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Bifurcation and Stability of Capillary-Gravity Waves

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Es werden dreidimensionale Kapillar-Schwerewellen bei der stationären Strömung einer Flüssigkeit endlicher Tiefe über ebenem Boden betrachtet. Für die kleinsten kritischen Froude-Zahlen zweigt eine zweidimensionale Welle von der trivialen Oberflächenform ab.

Исследуются трёхмерные гравитационные капиллярные волны в стационарном потоке жидкости конечной глубины над плоским дном. Для наименьших критических чисел Фроуда возникает двумерная волна разветвляющаяся от тривиальной конфигурации поверхности.

Three-dimensional capillary-gravity waves of a fluid of finite depth are considered. The fluid moves stationary over a plane bottom. For smallest critical Froude numbers there is a two-dimensional wave bifurcating from the trivial surface configuration.

1. Introduction

1.1. Introductory remarks. Starting with Zeidler [7], where a large class of wave problems in two dimensions is studied by one method (conformal mapping on the unit circle), we will give an example treating uniformly three-dimensional problems. Consider the three-dimensional irrotational stationary motion of an inviscid incompressible fluid of constant density ϱ and of finite depth B in the presence of gravity $(0, 0, -g)$ with surface tension β acting at the free surface Γ. A trivial configuration of Γ for constant velocity $(U, 0, 0)$ is a horizontal plane, which we take as $Z = 0$. We are seeking for periodic and small amplitude functions $Z = H(X, Y)$ as solutions for Γ. To do this, we formulate the variational problem for the boundary value problem with given surface Γ. By normalization the solution Φ is uniquely determined. Inserting Φ in the energy functional $\bar{E}: H \to \bar{E}(H)$ of the system with free surface, we get equilibrium states by minimization of $\bar{E}$ in dependence on H. We will see that stationary points of $\bar{E}$ satisfy the Bernoulli equation of the boundary value problem with free Γ. With vanishing second variation we get the set of critical *Froude numbers* F_c and critical *Bond numbers* b_c. In two dimensions, this set was studied by Kirchgässner [1]. For uncritical Froude numbers F and Bond numbers b, the problem in three dimensions but in the case of periodic bottom was studied by Shinbrot [5]. We are studying the smallest critical Froude numbers. With the results of Beyer [2] we show: in the corresponding Sobolev spaces the minimum of $\bar{E}$ and its first and second variation are analytic in a small neighbourhood of critical points. We get the bifurcation equation via Ljapunov-Schmidt procedure. The symmetries underlying the physical problem give a nice structure of the equation. So we can solve it only with the Implicit Function Theorem. The solution is a two-dimensional wave $H(X) = a(\varepsilon, \mu) \cos (mX/l + \delta)$ with a depending on small parameters describing a neighbourhood of b_c, $b = b_c(1 - \mu)$, and F_c, $F = F_c(1 + \varepsilon)$; m is determined by b_c, l follows from the starting periodicity $2\pi l$ of $H(X, Y)$, and δ is free as the consequence of symmetry properties. But our variational approach also gives local stability results. We find that the second variation is positive if $|\varepsilon|$ and $|\mu|$ are small enough and if the wave-length l/m is less than the product of the mean depth B and a constant $c \approx 4/5$.

1.2. The boundary value problem. We have to solve the boundary value problem ($\Delta\Phi = \Phi_{XX} + \Phi_{YY} + \Phi_{ZZ}$)

$$\begin{aligned} &\Delta\Phi = 0 \quad \text{for } -B < Z < H(X, Y), \\ &\Phi_X H_X + \Phi_Y H_Y = \Phi_Z \text{ on } \Gamma, \quad \text{and} \quad \Phi_Z = 0 \text{ for } Z = -B. \end{aligned} \tag{1}$$

Moreover, on Γ we have to fulfil the Bernoulli equation ($\nabla = (\partial/\partial X, \partial/\partial Y, \partial/\partial Z)$ and $\nabla' = (\partial/\partial X, \partial/\partial Y)$)

$$|\nabla\Phi|^2/2g + H - \operatorname{div}\left(\nabla' H/\sqrt{1 + |\nabla' H|^2}\right)\beta/\varrho g = \text{const.} \tag{2}$$

Let $\mathbb{Z}$ denote the integers and $\mathbb{R}^+$ the positive real numbers. For fixed $l, k \in \mathbb{R}^+$ we define a lattice $\bar{\Lambda}' = \{\omega' = k_1\omega_1' + k_2\omega_2' : k_1, k_2 \in \mathbb{Z}\}$, $\omega_1' = 2\pi l(1, 0)$ and $\omega_2' = 2\pi k(0, 1)$. Let R' be the rectangle which is formed by ω_1' and ω_2' and let $|R'|$ be its area. The dual lattice $\bar{\Lambda}$ of $\bar{\Lambda}'$ is given by $\bar{\Lambda} = \{\omega : \omega'\omega/2\pi \in \mathbb{Z} \text{ for all } \omega' \in \bar{\Lambda}'\}$. A basis in $\bar{\Lambda}$ is

$$\bar{\omega}_1 = (0, 1)/k \quad \text{and} \quad \bar{\omega}_2 = (1, 0)/l. \tag{3}$$

$\bar{\mathbf{H}}_m$ denotes the Sobolev space of $\bar{\Lambda}'$-periodic functions $H(X, Y) = \sum_{\omega \in \Lambda} H_\omega e^{i\omega x}$ where $\omega x = \omega_1 X + \omega_2 Y$ for $\omega = (\omega_1, \omega_2)$, with finite norm $\|H\|_m^2 = |H_0|^2 + \sum_{\omega \in \bar{\Lambda}} |\omega|^2 |H_\omega|^2$. For the sake of incompressibility we assume

$$\int_{R'} H(X, Y)\, dX\, dY = 0. \tag{4}$$

Let $\Phi(X, Y, Z) = U\big(\tilde{\Phi}(X, Y, Z) + X\big)$ and assume $\tilde{\Phi}$ to be $\bar{\Lambda}'$-periodic, which means that the X-axis has common direction with the mean flow $\mathsf{U} = \int_{R'} \nabla\Phi|_{Z=H(X,Y)}\, dX\, dY / |R'|$, hence, $U = |\mathsf{U}|$. So we consider our problem only in $\Omega = R' \times \{-B < Z < H(X, Y)\}$. By the transformation $X = Bx_1$, $Y = Bx_2$, $Z = B\big(x_3 + v(x_1, x_2, x_3)\big)$, where

$$\begin{aligned} &v(x_1, x_2, x_3) = \sum h_\omega e^{i\omega x} \sinh(|\omega| + |\omega| x_3)/\sinh|\omega|, \\ &H(X, Y) = B\, h(x_1, x_2), \qquad \tilde{\Phi}(X, Y, Z) = B\, \varphi(x_1, x_2, x_3), \end{aligned} \tag{5}$$

we map the fluid region Ω on a region $S = R \times \{-1 < x_3 < 0\}$ (R denotes the rectangle formed by $\omega_1'B$ and $\omega_2'B$) with fixed boundary and handle with dimensionless quantities. We remark that v belongs to some function class (later see the proof of Lemma 2), whereas a simpler transformation $v(x_1, x_2) = \sum h_\omega e^{i\omega x}$, for instance, would not have this property. The transformed dual lattice we call $\Lambda = \{\omega = (m/l, n/k)/B : m, n \in \mathbb{Z}\}$. Then h belongs to $\mathbf{H}_m$, the corresponding Sobolev space over the transformed lattice Λ'. We force the uniqueness of the solution of (1) by normalizing $\int_R \varphi(x_1, x_2, 0)\, dx_1\, dx_2 = 0$ (SHINBROT [5]). Finally we define the Bond number $b = \beta/\varrho g B^2$ and the Froude number $F = U^2/gB$.

2. Energy functional, properties and variations

2.1. The equivalent variational problem. The potential energy of our physical system is $\varrho g\bar{E}$ with

$$\bar{E} = bB^2 \int_{R'} \sqrt{1 + |\nabla' H|^2}\, dX\, dY + \int_{R'} \frac{H^2}{2}\, dX\, dY + \frac{FB}{2U^2} \int_{\Omega} |\nabla\Phi|^2\, dZ\, dX\, dY,$$

where $\Phi(X, Y, Z) = U(\tilde{\Phi}(X, Y, Z) + X)$ is assumed to be $\bar{\Lambda}'$-periodic and has to solve the boundary value problem (1) for given $H \in \bar{\mathbf{H}}_m$. Then $\bar{E}$ defines a functional over $\bar{H}_m$. For $t \in \mathbb{R}$ we consider a family of surfaces $\Gamma_t: Z = H(X, Y) + t\bar{\zeta}(X, Y)$ with $\Gamma_0 = \Gamma$, where $\bar{\zeta}$ is $\bar{\Lambda}'$-periodic and satisfies (4). The potentials we call Φ again and write $\partial\Phi(X, Y, Z, t)/\partial t = \dot{\Phi}(X, Y, Z, t)$. As first variation of $\bar{E}$ we then have ($\bar{E}_H$ denoting the Gateaux derivative)

$$\begin{aligned}\langle \bar{E}_H, \bar{\zeta}\rangle \equiv d\bar{E}(H + t\bar{\zeta})/dt|_{t=0} &= bB^2 \int_{R'} \nabla' H\, \nabla'\bar{\zeta}/\sqrt{1 + |\nabla' H|^2}\, dX\, dY \\ &+ \int_{R'} H\bar{\zeta}\, dX\, dY + FB/(2U^2) \int_{R'} \bar{\zeta}\, |\nabla\Phi|^2\, dX\, dY \\ &+ FB/U^2 \int_{\Omega} \nabla\Phi\, \nabla\dot{\Phi}\, dZ\, dX\, dY.\end{aligned}$$

Because

$$\begin{aligned}0 = \int_{R'} \partial/\partial X \int_{-B}^{H(X,Y)} \dot{\Phi}_X \Phi\, dZ\, dX\, dY &= \int_{\Omega} (\dot{\Phi}_{XX}\Phi + \dot{\Phi}_X \Phi_X)\, dV \\ &+ \int_{R'} H\dot{\Phi}_X \Phi_X\, dX\, dY,\end{aligned}$$

partial integration gives

$$\begin{aligned}\langle \bar{E}_H, \bar{\zeta}\rangle &= \int_{R'} \bar{\zeta} \left(-bB^2 \operatorname{div}\left(\nabla' H/\sqrt{1 + |\nabla' H|^2}\right) + H + \left(FB/(2U^2)\, |\nabla\Phi|^2\right)\right) dX\, dY \\ &- (FB/U^2)\left(\int_{\Omega} \dot{\Phi}\Delta\Phi\, dZ\, dX\, dY + \int_{R'} \dot{\Phi}\, \partial\Phi/\partial n\, dX\, dY - \int_{R'} \dot{\Phi}\Phi_Z|_{Z=-B}\, dX\, dY\right)\end{aligned}$$

where $\partial/\partial n$ denotes the derivative in normal direction. Since Φ solves (1), the integrals containing $\dot{\Phi}$ are vanishing. So by suitable choice for $\bar{\zeta}$ the equation $\langle \bar{E}_H, \bar{\zeta}\rangle = 0$ implies (2) (see also WHITHAM [6: p. 435ff.]).

Finally we note E, the transformed energy functional divided by $\varrho g B^4$. In the following we write $\partial f/\partial x_i = f_{,i}$ for $i = 1, 2, 3$. Since there can be no confusion, we call $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) = \nabla$ and $(\partial/\partial x_1, \partial/\partial x_2) = \nabla'$ again. The relation (5) transforms the problem $\bar{E} \to \min$ into the problem

$$E = b \int_{R'} \sqrt{1 + |\nabla' h|^2}\, dx_1\, dx_2 + \frac{1}{2} \int_{R'} h^2\, dx_1\, dx_2 + FJ \to \min, \tag{6}$$

where

$$\begin{aligned}2J &= \int_S \left(|\nabla\varphi|^2 (1 + v_{,3}) - 2\nabla v\, \nabla\varphi\, \varphi_{,3} + |\nabla v|^2 \varphi_{,3}^2 \,|\, (1 + v_{,3})\right) dV \\ &- 2 \int_R \varphi(x_1, x_2, 0)\, h_{,1}\, dx_1\, dx_2\end{aligned}$$

with $dV = dx_1\, dx_2\, dx_3$. Taking (1) into account with $\bar{\zeta}(X, Y) = B\zeta(x_1, x_2)$, the transformed first variation is

$$\langle E_h, \zeta\rangle = \int_R \zeta \left(-b \operatorname{div}\left(\nabla' h/\sqrt{1 + |\nabla' h|^2}\right) + h\right) dx_1\, dx_2 + F\langle J_h, \zeta\rangle. \tag{7}$$

2.2. Critical points. We compute the second variation of E at $h = 0$. From the stability theory it is clear that bifurcations may occur at points where the second variation looses its positivity. We have

$$E_{hh}(\zeta, \zeta)|_{h=0} = \int_R (b\,|\nabla\zeta|^2 + \zeta^2)\,dx_1\,dx_2 - FJ_{hh}(\zeta, \zeta)|_{h=0}$$

with

$$J_{hh}(\zeta, \zeta)|_{h=0} = \int_S |\nabla\dot{\varphi}|^2\,dV - 2\int_R \dot{\varphi}(x_1, x_2, 0)\,\zeta_{,1}\,dx_1\,dx_2, \tag{8}$$

where $\dot{\varphi}$ is determined by the Euler equation $-\int_S \psi\,\Delta\dot{\varphi}_1\,dV + \int_R \psi(\dot{\varphi}_{1,3} - \zeta_{,1})\,dx_1\,dx_2$, whose solution is $\dot{\varphi}_1 = \sum \dot{\varphi}_{1\omega}\,e^{i\omega x}$, $\dot{\varphi}_{1\omega} = i\omega_1\zeta_\omega \cosh(|\omega|\,x_3 + |\omega|)/(|\omega| \sinh|\omega|)$, and from now on $\omega x = \omega_1 x_1 + \omega_2 x_2$. This we put into (8), obtaining

$$E_{hh}(\zeta, \zeta)|_{h=0} = \sum_{\omega\in\Lambda} \gamma(b, F, \omega)\,|\zeta_\omega|^2$$

with $\gamma(b, F, \omega) = 1 + b\,|\omega|^2 - F\omega_1^2(\coth|\omega|)/|\omega|$. Denote $\tilde{F} = |\omega|\,(1 + b\,|\omega|^2) \times (\tanh|\omega|)/\omega_1^2$, then $E_{hh}(\zeta, \zeta)|_{h=0}$ remains positively definite, until $F < F_c = \min\{\tilde{F}(\omega) : \omega \in \Lambda \setminus \{0\}\}$. For these F the trivial solution of (6), $h \equiv 0$, is stable with respect to the Λ'-periodic perturbations. Since the linear map $L \equiv E_{hh}|_{F=\tilde{F}}$ may possess nontrivial kernel $N(L)$, we call the $\tilde{F}$ critical. Here we are only studying F_c described by

Lemma 1: *The following assertions are true:*

(i) *If $b \geqq 1/3$, $N(L)$ is trivial and $F_c = 1$.*

(ii) *If $b < 1/3$, $\tilde{F}$ attains its minimum only at points $(r, 0)$, $r = m/(Bl)$, $m \in \mathbb{Z}$. Every $r \neq 0$ uniquely determines the critical points.*

$$b_c = \frac{\sinh(2r) - 2r}{r^2(\sinh(2r) + 2r)}, \quad F_c = \frac{4\sinh^2 r}{r(\sinh(2r) + 2r)}, \tag{9}$$

and $h_1 = e^{irx_1}$ and $h_2 = e^{-irx_1}$ are a basis in $N(L)$.

(iii) *If $|\omega| \neq r$, then $\gamma(|\omega|, b_c(r), F_c(r)) > 0$.*

Proof: $\tilde{F}$ attains its minimum for $\omega_1^2 = |\omega|^2$. So we are studying the case $\omega = (s, 0)$, $s \in \mathbb{R}$. Consider

$$\tilde{F}_s = \partial\tilde{F}/\partial s = \big((bs^2 - 1)\sinh(2s) + (bs^2 + 1)\,2s\big)/(2s^2\cosh^2 s).$$

$\tilde{F}(0)$ is a minimum of $\tilde{F}$ if $\tilde{F}_{ss}(0) = 2(b - 1/3) > 0$. So $F_c = \tilde{F}(0, b) = 1$ for all $b \geqq 1/3$. $N(L)$ has the basic elements $e^{i\omega x}$ with $\omega \in \Lambda$ and $\gamma(b_c, F_c, \omega) = 0$.

a) $b \geqq 1/3$: We have $\gamma(b, 1, 0) = 0$. Taking the power series expansion of the hyperbolic functions, we get $\gamma(b, 1, (\omega_1, \omega_2)) \sinh r \geqq \gamma(b, 1, (\omega_1, 0)) \sinh r \geqq \sum |\omega_1|^2 \times \big((4n^2 + 10n + 6)\,b - 2n - 1\big)/(2n + 3)! \geqq \sum |\omega_1|^2\,(4n^2 + 4n + 3)/\big(3(2n + 3)!\big)$. So $\omega_1 = 0$ is the only solution of $\gamma(b; 1, (\omega_1, \omega_2)) = 0$. Remarking that the incompressibility condition (4) holds we have (i).

b) $b < 1/3$: Vanishing $\tilde{F}_s$ for $s \neq 0$ determines $b = b_c$, whence $\tilde{F} = F_c$ follows, and power series expansion of the second derivative gives $\tilde{F}_{ss}(s) > 0$. In correspondence with (i) the limit case $s \to 0$ gives $b_c = 1/3$ and $F_c = 1$. In order to get $N(L)$ we set $|\omega| = w$. Hence, $f(w) \equiv \gamma(b_c, F_c, \omega) \geqq (\text{const})^2\,\big(w^2(\sinh(2s) - 2s) + s^2(\sinh(2s) + 2s) + 2ws(1 - \cosh(2s))\coth w\big)$. But $\gamma(b_c, F_c, s) = 0$ and $f_w = \big(g(s) - g(w)\big)/w$ with $g(w) = (\text{const})^2\,\big(\sinh(2w) - 2w\big)/(w\sinh^2 w)$. Since g is a strictly monotone decreasing function, $s = w$ is the only solution of $\gamma(b_c, F_c, \omega) = 0$. Further, $F_c(s)$ is a strictly monotone decreasing function. So different s really give different F_c. This completes (ii). Since g is decreasing, f is increasing (decreasing) for $w > s$ ($w < s$). Therefore, $\gamma\big(b_c, F_c(s), \omega\big) > 0$ if $w \neq s$. The restriction $\omega = (s, 0) \in \Lambda$ gives $s = m/(Bl)$ with $m \in \mathbb{Z}$ ∎

Remark: Consider r in (9) as a continuous parameter. Then (9) describes a curve C parametrized with respect to r (Figure 1). Below C the trivial solution $h = 0$ of (1), (2) is stable, no bifurcations occur. At points $F_c = 1$, $b_c > 1/3$ by our additional incompressibility condition (4) the bifurcating waves $h = \text{const}$ are eliminated. So the bifurcation points $F_c < 1$ and $b_c < 1/3$ have to be studied.

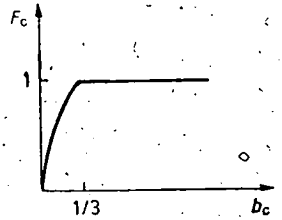

Fig. 1

2.3. Properties of the energy functional. Studying the energy functional and its first and second variation as maps in Sobolev spaces, we can use some results of BEYER [1]. We need some definitions given also in [1: § 3]. Let $\dot{\mathbf{H}}_m$ be the subspace of $\mathbf{H}_m$ whose functions satisfy (4). For any open interval $\mathbf{I}$ on the x_3-axis we denote by $\|\varphi\|_{\mathbf{I}}$ the norm of φ in $L_2(\mathbf{I})$. Let

$$W_{m,\mathbf{I}} = \{\varphi \in L_2(\mathbf{I}, \mathbf{H}_m) : \varphi^{(m)} = \partial^m\varphi/\partial x_3{}^m \in L_2(\mathbf{I}, \mathbf{H}_0)\}$$

be the Sobolev space of Λ'-periodic functions $\varphi(x_1, x_2, x_3) = \sum \varphi_\omega(x_3)\, e^{i\omega x}$ with distributional derivatives up to order m in $L_2(R \times \mathbf{I})$. The derivatives up to order $m - 1$ should be Λ'-periodic, too. Let $\|\varphi\|_{m,\mathbf{I}}^2 = \|\varphi_0\|_{\mathbf{I}}^2 + \sum_{\omega \in \Lambda} (|\omega|^{2m} \|\varphi_\omega\|_{\mathbf{I}}^2 + \|\varphi_\omega{}^{(m)}\|_{\mathbf{I}}^2)$ be the norm in $W_{m,\mathbf{I}}$. If $m \geqq 1$, we further define

$$V_{m,\mathbf{I}} = \{\varphi \in \mathscr{D}'(\mathbf{I}, \mathbf{H}_m) : \varphi', \varphi^{(m)} \in L_2(\mathbf{I}, \mathbf{H}_0)\}$$

with the norm $|\varphi|_{m,\mathbf{I}}^2 = \|\varphi_0'\|_{\mathbf{I}}^2 + \sum_{\omega \in \Lambda} (|\omega|^2 \|\varphi_\omega\|_{\mathbf{I}}^{2m} + \|\varphi_\omega{}^{(m)}\|_{\mathbf{I}}^2)$. Henceforth, we set $b = b_c(1 - \mu)$ and $F = F_c(1 + \varepsilon)$. So we set $E = E(h, \varepsilon, \mu)$.

Lemma 2: *Assume $s \geqq 5/2$. Then*

(i) *$E(h, \mu, \varepsilon)$ maps a neighbourhood of $(0, 0, 0) \in \mathbf{H}_s \times \mathbb{R}^2$ analytically into $\mathbb{R}$,*

(ii) *$E_h(h, \mu, \varepsilon)$ considered as a map from $\dot{\mathbf{H}}_s \times \mathbb{R}^2$ into $\dot{\mathbf{H}}_{s-2}$ is analytic at $(0, 0, 0)$,*

(iii) *$E_{hh}(h, \mu, \varepsilon)$ — originally considered as a map on $\dot{\mathbf{H}}_s \times \dot{\mathbf{H}}_s$ — is continuous on $\dot{\mathbf{H}}_1 \times \dot{\mathbf{H}}_1$, and its continuous extension on $\dot{\mathbf{H}}_1 \times \dot{\mathbf{H}}_1$ as a map from $\dot{\mathbf{H}}_s \times \mathbb{R}^2$ into $\mathscr{L}(\dot{\mathbf{H}}_1 \times \dot{\mathbf{H}}_1, \mathbb{R})$ is analytic at $(0, 0, 0)$.*

Proof: Setting

$$f' = -\nabla\varphi v_{,3} + \varphi_{,3} \nabla v + \big(\nabla v\, \nabla\varphi - |\nabla v|^2 \varphi_{,3}/(1 + v_{,3})\big)\,(0, 0, 1)$$

and transforming the variational equation $\int_\Omega \nabla\Phi \nabla\psi \, dZ \, dX \, dY = 0$ we get

$$\int_S \nabla\psi f' \, dV = \int_S \nabla\varphi \nabla\psi \, dV - \int_R \psi(0) h_{,1} \, dx_1 \, dx_2 \quad \text{for all } \psi \in V_1.$$

If we take $\varphi = \varphi_1 + \tilde{\varphi}$ with $\Delta\varphi_1 = 0$ in S, $\varphi_{1,3} = h_{,1}$ at $x_3 = 0$, and $\varphi_{1,3} = 0$ at $x_3 = -1$, then φ_1 is given by $\hat{\varphi}_1$ in Subsection 2.2 and

$$\int_S \nabla\psi f \, dV = \int_S \nabla\tilde{\varphi} \nabla\psi \, dV \quad \text{for all } \psi \in V_1, \text{ with } f = f'(\varphi_1) + f'(\tilde{\varphi}). \tag{10}$$

For the Fourier series $f = \sum f_\omega(x_3) \, e^{i\omega x}$ this implies

$$\int_I (\tilde{\varphi}_{\omega,3}\overline{\psi}_{\omega,3} + |\omega|^2 \tilde{\varphi}_\omega \overline{\psi}_\omega) \, dx_3 = \int_I (f_\omega \overline{\psi}_{\omega,3} - i\omega f_\omega \overline{\psi}_\omega) \, dx_3. \tag{11}$$

Choosing $\psi_\omega = \tilde{\varphi}_\omega$ and applying Schwarz's inequality, we obtain

$$\|\tilde{\varphi}_{\omega,3}\|^2 + |\omega|^2 \|\tilde{\varphi}_\omega\|^2 \leqq \|f_\omega\| (\|\tilde{\varphi}_{\omega,3}\| + |\omega| \|\tilde{\varphi}_\omega\|),$$

hence $\|\tilde{\varphi}_{\omega,3}\|^2 + |\omega|^2 \|\tilde{\varphi}_\omega\|^2 \leqq 2 \|f_\omega\|^2$. So, for $m = 0$ we have shown that the unique solution $\tilde{\varphi} \in V_1$ of (10) belongs to V_{m+1} and satisfies $|\tilde{\varphi}|_{m+1} \leqq c \|f\|_m$, with c independent of f. Differentiating the Euler equation of (11) $(m-1)$-times, we have $-\tilde{\varphi}_\omega^{(m+1)} + |\omega|^2 \tilde{\varphi}_\omega^{(m-1)} = -i\omega f_\omega^{(m-1)} - f_\omega^{(m)}$ for $m > 0$, whence

$$\|\tilde{\varphi}_\omega^{(m+1)}\| \leqq \| \, |\omega|^2 \tilde{\varphi}_\omega^{(m-1)}\| + \|i\omega^2 f_\omega^{(m-1)}\| + \|f_\omega^{(m)}\|$$

follows. With the same argument as for $m = 0$ it follows that

$$\|\tilde{\varphi}_\omega^{(m+1)}\|^2 \leqq 3(|\omega|^4 \|\tilde{\varphi}_\omega^{(m+1)}\|^2 + |\omega|^2 \|f_\omega^{(m-1)}\|^2 + \|f_\omega^{(m)}\|^2).$$

Taking notice of $\|\tilde{\varphi}_\omega^{(m)}\|^2 \leqq \text{const} \, (\tilde{\varepsilon}^k \|\tilde{\varphi}_\omega^{(m+k)}\|^2 + \|\tilde{\varphi}_\omega^{(l)}\|^2/\tilde{\varepsilon}^{m-l})$ for $\tilde{\varepsilon} > 0$, we get the proposition for $m > 0$.

Let $h \in \dot{\mathbf{H}}_{m+1/2}$. Obviously our transformation function v in (5) belongs to V_{m+1} and as map of h it is analytic in 0. Notice that $|\varphi_1|_{m+1} \leqq \sqrt{2} \|h\|_{m+1/2}$. Since the W_s are Banach algebras if $s > 3/2$,

$$f: (h, \varphi_1, \tilde{\varphi}) \in \dot{\mathbf{H}}_{m+1/2} \times V_{m+1} \times V_{m+1} \to f(h, \varphi_1, \tilde{\varphi}) \in W_m$$

is analytic at $(0, 0, 0)$ if $m \geqq 2$. So we can solve (10) via a fixed point theorem for $\tilde{\varphi} \in V_{m+1}$, obtaining $\tilde{\varphi}(h)$ analytic in 0, and then φ is analytic. Since J in (6) depends analytically on $(\varphi, v) \in V_1 \times V_{m+1}$, power series expansion of φ and v gives the analyticity of J in $h = 0$. Obviously the other terms in E are analytic, too, which gives (i). Now it is time to justify (7). The integrand in J is an analytic function in h, $\nabla\varphi|_{x_3=0}$ and $v|_{x_3=0}$. Since $\nabla\varphi|_{x_3=0}$, $\nabla v|_{x_3=0} \in \dot{\mathbf{H}}_{m-1/2}$, the integrand belongs to $\dot{\mathbf{H}}_{m-1/2}$, too. The term due to the Bond number belongs to $\dot{\mathbf{H}}_{m-3/2}$. This completes (ii). Proposition (iii) is independent of the concrete problem. It follows solely from the analyticity of E_h. Proposition (iii) is proved by estimating the coefficients of the power series expansion of E via interpolation theory. The proof is given in BEYER [1: Corollary 2.2], so we omit it ∎

3. The bifurcation equations

3.1. The Ljapunov-Schmidt procedure. Let $h \in \dot{\mathbf{H}}_s$. In a small neighbourhood of $(h, \mu, \varepsilon) = (0, 0, 0)$ and for all $\zeta \in \dot{\mathbf{H}}_s$ we have to solve $\langle E_h(h, \mu, \varepsilon), \zeta \rangle = 0$. With (4) we get

$$E_h(h, \mu, \varepsilon) = Lh + N(h, \mu, \varepsilon), \tag{12}$$

where $N(h, \mu, \varepsilon)$ denotes the nonlinear part of $E_h(h, \mu, \varepsilon)$. Let Q be the projection from $\mathbf{H}_s$ onto the range of L. Then $I - Q$, with I being the identity, projects onto the orthogonal complement of the range of L, that means onto $N(L^*)$. We have $I - Q = \langle \cdot, h_1 \rangle h_1^* + \langle \cdot, h_2 \rangle h_2^*$ with $h_1, h_2 \in N(L)$ and $h_1^*, h_2^* \in N(L^*)$. Recall that the norms in $N(L)$ and $N(L^*)$ are different. But, since L is formally self-adjoint, the basic elements in $N(L)$ and $N(L^*)$ are the same ($h_1 = h_1^*$ and $h_2 = h_2^*$). Now, (12) is equivalent to

$$QE_h(h, \mu, \varepsilon) = 0 \tag{13}$$

and

$$(I - Q)\, E_h(h, \mu, \varepsilon) = 0. \tag{14}$$

Decomposing $h = h' + \eta$ with $h' \in N(L)$ and $\eta \in N(L)^\perp$, we find that (13) has the unique solution $\eta(h', \mu, \varepsilon)$. Recall from Lemma 2 that η is analytic. Inserting this in (14), together with the linear independence of the h_i^* we get $\left\langle E_h\left(h' + \eta(h', \mu, \varepsilon), \mu, \varepsilon\right), \zeta_i\right\rangle = 0$ for $\zeta_i \in N(L)$ with $i = 1, 2$. Then, regarding $\langle L\eta, \zeta_i\rangle = 0$ for $i = 1, 2$ and (12) we get the bifurcation equations

$$G_i \equiv \left\langle N\left(h' + \eta(h', \mu, \varepsilon), \mu, \varepsilon\right), \zeta_i\right\rangle = 0 \text{ for } \zeta_i \in N(L),\ i = 1, 2, \tag{15}$$

where η has to be determined from

$$\langle L\eta, \zeta\rangle = -\langle QN(h' + \eta, \mu, \varepsilon), \zeta\rangle \qquad \text{for all } \zeta \in N(L)^\perp. \tag{16}$$

3.2. Symmetries. Since we are looking for real h', set $h' = z\,\mathrm{e}^{\mathrm{i}\omega x} + \bar{z}\,\mathrm{e}^{-\mathrm{i}\omega x}$, where $z = a\,\mathrm{e}^{\mathrm{i}\delta}$ with $a, \delta \in \mathbb{R}$. Taking into consideration the symmetries underlying the physical problem, we are able to say what terms $a^m\,\mathrm{e}^{\mathrm{i}\delta n}$, $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, only can occur in our equation. Here we suppress the dependence on the parameters in E_h.

Definition: We call $\langle E_h(h), \zeta\rangle = 0$, E_h acting in a Hilbert space, *covariant with respect to a unitary representation $T_{\mathbf{g}}$ of a compact group* $\mathbf{G}$ if $\langle E_h(h), \zeta\rangle = \langle E_h(T_{\mathbf{g}}h), T_{\mathbf{g}}\zeta\rangle$ for all $\mathbf{g} \in \mathbf{G}$.

In our case we consider the translations in the (x_1, x_2)-plane via a vector $\mathbf{a}$ with the representation $T_{\mathbf{a}}$ and the rotation through π with the representation T_π. If $\mathbf{g}$ is an element of the translation or rotation group, we take the representation $(T_{\mathbf{g}}h)\,(x_1, x_2) = h\mathbf{g}^{-1}(x_1, x_2)$. Consider (12) with E_h as analytic operator over a Hilbert space. We have $E_h(0, 0, 0) = 0$. L is a Fredholm operator of zero index. So we have the following.

Lemma 3: [4: Theorem 4.4]. *Let $\langle E_h(h), \zeta\rangle = 0$ be covariant with respect to $T_{\mathbf{g}}$, then*

(i) *$T_{\mathbf{g}}$ leaves $N(L)$ invariant;*

(ii) *L commutes with $T_{\mathbf{g}}$;*

(iii) *the bifurcation equations are covariant with respect to the finite-dimensional representation $T_{\mathbf{g}}$ restricted to $N(L)$.*

This we apply to prove the following

Lemma 4: *The bifurcation equations* (15) *are reduced to one equation*

$$G_1 = \sum_{m+k+j>0} c_{mkj}\mu^j\varepsilon^k a^{2m+1} = 0, \qquad c_{mkj} \text{ constants.}$$

Proof: Because of the analyticity of E_h and $\eta(h')$ our bifurcation equations (15) are

$$G_i = \sum_{n>0} \sum_{k+m=n} g^i_{mk}(\mu, \varepsilon)\, z^m \bar{z}^k = 0 \qquad (i = 1, 2). \tag{17}$$

The action of $T_{\mathbf{a}}$ restricted to $N(L)$ is $T_{\mathbf{a}}h_1 = e^{i\omega a}h_1$ and $T_{\mathbf{a}}h_2 = e^{-i\omega a}h_2$. Applying this to the terms of nth order in (17), we get

$$e^{i\omega a} \sum_{k+m=n} g^j_{mk}(\mu, \varepsilon)\, z^m \bar{z}^k = \sum_{k+m=n} g^j_{mk}(\mu, \varepsilon)\, e^{i\omega a(m-k)}\, z^m \bar{z}^k.$$

This must hold for all $\mathbf{a} \in \mathbb{R}^2$, whence $m = k + 1$ follows. So we have $G_i = \sum_{k\geq 0} g_k{}^i(\mu, \varepsilon) \times a^{2k+1} = 0$ $(i = 1, 2)$ with some $g_k{}^i$. The covariance under T_π gives a relation between G_1 and G_2. The action of T_π restricted to $N(L)$ is $T_\pi h_1 = h_2$ and $T_\pi h_2 = h_1$. So Lemma 4/(iii) gives $G_2(h_1, h_2) = \langle N(T_\pi h_1, T_\pi h_2), T_\pi \zeta_2\rangle = G_1(h_2, h_1)$, which completes the proof ∎

3.3. Concrete coefficients of the bifurcation equation. Here we will solve the bifurcation equation. We will see that it is enough to know the coefficients c_{mkj} up to $m + k + j = 4$. At first we have to compute some Fourier coefficients of the next approximations of $\varphi = \sum_{k>0} \varphi_k(h^k)$.

Lemma 5: *Let*

$$h = h' + \eta_2 = z\, e^{i\omega x} + \bar{z}\, e^{-i\omega x} + \eta'(z^2\, e^{2i\omega x} + \bar{z}^2\, e^{-2i\omega x}) \tag{18}$$

and $|x_3| = u$. *Then for the power series expansion*

$$\varphi(h) = \varphi_1(h') + \varphi_2(\eta_2) + \varphi_2(h'^2) + \varphi_3(\eta_2 h') + \varphi_3(h'^3) + \sum_{2k+m>3} \varphi(\eta_2{}^k h'^m)$$

we get

(i) $\varphi_1(h') = \varphi_\omega\, e^{i\omega x} + \bar{\varphi}_\omega\, e^{-i\omega x}$ *with* $\varphi_\omega = i\omega \cosh(u + r)/\sinh r$, $\varphi_2(\eta_2) = \varphi_{2\omega}\, e^{2i\omega x} + \bar{\varphi}_{2\omega}\, e^{-2i\omega x}$, $\varphi_2(h'^2) = 0$;

(ii) $\varphi_3(\eta_2 h')_\omega = 0$ *as Fourier coefficient of* $e^{i\omega x}$ *in the Fourier expansion of* φ_3;

(iii) *the Fourier coefficient of* $e^{i\omega x}$ *at* $x_3 = 0$

$$\varphi_3(h'^3)_\omega|_{x_3=0} = i z^2 \bar{z} \omega^2 \left(-\frac{13}{8} \sinh(4r) + \frac{3}{4} \sinh(2r) + 2r\right) \Big/ \sinh^4 r.$$

Proof: Transformation of (1) with (5) leads to (h.o.t. denotes higher order terms) $\Delta\varphi = A(\varphi v) + B(v^3) +$ h.o.t. in S with

$$A(\varphi v) = 2v_{,13}\varphi_{,1} + 2v_{,33}\varphi_{,3} + 2v_{,1}\varphi_{,13} - \varphi_{,11}v_{,3} + v_{,3}\varphi_{,33},$$
$$B(v^3) = 4v_{,1}v_{,3}v_{,33} + 5v_{,1}^2 v_{,13} + v_{,3}^2 v_{,13},$$

to $\varphi_{,3} = h_{,1} + C(\varphi v) + D(v^3) +$ h.o.t. on $x_3 = 0$ with $C(\varphi v) = 2\varphi_{,1}v_{,1} + \varphi_{,3}v_{,3}$ and $D(v^3) = -2v_{,1}(v_{,1}^2 + v_{,3}^2)$, and to $\varphi_{,3} = 0$ on $x_3 = -1$. The first approximation of φ with linear right sides is given in (11). We get $\varphi_2(h'^2)$ from $\Delta\varphi_2(h'^2) = A(\varphi_1(h')\, v(h'))$ in S and $\varphi_{2,3}(h'^2) = C(\varphi_1(h')\, v(h'))$ on $x_3 = 0$. This implies (i) since the right sides are zero.

With (18) equation (5) reads $v(h) = v_1(h') + v_2(\eta_2) +$ h.o.t. Solving

$$\Delta\varphi_3(h'\eta_2) = A(\varphi_2(\eta_2)\, v_1(h') + \varphi_1(h')\, v_2(\eta_2)) \quad \text{in } S,$$
$$\varphi_3(h'\eta_2)_{,3} = C(\varphi_3(\eta_2)\, v_1(h') + \varphi_1(h')\, v_2(\eta_2)) \quad \text{on } x_3 = 0,$$

notice that $\varphi_{i,1} = -v_{i,3}$ and $\varphi_{i,3} = v_{i,1}$ for $i = 1, 2$. This implies in that case $A = C = 0$, which gives (ii).

Finally the solution of

$$\Delta\varphi_3(h'^3) = B(v^3) = i\omega^4 z^2 \bar{z}(5 \cosh(3u + 3r) - 4 \cosh(u + r))/\sinh^3 r$$

in S and $\varphi_3(h'^3)_{,3} = D(v^3) = -2i\omega z^2 \bar{z}(3 + \coth^2 r)$ on $x_3 = 0$ gives (iii) ∎

Studying in which manner η from (16) contributes to the lower terms in the bifurcation equations (15) we will also justify (18).

Lemma 6: *The solution* $\eta = \sum_{k+l+m\geq 0} \eta_{klm}(h'^{k+2}\varepsilon^l\mu^m)$ *of* (16) *contains the terms*

$$\eta_2 = \eta_{200} = \eta'(z^2 e^{2i\omega x} + \bar{z}^2 e^{-2i\omega x}), \quad \eta' = -\omega \frac{\cosh(2r)}{\sinh r \cosh r + 2\tanh r - 3r}.$$

The nontrivial Fourier coefficients in η_{210} *and* η_{201} *are only* $(\eta_{210})_{2\omega}$ *and* $(\eta_{201})_{2\omega}$.

Proof: Let $\tilde{\eta}$ be the approximation of η, depending on h'^2. Then $\varphi = \sum_{k>0} \varphi_k(h^k)$ and $h = h' + \tilde{\eta} + \sum_{k+l+m\geq 0} \eta_{klm}(h'^{k+3}\varepsilon^l\mu^m)$ implies $\varphi = \varphi_1(h') + \varphi_2(\tilde{\eta}) + \sum_{k\geq 0} \varphi_k(h'^k)$. Recall from (7) that (16) is

$$\langle L\eta, \zeta\rangle = -\frac{1}{2} F_c(1 + \varepsilon) \langle (|\nabla\varphi_1|^2 - 2\varphi_{1,3}h'_{,1}), \zeta\rangle - \varepsilon F_c\langle \varphi_{2,1}, \zeta\rangle$$
$$+ \mu b_c\langle \Delta\eta, \zeta\rangle + \text{h.o.t.}$$

Lemma 5/(i) yields

$$\langle L\tilde{\eta}, \zeta\rangle = \left(-F_c\omega^2 \cosh(2r)/(2\sinh^2 r) + \theta(r, \varepsilon, \mu)\right) \langle z^2 e^{2i\omega x}$$
$$+ \bar{z}^2 e^{-2i\omega x}, \zeta\rangle + \text{h.o.t.}$$

with some θ depending on ε and μ. So, θ determines only $(\tilde{\eta}_{201})_{2\omega}$ and $(\tilde{\eta}_{210})_{2\omega}$. The last equation must hold for all $\zeta \in N(L)^{\perp} \cap \dot{\mathbf{H}}_s$. Choosing $\zeta = \zeta_{2\omega} e^{2i\omega x} + \bar{\zeta}_{2\omega} e^{-2i\omega x}$ we get together with $L(\tilde{\eta}_{200})_{2\omega} = \left(1 + 4\omega^2 b_c - 2\omega F_c \coth(2r)\right) (\tilde{\eta}_{200})_{2\omega}$ and with (9) $\tilde{\eta}_{200} = \eta_{200}$ ∎

At this time we have determined all unknown functions which are important for our bifurcation equation.

Theorem 1: *The following assertions are true:*

(i) *The bifurcation equation* (15) *reads as*

$$-(\mu' + \varepsilon')\, a + \left(w(r) + 4\eta'\right) a^3 + \sum_{k+l+m>0} c_{klm} a^{2k+3}\varepsilon^l\mu^m = 0, \tag{19}$$

where $\mu' = \left(\sinh(2r) - 2r\right)/\omega^2$, $\varepsilon' = 2\sinh(2r)/\omega^2$, *and*

$$w(r) = -\frac{3}{2}\left(\sinh(2r) - 2r\right) + \left(7\sinh(4r) - 4\sinh(2r) - 8r\right)/\sinh^2 r.$$

(ii) *The nontrivial solution of* (19), which only exists if $r > r_0$ *for some* $r_0 \approx 0.8$, *is*

$$a^2 = (\varepsilon' + \mu')/\left(w(r) + 4\eta'\right) + \sum_{i+j>1} a_{ij}\varepsilon^i\mu^j. \tag{20}$$

Proof: The bifurcation equation (15) with the concrete terms from (7) reads as

$$\langle (b_c\mu\, \Delta h' + \varepsilon F_c\varphi_{1,1}), \zeta_1\rangle + 3b_c\langle (h_1'^2, h'_{1,1}), \zeta_1\rangle$$
$$= -F_c\langle (\varphi_{3,1}^2 + \nabla\varphi_1\, \nabla\varphi_2 - v_{1,1}\varphi_{2,3} - v_{2,1}\varphi_{1,3} + v_{1,1}^2 v_{1,3}), \zeta_1\rangle + \text{h.o.t.}, \tag{21}$$

where h.o.t. denotes higher order terms in a^k with $k \geq 4$. A simple computation with the functions given in Lemma 5 and Lemma 6 and the b_c and F_c from (9) together with Lemma 4 gives (i). The Implicit Function Theorem together with the condition $w(r) > 0$ finally gives (ii) ∎

Remark: We have considered our problem in three dimensions. However, we have shown that at smallest critical Froude numbers the bifurcating wave is a two-dimensional one. Indeed, it is that wave which was derived in ZEIDLER [7: Chapter 4]. Therefore, we have also justified the two-dimensional approach of [7].

Now, in analogy to Beyer [2], the variational approach gives the stability of the wave in a natural way.

4. Stability

4.1. The second variation of E. Stability intervals we get by the positive definiteness of the second variation of the potential energy. By Lemma 2, E_{hh} has the structure $E_{hh}(\zeta, \zeta) = \langle L\zeta, \zeta\rangle + \sum_{i>0} E_i(\zeta^2 h'^i)$. If we set $\zeta = \zeta_1 + \zeta_2$, where $\zeta_1 \in N(L)$ and $\zeta_2 \in N(L)^\perp$, this yields

Lemma 7: *For $|\varepsilon|$ and $|\mu|$ small enough the stability of the solution (20) is determined by the positive definiteness of*

$$E_0 = \langle L\zeta_2, \zeta_2\rangle + E_1(\zeta_1\zeta_2 h') + E_2(\zeta_1^2 h'^2),$$

where

$$\zeta_1 = X_1(e^{i\omega x + i\delta} + e^{-i\omega x - i\delta}) \text{ and } \zeta_2 = X_2(e^{2i(\omega x+\delta)} + e^{-2i(\omega x+\delta)}).$$

Proof: 1. The inequality $\langle L\zeta, \zeta\rangle \geqq \|\zeta_2\|^2$ follows by Lemma 1/(iii). 2. Recall from (21) that $h' = O(\varepsilon, \mu)$ as $\varepsilon, \mu \to 0$. Since $\|\zeta_1\|\,\|\zeta_2\| \leqq \lambda\,\|\zeta_1\|^2 + \|\zeta_2\|^2/\lambda$, where we take $\lambda = \varepsilon$ or $\lambda = \mu$, respectively, together with Part 1 we get $|E_2(\zeta^2 h'^2)| \leqq c'\,\|\zeta_1\|^2\,(|\varepsilon| + |\mu|)$. The same argument yields $|E_1(\zeta^2 h')| \leqq c''(|\varepsilon| + |\mu|)\,(\|\zeta_1\|^2 + \|\zeta_1\|\,\|\zeta_2\|)$. But integration of the $e^{ik\omega x}$ terms $(k = 1, 3)$ shows $E_1(\zeta_1^2 h') = 0$. Hence, $E_{hh}(\zeta, \zeta)$ is given by the sum of a positive part $\langle L\tilde\zeta, \tilde\zeta\rangle$, $\tilde\zeta \perp \zeta_2$ and $\tilde\zeta \in N(L)^\perp$, of E_0, and of other terms of higher order if $|\varepsilon|$ and $|\mu|$ are small enough. 3. With respect to translations through any $\mathbf{a} = (a_1, a_2)$ in the (x_1, x_2)-plane the covariance of $E_h(h)$ was given by $T_{\mathbf{a}}E_h(h) = E_h(T_{\mathbf{a}}h) = 0$. For $i = 1$ or $i = 2$ differentiating with respect to the parameters a_i yields $E_{hh}(T_{\mathbf{a}}h)\,\partial(T_{\mathbf{a}}h)/\partial a_i = 0$. For $a_i = 0$ this is $E_{hh}(h)\,h_{,i} = 0$. But that is the well-known fact (see SATTINGER [3]) that the covariance of E_h with respect to a two-parameter Lie group yields two zero eigenvalues for E_{hh}, so stability in our case can only mean orbital stability. That is stability over a subspace of all $\zeta \in \mathbf{H}_s$ with $\langle\zeta, h_{,i}\rangle = 0$ for $i = 1, 2$ and h from (18). Let $\zeta_{1\omega} = X_1\,e^{i\delta'}$ and $\zeta_{2\omega} = X_2\,e^{2i\delta''}$. Computing the scalar product we find $\delta = \delta' = -\delta''$ ∎

4.2. The stability interval. Next we compute E_0. We know from the splitted Euler equation (15) and (16) that

$$\langle E_h, \zeta\rangle = \langle L\eta, \zeta\rangle + \langle N(h' + \eta), \zeta\rangle + \langle N(h' + \eta), \zeta_1\rangle,$$

where $\zeta \in N(L)^\perp$. (19) and (21) taken into account, this reads as

$$\frac{1}{2}\langle E_h, \zeta\rangle = \big(\sinh(2r) - 6r + 4\tanh r\big)\,\eta' a^2 X_2 + 2\cosh(2r)\,a^2 X_2 - (\varepsilon' + \mu')\,aX_1 + w(r)\,a^3 X_1 + 4\eta' a^3 X_1 + h.o.t.$$

But variating h really means variating every Fourier coefficient. So we have to take $a(t) = a + tX_1$ and $\eta'(t) = \eta' + tX_2$, which yields $E_0 = E_{ha}X_1 + E_{h\eta'}X_2$. Therefore, $E_0 = C_1X_1^2 + 2C_3X_1X_2 + C_2X_2^2$ with $C_1 = 4(\varepsilon' + \mu')$, $C_2 = 2a^2\big(\sinh(2r) - 6r + 4\tanh r\big)$, and $C_3 = 4a^3$. This quadratic form is positive definite if $C_1 + C_2 > 0$

and $C_1C_2 - C_3^2 > 0$. The first condition is fulfilled for all $\varepsilon, \mu > 0$, the second means $\left((\sinh(2r) - 2r)/\sinh(2r)\right)\mu + 2\varepsilon < g(r)$ with a function g, $g(r) > 0$ for $r > 0$ and $g(0) = 0$, with $r = m/(Bl)$ where B is the mean depth and m/l the wave-length determined by the choice of the critical parameters. The inequality shows that for small wave-length the dependence on μ becomes important. This finally yields

Theorem 2: *The bifurcating wave $h = a(\varepsilon, \mu) \cos\left(x_1 m/(Bl) + \delta\right)$ is stable with respect to perturbations, which belong to $\dot{\mathrm{H}}_m$, if $|\varepsilon'|$ and $|\mu'|$ are small enough and if $r > r_0$.*

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