and T. RUNST

Wir beschäftigen uns mit der Lösbarkeit eines semilinearen Randwertproblems mit superlinearer Nichtlinearität in Räumen vom Besov- und Triebel-Lizorkin-Typ.

Мы исследуем разрешимость одной полулинейной эллиптической краевой задачи с суперлинейной нелинейностью в пространствах типа Бесова и Трибель-Лизоркина.

We study the solvability of a semilinear elliptic boundary value problem with superlinear nonlinearity in spaces of Besov and Triebel-Lizorkin type.

1. Introduction

The aim of this paper is the study of the following superlinear problem

$$
Lu = -g(u) + f(x) + t \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,
$$

where L is a uniformly elliptic second order differential operator and $t \in \mathbb{R}_1$. Moreover, let g be a sufficiently smooth function defined on R_1 and satisfying the following conditions:

 $g(x) \ge 0$ if $x \ge 0$ and $g(0) = 0$, (\mathbf{I}) The function $g(x) + \lambda_1 x$, where $\lambda_1 > 0$ is the first eigenvalue of L, is bounded (II) from below.

We consider problems of this form in the framework of real Besov spaces $\bar{B}_{p,q}^s(Q)$ and Triebel-Lizorkin spaces $\tilde{F}_{p,q}^s(\Omega)$ on bounded C^{∞} -domains $\Omega \subset \mathbb{R}_n$, where s large enough. Roughly speaking, we shall prove that for each fixed f there exists a $t_0(f) \in \mathbf{R}_1$ such that

(1) has at least one solution if $t > t_0(f)$;

(1) has no solution if
$$
t < t_0(f)
$$
.

Furthermore, we are interested in non-negative solutions of (1). Problems of this type were considered by many authors (see e.g. H. AMANN [1] and the references given there) and arise for example in the theory of nonlinear diffusion processes and in reactor theory. Questions of existence and smoothness of solutions of (1) are reduced to problems involving nonlinear mappings in the spaces considered here (§ 2.3) and a maximum principle (§ 2.2). $\,$

Nonlinear elliptic equations in the framework of Besov and Triebel-Lizorkin spaces were first considered by H. TRIEBEL [17]. Further results in this direction may be found in D. E. EDMUNDS and H. TRIEBEL [2], and J. FRANKE and T. RUNST [5, 6].

2. Preliminaries

2.1 Spaces. Let \mathbf{R}_n be the real Euclidean *n*-spaces. The theory of the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ was developed in H. Triesel [16]. We do not need the full theory, but only some properties, which we list in the sequel.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \mathbf{R}_n , S' the set of all tempered distributions on \mathbf{R}_n , F and \mathcal{F}^{-1} the Fourier transform and its inverse on $S'(\mathbf{R}_n)$, respectively. Now let $\varphi \in S$ be a real-valued function such that $\varphi(x) = \varphi(-x)$ if $x \in \text{supp } \varphi \subset \{y \in \mathbb{R}_n | |y| \leq 2\}$ and $\dot{\varphi}(x) = 1$ if $|x| \leq 1$. Then we define a sequence $\{\varphi_i\}_{i=0}^{\infty}$ of functions by

$$
\varphi_0(x) = \varphi(x), \qquad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x) \qquad (j \ge 1)
$$

for each $x \in \mathbf{R}_n$. We have $\varphi_0(x) + \varphi_1(x) + \ldots = 1$ $(x \in \mathbf{R}_n)$. If $-\infty < s < \infty$. $0 < p, q < \infty$, then by definition

$$
B_{p,q}^{s}(\mathbf{R}_{n})=\left\{f\in S'\bigg|\bigg(\sum_{j=0}^{\infty}2^{j\epsilon q}\left|\left|\mathcal{F}^{-1}\varphi_{j}\mathcal{F}f\right|\right)L_{p}(\mathbf{R}_{n})\right|^{q}\bigg|^{1/q}<\infty\right\}
$$
(2)

and)

$$
F_{p,q}^s(\mathbf{R}_n) = \left\{f \in S' \left| \left| \left| \left(\sum_{j=0}^\infty 2^{jsq} |\mathcal{F}^{-1} \varphi_j \mathcal{F} f(\cdot)|^q \right)^{1/q} \right| L_p(\mathbf{R}_n) \right| \right| < \infty \right\} \qquad (3)
$$

(usual modification if $p = \infty$ and/or $q = \infty$).

It can be shown (cf. H. TRIEBEL [16]) that with the respective quasi-norms $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^{s}(R_n)$ are quasi-Banach spaces; moreover, they are independent of the particular choice of φ (equivalent quasi-norms).

 \therefore Remark 1: By means of the fact that φ is a real-valued even function we can introduce the real part of the spaces $B_{p,q}^s(\mathbf{R}_n)$, etc., denoted by $\tilde{B}_{p,q}^s(\mathbf{R}_n)$, ... (for exact definitions see J. FRAN-KE and T. RUNST [5: Subsection 3.2]).

Remark 2: These two scales of function spaces include many well known classical spaces. Equivalent quasi-norms for these spaces may be found in H. TRIEBEL [16], for instance $B_{\infty,\infty}^{s}(\mathbf{R}_{n}) = \mathscr{C}^{s}(\mathbf{R}_{n})$ (Zygmund spaces) if $s > 0$.

Next we define the corresponding spaces on open, sets. Let Ω be a bounded C^{∞} domain with boundary $\partial\Omega$. Then one can introduce the spaces $B_{p,q}^s(\partial\Omega)$ and $F_{p,q}^s(\partial\Omega)$ by standard procedure via local charts, cf. H. TRIEBEL [16: Subsection 3.2.2]. The spaces $B_{p,q}^{s}(\Omega)$ and $F_{p,q}^{s}(\Omega)$ are defined as usual by the restriction method, cf. H. **TRIEBEL** [16: Subsection 3.2.2]: if $-\infty < s < \infty$ and $0 < p, q \le \infty$, then for instance

$$
B_{p,q}^{s}(\Omega) = \{f \in D'(\Omega) \mid \exists g \in B_{p,q}^{s}(\mathbf{R}_n) \text{ with } g \mid \Omega = f\},\
$$

$$
||f \mid B_{p,q}^{s}(\Omega)|| = \inf ||g|| B_{p,q}^{s}(\mathbf{R}_n)||,
$$
 (4)

where the infimum is taken over all $g \in B_{p,q}^s(\mathbf{R}_n)$ in the sense of (4). Similarly one can define the spaces $F_{p,q}^s(\Omega)$.

2.2 Traces and linear elliptic differential operators. Let Ω be a bounded C^{∞} -domain in \mathbf{R}_n and let f be a function defined in Ω and belonging to some function spaces of the above type. A denotes the restriction operator given by $\mathcal{R}f = f | \partial\Omega$. The following results are well-known for $0 < q \le \infty$ and $s > (n-1)$ $(1/\min(p, 1) - 1) + 1/p$. (for the proof see J. FRANKE [4] and H. TRIEBEL [16: Subsection 3.3.3]):

If $0 < p \leq \infty$, then *R* is a linear and continuous mapping from $B_{p,q}^s(\Omega)$ onto $B_{p,q}^s(\Omega)$. If $0 < p < \infty$, then \mathcal{R}_t is a linear and continuous mapping from $F^s_{p,q}(\Omega)$ onto $\begin{array}{r} \text{if } 0 < p, \ B^{s-1/p}_{p,q}(\partial \Omega) \longrightarrow \text{if } 0 < p, p \in \mathbb{R}^n, p \in \mathbb{R}^n, \ \text{As above} \end{array}$ On the E
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 $p < \infty$, then \Re , is a linear and continuous r
 p, Q denotes a bounded C^{∞} -domain in \mathbf{R}_n with $\Re(x) = -\sum a_n(x) D^2 u(x) + \sum a_n(x) D^2 u(x)$. *i*
 A is a linear and continuous mapping from $B_{p,q}^s(\Omega)$ onto
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 $c \infty$, then \mathcal{R}_i is a linear and continuous mapping from $F_{p,q}^s(\Omega)$
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If $0 < p < \infty$, then \mathcal{R}_i is a linear and continuous mapping from $F_{p,q}^s(\Omega)$
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 $B_{p,p}^{s-1/p}(\partial\Omega)$.

As above, Ω denotes a bounded C^{∞} -domain in \mathbf{R}_n with boundary $\partial\Omega$. Let L ,

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Lu(x) = -\sum_{|\alpha|=2} a_{\alpha}(x) D^{\alpha}u(x) + \sum_{|\alpha|=1} a_{\alpha}(x) D^{\alpha}u(x),
$$

As above, Ω denotes a bou
 $Lu(x) = -\sum_{|\mathfrak{a}|=2} a_{\mathfrak{a}}(x)$
 $x \in \Omega$, $a_{\mathfrak{a}} \in \tilde{C}^{\infty}(\overline{\Omega})$ if $|\alpha| \le$

uniformly elliptic operator: $x \in \Omega$, $a_a \in \tilde{C}^{\infty}(\overline{\Omega})$ if $|\alpha| \leq 2$ (i.e. the a_a are real C^{∞} -functions), be a second order uniformly elliptic operator:

$$
\sum_{|\alpha|=2} a_{\alpha}(x) y^{\alpha} \geq c |y|^2 > 0, \qquad y \in \mathbf{R}_n, y \neq 0
$$

In this paper we only consider the corresponding homogeneous Dirichlet problem As above, Ω denotes a bounded C^{∞} -domain in \mathbf{R}_n with boundary $\partial\Omega$
 $Lu(x) = -\sum_{|\mathbf{a}|=2} a_{\mathbf{a}}(x) D^{\mathbf{a}}u(x) + \sum_{|\mathbf{a}|=1} a_{\mathbf{a}}(x) D^{\mathbf{a}}u(x)$,
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We introduce the notations (for admissible couples (s, p))
 $B_{p,q,0}^{s+2}(\Omega) = \{f \in B_{p$

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B_{p,q,0}^{s+2}(\Omega) = \{f \in B_{p,q}^{s+2}(\Omega) | f|\partial\Omega = 0\}
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F_{p,q,0}^{s+2}(\Omega) \triangleq \{f \in F_{p,q}^{s+2}(\Omega) | \quad f | \partial \Omega = 0 \}.
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*i*₄ $\in \tilde{C}^{\infty}(\overline{\Omega})$ if $|\alpha| \leq 2$ (i.e. the a_a are real C^{∞} -fully elliptic operator:
 $\sum_{|\alpha|=2} a_a(x) y^{\alpha} \geq c |y|^2 > 0$, $y \in \mathbb{R}_n, y \neq 0$.

Super we only consider the corresponding homogentroduce the notat est eigenvalue of $L\left|_{\tilde{B}_{2,2,0}^1(\Omega)}\right|$ with Dirichlet condition, then $\lambda_1 > 0$. Furthermore, $B_{p,q,0}^{s+2}(\Omega) = \{f \in B_{p,q}^{s+2}(\Omega) | \text{ } | \partial \Omega = 0 \}$

and
 $F_{p,q,0}^{s+2}(\Omega) \geq \{f \in F_{p,q}^{s+2}(\Omega) | \text{ } | \partial \Omega = 0 \}$.

By S. Fučnk [7: Theorem 34.10] we obtain the following result. If λ_1 denotes the smast eigenvalue of L

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$$
\sum a_a(x) y^2 \ge c |y|^2 > 0, \quad y \in \mathbb{R}_n, y \neq 0.
$$
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\n- \n In this paper we only consider the corresponding homogeneous Dirichlet problem. We introduce the notations (for admissible couples (s, p))\n $B_{p,q,0}^{s+2}(Q) = \{f \in B_{p,q}^{s+2}(\Omega) | f | \partial \Omega = 0\}$ \n and\n $F_{p,q,0}^{s+2}(Q) \ge \{f \in F_{p,q}^{s+2}(\Omega) | f | \partial \Omega = 0\}$ \n
\n- \n By S. FuČrK [7: Theorem 34.10] we obtain the following result. If λ_1 denotes the smallest eigenvalue of $L\left| \frac{1}{B_{\lambda_2,0}(a)}$, with Dirichlet condition, then $\lambda_1 > 0$. Furthermore, λ_1 there exists a unique normed positive eigenfunction $\varphi^* \in \tilde{C}^{\infty}(\overline{\Omega})$ to λ_1 with\n $\int \varphi^*(x)^2 dx = 1, \quad \varphi^*(x) \geq 0$ if $x \in \overline{\Omega},$ \n $\varphi^* \mid \partial \Omega = 0$,\n $L\varphi^* = \lambda_1 \varphi^*$ \n
\n- \n (cf. e.g. M. G. KREY and M. A. RUTMAX [9], M. A. KRASNOSELSK [8]). Then the following may be found in J. FRAXKE [4] (see also H. TRIEEL [16: Subsection 3.3.3]) for $0 < q \leq \infty$ and $s > (n-1) \left(\frac{1}{\min(p, 1) - 1} \right) + \frac{1}{p} - 2$.\n If $0 < p \leq \infty$, then L yields an isomorphic mapping.\n $f \text{ from } B_{p,q,0}^{s+2}(\Omega) \text{ onto } B_{p,q}^s(\Omega).$ \n
\n- \n In order to prove our main result we need some results about subsolutions and

(cf. e.g. M. G. KREIN and M. A. RUTMAN ^[9], M. A. KRASNOSELSKI ^[8]). Then the (cf. e.g. M. G. KREIN and M. A. KUTMAN [9], M. A. KRASNOSELSKI [9]). Then the
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If $0 < p$ **for** $0 < q \leq \infty$ and $s > (n-1) ((1/\min(p, 1) - 1)) + 1/p - 2$: $y^2 + (x + 2) = 0$. $L\varphi^* = \lambda_1 \varphi^*$

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following may be found in J. FRANKE [4] (see also H. TRIEBEL [16: S

for $0 < q \le \infty$ and $s > (n - 1) ((1/\min (p, 1) - 1)) + 1/p - 2$:

If
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, then L yields an isomorphic mapping from $B_{p,q,0}^{s+2}(\Omega)$ onto $B_{p,q}^s(\Omega)$.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nfrom $F_{p,q,0}^{s+2}(\Omega)$ onto $F_{p,q}^s(\Omega)$.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic mapping.\n\nIf $0 < p < \infty$ then L yields an isomorphic

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\nIf $0 < p < \infty$ then L yields an isomorphic mapping.

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Definition 1: A distribution $\varphi \geq 0$.

Here $D(\Omega) = C_0^{\infty}(\Omega)$ de

Definition 1: A distribution $\psi \in \tilde{D}(Q)$ is said to be *non-negative if* $\psi(\varphi) \ge 0$ for any $\varphi \in \tilde{D}(Q)$ with $\varphi \ge 0$.

Here $D(Q) = C_0^{\infty}(\Omega)$ denotes as usual the collection of all complex-valued infinitely dif Here $D(\Omega) = C_0^{\infty}(\Omega)$ denotes as usual the collection- of all complex-valued in- $\mathcal{L}=\left\{ \begin{array}{ll} \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) & \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) & \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) \\ \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) & \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) & \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal{L}}\right) \\ \mathcal{L}_{\mathcal{L}}\left(\mathcal{L}_{\mathcal$ mapping

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The set of non-neg There $D(\Omega) = C_0^{\infty}(\Omega)$ denotes as usual the collection of all complex-valued in-
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supersolutions.

Definition 1: A distribution $\psi \in \tilde{D}'(\Omega)$ is said to be *non-negative* if $\psi(\varphi) \ge 0$ for

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· Definition 2: A function $u \in \tilde{C}(\overline{\Omega})$ is said to be a

(1) if $Lu \leq -g(u) + j(x) + t$ $(Lu \geq -g(u) + j(x) + t)$
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 $\exp \varphi \in \tilde{D}(\Omega)$ with $\varphi \ge 0$

Furthermore, $B_{p,q}^s(\Omega)$ denotes the completion of $D(\Omega)$ in $B_{p,q}^s(\Omega)$. In Section 3 we \sim shall use the following

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Lemma 1: Let $v \in \bigcup_{\alpha} \overline{B}_{\infty,\infty}^{\alpha}(\Omega)$: $\varepsilon > 0$ and $\mu > 0$. If $v \mid \partial\Omega = 0$ and $Lv + \mu v \geq 0$ *(in the sense of distributions), then* $v \geq 0$.

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Lemma 1: Let $v \in \cup \{\tilde{B}_{\infty,\infty}^{\epsilon}(\Omega) : \epsilon > 0\}$ and $\mu > 0$. If $v \mid \partial\Omega = 0$ and $Lv + \mu v \geq 0$

i the sense of distributions), then $v \geq 0$.

Proof: Step 1. Let $w \in \tilde{B}_{\infty,\infty}^{\epsilon-2}$, $0 < \epsilon < 1$ then $\psi \in \tilde{B}_{1,1}^{2-\epsilon}(\Omega)$. We prove that if ψ is non-negative, then ψ can be approximated in $\tilde{B}_{1,1}^{2-\epsilon}(\Omega)$ by non-negative \tilde{C}_0^{∞} -functions. To do this, we apply the method used in the proof of H. TRIEBEL [16: Subsect. 3.4.3]: Without loss of generality we may assume that Proof: $Step 1$. Let $w \in \tilde{B}_{\infty, \infty}^{2}$, $0 < \varepsilon < 1$, be non-negative. If $\psi \in \tilde{C}^{\infty}(\overline{\Omega})$, $\psi \mid \partial \Omega = 0$,
then $\psi \in \tilde{B}_{1.1}^{2}(\Omega)$. We prove that if ψ is non-negative, then ψ can be approximated in
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 $\sum_{n} [16$: Subsect. 3.4.3]. Without loss of generality we may assume that
 $\sum_{n} \binom{n}{x} (x', x_n)$ if $x_n > 0$. We define the extension operator

$$
\Omega = \mathbf{R}_n + \{x \in \mathbf{R}_n \mid x = (x', x_n), x_n > 0\}.
$$
 We ded

$$
\mathcal{S} \psi(x', x_n) = \begin{cases} \psi(x', x_n) & \text{if } x_n > 0 \\ \psi(x', -x_n) & \text{if } x_n < 0. \end{cases}
$$

By our assumptions it holds that the characteristic function is a multiplier in $\bar{B}_{1,1}^{1-\epsilon}(\Omega)$ (cf. H. **TRIEBEL** [16: Subsection 2.8.7]: hence we obtain By our assumption

(cf. H. TRIEBEL [1]
 $|\mathcal{S}\psi| \tilde{B}_{1}^{2}$.

(Now let $|x| \approx$ by

$$
\|\mathscr{S}\psi\mid \tilde{B}_{1,1}^{2-\epsilon}(\mathbf{R}_n)\| \sim \|\mathscr{S}\psi\mid \tilde{B}_{1,1}^{1-\epsilon}(\mathbf{R}_n)\| + \|\nabla \mathscr{S}\psi\mid \tilde{B}_{1,1}^{1-\epsilon}(\mathbf{R}_n)\| \leq c \|\psi\mid \tilde{B}_{1,1}^{2-\epsilon}(\mathbf{R}_n^*)\|
$$

By our assumptions it holds that the characteristic function is a multiplier in $\tilde{B}_{1,1}^{1-\epsilon}(\Omega)$.

(cf. H. TRIEBEL [16: Subsection 2.8.7]: Hence we obtain
 $||\mathcal{F}\psi||\tilde{B}_{1,1}^{2-}(\mathbf{R}_n)|| \sim ||\mathcal{F}\psi||\tilde{B}_{1,1}^{1-}(\mathbf{$ where δ is the Dirac distribution. Then the system $\{\psi_j\}$, $\psi_j = \mathscr{S}\psi * \varphi'_j$, is the desired approximation.

Step 2. Thus $\psi(w)$ is well-defined (for the dual space of $\tilde{B}_{1,1}^{2-}(Q)$; $(\tilde{B}_{1,1}^{2-}(Q))'$) $\mathcal{B}_{\mathcal{B},\mathcal{B}}^{i}(\Omega)$ and non-negative. The result $(\hat{B}_{\mathcal{B},q}^{\mathcal{B}}(\Omega))' = B_{\mathcal{B},q}^{-s}(\Omega)$, where $1 < p < \infty$, approximation.

Step 2. Thus $\psi(w)$ is well-defined (for the dual space of $\tilde{B}_{1,1}^{2-}(\Omega)$; $(\tilde{B}_{1,1}^{2-}(\Omega))'$
 $= \tilde{B}_{\infty,\infty}^{(-1)}(\Omega)$ and non-negative. The result $(\tilde{B}_{p,q}^{s}(\Omega))' = B_{p,q}^{-s}(\Omega)$, where $1 \leq p < \infty$,

$$
L^* = -\sum_{|a|=2} D^a a_a(x) - \sum_{|a|=1} D^a a_a(x)
$$

By our assumptions it holds that the characteristic function is a multiplier in $\tilde{B}_{1,1}^{1-}(Q)$

(cf. H. TRIEBEL [16: Subsection 2.8.7]: Hence we obtain
 $||\mathcal{F}\psi||\tilde{B}_{1,1}^{2-}(\mathbf{R}_n)|| - ||\mathcal{F}\psi||\tilde{B}_{1,1}^{1-}(\mathbf{R}_n)||$ (cf. (5)). If $f \in \tilde{C}_0^{\infty}(\Omega)$ is non-negative, then there exists a non-negative $g \in \tilde{C}^{\infty}(\overline{\Omega})$ with $g \mid \partial \Omega = 0$, $Lg + \mu g = f$ (cf. S. Fučin [7: Chapter 34]). According to $\int \psi(x) f(x) dx$ (cf. (5)). If $f \in \tilde{C}_0^{\infty}(\Omega)$ is non-negative, then there exists a non-negative $g \in \tilde{C}^{\infty}(\overline{\Omega})$ with $g \mid \partial \Omega = 0$, $Lg + \mu g = f$ (cf. S. Fučik [7: Chapter 34]). According to $\int_{\Omega} \psi(x) f(x) dx$
= $\int_{\Omega} \varphi(x) g(x) dx$ w tion in the transformal probability of $L^* = -\sum_{|a|=2} D^a a_a(x) - \sum_{|a|=1} L^a$

(cf. (5)). If $f \in \tilde{C}_0^{\infty}(\Omega)$ is non-negative
 $g \mid \partial \Omega = 0$, $Lg + \mu g = f_i$ (cf. S. Furrelly, $f \in \mathcal{L}$
 $= \int \varphi(x) g(x) dx$ we obtain the form of tion ψ of $L^* \psi + \mu \psi = \varphi$, $\psi \mid \partial \Omega = 0$ is non-negative if φ is. $f = f$
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Step 3. Let v be the same as in the formulation of Lemma 1. Let $\varphi \in \tilde{C}_0^{\infty}(\Omega)$ *be* non-negative, $\varphi = L^* \psi + \mu \psi$ with $\psi \in \tilde{C}^{\infty}(\bar{\Omega})$ non-negative and $\psi | \partial \Omega = 0$. Then an easy limiting argument proves

iting argument proves
\n
$$
\int_{\Omega} \varphi(x) v(x) dx = \int_{\Omega} \left\{ (L^* + \mu) \psi(x) \right\} v(x) dx = \int_{\Omega} \psi(x) \left(Lv(x) + \mu v(x) \right) dx \ge 0 \quad \blacksquare
$$

2.3 Mapping properties. In this subsection we list some results which may be found in T. RUNST [14: Subsection 5.4]. In the following, let Ω be a bounded C^{∞} -domain in \mathbf{R}_n , while C^e denotes as usual the classical Hölder space if $\rho > 0$ is not an integer and the well-known Banach space of differentiable functions if $\rho > 0$ is an integer.
As mentioned above, $B_{\infty,\infty}^{\rho}(\mathbf{R}_n) = \mathcal{E}^{\rho}(\mathbf{R}_n)$ if $\rho > 0$. For real s we put $s = [s]_+ + \{s\}_+$, As mentioned above, $B_{\infty} \infty(\mathbb{R}_n) = \{e^{\alpha}(\mathbb{R}_n) \text{ if } \rho > 0, \text{ is an integer and } \rho > 0\}$.

As monomogative, $\phi = L^* \psi + \mu \psi = \varphi, \psi \mid \partial \Omega = 0$ is non-negative if φ is.

Step 3. Let *v* be the same as in the formulation of Lemma **2.3 Mapping properties.** In
in T. RUNST [14: Subsection \mathbf{R}_n , while C^e denotes as
and the well-known Banac
As mentioned above, $B^e_{\infty, \infty}$
[*s*]₋ integer, $0 < \{s\}_+ \leq 1$
Lemma 2 (T. RUNST [15] Hom-higative, $\varphi = L^2 \psi + \mu \psi$ with $\psi \in C^{\infty}(3)$ hom-higative and $\psi \mid \delta\Omega =$

easy limiting argument proves
 $\int \varphi(x) v(x) dx = \int ((L^* + \mu) \psi(x)) v(x) dx = \int \psi(x) (Lv(x) + \mu v(x))$

2.3 **Mapping properties.** In this subsection we list some **Example properties.** In this subsection we has some results which may be found
in **R**_n, while C^{*e*} denotes as usual the classical Hölder space if $\varrho > 0$ is not an integer
and the well-known Banach space of differen

Lemma 2 (T. RUNST [14: Subsection 5.4]): *Let* $0 < q \le \infty$, $s > n(1/\min (p, 1) - 1)$, $\rho > \max (1, s)$ and $g \in \tilde{C}^e(u(\Omega))$. *Then .(i) if* $0 < p < \infty$, then

$$
\text{Lemma 2 (T. Ruustr [14: Subsection 5.4]):} \text{ Let } 0 < q \leq \infty, s > n \text{ (1/min (p, -1), } \varrho > \max(1, s) \text{ and } g \in \tilde{C}^{\varrho}(\overline{u(\Omega)}) \text{. Then}
$$
\n
$$
\text{(i) if } 0 < p < \infty, \text{ then}
$$
\n
$$
\|g(u) + F_{p,q}^{s}\| \leq \|g(u) + L_p\| + c' \left(\sum_{l=1}^{\lfloor \varrho \rfloor} \sup_{x \in \Omega} |g^{(l)}(u(x))| \, \|u + F_{p,q}^{s}\| \, \|u + L_{\infty}\|^{l-1} + \|g + C^{\varrho}(\overline{u(\Omega)})\| \, \|u + F_{p,q}^{s}\| \, \|u + L_{\infty}\|^{l-1} + \|g + C^{\varrho}(\overline{u(\Omega)})\| \, \|u + F_{p,q}^{s}\| \, \|u + L_{\infty}\|^{l-1} + \|g + C^{\varrho}(\overline{u(\Omega)})\| \, \|u + F_{p,q}^{s}\| \, \|u + L_{\infty}\|^{l-1}
$$

$$
\qquad \qquad and
$$

 \vert \vert

$$
\text{On the Existence of Solutions} \qquad 100
$$
\n
$$
\|g(u) \mid L_p\| \le \|g\| C^1(\overline{u(\Omega)})\| \|u\| L_p\| \qquad \left(u \in \tilde{F}_{p,q}^s(\Omega) \cap L_\infty(\Omega)\right);
$$
\n
$$
\text{(ii) } \tilde{i}f \cup p \le \infty, \text{ then}
$$
\n
$$
\|g(u) \mid B_{p,q}^s\| \le \|g(u) \mid L_p\| + c \left(\sum_{l=1}^{\lfloor \Omega \rfloor} \sup_{x \in \Omega} |g^{(l)}(u(x))| \|u\| B_{p,q}^s\| \|u\| L_\infty\|^{l-1} + \left\|g\| C^s(\overline{u(\Omega)})\| \|u\| B_{p,q}^s\| \right\| \right)
$$

On the Existence of Solution
\n
$$
\text{and}
$$
\n
$$
||g(u)||L_p|| \le ||g||C^1(u(\Omega))|| ||u||L_p|| \qquad (u \in \tilde{F}_{p,q}^s(\Omega) \cap L_{\infty}(\Omega));
$$
\n(ii) $if 0 < p \le \infty$, then
\n
$$
||g(u)||B_{p,q}^s|| \le ||g(u)||L_p|| + c \left(\sum_{l=1}^{\lfloor \frac{d}{2} \rfloor} \sup_{x \in \Omega} |g^{(l)}(u(x))|| ||u||B_{p,q}^s|| ||u||L_{\infty}||^{l-1} + ||g||C^e(u(\Omega))|| ||u||B_{p,q}^s|| ||u||L_{\infty}||^{l-1} + ||g||C^e(u(\Omega))|| ||u||B_{p,q}^s|| ||u||L_{\infty}||^{l-1} + ||g||C^e(u(\Omega))|| ||u||
$$
\nand
\n
$$
||g(u)||L_p|| \le ||g||C^1(u(\Omega))|| ||u||L_p|| \qquad (u \in \tilde{B}_{p,q}^s(\Omega) \cap L_{\infty}(\Omega)).
$$
\nThe following result is a' consequence of Lemma 2.

 $F_{\mathcal{F}}\left[\|g\|C^{e}(\overline{u(\Omega)})\|\|u\|B_{p,q}^{s}\| \|u\|L_{\infty} \|^{q-1}\right] + \|g\|C^{e}(\overline{u(\Omega)})\|\|u\|B_{p,q}^{s}\| \right)$

$$
|g(u)| \geq ||g(u)|| + c \left(\sum_{l=1}^{n} \sup_{x \in \Omega} |g^{(l)}(u(x))| ||u|| D_{p,q}|| ||u|| D_{\infty}||
$$

+
$$
||g| C^{q}(\overline{u(\Omega)})|| ||u|| D_{p,q}|| ||u|| L_{\infty}||^{q-1} + ||g| C^{q}(\overline{u(\Omega)})|| ||u||
$$

$$
||g(u) || L_p|| \leq ||g| C^{q}(\overline{u(\Omega)})|| ||u|| L_p|| \qquad (u \in \tilde{B}_{p,q}^{s}(\Omega) \cap L_{\infty}(\Omega)).
$$

allowing result is a consequence of Lemma 2.

The following result is a consequence of Lemma 2.

d
 $||g(u)|| L_p|| \leq ||g|| C^1(u(\Omega))|| ||u|| L_p||$ $(u \in \tilde{B}_{p,q}^s(\Omega) \cap L_\infty(\Omega)).$

The following result is a consequence of Lemma 2.

Corollary: Let $0 < p < \infty$ (0 $< p \leq \infty$), $0 < q \leq \infty$ and $s > n/p$.

(i) Let $g \in \tilde{C}^e$, $\rho > \max(1, s)$. Then *(i)* $|g| \leq C^e(\overline{u(\Omega)})|| ||u|| |B_{p,q}^s|| ||u|| |L_{\infty}||^{e-1} + ||g|| C^e(\overline{u(\Omega)})|| ||u|| |B_{p,q}^s||$
 (i) $|L_p|| \leq ||g|| C^1(\overline{u(\Omega)})|| ||u|| |L_p||$ $(u \in \tilde{B}_{p,q}^s(\Omega) \cap L_{\infty}(\Omega))$.

The following result is a consequence of Lemma 2.

Corollary: Let $,\infty$), *w* $||g(u)|| L_p|| \le ||g|| C'(u(\Omega))|| ||u|| L_p||$ $(u \in F_{p,q}^s(\Omega) \cap L_{\infty}$
 $||g(u)|| B_{p,q}^s|| \le ||g(u)|| L_p|| + c \left(\sum_{l=1}^{[p]} \sup_{x \in \Omega} |g^{(l)}(u(x))|| ||u|| B_{p,q}^s|| ||u|| L_{\infty}$
 $+ ||g|| C^e(u(\Omega))|| ||u|| B_{p,q}^s|| ||u|| L_{\infty} |e^{-t} + ||g|| C^e(u(\Omega))$

and
 $||g(u)|| L_p|| \le ||g|| C'(u(\Omega))|| ||u|| L_p||$ Howing result is a consequence of Lemma 2.
 $\text{Iary:} \text{ Let } 0 < p < \infty \ (0 < p \leq \infty), \ 0 < q \leq \infty \ and \ s$
 $g \in \tilde{C}^e, \ 0 > \max \ (1, s)$. Then there exists a function γ_g, γ

independent of u such that
 $\|g(u) \pm F^s_{p,q}\| \leq \gamma$ (i) Let $g \in C$, $g > \text{max}(1, s)$. Then there exists a function γg , γg , γg , γg , ζ ,

$$
||g(u)||F_{p,q}^s|| \leq \gamma_g(||u||F_{p,q}^s||) ||u||F_{p,q}^s|| \text{ for all } u \in \overline{F}_{p,q}^s(\Omega)
$$

$$
\left(||g(u)||B_{p,q}^s||\leqq \gamma_q(||u||B_{p,q}^s||) ||u||B_{p,q}^s|| \text{ for all } u\in \dot{B}_{p,q}^s(\Omega) \right).
$$

 $\tilde{F}_{p,q}^{s}(\Omega)$ *into* $\tilde{F}_{p,q}^{s}(\Omega)$ (*from* $\tilde{B}_{p,q}^{s}(\Omega)$ *into* $\tilde{B}_{p,q}^{s}(\Omega)$).

Proof: *Step* **1.** We prove (i).,Tliis is a consequence of Lemma **2 and** the continuous imbedding $\tilde{F}_{p,q}^s(\Omega) \hookrightarrow \tilde{L}_{\infty}(\Omega) (\tilde{B}_{p,q}^s(\Omega) \hookrightarrow \tilde{L}_{\infty}(\Omega))$ if $s > n/p$.

Step 2. We can extend Lemma 2 and consequently part (i) of the above, corollary to the case $G \in \overline{C}^e$, $G: \mathbb{R}_m \to \mathbb{R}_1$, $m = 1, 2, \ldots$ Let $g: u \to g(u)$ for $u \in \overline{F}_{p,q}^s(\Omega)$. We put $b(x, y) = (g(x) - g(y))/(x - y)$ (x, $y \in \mathbb{R}_1$). Then for *b* $g \in \tilde{C}e^{+1}$, $g > \max(1, s)$. Then $u \rightarrow g(u)$ is a continuous mappin
 bto $\tilde{F}_{p,q}^s(\Omega)$ (from $\tilde{B}_{p,q}^s(\Omega)$ into $\tilde{B}_{p,q}^s(\Omega)$).
 \therefore Step 1. We prove (i). This is a consequence of Lemma 2 and the con
 $g \$ *y*). Then $u \rightarrow g(u)$ is a continuous mapping fr

(*y*) into $\tilde{B}_{p,q}^s(\Omega)$).

This is a consequence of Lemma 2 and the continu
 ${}_{p,q}^s(\Omega) \hookrightarrow \tilde{L}_{\infty}(\Omega)$ if $s > n/p$.

(a 2 and consequently part (i) of the above coroll

$$
b(u, v) (\cdot) = b(u(\cdot), v(\cdot)) = \frac{g(u) - g(v)}{u - v} (\cdot)
$$

bo the case $c_i \in C^2$, $G: \mathbf{h}_m \to \mathbf{h}_1$, $m \equiv 1, 2, ...$ Let $g: u \to g(u)$ for $u \in F_{p,q}(s_2)$. We
put $b(x, y) = (g(x) - g(y))/(x - y)$ $(x, y \in \mathbf{R}_1)$. Then for
 $b(u, v) (\cdot) = b(u(\cdot), v(\cdot)) = \frac{g(u) - g(v)}{u - v} (\cdot)$
we obtain that $||b(u, v) | F_{p,q}^s|| \le \$ $\begin{split} b(u,v)\ (\cdot) & = \partial(u(\cdot),v(\cdot))\ b(u,v)\ |\ F_{p,q}^s\| \ (y),&~x,~y\geqq 0\} \rightarrow [0,\infty) \ \mathrm{i}\ ||g(u)-g(v)\ |\ F_{p,q}^s\| \leqq \ \mathrm{with~} b\ \mathrm{arithmetic}(0). \end{split}$

$$
||g(u) - g(v) | F_{p,q}^s|| \leq \gamma'||u | F_{p,q}^s||, ||v| | F_{p,q}^s|| ||u - v | F_{p,q}^s||, \quad \sigma
$$
 (9)

which yields the continuity. (9) follow from the fact that $\tilde{F}_{p,q}^s(Q)$ is a multiplication algebra if *s* > *n/p,* cf. H. **TRIEBEL [16: Subsection 33.2],** J. FANKE **[3:** Subsection 3.3], and T. RUNST [14: Subsection 5.3]. The proofs in the case $B_{p,q}^s$ are almost the same **B** algebra if $s > n/p$, cf. H. TRIEBEL [16: Subsection 3.3.2], J. FRA:
3.3], and T. RUNST [14: Subsection 5.3]. The proofs in the case.
same **1**
Remark 3: Results in this direction were also proved by J. PETRE
 $s > n/p$, $1 \le p, q \$ $||g(u) - g(v)| + F_{p,q}|| \leq y'(||u| + F_{p,q}||, ||v| + F_{p,q}||) ||u - v| + F_{p,q}||,$ **c** (9)

Elds the continuity (9) follow from the fact that $\tilde{F}_{p,q}^s(Q)$ is a multiplication

is $> n/p$, cf. H. TRIEBEL [16: Subsection 3.3.2], **J. FRANKE** [3: Su

Remark 3: Results in this direction were also proved by J. PEETRE [13] in the case $\tilde{B}_{n,q}^s$, **3.3], and T. RUNST [14: Subsection 5.3]. The proofs in the case** $B_{p,q}^s$ **are almost the same** \blacksquare **
Remark 3: Results in this direction were also proved by J. PEETRE [13] in the case** $\tilde{B}_{p,q}^s$ **,** $s > n/p$ **,** $1 \le p$ **, q \le \ ZAKI** [18] and **J.MAR5CHALL [10]. Solution Solution** is this direction were also proved by J. PEETH $\{p, 1 \leq p, q \leq \infty, \text{ by } Y. \text{ Meygen } [11, 12] \text{ in the case } \tilde{H}_p^{\delta}, 1 < p < 18] \text{ and J. MasseHALL } [10].$ algebra if $s > n/p$, cf. H. TRIEBEL [16: Subsection 3.3.2], J. FRANKE [3: Subsection
3.3.2], J. FRANKE [3: Subsection
same **i**
B. and T. Rowser [14: Subsection 5.3]. The proofs in the case $B_{p,q}^s$ are almost the
same **i**

3. The main result

Let Ω , be a bounded C^{∞} -domain in \mathbf{R}_n and let

- - . -

$$
Lu = -g(u) + f(x) + t \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega
$$

be a second order elliptic boundary'value problem, where *L* is a uniformly elliptic second order differential operator as described by (5) and $t \in \mathbb{R}_1$.

. Theorem: Let $0 < p, q \leq \infty$, $s > n/p$, $t \in \mathbb{R}_1$ and let $g \in \tilde{C}^{e+1}(\mathbb{R}_1)$, $\varrho > \max(1, s)$, **i** 110 P. DRABEK and T. RUNST
 \therefore Theorem: Let $0 < p, q \le \infty$, $s > n/p$, $t \in \mathbb{R}_1$ and let $g \in \tilde{C}^{e+i}(\mathbb{R}_1)$, $\varrho > \max(1, s)$, satisfy the conditions $(I) g(x) \ge 0$ if $x \ge 0$ and $g(0) = 0$, (II) the function $g(x) + \lambda_$ *satisfy the conditions* (I) $g(x) \ge 0$ *if* $x \ge 0$ *and* $g(0) = 0$, (II) the function $g(x) + \lambda_1 x$, where $\lambda_1 > 0$ *is the first eigenvalue of L*, *is bounded from below.* **a** if $Let \ 0 < p, q \leq \infty, s > n/p, t \in \mathbf{R}_1$ and let $g \in \tilde{C}^{e+i}(\mathbf{R}_1), e \geq \max(1, s)$,
 a conditions (I) $g(x) \geq 0$ if $x \geq 0$ and $g(0) = 0$, (II) the function $g(x) + \lambda_i x$,
 > 0 is the first eigenvalue of L , is bou

(i) Let $p < \infty$ and $f \in \tilde{F}_{p,q}^{s-2}(\Omega) \cap L_{\infty}(\Omega)$. Then there exists a $t_0(f) \in \mathbb{R}_1$ such that problem (10) has at least one solution $u \in \tilde{F}_{p,q}^s(\Omega)$ if $t > t_0$ and has no solution if $t < \tilde{t}_0$.

(ii) Let $f \in \tilde{B}_{p,q}^{s-2}(\Omega) \cap L_{\infty}(\Omega)$. Then there exists a $t_0(f) \in \mathbf{R}_1$ such that problem (10) has.

at least one solution $u \in \tilde{B}_{p,q}^s(\Omega)$, if $t > t_0$ and has no solution if $t < t_0$.

Proof: We prove (i). The proof of (ii) is the same.

Step 1. First of all we remark that the Dirichlet problem $Lu = \lambda u$ in Ω , $u = 0$ on $\partial\Omega$ has a unique normed positive eigenfunction φ^* corresponding to the smallest **Example 10** Construction $\mu \in T_{p,q}(\Sigma)$ of $\iota > \iota_0$ and has no solution $\iota_1 \iota < \iota_0$.

(ii) Let $f \in \tilde{B}_{p,q}^* (\Omega) \cap L_{\infty}(\Omega)$. Then there exists a $t_0(f) \in \mathbf{R}_1$ such that problem (10) has.
 Least one solutio eigenvalue $\lambda_1 > 0$, see (6). It is known that $\varphi^* > 0$ in Ω and

$$
\partial \varphi^* / \partial \nu < c < 0 \text{ on } \partial \Omega. \tag{11}
$$

satisfy the conditions (1) $g(x) \geq 0$ if $x \geq 0$ and $g(0) = 0$, (11) the function $g(x) + \lambda_1 x$,
where $\lambda_1 > 0$ is the first eigenvalue of L, is bounded from below.
(i) Let $p < \infty$ and $f \in \tilde{F}_{p,q}^{s-2}(\Omega) \cap L_{\infty}(\Omega)$. T *Step* 2. We have $f \in \tilde{F}_{p,q}^{s-2}(\Omega) \cap L_{\infty}(\Omega)$. Hence we can choose $t > 0$ so large that $f(x) + t > 0$ holds for $x \in \Omega$. Then $u_1 = 0$ is a subsolution of (10). Here we used the fact that $g(0) = 0$. We fix such a *t*.
Step 3. Because of $f \in \tilde{F}_{p,q}^{s-2}(\Omega) \cap L_{\infty}(\Omega)$ and the properties of *L* (see (7), (8)) there

(1) Let $p < \infty$ and $f \in F_{p,q}^*(\Omega) \cap L_{\infty}(\Omega)$. Then there exists
problem (10) has at least one solution $u \in \tilde{F}_{p,q}^*(\Omega)$ if $t > t_0$ and h
(ii) Let $f \in \tilde{B}_{p,q}^*(\Omega) \cap L_{\infty}(\Omega)$. Then there exists a $t_0(f) \in \mathbb{R}_1$ s exists a function $w \in \tilde{F}_{p,q}^s(\Omega)$ such that $Lw > f(x) + t$ in Ω and $w = 0$ on $\partial\Omega$. Notice that $\tilde{F}_{p,q}^s(\Omega) \hookrightarrow \tilde{C}(\overline{\Omega})$ if $s > n/p$. Now we choose $r > 0$ such that $w + r\varphi^* > 0$ in Ω holds. Therefore we apply (11) and the same arguments as in S. Fučik $[7: Subsec$ tions 34.11 and 34.12]. Then $u_2 = w + r\varphi^*$ is a supersolution of (10) because of $\mu_2 > -g(u_2) + f(x) + t$ in Ω , $u_2 = 0$ on $\partial\Omega$. Here we used property (1) of the function.g. We have $u_2 > u_1$ in Ω . on $w \in \tilde{F}_{p,q}^* (\Omega)$ such that $Lw > f(x) + t$ in Ω and $w = 0$
 $\geq \tilde{C}(\overline{\Omega})$ if $s > n/p$. Now we choose $r > 0$ such that $w +$

ore we apply (11) and the same arguments as in S. Fuči

id 34.12]. Then $u_2 = w + r\varphi^*$ is a

Step 4. In what follows we show: If u_1 is a subsolution and u_2 a supersolution of (10) and $u_1(x) \leq u_2(x)$, $x \in \Omega$, then there exists a function $u \in \tilde{F}_{p,q}^s(\Omega)$ such that $u_1 \leq u_2$ $\leq u_2$ in *Q* and (10) holds. This result is a, generalization of H. AMANN [1], see also **S. FUóIK** [7: Subsection 34.7]. We apply the (same argumens_asJ. **FRANKE** and.' T. RUNST [6: Subsection 3.4]. The above conditions yield $u_{1,2} \in \tilde{C}(\bar{\Omega})$. Let $\omega > 0$ be. and $u_1(x) \le u_2(x)$, $x \in \Omega$, then there exists a function $u \in \tilde{F}_{p,q}^s(\Omega)$ such that $u_1 \le u_2$
 $\le u_2$ in Ω and (10) holds. This result is a generalization of H. AMANN [1], see also

S. FUCKIK [7: Subsection 34.7] ctic
ctic
-**Example 1.** In what follows we show: If u_1 is a subsolution and u_2 a supersolution'of (10) and $u_1(x) \le u_2(x)$, $x \in \Omega$, then there exists a function $u \in \tilde{F}_{p,q}^s(\Omega)$ such that $u_1 \le u \le u_2$ in Ω and (10) hold **Lv** and 34.12]. Then $u_2 = w + r\varphi^*$ is a supersolution of (10) because of $g(u_2) + f(x) + t$ in Ω , $u_2 = 0$ on $\partial\Omega$. Here we used property (1) of the func-
 Le have $u_2 > u_1$ in Ω .

In what follows we show: If u_1 *Step 4.* In what \mathbf{f}

and $u_1(x) \leq u_2(x)$,
 $\leq u_2$ in Ω and (10
 S. FUCKIE [7: Subs

T. RUNST [6: Subs

such that
 $\omega - g'(\xi)$

Let *T* be the op
 $v \in \bigcup {\{\tilde{B}_{\infty,\infty}^{\epsilon}(\Omega) : \varepsilon : \xi \in \mathbb{R}^n : \xi \in \mathbb{R}^n \text{$

Q

Let *T* be the operator which sassigns to each $u \in \tilde{C}(\overline{\Omega})$ the unique solution $v \in \bigcup {\{\tilde{B}_{\infty,\infty}^{\epsilon}(\Omega) : \epsilon > 0\}}$ of

$$
Lv + \omega v = f(x) + t - g(u) + \omega u \text{ in } \Omega, \qquad v = 0 \text{ on } \partial \Omega.
$$

The definition of T is correct with respect to $\omega > 0$ and the properties of *L*. We put $u_1^{(k)} = T^k u_1$ and $u_2^{(k)} = T^k u_2$, where as usual $T^1 = T$, $T^{k+1} = T^1 T^k$. In analogy to S. FUCIK [7: Subsection 34.7] we can show that *T* is monotone, i.e. if $u \le v$ in Ω , then $Tu \leq Tv$ in Ω . For this we apply Lemma 1 (maximum principle in the sense of distributions). By induction we get a monotonically decreasing sequence Let T be the opera
 $v \in \bigcup {\{\tilde{B}^{\epsilon}_{\infty,\infty}(\Omega): \epsilon > 0} \$
 $- Lv + \omega v = f$

The definition of T is
 $u_1^{(k)} = T^k u_1$ and $u_2^{(k)}$

S. FUCK [7: Subsection

then $T u \leq Tv$ in Ω . F

distributions). By indued
 $u_1 \leq u_1^{(1)} \leq$

$$
u_1 \leq u_1^{(1)} \leq u_1^{(2)} \leq \cdots \leq u_2^{(2)} \leq u_2^{(1)} \leq u_2.
$$
 (13)

From (8), the Corollary in Subsection 2.3 and $u_{1,2} \in L_{\infty}(\Omega)$ we deduce by (12) the $u_1 \leq u_1^{(1)} \leq u_1^{(2)} \leq \cdots \leq u_2^{(2)} \leq u_2^{(1)} \leq u_2.$ (13)

From (8), the Corollary in Subsection 2.3 and $u_{1,2} \in L_{\infty}(\Omega)$ we deduce by (12) the

'inequality $||u_{1,2}^{(k+1)}F_{p,q}^s|| \leq c_1 + c_2 ||u_{1,2}^{(k)}[F_{p,q}^{s-1}||$ $\leq Tv$ in Ω . For this we apply Lemma 1 (nois). By induction we get a monotonicall:
 $u_1 \leq u_1^{(1)} \leq u_1^{(2)} \leq \cdots \leq u_2^{(2)} \leq u_2^{(1)} \leq u_1$,

the Corollary in Subsection 2.3 and $u_{1,2}$
 $|u_{1,2}^{(k+1)}F_{p,q}^s| \leq c_$ *i* and the property $||u_{1,2}^{(k+1)}F_{p,q}^s|| \leq c_1 + c_2 ||u_{1,2}^{(k)}| \cdot F_{p,q}^{s-1}||$. The imbedding $\bar{C}(\Omega) \hookrightarrow \bar{F}_{p,q}^{-(\Omega)}(\Omega)$ and

13) yield $||u_{1,2}^{(k)}| \cdot F_{p,q}^{-\epsilon}|| \leq c \max(|u_1| |C|, ||u_2| |C|) < c_3$.

$$
||u_{1,2}^{(k)} | F_{p,q}^{-\epsilon} || \leq c \max_{\alpha} (||u_1 || C||, ||u_2 || C||) < c_3.
$$

•

 $\frac{1}{2}$

 (14)

 (15)

Applying the well-known inequality $||w|F_{p,q}^{s-\epsilon}|| \leq \delta ||w|F_{p,q}^{s}|| + c_{\delta} ||w|F_{p,q}^{-1}||$, it follows that $||u_{1,2}^{(k+1)}||F_{p,q}^s|| \leq c_4 + 2^{-1} ||u_{1,2}^{(k)}||F_{p,q}^s||$. By induction we get

$$
||u_{1,2}^{(k)}||F_{p,q}^{s}|| \leq C, \qquad C > \max(2c_4, ||u_1||F_{p,q}^{s}||, ||u_2||F_{p,q}^{s}||).
$$

Because of $\tilde{F}_{p,q}^s(Q) \hookrightarrow \tilde{C}(\overline{Q}), s > n/p$, the pointwise limits

$$
u_{\pm}(x)=\lim u_{1.2}^k(x)
$$

both exist. Now we apply the same arguments as J. FRANKE and T. RUNST [6: Subsection 3.4]. Let $\mathscr S$ be the coretraction constructed in J. FRANKE [3: Subsection 4.1]. We may assume that supp $\mathscr{S}\varphi$ is uniformly bounded for all φ . The construction of \mathscr{S} yields $\mathcal{J}u_{1,2}^{(k)}(x) \to \mathcal{J}u_{\pm}(x)$ pointwise. L'ebesgue's theorem proves that this holds also for the weak $\sigma(S(\mathbf{R}_n), S'(\mathbf{R}_n))$ -topology. Applying the Fatou property (cf. J. FRANKE [3: Subsection 2.6]) we get $u_{\pm} \in \tilde{F}_{p,q}^s(\Omega)$. By means of $\tilde{F}_{p,q}^s(\Omega) \hookrightarrow \tilde{C}(\overline{\Omega})$ if $s > n/p$ it follows that $u_{\pm} \in \tilde{C}(\bar{\Omega})$. Finally Dini's theorem yields that (14) holds in $\tilde{C}(\bar{\Omega})$. Now it is not hard to check that u_{\pm} are solutions of (10).

Step 5. Here we prove the following: If (10) is solvable for some $t_1 \in \mathbb{R}_1$, then it is also solvable for all $t > t_1$. Let $t_2 > t_1$. The solution of (10) with $t = t_1$ is denoted by u_1 . Then $Eu_1 < -g(u_1) + f(x) + t_2$ in Ω and $u_1 = 0$ on $\partial\Omega$. Hence u_1 is a subsolution of (10) with $t = t_2$. Let us choose $v \in \tilde{F}_{p,q}^s(\Omega)$ such that $Lv > f(x) + t_2$ in Ω and $v = 0$ on $\partial\Omega$ and $\tilde{r} > 0$ such that $v + \tilde{r}\varphi^* > u_1$ in Ω . Therefore we use the same arguments as in Step 3. Then $u_2 = v + \tilde{r}\varphi^*$ is a supersolution of (10) with $t = t_2$. In analogy to Step 4 there exists a $u \in \tilde{F}_{p,q}^s(\Omega)$ satisfying $u_1 \leq u \leq u_2$ and $Lu = -g(u) + f(x) + t_2$ in Ω , $u = 0$ on $\partial\Omega$.

Step 6. We put $t_0 = \inf \{t \in \mathbb{R}_1 \mid (10) \text{ is solvable}\}\.$ We remark that, by Step 4, $t_0 < \infty$.

Step 7. We show that $t_0 > -\infty$ holds. According to the properties of g the function

$$
s \to -g(s) - \lambda_1 s
$$

is bounded from above. Let u be a solution of (10) for some $t \in \mathbb{R}_1$. Then we get

$$
Lu - \lambda_1 u = -g(u) - \lambda_1 u + f(x) + t \text{ in } \Omega. \tag{16}
$$

Note that the adjoint problem to $L\varphi - \lambda_1 \varphi = 0$ in Ω , $\varphi = 0$ on $\partial \Omega$ has also only a one-dimensional space of solutions generated by a positive function $\psi^* \in C^{\infty}(\overline{\Omega})$ with $\int \psi^*(x)^2 dx = 1$ and $\psi^*(x) \ge 0$ if $x \in \overline{\Omega}$ (cf. S. Fučik [7: Subsection 34.12]).

Hence it follows from (16) that

$$
0 = \int_{\Omega} (Lu - \lambda_1 u) (x) \psi^*(x) dx
$$

$$
\angle = \int_{\Omega} (-g(u) - \lambda_1 u) (x) \psi^*(x) dx + \int_{\Omega} f(x) \psi^*(x) dx + t \int_{\Omega} \psi^*(x) dx,
$$

which together with the boundedness of (15) implies that $t = t(f)$ holds \blacksquare

Remarks: 4. A corresponding result holds also in the case $Lu = g(u) + f(x) + t$ in Ω , $u = 0$ on $\partial\Omega$. 5. If $u_i \in \tilde{F}_{p,q}^s(\Omega)$ are solutions of $Lu = -g(u) + f_i(x)$ in Ω , $u = 0$ on $\partial\Omega$, $i=1, 2$, then $f_1 \leq f_2$ implies $u_1 \leq u_2$. 6. Note that the above proof yields the following result: for each $f \in \widetilde{F}_{p,q}^s(Q)$ in $\widetilde{L}_{\infty}(\Omega)$ there exists a $t_1(f) \geq t_0(f)$ such that (10) has a positive solution for all $t > t_1(f)$.

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