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On the Existence of Conjugate Points for Sturm-Liouville Differential Equations

E. MÜLLER-PFEIFFER and TH. SCHOTT

Mit Hilfe von Integralbedingungen für die Koeffizienten p und q Sturm-Liouvillescher Differentialgleichungen $-(p(x)\,u')' + q(x)\,u = 0$ wird der Abstand benachbarter Nullstellen nichttrivialer Lösungen u nach oben abgeschätzt.

При помощи интегральных условий для коэффициентов p и q дифференциального уравнения Штурма-Лиувилля $-(p(x)\,u')' + q(x)\,u = 0$ расстояние соседних нулей нетривиальных решений u оценивается сверху.

By means of integral conditions for the coefficients p and q of Sturm-Liouville differential equations $-(p(x)\,u')' + q(x)\,u = 0$ the distance between consecutive zeros of non-trivial solutions u will be estimated from above.

There are various results in the literature on estimating the distance between consecutive zeros of solutions of second order differential equations (cf. [10], for instance). The following investigation is devoted to this problem. We consider the Sturm-Liouville differential equation on a bounded interval,

$$-\big(p(x)\,u'\big)' + q(x)\,u = 0, \qquad (-a \leqq x \leqq a < \infty;\ p, q \in C[-a, a]) \tag{1}$$

and suppose that p is positive and piecewise continuously differentiable on $[-a, a]$. The points x_1, x_2, $-a \leqq x_1 < x_2 \leqq a$, are said to be *conjugate* with respect to the equation (1) if there exists a nontrivial solution u to (1) with $u(x_1) = 0 = u(x_2)$. Solutions to (1) are always real-valued functions belonging to $C^1[-a, a]$ (cf. [10, p. 25]). Set, for $0 \leqq s < a$,

$$Q(s) = \sup_{s<h<a} \frac{1}{2h} \int_{-h}^{h} q\,dx, \qquad \overline{p}(s) = \frac{1}{2(a-s)} \left(\int_{-a}^{-s} p\,dx + \int_{s}^{a} p\,dx \right).$$

Theorem 1: *If there exists a number s, $0 \leqq s < a$, such that*

$$\frac{3}{(a-s)\,(a+2s)}\,\overline{p}(s) + Q(s) \leqq 0, \tag{2}$$

then there exists a pair of conjugate points on $(-a, a)$ with respect to (1) and the constant $3(a-s)^{-1}\,(a+2s)^{-1}$ in (2) is the best possible one.

Proof: Consider the sesquilinear form

$$t[f, g] = \int_{-a}^{a} (pf'\overline{g}' + qf\overline{g})\,dx \qquad (f, g \in D(t)) \tag{3}$$

to (1). The domain $D(t)$ of this form is identical with the Sobolev space $\mathring{W}_2^1(-a, a)$. In the following the form is estimated by means of the test function

$$v(x) = \begin{cases} a - |x|, & s \leqq |x| \leqq a, \\ a - s, & |x| \leqq s, \end{cases} \tag{4}$$

which belongs to $D(t)$. By Fubini's theorem we obtain

$$\begin{aligned} t[v, v] &= \int_{-a}^{a} \left(p(v')^2 + qv^2\right) dx = 2(a - s)\,\overline{p}(s) + \int_{-a}^{a}\int_{0}^{v^2(x)} q(x)\,dy\,dx \\ &= 2(a - s)\,\overline{p}(s) + 2\int_0^{(a-s)^2} \left(\frac{1}{2h(y)} \int_{-h(y)}^{h(y)} q(x)\,dx\right) h(y)\,dy \\ &\leqq 2(a - s)\,\overline{p}(s) + 2Q(s) \int_0^{(a-s)^2} h(y)\,dy, \end{aligned} \tag{5}$$

where $h(y) = a - \sqrt{y}$, $0 \leqq y \leqq (a - s)^2$. Hence, in view of (2), there follows that $t[v, v] \leqq 2(a - s)\,\overline{p}(s) + \frac{2}{3} Q(s)\,(a - s)^2\,(a + 2s) \leqq 0$. Consequently, we have $\inf\{t[f, f] : f \in D(t), \|f\| = 1\} \leqq 0$, where $\|\cdot\|$ denotes the norm in the Hilbert space $L_2(-a, a)$. If this infimum is less than zero, then there exists a nontrivial solution u to (1) having at least two zeros on $(-a, a)$ (cf. [8]). If the infimum is equal to zero, the (normalized) test function v is realizing the infimum and, consequently, it is a solution to (1) (cf. [9]). This, however, is impossible, because a solution to (1) belongs to the Sobolev space $W_2^2(-a, a)$ (cf. [2]). The function v, however, does not belong to $W_2^2(-a, a)$.

We prove now that the constant $\sigma = 3(a - s)^{-1}\,(a + 2s)^{-1}$ is the best possible one. Let us discuss the case $0 < s < a$. We prove that for any $\varepsilon > 0$ there exist functions p_ε and q_ε with

$$(\sigma - \varepsilon)\,\overline{p}_\varepsilon(s) + Q_\varepsilon(s) \leqq 0, \tag{6}$$

where

$$\overline{p}_\varepsilon(s) = \frac{1}{2(a - s)} \left(\int_{-a}^{-s} p_\varepsilon\,dx + \int_s^a p_\varepsilon\,dx\right), \qquad Q_\varepsilon(s) = \sup_{s<h<a} \frac{1}{2h} \int_{-h}^{h} q_\varepsilon\,dx$$

such that there does not exist a pair of conjugate points on $(-a, a)$ with respect to the differential equation

$$-\left(p_\varepsilon(x)\,u'\right)' + q_\varepsilon(x)\,u = 0, \qquad (-a \leqq x \leqq a). \tag{7}$$

Obviously, it suffices to assume that $\varepsilon < \sigma$ (if $\varepsilon \geqq \sigma$, choose $p_\varepsilon \equiv 1$ and $q_\varepsilon \equiv 0$). Choose $q_\varepsilon = \varepsilon - \sigma$ and

$$p_\varepsilon(x) = (\varepsilon - \sigma) \begin{cases} \left(x^2/2 - \left(a + 2t^2(a + s)^{-1}\right)|x| + A(t)\right), & |x| \geqq s \\ \left(t^2 - x \sinh^{-1}(x/t)\,B(t)\right), & |x| \leqq s, \end{cases}$$

where, for $0 < t < \infty$,

$$A(t) = \frac{s^2}{2} + t^2 - st \coth\frac{s}{t}, \qquad B(t) = \left(a - s + \frac{2t^2}{a + s}\right)\sinh\frac{s}{t} + t\cosh\frac{s}{t}.$$

Here t is a parameter which will be fixed later. The value $p_\varepsilon(0)$ is defined by the limit $(\varepsilon - \sigma)\big(t^2 - tB(t)\big)$ of $p_\varepsilon(x)$ for $x \to 0$. Clearly, p_ε is an even function. Further, it can easily be verified that p_ε is continuous and piecewise continuously differentiable on $[-a, a]$. We have

$$p_\varepsilon'(x) = (\sigma - \varepsilon)\begin{cases}\big(a + 2t^2(a+s)^{-1} - x\big), & s < x \leqq a,\\ \sinh^{-1}(x/t)\, B(t)\,\big(1 - (x/t)\coth(x/t)\big), & 0 < x < s,\end{cases}$$

$p_\varepsilon'(0) = 0$, and $p_\varepsilon'(x) > 0$ $(s < x \leqq a)$, $p_\varepsilon'(x) < 0$ $(0 < x < s)$. Hence

$$\begin{aligned}\min_{0 \leqq x \leqq a} p_\varepsilon(x) &= p_\varepsilon(s) = (\varepsilon - \sigma)\big(s^2/2 - \big(a + 2t^2(a+s)^{-1}\big)s + A(t)\big)\\ &= (\varepsilon - \sigma)\,(s^2 + t^2 - st\coth(s/t) - (a + 2t^2(a+s)^{-1})\,s)\\ &> (\sigma - \varepsilon)\,t^2\big((s/t)\coth(s/t) - 1\big) > 0.\end{aligned}$$

Thus, it follows that $p_\varepsilon(x) > 0$, $-a \leqq x \leqq a$, whenever $0 < t < s$. Next we prove that (6) holds if t is chosen sufficiently small. The inequality (6) is equivalent to

$$(a - s)^{-1}\int_s^a p_\varepsilon\,dx \leqq 1. \tag{8}$$

An easy calculation shows that

$$(a - s)^{-1}\int_s^a p_\varepsilon\,dx = 1 - \varepsilon\sigma^{-1} + (\sigma - \varepsilon)\,st\coth(s/t).$$

A value $t = t_\varepsilon$, $0 < t_\varepsilon < s$, can be chosen so small that $t_\varepsilon \coth(s/t_\varepsilon) \leqq \varepsilon/\sigma(\sigma - \varepsilon)\,s$. By such choice the inequalities (8), and, consequently, (6) are fulfilled. It is easily seen that the function

$$u(x) = \begin{cases} a + 2t_\varepsilon^2(a+s)^{-1} - |x|, & |x| \geqq s,\\ \sinh^{-1}(s/t_\varepsilon)\,\big(B(t_\varepsilon) - t_\varepsilon\cosh(x/t_\varepsilon)\big), & |x| \leqq s,\end{cases} \tag{9}$$

belongs to $C^1[-a, a]$ and is positive. The function $p_\varepsilon u'$ belongs also to $C^1[-a, a]$ and by calculation one can prove that $(p_\varepsilon u')' = q_\varepsilon u$. The function u is a positive solution to (7). Finally, assume that there exists a nontrivial solution u_0 to (7) possessing at least two zeros x_1 and x_2 on $(-a, a)$. Then, by Sturm's comparison theorem, each solution to (7) has a zero between x_1 and x_2 or is a constant multiple of u_0. The solution (9), however, contradicts this conclusion. Hence, there cannot exist a pair of conjugate points on $(-a, a)$ with respect to (7). This proves the theorem in the case $0 < s < a$. Similarly, one can prove that the constant $3a^{-2}$ is best possible in the case $s = 0$ ∎

The mean value $(1/2h)\int_{-h}^{h} q\,dx$, $s \leqq h \leqq a$[1]), is an increasing or a decreasing function of h if its derivative is non-negative or non-positive, i.e. if

$$\frac{1}{2h}\int_{-h}^{h} q\,dx \leqq \frac{1}{2}\big(q(-h) + q(h)\big) \quad\text{or}\quad \frac{1}{2h}\int_{-h}^{h} q\,dx \geqq \frac{1}{2}\big(q(-h) + q(h)\big),$$

[1]) Set $q(0)$ for the mean value when $s = h = 0$.

respectively. Then, the corresponding suprema $Q(s)$ are $Q(s) = \frac{1}{2a}\int_{-a}^{a} q\,dx$ and $Q(s) = \frac{1}{2s}\int_{-s}^{s} q\,dx$, respectively. Thus we obtain the following corollary.

Corollary 1: *If there exists a number s, $0 \leqq s < a$, such that*

$$\frac{1}{2h}\int_{-h}^{h} q\,dx \ (\gtreqless)\ \frac{1}{2}\left(q(-h) + q(h)\right) \qquad (s \leqq h \leqq a), \tag{10}$$

then there exists a pair of conjugate points on $(-a, a)$ with respect to (1) if

$$\frac{3}{(a - s)(a + 2s)}\,\overline{p}(s) + \frac{1}{2a}\int_{-a}^{a} q\,dx \leqq 0 \quad \left(\frac{3}{(a - s)(a + 2s)}\,\overline{p}(s) + \frac{1}{2s}\int_{-s}^{s} q\,dx \leqq 0\right).$$

In each case the factor $3(a - s)^{-1}(a + 2s)^{-1}$ is best possible.

If q can be written as a sum $q = q_1 + q_2$ where q_1 is an odd function and q_2 is monotone decreasing on $[-a, -s]$ and monotone increasing on $[s, a]$, then one can easily prove that $\frac{1}{2s}\int_{-s}^{s} q\,dx \leqq \frac{1}{2}\left(q(-s) + q(s)\right)$ implies (10) written with the sign $\leqq$. If q_2 is monotone increasing on $[-a, -s]$ and monotone decreasing on $[s, a]$, then $\frac{1}{2s}\int_{-s}^{s} q\,dx \geqq \frac{1}{2}\left(q(-s) + q(s)\right)$ implies (10) written with the sign $\geqq$. Hence, we obtain the following corollary from Corollary 1.

Corollary 2: *Suppose that there exists a point s, $0 \leqq s < a$, such that q can be written as a sum $q = q_1 + q_2$ in $C[-a, a]$ with q_1 odd and $q_2(x_1)\ (\gtreqless)\ q_2(x_2)$ $(x_1 < x_2 \leqq -s)$, $q_2(x_1)\ (\lesseqgtr)\ q_2(x_2)$ $(s \leqq x_1 < x_2)$, and suppose further that*

$$\frac{1}{2s}\int_{-s}^{s} q(x)\,dx \ (\lesseqgtr)\ \frac{1}{2}\left(q(-s) + q(s)\right). \tag{11}$$

Then there exists a pair of conjugate points on $(-a, a)$ with respect to (1) if

$$\frac{3}{(a - s)(a + 2s)}\,\overline{p}(s) + \frac{1}{2a}\int_{-a}^{a} q\,dx \leqq 0 \quad \left(\frac{3}{(a - s)(a + 2s)}\,\overline{p}(s) + \frac{1}{2s}\int_{-s}^{s} q\,dx \leqq 0\right). \tag{12}$$

In the special case $s = 0$ the hypotheses (11) are always satisfied and the conditions (12) call

$$\frac{3}{a^2}\int_{-a}^{a} p\,dx + \int_{-a}^{a} q\,dx \leqq 0 \quad \left(\frac{3}{2a^3}\int_{-a}^{a} p\,dx + q(0) \leqq 0\right). \tag{13}$$

Corollary 3: *Suppose that q belongs to* $C^2[-a, a]$[2]) *and is convex (concave) on* $[-a, a]$. *If the hypotheses* (13) *are satisfied, respectively, then there exists a pair of conjugate points on* $(-a, a)$ *with respect to* (1). *The constants* $3/a^2$ $(3/2a^3)$ *are best possible.*

Proof: Set $q(x) = q(0) + q'(0)\,x + r(x)$, $-a \leqq x \leqq a$, and note that r has the properties $r'(0) = 0$ and $r'' \geqq 0$ $(r'' \leqq 0)$. Hence, we have $r'(x) \leqq 0$ $\left(r'(x) \geqq 0\right)$ on $[-a, 0]$ and $r'(x) \geqq 0$ $\left(r'(x) \leqq 0\right)$ on $[0, a]$. Further, $q_1(x) = q'(0)\,x$ is an odd function and $q_2(x) = q(0) + r(x)$ has the concerned properties formulated in Corollary 2 (with $s = 0$). The corollary now follows from Corollary 2 ∎

Corollary 4: *Let p be positive and piecewise continuously differentiable on* $[0, X)$, $X \leqq \infty$, *and assume that* $q\left(\in C^2[0, X)\right)$ *is convex (concave). Let u be a nontrivial solution to* (1) *considered on* $[0, X)$ *with* $u(0) = 0$. *The first conjugate point to* $x = 0$ *is smaller than b, where b is the smallest positive root of the equation*

$$12 \int_0^b p\,dx + b^2 \int_0^b q\,dx = 0 \qquad \left(12 \int_0^b p\,dx + b^3 q(b/2) = 0\right),$$

when b exists.

Proof: Use Corollary 3 and Sturm's comparison theorem ∎

Set $s = 0$ in the estimate (5) for $t[v, v]$ and suppose that q is monotone increasing on $[-a, a]$. Then we obtain

$$t[v, v] \leqq \int_{-a}^{a} p(x)\,dx + \int_0^{a^2} \left(\frac{1}{h(y)} \int_{-h(y)}^{0} q(x)\,dx\right) h(y)\,dy + \int_0^{a^2} \left(\frac{1}{h(y)} \int_0^{h(y)} q(x)\,dx\right) h(y)\,dy$$

$$\leqq \int_{-a}^{a} p(x)\,dx + q(0) \int_0^{a^2} h(y)\,dy + \left(\frac{1}{a} \int_0^{a} q(x)\,dx\right) \int_0^{a^2} h(y)\,dy$$

$$= \int_{-a}^{a} p(x)\,dx + \frac{1}{3}\,a^3 q(0) + \frac{1}{3}\,a^2 \int_0^{a} q(x)\,dx.$$

Analogously, if q is monotone decreasing on $[-a, a]$ we have

$$t[v, v] \leqq \int_{-a}^{a} p\,dx + \frac{1}{3}\,a^3 q(0) + \frac{1}{3}\,a^2 \int_{-a}^{0} q\,dx.$$

These estimates yield analogous corollaries to the Corollaries 3 and 4. The following corollary corresponds to Corollary 4.

Corollary 5: *Let p be positive and piecewise continuously differentiable on* $[0, X)$, $X \leqq \infty$, *and assume that q is monotone increasing (decreasing) on* $[0, X)$ *with* $u(0) = 0$. *The first conjugate point to* $x = 0$ *is smaller than b where b is the smallest positive root of the equation*

$$12 \int_0^b p\,dx + \frac{b^3}{2}\,q\left(\frac{b}{2}\right) + b^2 \int_{b/2}^{b} q\,dx = 0 \quad \left(12 \int_0^b p\,dx + \frac{b^3}{2}\,q\left(\frac{b}{2}\right) + b^2 \int_0^{b/2} q\,dx = 0\right),$$

when b exists.

[2]) The hypothesis $q \in C^2$ can be weakened.

Corollary 6: *If there exists a nontrivial solution u to* (1) *with $u(-a) = 0 = u(a)$ and $u(x) \neq 0$, $-a < x < a$, then*

$$\int_{-a}^{a} p(x)\,dx + \int_{-a}^{0} (a + x)^2 q(x)\,dx + \int_{0}^{a} (a - x)^2 q(x)\,dx > 0. \tag{14}$$

Proof: By assuming the contrariety of (14) there follows from (5) that $t[v, v] \leqq 0$ where v is defined by (4). This implies that there exists a pair of conjugate points on $(-a, a)$ with respect to (1). This, however, is impossible because by Sturm's comparison theorem the solution u would have a zero in $(-a, a)$ ∎

In the special case $p \equiv 1$ and $q \leqq 0$ the condition (14) is due to LOVELADY [7].

By using the test function $v(x) = \cos(\pi x/2a)$, $-a \leqq x \leqq a$, in place of (4) and setting

$$P = \sup_{0<h<a} \frac{1}{2(a-h)} \left(\int_{-a}^{-h} p\,dx + \int_{h}^{a} p\,dx \right); \quad Q = \sup_{0<h<a} \frac{1}{2h} \int_{-h}^{h} q\,dx$$

we obtain the following theorem.

Theorem 2: *If*

$$(\pi^2/4a^2)\,P + Q \leqq 0, \tag{15}$$

then there exists a pair of conjugate points on $[-a, a]$ with respect to (1). *The constant $\pi^2/4a^2$ is the best possible one.*

Proof: The proof is analogous to that of Theorem 1. By means of Fubini's theorem and setting

$$\sigma(x) = \sin^2(\pi x/2a), \qquad \gamma(x) = \cos^2(\pi x/2a), \qquad -a \leqq x \leqq a,$$
$$h(y) = (2a/\pi) \arcsin \sqrt{y}, \qquad k(y) = (2a/\pi) \arccos \sqrt{y}, \qquad 0 \leqq y \leqq 1,$$

we obtain

$$\begin{aligned}
t[v, v] &= \int_{-a}^{a} p(x) \left(v'(x)\right)^2 dx + \int_{-a}^{a} q(x)\,v^2(x)\,dx \\
&= \frac{\pi^2}{4a^2} \int_{-a}^{a} p(x) \sin^2\left(\frac{\pi}{2a}\,x\right) dx + \int_{-a}^{a} q(x) \cos^2\left(\frac{\pi}{2a}\,x\right) dx \\
&= \frac{\pi^2}{4a^2} \int_{-a}^{a} \int_{0}^{\sigma(x)} p(x)\,dy\,dx + \int_{-a}^{a} \int_{0}^{\gamma(x)} q(x)\,dy\,dx \\
&= \frac{\pi^2}{2a^2} \int_{0}^{1} \frac{1}{2(a - h(y))} \left(\int_{-a}^{-h(y)} p(x)\,dx + \int_{h(y)}^{a} p(x)\,dx \right) \left(a - h(y)\right) dy \\
&\quad + 2 \int_{0}^{1} \frac{1}{2k(y)} \left(\int_{-k(y)}^{k(y)} q(x)\,dx \right) k(y)\,dy \\
&\leqq \frac{\pi^2}{2a^2}\,P \int_{0}^{1} \left(a - h(y)\right) dy + 2Q \int_{0}^{1} k(y)\,dy = \frac{\pi^2}{4a}\,P + aQ.
\end{aligned} \tag{16}$$

It follows from (15) that $l[v, v] \leqq 0$. Hence, there exists a pair of conjugate points on $[-a, a]$. To prove that the constant $\pi^2/4a^2$ is best possible observe that $\cos(\pi x/2\alpha)$, $\alpha > a$, is a positive solution to the differential equation

$$-u'' - (\pi^2/4\alpha^2)\, u = 0 \qquad (-a \leqq x \leqq a). \tag{17}$$

The coefficients $p \equiv 1$ and $q \equiv -\pi^2/4\alpha^2$ satisfy $(\pi^2/4\alpha^2)\, P + Q = 0$, and, on the other hand, there does not exist a pair of conjugate points on $[-a, a]$ with respect to (17). Therefore, the constant $\pi^2/4a^2$ in (15) cannot be chosen smaller ∎

In the special case $p \equiv 1$ the inequality (15) calls

$$\sup_{0<h<a} \frac{1}{2h} \int_{-h}^{h} q(x)\, dx \leqq -\frac{\pi^2}{4a^2}.$$

This condition for the existence of conjugate points with respect to the equation $-u'' + q(x)\, u = 0$ $(-a \leqq x \leqq a)$ is due to Leighton [5, 6].

Assume now that the coefficients p and q can be written as $p = p_1 + p_2$, $q = q_1 + q_2$, on $[-a, a]$, where the following possibilities are to be discussed:

(1) p_1, q_1 are odd functions;
(2a) $p_2(x_1) \geqq p_2(x_2)$ $(x_1 < x_2 \leqq 0)$, $p_2(x_1) \leqq p_2(x_2)$ $(0 \leqq x_1 < x_2)$;
(2b) $p_2(x_1) \leqq p_2(x_2)$ $(x_1 < x_2 \leqq 0)$, $p_2(x_1) \geqq p_2(x_2)$ $(0 \leqq x_1 < x_2)$;
(3a) $q_2(x_1) \geqq q_2(x_2)$ $(x_1 < x_2 \leqq 0)$, $q_2(x_1) \leqq q_2(x_2)$ $(0 \leqq x_1 < x_2)$;
(3b) $q_2(x_1) \leqq q_2(x_2)$ $(x_1 < x_2 \leqq 0)$, $q_2(x_1) \geqq q_2(x_2)$ $(0 \leqq x_1 < x_2)$.

Corollary 7: *There exists a pair of conjugate points on $[-a, a]$ with respect to (1) in each of the following cases.*

(i) (1), (2a), (3a) *are fulfilled (this is so, for instance, if the functions p, q are convex) and* $(\pi^2/4a)\left(p(-a) + p(a)\right) + \int_{-a}^{a} q\, dx \leqq 0$.

(ii) (1), (2a) (3b) *are fulfilled (this is so, for instance, if p is convex and q is concave) and* $(\pi^2/8a^2)\left(p(-a) + p(a)\right) + q(0) \leqq 0$.

(iii) (1), (2b), (3a) *are fulfilled (this is so, for instance, if p is concave and q is convex) and* $(\pi^2/4a^2) \int_{-a}^{a} p\, dx + \int_{-a}^{a} q\, dx \leqq 0$.

(iv) (1), (2b), (3b) *are fulfilled (this is so, for instance, if p, q are concave) and* $(\pi^2/8a^3) \int_{-a}^{a} p\, dx + q(0) \leqq 0$.

The constants $\pi^2/4a$ and so on are all best possible.

The following corollary follows directly from Corollary 7.

Corollary 8: *An upper bound b for the first conjugate point of $x = 0$ with respect to (1) considered on $[0, X)$, $X \leqq \infty$, is given by the first roots of the following equations:*

(i) $(\pi^2/2b)\left(p(0) + p(b)\right) + \int_0^b q\, dx = 0$ *when p, q are convex;*

(ii) $(\pi^2/2b^2)\left(p(0) + p(b)\right) + q(b/2) = 0$ *when p is convex and q is concave;*

(iii) $(\pi^2/b^2) \int_0^b p\,dx + \int_0^b q\,dx = 0$ *when* p *is concave and* q *is convex;*

(iv) $(\pi^2/b^3) \int_0^b p\,dx + q(b/2) = 0$ *when* p, q *are concave.*

In the special case that $p \equiv 1$, $q \leqq 0$, q is monotone and convex the conditions (i) and (iii) are due to Leighton [6]. These conditions also improve a condition by Fink [3, Th. 3]. In the special case that $p \equiv 1$, $q \leqq 0$, q is monotone and concave the conditions (ii) and (iv) are due to Leighton and Oo Kian Ke [4, 6].

Finally, let p and q be monotone functions. If both are monotone increasing on $[-a, a]$ the estimate (16) can be modified as follows:

$$
\begin{aligned}
t[v, v] &= \frac{\pi^2}{4a^2} \int_0^1 \left(\frac{1}{a - h(y)} \int_{-a}^{-h(y)} p(x)\,dx \right) \big(a - h(y)\big)\,dy \\
&\quad + \frac{\pi^2}{4a^2} \int_0^1 \left(\frac{1}{a - h(y)} \int_{h(y)}^{a} p(x)\,dx \right) \big(a - h(y)\big)\,dy \\
&\quad + \int_0^1 \left(\frac{1}{k(y)} \int_{-k(y)}^{0} q(x)\,dx \right) k(y)\,dy + \int_0^1 \left(\frac{1}{k(y)} \int_0^{k(y)} q(x)\,dx \right) k(y)\,dy \\
&\leqq \frac{\pi^2}{4a^3} \left(\int_{-a}^{0} p(x)\,dx \right) \left(\int_0^1 \big(a - h(y)\big)\,dy \right) + \frac{\pi^2}{4a^2} p(a) \int_0^1 \big(a - h(y)\big)\,dy \\
&\quad + q(0) \int_0^1 k(y)\,dy + \left(\frac{1}{a} \int_0^a q(x)\,dx \right) \left(\int_0^1 k(y)\,dy \right) \\
&= \frac{\pi^2}{8a^2} \left(ap(a) + \int_{-a}^{0} p(x)\,dx \right) + \frac{1}{2} \left(aq(0) + \int_0^a q(x)\,dx \right).
\end{aligned}
$$

This estimate proves the assertion of the following corollary under the hypothesis (i). The other assertions can analogously be proved.

Corollary 9: *An upper bound* b *for the first conjugate point of* $x = 0$ *with respect to the equation* (1) *considered on* $[0, X)$, $X \leqq \infty$, *is given by the smallest roots of the following equations:*

(i) $\frac{\pi^2}{b^2} \left(\frac{b}{2} p(b) + \int_0^{b/2} p\,dx \right) + \frac{b}{2} q\left(\frac{b}{2}\right) + \int_{b/2}^{b} q\,dx = 0$ *when* p, q *are monotone increasing;*

(ii) $\frac{\pi^2}{b^2}\left(\frac{b}{2}p(b) + \int\limits_0^{b/2} p\,dx\right) + \frac{b}{2}q\left(\frac{b}{2}\right) + \int\limits_0^{b/2} q\,dx = 0$ *when p is monotone increasing and q is monotone decreasing;*

(iii) $\frac{\pi^2}{b^2}\left(\frac{b}{2}p(0) + \int\limits_{b/2}^{b} p\,dx\right) + \frac{b}{2}q\left(\frac{b}{2}\right) + \int\limits_{b/2}^{b} q\,dx = 0$ *when p is monotone decreasing and q is monotone increasing;*

(iv) $\frac{\pi^2}{b^2}\left(\frac{b}{2}p(0) + \int\limits_{b/2}^{b} p\,dx\right) + \frac{b}{2}q\left(\frac{b}{2}\right) + \int\limits_0^{b/2} q\,dx = 0$ *when p, q are monotone decreasing.*

Examples: **a)** Consider the differential equation

$$-u'' + (x^2 - 7)\,u = 0 \qquad (0 \leqq x < \infty).$$

The interesting solution defined by the boundary condition $u(0) = 0$ is the Hermite function $H_3(x) = c\,e^{x^2/2}(e^{-x^2})^{(3)}$. The first conjugate point of $x = 0$ is $x_1 = \sqrt{3/2} = 1{,}2247\ldots$. By means of Corollary 8/(i) we obtain the upper bound $b = 1{,}2328\ldots$ for x_1.

b) Consider the equation

$$-[(1 - x^2)\,u']' + \left(\frac{1}{1 - x^2} - 12\right)u = 0 \quad (-a \leqq x \leqq a < 1). \qquad (18)$$

p is concave and q is convex. By applying Corollary 7/(iii), it is seen that there exists a pair of conjugate points on $[-a, a]$ with $a = 0{,}4584\ldots$ (18) is a Legendre differential equation the interesting solution of which is equal to

$$P_3^1(x) = c_1\sqrt{1 - x^2}\,((1 - x^2)^3)^{(4)} = c_2\sqrt{1 - x^2}\,(5x^2 - 1)$$

(cf. [1, pp. 94, 96, 344]) with zeros at $x_{1,2} = \pm 5^{-1/2} = \pm 0{,}4472\ldots$

Both examples show that the calculated upper bounds are good approximate values for the exact numbers.

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VERFASSER:

Prof. Dr. ERICH MÜLLER-PFEIFFER and THOMAS SCHOTT
Sektion Mathematik/Physik
Pädagogische Hochschule „Dr. Th. Neubauer" Erfurt/Mühlhausen
Nordhäuser Str. 63
DDR-5010 Erfurt