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On the Existence of Conjugate Points for Sturm-Liouville Differential Equations

E. MÜLLER-PFEIFFER and TH. SCHOTT

Mit Hilfe von Integralbedingungen für die Koeffizienten p und q Sturm-Liouvillescher Differentialgleichungen – $(p(x) u') - q(x) u = 0$ wird der Abstand benachbarter Nullstellen nichttrivialer Lösungen u nach oben abgeschätzt.

При помощи интегральных условий для коэффициентов р и q дифференциального уравнения Штурма-Лиувилля $-(p(x) u')' + q(x) u = 0$ расстояние соседних нулей нетривиальных решений и оценивается сверху.

By means of integral conditions for the coefficients p and q of Sturm-Liouville differential equations $-(p(x) u')' + q(x) u = 0$ the distance between consecutive zeros of non-trivial solutions u will be estimated from above.

There are various results in the literature on estimating the distance between consecutive zeros of solutions of second order differential equations (cf. [10], for instance). The following investigation is devoted to this problem. We consider the Sturm-Liouville differential equation on a bounded interval,

$$
-(p(x) u')' + q(x) u = 0, \qquad (-a \le x \le a < \infty; p, q \in C[-a, a]) \quad (1)
$$

and suppose that p is positive and piecewise continuously differentiable on $[-a, a]$. The points $x_1, x_2, -a \le x_1 < x_2 \le a$, are said to be *conjugate* with respect to the equation (1) if there exists a nontrivial solution u to (1) with $u(x_1) = 0 = u(x_2)$. Solutions to (1) are always real-valued functions belonging to $C^T[-a, a]$ (cf. [10, p. 25]). Set, for $0 \leq s < a$,

$$
Q(s) = \sup_{s < h < a} \frac{1}{2h} \int_{-h}^{h} q \ dx, \qquad \overline{p}(s) = \frac{1}{2(a-s)} \left(\int_{-a}^{-s} p \ dx + \int_{s}^{a} p \ dx \right)
$$

Theorem 1: If there exists a number s, $0 \leq s < a$, such that

$$
\frac{3}{(a-s)(a+2s)}\bar{p}(s) + Q(s) \leq 0,
$$
\n(2)

then there exists a pair of conjugate points on $(-a, a)$ with respect to (1) and the constant $3(a - s)^{-1} (a + 2s)^{-1}$ in (2) is the best possible one.

Proof: Consider the sesquilinear form

$$
t[f, g] = \int\limits_{-a}^{a} (p f' \bar{g}'_e + q f \bar{g}) dx \qquad (f, g \in D(t))
$$

to (1). The domain $D(t)$ of this form is identical with the Sobolev space $\ddot{W}_2^{-1}(-a, a)$. In the following the form is estimated by means of the test function

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to (1). The domain
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D(t)
$$
 of this form is identical
In the following the form is estimated by m

$$
v(x) = \begin{cases} a - |x|, & s \leq |x| \leq a, \\ a - s, & |x| \leq s, \end{cases}
$$
which belongs to $D(t)$. By Fubini's theorem

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\nIn the following the form is estimated by means of the test function
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$$
v(x) = \begin{cases} a - |x|, & s \le |x| \le a, \\ a - s, & |x| \le s, \end{cases}
$$
\nwhich belongs to $D(t)$. By Fubini's theorem we obtain
\n
$$
t[v, v] = \int_{-a}^{a} (p(v')^2 + qv^2) dx = 2(a - s) \overline{p}(s) + \int_{-a}^{a} \int_{-a}^{b(t)} q(x) dy dx
$$
\n
$$
= 2(a - s) \overline{p}(s) + 2 \int_{0}^{a(-s)^2} \left(\frac{1}{2h(y)} \int_{-h(y)}^{h(y)} q(x) dx \right) h(y) dy
$$
\n
$$
\leq 2(a - s) \overline{p}(s) + 2Q(s) \int_{0}^{b(t)} h(y) dy,
$$
\nwhere $h(y)' = a - \sqrt{y}, 0 \le y \le (a - s)^2$. Hence, in view of (2), there
\n $t[v, v] \leq 2(a - s) \overline{p}(s) + \frac{2}{2}Q(s) (a - s)^2 (a + 2s) \leq 0$. Consequently,

 s ². Hence, in view, of (2) , there follows that $\leq 2(a-s) \bar{p}(s) + 2Q(s) \int h(y) dy,$ (5)

where $h(y) = a - \sqrt{y}$, $0 \leq y \leq (a-s)^2$. Hence, in view of (2), there follows that
 $\forall [v, v] \leq 2(a-s) \bar{p}(s) + \frac{2}{3} Q(s) (a-s)^2 (a + 2s) \leq 0$. Consequently, we have $\leq 2(a-s) \bar{p}(s) + 2Q(s) \int_0^b h(y) dy,$ (5)

where $h(y) = a - \sqrt{y}, 0 \leq y \leq (a-s)^2$. Hence, in view of (2), there follows that
 $\{[v, v] \leq 2(a-s) \bar{p}(s) + \frac{2}{3} Q(s) (a-s)^2 (a+2s) \leq 0$. Consequently, we have
 $\inf_{L_2(-a, a)} [t[f, f]; f \in D(t), ||f||$ **•** $\begin{array}{c} \text{2}(a - s) \overline{p}(s) + 2Q(s) \int h(y) dy, \\ \text{where } h(y) = a - \sqrt{y}, 0 \leq y \leq (a - s)^2. \end{array}$ Hence, in view of (2), there follows that $\{[v, v] \leq 2(a - s) \overline{p}(s) + \frac{2}{3}Q(s) (a - s)^2 (a + 2s) \leq 0. \end{array}$ Consequently, we have inf $\{[t], f] :$ $L_2(-a, a)$. If this infimum is less than zero, then there exists a nontrivial solution u to (1) having at least two zeros on $(-a, a)$ (cf. [8]). If the infimum is equal to zero, the $(i$ ^hormalized) test function v is realizing the infimum and, consequently, it is a solution to (1) (cf. $[9]$). This, however, is impossible, because a solution to (1) belongs to the Sobolev space $W_2^2(-a, a)$ (cf. [2]). The function *v*, however; does not belong to $W_2^2(-a, a)$. $\inf \{t[f,f]: f \in D(t), ||f|| = 1\} \leq 0$, where $||\cdot||$ denotes the norm in the Hilbert space $(y)' = a - \sqrt{y}$, $0 \le y \le (a - s)^2$. Hence, in vi
 $2(a - s) \overline{p}(s) + \frac{2}{3} Q(s) (a - s)^2 (a + 2s) \le$
 $\therefore f \in D(t), ||f|| = 1) \le 0$, where $||\cdot||$ denotes th

). If this infimum is less than zero, then there

ring at least two zeros on $(-a, a)$ $\langle u, v \rangle \leq 2(a-s) \bar{p}(s) + \frac{2}{2} \frac{3}{2} \cdot (s-a)^2 \cdot (a-s)^2 \cdot (a+2s) \leq 0$. Consequently, we have
inf $\langle u|f_i, f_i \rangle \neq \bar{c} D(i), ||j|| = 1 \leq 0$, where $\frac{1}{2} \langle e_1, e_2, e_3 \rangle \leq 0$. Consequently, we have
 $L_2(-a, a)$. If this infimum is *fp dx ^fdx,* Q) = sup *fqedx* • 2(a - *.\$)* • Finere exists

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 $\frac{1}{a}$ $(a + 2s)^{-1}$ if

for any $\varepsilon >$
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We prove now that the constant $\sigma = 3(a - s)^{-1} (a + 2s)^{-1}$ is the best possible one. p_{ϵ} and q_{ϵ} with

$$
(\sigma - \varepsilon) \overline{p}_\varepsilon(s) + Q_\varepsilon(s) \leq 0, \tag{6}
$$

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We prove now that the constant
$$
\sigma = 3(a - s)^{-1} (a + 2s)^{-1}
$$
 is the best possible one.
\nLet us discuss the case $0 < s < a$. We prove that for any $\varepsilon > 0$ there exist functions.
\n p_{ε} and q_{ε} with
\n
$$
(\sigma - \varepsilon) \overline{p}_{\varepsilon}(s) + Q_{\varepsilon}(s) \leq 0,
$$
\nwhere
\n
$$
\overline{p}_{\varepsilon}(s) = \frac{1}{2(a - s)} \left(\int_{-a}^{s} p_{\varepsilon} dx + \int_{s}^{a} p_{\varepsilon} dx \right), \qquad Q_{\varepsilon}(s) = \sup_{s < h < a} \frac{1}{2h} \int_{-a}^{h} q_{\varepsilon} dx
$$
\nsuch that there does not exist a pair of conjugate points on $(-a, a)$ with respect to the differential equation
\n
$$
-(p_{\varepsilon}(x) u')' + q_{\varepsilon}(x) u = 0, \qquad (-a \leq x \leq a).
$$
\n(7)
\nObviously, it suffices to assume that $\varepsilon < \sigma$ (if $\varepsilon \geq \sigma$, choose $p_{\varepsilon} \equiv 1$ and $q_{\varepsilon} \equiv 0$):
\nChoose $q_{\varepsilon} = \varepsilon - \sigma$ and
\n
$$
p_{\varepsilon}(x) = (\varepsilon - \sigma) \left\{ \begin{cases} (x^2/2 - (a + 2t^2(a + s)^{-1}) |x| + A(t)), & |x| \geq s \\ (t^2 - x \sinh^{-1} (x/t) B(t)), & |x| \leq s, \end{cases} \right\}
$$
\nwhere, for $0 < t < \infty$,

such that there does not exist a pair of conjugate points on $(-a, a)$ with respect to such that there does not exist a pair of conjugate points on $(-a, a)$ with respective differential equation
 $-(p_{\epsilon}(x) u')' + q_{\epsilon}(x) u = 0, \quad (-a \le x \le a).$

Obviously, it suffices to assume that $\epsilon < \sigma$ (if $\epsilon \ge \sigma$, choose $p_{\epsilon} \$

$$
-(p_{\iota}(x) u')' + q_{\iota}(x) u = 0, \qquad (-a \leq x \leq a). \qquad (7)
$$

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 $\rho_{\epsilon} \equiv 1 \text{ and } q_{\epsilon} \equiv 0$. γ , it s
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$$
-(p_{\epsilon}(x) u')' + q_{\epsilon}(x) u = 0, \quad (-a \le x \le a).
$$
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$$
y, \text{ it suffices to assume that } \epsilon < \sigma \text{ (if } \epsilon \ge \sigma, \text{ choose } p_{\epsilon} \equiv 1 \text{ and } q_{\epsilon} \equiv \epsilon - \sigma \text{ and}
$$
\n
$$
p_{\epsilon}(x) = (\epsilon - \sigma) \left\{ \begin{aligned} &\left(x^{2}/2 - (a + 2t^{2}(a + s)^{-1}) |x| + A(t) \right), &|x| \ge s \\ &\left(t^{2} - x \sinh^{-1} (x/t) B(t) \right), &|x| \le s, \end{aligned} \right.
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y, \text{ if } \epsilon > \epsilon \equiv 1 \text{ and } \epsilon
$$
\n
$$
p_{\epsilon}(x) = (\epsilon - \sigma) \left\{ \begin{aligned} &\left(x^{2}/2 - (a + 2t^{2}(a + s)^{-1}) |x| + A(t) \right), &|x| \ge s, \\ &|x| \le s, &|\epsilon| \le s, \end{aligned} \right.
$$

$$
\overline{p}_\epsilon(s) = \frac{1}{2(a-s)} \left(\int_{-a}^{\infty} p_\epsilon dx + \int_{s}^{\infty} p_\epsilon dx \right), \qquad Q_\epsilon(s) = \sup_{s < h < a} \frac{1}{2h} \int_{s}^{\infty} q_\epsilon dx
$$
\nsuch that there does not exist a pair of conjugate points on $(-a, a)$ with respect to differentiable equation

\n
$$
-(p_\epsilon(x) u')' + q_\epsilon(x) u = 0, \qquad (-a \le x \le a).
$$
\nObviously, it suffices to assume that $\epsilon < \sigma$ (if $\epsilon \ge \sigma$, choose $p_\epsilon \equiv 1$ and $q_\epsilon \equiv$ Choose $q_\epsilon = \epsilon - \sigma$ and

\n
$$
p_\epsilon(x) = (\epsilon - \sigma) \left\{ \left(t^2/2 - (a + 2t^2(a + s)^{-1}) |x| + A(t) \right), \quad |x| \ge s \right\}
$$
\nwhere, for $0 < t < \infty$,

\n
$$
A(t) = \frac{s^2}{2} + t^2 - st \coth \frac{s}{t}, \qquad B(t) = \left(a - s + \frac{2t^2}{a + s} \right) \sinh \frac{s}{t} + t \cosh \frac{s}{t}.
$$

 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

-

Here t is a parameter which will be fixed later. The value $p_e(0)$ is defined by the limit Here *t* is a parameter which will be fixed later. The value $p_c(0)$ is defined by the limit $(\varepsilon - \sigma) (t^2 - tB(t))$ of $p_c(x)$ for $x \to 0$. Clearly, p_c is an even function. Further, it can easily be verified that p_c is co easily be verified that p_t is continuous and piecewise continuously differentiable on $[-a, a]$. We have Conjugate Points for Sturm Liouville D

Here t is a parameter which will be fixed later. The value $p_i(0)$ is def
 $(\varepsilon - \sigma)(t^2 - tB(t))$ of $p_i(x)$ for $x \to 0$. Clearly, p_i is an even function

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eter which will be fixed later. The v

(a) of $p_t(x)$ for $x \to 0$. Clearly, p_t is

that p_t is continuous and piecewi
 $(\sigma - \varepsilon) \begin{cases} (a + 2t^2(a + s)^{-1} - x), \\ \sinh^{-1} (x/t) B(t) (1 - (x, t)) \end{cases}$

Conjugate Points for Sturm-Liouville Diff. Equ. 1
\nHere t is a parameter which will be fixed later. The value
$$
p_c(0)
$$
 is defined by the lin
\n $(e - i \sigma) (i^2 - iB(t))$ of $p_c(x)$ for $x \to 0$. Clearly, p_c is an even function. Further, it
\neasily be verified that p_c is continuous and piecewise continuously differentiable
\n $[-a, a]$. We have
\n $p_c'(x) = (\sigma - \varepsilon) \begin{cases} (a + 2t^2(a + s)^{-1} - x), & s < x \leq a, \\ \sinh^{-1}(x/t) B(t) (1 - (x/t) \coth (x/t)), & 0 < x < s, \end{cases}$
\n $p_c'(0) = 0$, and $p_c'(x) > 0$ ($s < x \leq a$), $p_c'(x) < 0$ ($0 < x < s$). Hence
\n $\min p_c(x) = p_c(s) = (\varepsilon - \sigma) (s^2/2 - (a + 2t^2(a + s)^{-1}) s + A(t))$
\n $= (\varepsilon - \sigma) (s^2 + t^2 - st \coth (s/t) - (a + 2t^2(a + s)^{-1}) s)$
\n $>(\sigma - \varepsilon) t^2((s/t) \coth (s/t) - 1) > 0.$
\nThus, it follows that $p_c(x) > 0$, $-a \leq x \leq a$, whenever $0 < t < s$. Next we pre
\nthat (6) holds if t is chosen sufficiently small. The inequality (6) is equivalent to
\n $(a - s)^{-1} \int p_c dx \leq 1$.
\nAn easy calculation shows that

Thus, it follows that $p_{\epsilon}(x) > 0, -a \leq x \leq a$, whenever $0 < t < s$. Next we prove that (6) holds if *1* is chosen sufficiently small. The inequality (6) is equivalent to

$$
(a-s)^{-1}\int\limits_{s}^{a}p_{s}\,dx\leq 1.
$$

An easy calculation shows that

$$
(a - s) \int_{s}^{a} P_{\epsilon} dx = 1 - \epsilon \sigma^{-1} + (\sigma - \epsilon) \text{ s to both } (s|t).
$$

A value $t = t_c$, $0 < t_c < s$, can be chosen so small that t_c coth $(s/t_c) \leq \varepsilon/\sigma(\sigma - \varepsilon) s$. By such choice the inequalities (8), and, consequently, (6) are fulfilled: It is easily seen that the function
 $u(x) = \begin{cases} a + 2t^2(a + s)^{-1} - |x|, & |x| \ge s, \\ \sinh^{-1} (s/t) (B(t_i) - t_i \cosh (x/t_i)), & |x| \le s, \end{cases}$

below the CII, and and is pos *l* $(a \text{ s.t. } (6) \text{ hold}$
 $(a \text{ s.t. } (6) \text{ hold})$
 $(a \text{ s.t. } (a \text{ s$ *s*
 $< t_{\epsilon} < s$, can be chosen so small that t_{ϵ} cot
 e inequalities (8), and, consequently, (6) ar

tion
 $a + 2t_{\epsilon}^2(a + s)^{-1} - |x|$, $|x| \geq s$,
 $\sinh^{-1} (s/t_{\epsilon}) (B(t_{\epsilon}) - t_{\epsilon} \cosh (x/t_{\epsilon}))$, $|x| \leq s$, *s, '-* $\left(\begin{matrix} 0 \\ 0 \end{matrix}\right)$

seen that the function

\n
$$
u(x) = \begin{cases} a + 2t \cdot (a + s)^{-1} - |x|, & |x| \ge s, \\ \sinh^{-1} \left(s/t \right) \left(B(t_\epsilon) - t_\epsilon \cosh \left(x/t_\epsilon \right) \right), & |x| \le s, \end{cases}
$$

belongs to $C^1[-a, a]$ and is positive. The function $p_i u'$ belongs also to $C^1[-a, a]$ and by calculation one can prove that $(p_eu')' = q_eu$. The function *u* is a positive solution to (7). Finally, assume that there exists a nontrivial solution u_0 to (7) possessing at least two zeros x_1 and x_2 on $(-a, a)$. Then, by Sturm's comparison theorem, each solution to (7) has a zero between x_1 and x_2 or is a constant multiple of u_0 . The solution (9), however, contradicts this conclusion. Hence, there cannot exist a pair of conjugate points on $(-a, a)$ with respect to (7). This proves the theorem in the case $0 < s < a$. Similarly, one can prove that the constant $3a^{-2}$ is best possible in the case belongs to $C^1[-a, a]$ and is positive. The function p_iu' belongs also to $C^1[-a, a]$ and
by calculation one can prove that $(p_iu')' = q_iu$. The function u is a positive solution
to (7). Finally, assume that there exists a belongs to $C^1[-a; a]$ and is positive. The function p_iu' belongs also to
by calculation one can prove that $(p_iu')' = q_iu$. The function u is a p
to (7). Finally, assume that there exists a nontrivial solution u_0 to (
 $u(x) = \begin{cases} \n\text{arg}\; \text{to} \; C^1[-a] \n\text{alculation on} \n\end{cases}$. Finally, as

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if its derivat Considered to α for α for the mean value (1/2*h*) $\int_{1}^{h} q \, dx$, $\alpha \leq h \leq \alpha$. Since α is a consistion (9), however, contradicts this conclusion. Hence, conjugate points on $(-\alpha, \alpha)$ with respect to (7). This p

 $S_{\rm{max}}$. So the set of $S_{\rm{max}}$ is the set of $S_{\rm{max}}$

Again value (1/2h)
$$
\int_{-h}^{h} q \, dx, s \leq h \leq a^1
$$
, is an increasing or a decreasing function:

\nderivative is non-negative or non-positive, i.e. if

\n
$$
\frac{1}{2h} \int_{-h}^{h} q \, dx \leq \frac{1}{2} \left(q(-h) + q(h) \right) \quad \text{or} \quad \frac{1}{2h} \int_{-h}^{h} q \, dx \geq \frac{1}{2} \left(q(-h) + q(h) \right),
$$

respectively. Then, the corresponding suprema $Q(s)$ are $Q(s)$ = $\int q \, dx$ and $Q(s)$ a $q dx$, respectively. Thus we obtain the following corollary. *MÜLLER-PEEIFFER* and TH. SCHOTT
 f. Then, the corresponding suprema $Q(s)$ are $Q(s) = \frac{1}{2a} \int_{a}^{a} q \, dx$ and $Q(s)$
 f. respectively. Thus we obtain the following corollary.²
 f 1 : *If there exists a number s*,

Corollary 1: *If there exists a numbers,* $0 \le s < a$, such that

$$
\frac{1}{2h}\int\limits_{-h}^{h}q\ dx\leqslant_{1}\frac{1}{2}\left(q(-h)+q(h)\right)\qquad (s\leq h\leq a),
$$
\n(10)

then there exists a pair of conjugate points on $(-a, a)$ *with respect to* (1) *if*

then there exists a pair of conjugate points on
$$
(-a, a)
$$
 with respect to (1) if\n
$$
\frac{3}{(a-s)(a+2s)}\overline{p}(s) + \frac{1}{2a}\int_{-a}^{a} q \, dx \leq 0 \left(\frac{3}{(a-s)(a+2s)}\overline{p}(s) + \frac{1}{2s}\int_{-s}^{s} q \, dx \leq 0\right).
$$

In each case the factor $3(a-s)^{-1} (a+2s)^{-1}$ *is best possible.*

•

If q can be written as a sum $q = q_1 + q_2$ where q_1 is an odd function and q_2 is monotone decreasing on $[-a, -s]$ and monotone increasing on $[s, a]$, then one can easily $\frac{1}{(a-s)(a+2s)}\overline{p}(s) + \frac{1}{2a}\int_{-a_s} q \,dx \le 0$ $\left(\frac{a}{(a-s)(a+2s)}\overline{p}(s) + \frac{1}{2s}\int_{-s} q \,dx \le 0\right)$.
 In each case the factor $3(a-s)^{-1}(a+2s)^{-1}$ *is best possible.*
 If q can be written as a sum $q = q_1 + q_2$ *where* q_1 q_2 is monotone increasing on $[-a, -s]$ and monotone decreasing on $[s, a]$, then $\int\limits_{s}$ q an be written as a sum $q = \dot{q}_1 + q_2$ where q_1 is an odd function and q_2 is moncorreasing on $[-a, -s]$ and monotone increasing on $[s, a]$, then one can easil hat $\frac{1}{2s} \int_{-s}^{s} q \, dx \leq \frac{1}{2} (q(-s) + q(s))$ implies (10) tain the following corollary from Corollary 1. If *q* can be written as a sum $q = q_1 + q_2$ where q_1 is an odd function and q_2 is monotone decreasing on $[-a, -s]$ and monotone increasing on $[s, a]$, then one can easily prove that $\frac{1}{2s} \int_{s}^{s} q \, dx \leq \frac{1}{2} \left(q(\frac{f_2}{g}\Big|\Big(q(-s) + q(s)\Big)$ implies (10) writter
 cowing corollary from Corollary 1.
 f $g = q_1 + q_2$ in $C[-a, a]$ with q_1 odd an
 $f(z)$, $(s \le x_1 < x_2)$, and suppose further there is $\int_a^s q(x) dx \leqslant \frac{1}{2} (q(-s) + q(s)).$ notone decreasing on [s, a], then

i with the sign \geq . Hence, we ob-
 $0 \leq s < a$, such that q can be writ-
 $0 d q_2(x_1) \geq q_2(x_2)$ ($x_1 < x_2 \leq -s$),

that

(11)

a) with respect to (1) if (s) $+ \frac{1}{2a} \int_a^b q \, dx \le 0 \left(\frac{3}{(a - s) (a + 2s)} \overline{p}(s) + \frac{1}{2s} \int_a^b q \, dx \le 0 \right)$
 $r 3(a - s)^{-1} (a + 2s)^{-1}$ is best possible.

as a sum $q = q_1 + q_2$ where q_1 is an odd function and q_2 is mono
 $-a, -s$ and monotone incre

Corollary 2: Suppose that there exists a point s, $0 \leq s < a$ *, such that q can be written as a sum* $q = q_1 + q_2$ *in C[-a, a] with* q_1 *odd and* $q_2(x_1) \geqslant q_2(x_2)$ $(x_1 < x_2 \leq -s)$ *,* $q_2(x_1) \geqslant q_2(x_2)$ $(s \leq x_1 < x_2)$ *, and suppose further that*

$$
\frac{1}{2s}\int_{-s}^{s}q(x)\,dx\leqslant \frac{1}{2}\,\left(q(-s)+q(s)\right). \tag{11}
$$

Then there exists a pair of conjugate points on $(-a, a)$ *with respect to.*(1) *if*

then as a sum
$$
q = q_1 + q_2
$$
 in $C[-a, a]$ with q_1 odd and $q_2(x_1) \in q_2(x_2)$ $(x_1 < x_2 \leq -s)$, $q_2(x_1) \in q_2(x_2)$ (s $\leq x_1 < x_2$), and suppose further that

\n
$$
\frac{1}{2s} \int_{-s}^{s} q(x) dx \leq \frac{1}{2} \left(q(-s) + q(s) \right).
$$
\nThen there exists a pair of conjugate points on $(-a, a)$ with respect to (1) if

\n
$$
\frac{3}{(a-s)(a+2s)} \overline{p}(s) + \frac{1}{2a} \int_{-a}^{a} q \, dx \leq 0 \left(\frac{3}{(a-s)(a+2s)} \overline{p}(s) + \frac{1}{2s} \int_{-s}^{s} q \, dx \leq 0 \right).
$$
\n(12)

\nIn the special case $s = 0$ the hypotheses (11) are always satisfied and the conditions

\n
$$
\frac{3}{a^2} \int_{-a}^{a} p \, dx + \int_{-a}^{a} q \, dx \leq 0 \left(\frac{3}{2a^3} \int_{-a}^{a} p \, dx + q(0) \leq 0 \right).
$$
\n(13)

$$
x \ge \frac{1}{2} |(q(-s) + q(s)) \text{ implies (10) written with the sign } \ge. \text{ Hence, we obtain}
$$
\n
$$
x \ge \frac{1}{2} |(q(-s) + q(s)) \text{ implies (10) written with the sign } \ge. \text{ Hence, we obtain}
$$
\n
$$
x \ge 2. \text{ Suppose that there exists a point } s, 0 \le s < a, \text{ such that } q \text{ can be written}
$$
\n
$$
x \ge 2. \text{ Suppose that there exists a point } s, 0 \le s < a, \text{ such that } q \text{ can be written}
$$
\n
$$
x_2(x_2) \cdot (s \le x_1 < x_2), \text{ and suppose further that}
$$
\n
$$
\frac{1}{2s} \int_{2s}^{s} q(x) \, dx \le \frac{1}{2} \cdot \left(q(-s) + q(s) \right).
$$
\n
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x \ge 2. \text{ Thus, we have}
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x \ge 2. \text{ Thus, we have}
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Corollary 3: Suppose that q belongs to $C^2[-a, a]^2$ and is convex (concave) on $[-a, a]$. *I/ the hypotheses (13) are satisfied, respectively, then there exists a pair of conjugate points on* $(-a, a)$ *with respect to (1). The constants* $3/a^2$ *(3/2a³) are best possible.*

Proof: Set $q(x) = q(0) + q'(0)x + r(x)$, $-a \le x \le a$, and note that *r* has the Corollary 3: Suppose that *q* belongs to $C^2[-a, a]^2$ and is convex (concave) on $[-a, a]$.

If the hypotheses (13) are satisfied, respectively, then there exists a pair of conjugate

points on $(-a, a)$ with respect to (1). T be a mypotheses (13) are satisfied, respectively, then there exists a pair of conjugue
 $\begin{array}{l}\n\text{for } n(-a, a) \text{ with respect to (1).} \text{ The constants } 3/a^2 \text{ (3/2a)} \text{ are best possible.}\n\end{array}$
 $\begin{array}{l}\n\text{for } n(-a, a) \text{ with respect to (1).} \text{ The constants } 3/a^2 \text{ (3/2a)} \text{ are best possible.}\n\end{array}$
 $[-a, 0]$ and $r'(x) \ge 0$ $(r'(x) \le 0)$ on $[0, a]$. Further, $q_1(x) = q'(0)$ *x* is an odd function and $q_2(x) = q(0) + r(x)$ has the concerned properties formulated in Corollary 2. (with $s = 0$). The corollary now follows from Corol Corollary *3*: Suppose that q belongs to $C^2[-a, a]^2$ and is convex (conce $I\{$ the hypotheses (13) are satisfied, respectively, then there exists a particular points on $(-a, a)$ with respect to (1). The constants $3/a^2$ *,.* Corollary 3: Suppose that q belongs to C²[-a, a]²) and is convex (concave) on [-
 If the hypotheses (13) are satisfied, respectively, then there exists a pair of concording points on $(-a, a)$ with respect to (1). The

Corollary 4: Let p be positive and piecewise continuously differentiable on $[0, X)$, $X \leq \infty$, and assume that $q(\in C^2[0, X))$ is convex (concave). Let u be a nontrivial solu *tion to (1) considered on [0, X) with u(0) = 0. The first conjugate point to x = 0.* $X \leq \infty$, and assume that $q(\in C^2[0, X))$ is convex (concave). Let u be a nontrivial solution to (1) considered on [0, X) with u(0) = 0. $\begin{array}{ccccc}\n & & \text{pre} & \[-1.2em] \hline\n & & \text{on} & \[-1.2em] \hline\n\end{array}$ and $q_2(x) = q(0) + r(x)$ has the concerned properties formulated in Corollary 2.

(with $s = 0$). The corollary now follows from Corollary 2.

Corollary 4: Let p be positive and piecewise continuously differentiable on $[0, X$ $(-a, 0]$ and $r(x) \le 0$ $(r(x) \le 0)$ on $[0, a]$. Fund $q_2(x) = q(0) + r(x)$ has the concerned

(with $s = 0$). The corollary now follows from

Corollary 4: Let p be positive and pieceu
 $X \le \infty$, and assume that $q(\in C^2[0, X))$ is

$$
12 \int\limits_{0}^{b} p\,dx + b^2 \int\limits_{0}^{b} q\,dx = 0 \qquad \left(12 \int\limits_{0}^{b} p\,dx + b^3 q(b/2) = 0\right),
$$

when b exist.

Proof: Use Corollary 3 and Sturm's comparison theorem.

$$
t[v, v] = \int_{-a}^{t} p(x) dx + \int_{0}^{a^{2}} \left(\frac{1}{h(y)} \int_{-h(y)}^{0} q(x) dx \right) h(y) dy + \int_{0}^{a^{2}} \left(\frac{1}{h(y)} \int_{0}^{h(y)} q(x) dx \right) h(y) dy
$$

\n
$$
\leq \int_{-a}^{a} p(x) dx + q(0) \int_{0}^{a^{2}} h(y) dy + \left(\frac{1}{a} \int_{0}^{a} q(x) dx \right) \int_{0}^{a^{2}} h(y) dy
$$

\n
$$
= \int_{-a}^{a} p(x) dx + \frac{1}{3} a^{3}q(0) + \frac{1}{3} a^{2} \int_{0}^{a} q(x) dx.
$$

\nAnalogously, if q is monotone decreasing on [-a, a] we have
\n
$$
t[v, v] \leq \int_{-a}^{a} p dx + \frac{1}{3} a^{3}q(0) + \frac{1}{3} a^{2} \int_{-a}^{0} q dx.
$$

Analogously, if *q is* monotone decreasing on [—a, *a]* we have

$$
t[v, v] \leq \int_{-a}^{a} p \ dx + \frac{1}{3} a^3 q(0) + \frac{1}{3} a^2 \int_{-a}^{b} q \ dx.
$$

These estimates yield analogous corollaries to the Corollaries 3 and 4. The followingcorollary corresponds to Corollary 4.

 $Corollary 5: Let p be positive and piecewise continuously differentiable on $[0, X)$,$ $X \leq \infty$, and assume that q is monotone increasing (decreasing) on $[0, X)$ with $u(0) = 0$.
The first conjugate point to $x = 0$ is smaller than b where b is the smallest positive root *•* $\iota[v, v] \leq \int p dx + \frac{1}{3} a^3 q(0) + \frac{1}{3} a^2 \int q dx$.

These estimates yield analogous corollaries to the Corollaries 3 and 4. The following corollary corresponds to Corollary 4.

Corollary 5: Let p be positive and piecewis *of the equation -*

Corollary 5: Let p be positive and piecewise continuously differentiable on
$$
[0, X)
$$
,
\n $X \le \infty$, and assume that q is monotone increasing (decreasing) on $[0, X)$ with $u(0) = 0$.
\nThe first conjugate point to $x = 0$ is smaller than b where b is the smallest positive root
\nof the equation
\n
$$
12 \int_{0}^{b} p \, dx + \frac{b^3}{2} q \left(\frac{b}{2}\right) + b^2 \int_{b/2}^{b} q \, dx = 0 \left(12 \int_{0}^{b} p \, dx + \frac{b^3}{2} q \left(\frac{b}{2}\right) + b^2 \int_{0}^{b/2} q \, dx = 0\right),
$$
\nwhen b exists.
\n
$$
2 \int_{0}^{b} \text{The hypothesis } q \in C^2 \text{ can be weakened.}
$$

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Corollary 6: *If there exists a nontrivial solution u to (1) with* $u(-a) = 0 = u(a)$ and $u(x) = 0$, $-a < x < a$, then **160 b**. MÜLLER-PFEIFFER and TH. SCHOT
 Corollary 6: If there exists a nontrivident $u(x) = 0$, $-a < x < a$, then
 $\int_a^a p(x) dx + \int_a^b (a + x)^2 q(x) dx$

2. MüLLER-PFEIFFER and TH. SCHOTT

\nary 6: If there exists a nontrivial solution u to (1) with
$$
u(-a) = 0 = u(a)
$$

\n $\neq 0, -a < x < a$, then

\n $\int_a^a p(x) \, dx + \int_{-a}^0 (a + x)^2 q(x) \, dx + \int_0^a (a - x)^2 q(x) \, dx > 0.$

\n(14)

\nBy assuming the contrariety of (14) there follows from (5) that $l[v, v] \leq 0$ defined by (4). This implies that there exists a pair of conjugate points on

Proof: By assuming the contrariety of (14) there follows from (5) that $t[v, v] \leq 0$ where *v* is defined by (4). This implies that there exists a pair of conjugate points on $(-a, a)$ with respect to (1). This, however, is impossible because by Sturm's comparison theorem the solution *u* would have a zero $(-a, a)$ with respect to (1). This, however, is impossible because by Sturm's comparison theorem the solution u would have a zero in $(-a, a)$ Proof: By assuming the contrariety of (14) there follows from (5) that $l[v, v] \le 0$
tere v is defined by (4). This implies that there exists a pair of conjugate points on
a, a, with respect to (1). This, however, is impo ary 6: If there exist
 $\neq 0$, $-a < x < a$, t
 $\leftarrow a$
 Level exists a nontrivial solution is to (1) with $u(-a)$
 $\frac{2}{x} \le a$, then
 $\int_{-a}^{0} (a + x)^2 q(x) dx + \int_{0}^{a} (a - x)^2 q(x) dx > 0$.
 $\frac{1}{a}$

ing the contrariety of (14) there follows from (5) th
 $\frac{1}{a}$ (4). This implies tha contrariety of (14) there follows from (5) that $l[v, v] \leq 0$
is implies that there exists a pair of conjugate points on
This, however, is impossible because by Sturm's compari-
would have a zero in $(-a, a) \parallel$.
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 \ldots

In the special case $p \equiv 1$ and $q \leq 0$ the condition (14) is due to LOVELADY [7].

setting

From theorem the solution
$$
u
$$
 would have a zero in $(-a, a)$.

\nIn the special case $p \equiv 1$ and $q \leq 0$ the condition (14) is due to LovELADY [7]. By using the test function $v(x) = \cos(\pi x/2a)$, $-a \leq x \leq a$, in place of (4) and setting

\n $P = \sup_{0 < h < a} \frac{1}{2(a-h)} \left(\int_{-a}^{h} p \, dx \right)$, $Q = \sup_{0 < h < a} \frac{1}{2h} \int_{-h}^{h} q \, dx$

\nwe obtain the following theorem.

\nTheorem 2: If

\n $(n^2/4a^2) P + Q \leq 0$,

\nthen there exists a pair of conjugate points on $[-a, a]$ with respect to (1). The constant $\pi^2/4a^2$ is the best possible one.

\nProof: The proof is analogous to that of Theorem 1. By means of Fubini's theorem, and setting

\n $\sigma(x) = \sin^2(\pi x/2a)$, $\gamma(x) = \cos^2(\pi x/2a)$, $-a \leq x \leq a$,

\n $h(y) = (2a/\pi) \arcsin \sqrt{y}$, $k(y) = (2a/\pi) \arccos \sqrt{y}$, $0 \leq y \leq 1$,

we obtain the following theorem. $\sqrt{ }$

Theorem 2:1/

$$
(\pi^2/4a^2)\ P\ +\ Q\leqq 0\,,
$$

then there exists a pair of conjugate points on $[-a, a]$ *with respect to (1). The constant* $\pi^2/4a^2$ *is the best possible one.*

 $\rm Proof:$ The proof is analogous to that of Theorem 1. By means of Fubini's theorem **b**
 a $\left(\frac{\pi^2}{4a^2}\right)P + Q \le 0,$
 a $\left(\frac{\pi^2}{4a^2}\right)$ *a* $\left(\frac{\pi^2}{4a^2}\right)$ *a* $\left(\frac{\pi^2}{4a^2}\right)$ *<i>a* $\left(\frac{\pi^2}{4a^2}\right)$ *a* $\left(\frac{\pi}{2}\right)$ *<i>a* $\left(\frac{\pi}{2}\right)$ *a* $\left(\frac{\pi}{2}\right)$ *a* $\left(\frac{\pi}{2}\right)$ *<i>a* $\left(\frac$ *h* exists a pair of conjugate points on $[-a, a]$ with respect to (1). The contract the best possible one.
 \therefore The proof is analogous to that of Theorem 1. By means of Fubini's theorem
 $\sigma(x) = \sin^2(\pi x/2a), \qquad \gamma(x) = \cos^2(\pi x/$

$$
\sigma(x) = \sin^2(\pi x/2a), \qquad \gamma(x) = \cos^2(\pi x/2a), \qquad -a \le x \le a,
$$

$$
\int_{0}^{0.6\lambda \le a} 2(a - h) \int_{-a}^{0.6\lambda \le a} \frac{y}{2a - h} \qquad \int_{0}^{0.6\lambda \le a} \frac{y}{2a - h} \qquad (15)
$$
\nwe obtain the following theorem.
\nTheorem 2: If
\n $(\pi^2/4a^2) P + Q \le 0$,
\n $(\pi^2/4a^2) P + Q \le 0$,
\n $\pi^2/4a^2 \text{ is the best possible one.}$
\nProof: The proof is analogous to that of Theorem 1. By means of Fubini's theorem
\nand setting
\n $\sigma(x) = \sin^2(\pi x/2a), \qquad y(x) = \cos^2(\pi x/2a), \qquad -a \le x \le a,$
\n $h(y) = (2a/\pi) \arcsin \sqrt{y}, \qquad k(y) = (2a/\pi) \arccos \sqrt{y}, \qquad 0 \le y \le 1,$
\nwe obtain
\n
$$
l(v, v) = \int_{-a}^{a} p(x) (v'(x))^2 dx + \int_{-a}^{a} q(x) w^2(x) dx
$$

\n
$$
= \frac{\pi^2}{4a^2} \int_{-a}^{a} p(x) \sin^2 \left(\frac{\pi}{2a} x\right) dx + \int_{-a}^{a} q(x) \cos^2 \left(\frac{\pi}{2a} x\right) dx
$$

\n
$$
= \frac{\pi^2}{4a^2} \int_{-a}^{a} \int_{0}^{a(x)} p(x) dy dx + \int_{-a}^{a} \int_{0}^{y(x)} q(x) dy dx
$$

\n
$$
= \frac{\pi^2}{2a^2} \int_{0}^{1} \frac{1}{2(a - h(y))} \left(\int_{-a}^{h(y)} p(x) dx + \int_{h(y)}^{h} p(x) dx\right) (a - h(y)) dy
$$

\n
$$
+ 2 \int_{0}^{1} \frac{1}{2k(y)} \left(\int_{k(y)}^{k(y)} q(x) dx\right) k(y) dy
$$

\n
$$
\le \frac{\pi^2}{2a^2} P \int_{0}^{1} (a - h(y)) dy + 2Q \int_{0}^{1} k(y) dy = \frac{\pi^2}{4a} P + aQ.
$$
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 $\mathbf{S} = \begin{pmatrix} \mathbf{S} & \mathbf{S} & \mathbf{S} \ \mathbf{S} & \mathbf{S} & \mathbf{S} \end{pmatrix}$

It follows from (15) that $t[v, v] \leq 0$. Hence, there exists a pair of conjugate points on $[-a, a]$. To prove that the constant $\pi^2/4a^2$ is best possible observe that cos $(\pi x/2\alpha)$, $\alpha > a$, is a positive solution to the differential equation **Conjugate Points for Sturm-Liouville Diff. Equ.**
 t follows from (15) that $l[v, v] \le 0$. Hence, there exists a pair of conjugate po
 -a, *a*, is a positive solution to the differential equation
 $-u'' - (n^2/4\alpha^2) u = 0 \t (-$ Conjugate Points for Sturm-Liouville Diff. Equ. 161

(2) that $l[v, v] \leq 0$. Hence, there exists a pair of conjugate points on

that the constant $\pi^2/4a^2$ is best possible observe that $\cos(\pi x/2\alpha)$,
 $\cos(\pi x/2\alpha)$,
 $\cos(\$

$$
-u'' - (\pi^2/4\alpha^2) u = 0 \t (-a \le x \le a). \t(17)
$$

The coefficients $p \equiv 1$ and $q \equiv -\pi^2/4\alpha^2$ satisfy $(\pi^2/4\alpha^2)$ $P + Q = 0$, and, on the $\alpha > a$, is a positive solution to the differential equation
 $-u'' - (\pi^2/4\alpha^2) u = 0 \quad (-a \le x \le a)$. (17)

The coefficients $p \equiv 1$ and $q \equiv -\pi^2/4\alpha^2$ satisfy $(\pi^2/4\alpha^2) P + Q = 0$, and, on the other hand, there does not exist a Conjugate Points for Sturm-Liouville Diff. Equ. 161

It follows from (15) that $\{v, v\} \le 0$. Hence, there exists a pair of conjugate points on $[-a, a]$. To prove that the constant $\pi^2/4a^2$ is best possible observe that In the special case $p \equiv 1$ the inequality (15) calls on the constant $\pi^2/4a^2$ is best possible observe that cos (π , α , is a positive solution to the differential equation
 $-u'' - (\pi^2/4\alpha^2) u = 0 \quad (-a \le x \le a)'$

i.e co It follows from (15) that $t[v, v]$
 $[-a, a]$. To prove that the co
 $x > a$, is a positive solution to
 $-u'' - (\pi^2/4\alpha^2) u =$

The coefficients $p \equiv 1$ and q

other hand, there does not exist

17). Therefore, the constant τ
 To prove that the constant $\pi^2/4a^2$ is best 1

a positive solution to the differential equal
 $-u'' - (\pi^2/4a^2) u = 0 \t (-a \le x \le a)$

ficients $p \equiv 1$ and $q \equiv -\pi^2/4a^2$ satisfy (

dd, there does not exist a pair of conjugate The coefficients $p \equiv 1$ and $q \equiv -\pi^2/4\alpha^2$ satisfy $(\pi^2/4\alpha^2)$ $P + Q = 0$, and
other hand, there does not exist a pair of conjugate points on $[-a, a]$ with re
(17). Therefore, the constant $\pi^2/4a^2$ in (15) cannot be

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$$
\sup_{0
$$

This condition for the existence of conjugate points with respect to the equation $-u'' + q(x) u$ $=0$ ($-a \le x \le a$) is due to LEIGHTON [5, 6].

Assume now that the coefficients p and q'can be written as $p = p_1 + p_2$, $q = q_1 + q_2$

on $[-a, a]$, where the following possibilities are to be discussed:

(1) p_1, q_1 are odd functions;

(2a) $p_2(x_1) \ge p_2(x_2)$ $(x_1 < x_2 \le$ \cdot + $q(x) u$
= $q_1 + q_2 v$ *i lities* are *V*

- $(p_1, q_1 \text{ are odd functions};$
- (1) Therefore, the constant $\pi^2/4a^2$ in (15) cannot be chosen smaller **I**

In the special case $p \equiv 1$ the inequality (15) cannot be chosen smaller **I**

In the special case $p \equiv 1$ the inequality (15) cannot be chosen (a) $\frac{1}{2}$ $\frac{1}{2}$ **c** $\frac{1}{2}$ (3b) $(-a \le x \le a)$ is due to Leterron [5, 6].

Assume now that the coefficients p and q'can be written as $p = p_1 + p_2$, $q = q$
 $[-a, a]$, where the following possibilities are to be discussed:

(1) p_1, q_1 are odd functions; (3b) $q_2(x_1) \leq q_2(x_2)$ $(x_1 < x_2 \leq 0)$, $q_2(x_1) \geq q_2(x_2)$ $(0 \leq x_1 < x_2)$.
Corollary 7: *There exists a pair of conjugate points on* $[-a, a]$ *with respect to* (1) $\sup_{0 \le h \le a} \frac{1}{2h} \int_{-h}^{h} q(x) dx \le -\frac{\pi^2}{4a^2}.$
This condition for the existence of conjugate points w
= $0(-a \le x \le a)$ is due to LETGRTON [5, 6].
Assume now that the coefficients p and q'can l
on $[-a, a]$, where the follo $a_0 = 0$ ($-a \le x \le a$) is due to Lefterfon [5, 6].

Assume now that the coefficients p and q'can be written $[-a, a]$, where the following possibilities are to be

(1) p_1, q_1 are odd functions;

(2a) $p_2(x_1) \ge p_2(x_2)$ $(x_$ (2a) $p_2(x_1) \leq p_2(x_2)$ $(x_1 < x_2 \leq 0)$, $p_2(x_1) \leq p_2(x_2)$ $(0 \leq x_1 < x_2)$

(2b) $p_2(x_1) \leq p_2(x_2)$ $(x_1 < x_2 \leq 0)$, $q_2(x_1) \leq q_2(x_2)$ $(0 \leq x_1 < x_2)$;

(3a) $q_2(x_1) \leq q_2(x_2)$ $(x_1 < x_2 \leq 0)$, $q_2(x_1) \le$

(2 a), (3 a) *are'* fulfilled (this is so, for instance, if the functions p , q are convex) $\begin{bmatrix} a \\ b \end{bmatrix}$
 $\begin{bmatrix} p(a-a) + p(a) \end{bmatrix} + \begin{bmatrix} q & d & r \end{bmatrix} \leq 0$ *V V*

(ii) (1), (2a) (3b) are fulfilled (this is so, for instance, if p is convex and q is concave) and $(\pi^2/8a^2)(p(-a) + p(a)) + q(0) \leq 0$. *and* $(\pi^2/4a) (p(-a) + p(a)) + \int q dx \le 0.$

(ii) (1), (2a) (3b) are fulfilled (this is so, for instance, if p is convex and q is concave)
 and $(\pi^2/8a^2) (p(-a) + p(a)) + q(0) \le 0.$

(iii) (1), (2b), (3a) are fulfilled (this is so, f

and $(\pi^2/4a^2)$ $\int p \, dx + \int q \, dx \le 0$.
 -a

(iv) (1), (2b), (3b) are fulfilled (this is so, for instance, if p, q are concave) and (ii) (1), (2a) (3b) are fulfilled (this is so, for instance, if *p* is convex and *q* is concave)
 $d (\pi^2/8a^2) (p(-a) + p(a)) + q(0) \leq 0$.

(iii) (1), (2b), (3a) are fulfilled (this is so, for instance, if *p* is concave and *q* Corollary 7: There exists a pair of conjugate point
in each of the following cases.
(i) (1), (2a), (3a) are fulfilled (this is so, for instance
and $(\pi^2/4a) (p(-a) + p(a)) + \int_a^a q dx \le 0$.
(ii) (1), (2a) (3b) are fulfilled (this *to* $f(x) = \frac{1}{2}$ $f(x) = \$ (iii) (1), (2b), (3a) are fulfilled (this is so, for instance, if p is concave and q is convex;
 $d (\pi^2/4a^2) \int_{0}^{b} p' dx + \int_{a}^{b} q dx \leq 0$.

(iv) (1), (2b), (3b) are fulfilled (this is so, for instance, if p, q are concave

The constants $\pi^2/4a$ *and so on are all best possible.*

The following corollary follows directly from Corollary 7 .

Corollary 8: *An upper bound b for the first conjugate point of* $x = 0$ *with respect* to (1) considered on [0, X), $X \leq \infty$, is given by the first roots of the following equations: The following corollary follows

The following corollary follows
 $C \text{orollary } 8: An upper bound
\n
$$
\begin{array}{l}\n\text{Corollary } 8: A \cdot n \text{ upper bound} \\
\text{for } n \geq 0, \quad 1, \quad 2 \leq n, \\
\text{(i) } (\pi^2/2b) \left(p(0) + p(b) \right) + \int_0^b q \text{ (ii) } (\pi^2/2b^2) \left(p(0) + p(b) \right) + q(b) \\
\text{for
$$$

-
- (ii) $(\pi^2 / 2b^2) (\r{p}(0) + p(b)) + q(b/2) = 0$ when p is convex and q is concave; $\frac{1}{\sqrt{2}}$ *V* $\frac{1}{\sqrt{2}}$ *V* $\frac{1}{\sqrt{2}}$ *V* $\frac{1}{\sqrt{2}}$

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 $\frac{1}{2}$

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 $\frac{1}{2}$

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\n1. (iii)
$$
(\pi^2/b^2)\int_0^b p\ dx + \int_0^b q\ dx = 0
$$
 when p is concave and q is convex;
\n
\n(ix) $(\pi^2/b^3)\int_0^b p\ dx + q(b/2) = 0$ when p, q are concave.

(iv)
$$
(\pi^2/\hat{b}^3)\int p\ dx + q(\hat{b}/2) = 0
$$
 when p, q are concave.

(iv) (π^2/\hat{b}^3) $\int_0^b p \ dx + q(b/2) = 0$ when p, q are concave.
In the special case that $p \equiv 1$, $q \le 0$, q is monotone and convex the conditions (i) and. (iii)
educ to LEIGHTON [6]. These conditions also improve a condi are due to LEIGHTON [6]. These conditions also improve a condition by FINK [3, Th. 3]. In the special case that $p \equiv 1, q \leq 0, q$ is monotone and concave the conditions (ii) and (iv) are due In the special case that $p \equiv 1, q \leq 0, q$ is monotone and convex the conditions (i) and (iii) are due to LEIGHTON [6]. These conditions also improve a condition by FINK [3, Th. 3]. In the special case that $p \equiv 1, q \leq 0, q$ • • **•• toJErnIIT0NandOO KlAN** Kn [4, 6]. .. ^S

Finally, let p and q be monotone functions. If both are monotone increasing on $[-a, a]$ the estimate (16) can be modified as follows:

6s' and 00 Kras Kg [4, 6].
\n7, let p and q be monotone functions. If both are monotone increasing on
\nthe estimate (16) can be modified as follows:
\n
$$
f[v, v] = \frac{\pi^2}{4a^2} \int_0^1 \left(\frac{1}{a - h(y)} \int_a^a p(x) dx \right) (a - h(y)) dy
$$
\n
$$
+ \frac{\pi^2}{4a^2} \int_0^1 \left(\frac{1}{a - h(y)} \int_a^a p(x) dx \right) (a - h(y)) dy
$$
\n
$$
+ \int_0^1 \left(\frac{1}{k(y)} \int_a^a q(x) dx \right) k(y) dy + \int_0^1 \left(\frac{1}{k(y)} \int_a^b q(x) dx \right) k(y) dy
$$
\n
$$
\leq \frac{\pi^2}{4a^3} \left(\int_a^a p(x) dx \right) \left(\int_0^1 (a - h(y)) dy \right) + \frac{\pi^2}{4a^2} p(a) \int_0^1 (a - h(y)) dy
$$
\n
$$
+ q(0) \int_0^1 k(y) dy + \left(\frac{1}{a} \int_0^a q(x) dx \right) \left(\int_0^1 k(y) dy \right)
$$
\n
$$
= \frac{\pi^2}{8a^2} \left(ap(a) + \int_0^a p(x) dx \right) + \frac{1}{2} \left(aq(0) + \int_0^a q(x) dx \right).
$$
\n
$$
= \frac{\pi^2}{8a^2} \left(ap(a) + \int_{-a}^a p(x) dx \right) + \frac{1}{2} \left(aq(0) + \int_0^a q(x) dx \right).
$$
\n
$$
= \frac{\pi^2}{8a^2} \left(ap(a) + \int_{-a}^a p(x) dx \right) + \frac{1}{2} \left(aq(0) + \int_0^a q(x) dx \right).
$$
\n
$$
= 0 \text{ with respect to } a, \text{ is given by the system of the hypothesis, there is a given by the smallest roots of the equations:
$$

This estimate proves the assertion of the following corollary under the hypothesis (i). The other assertions can analogously be proved.

 C _{orollary 9: An upper bound b for the first conjugate point of $x = 0$ with respect} This estimate proves the assertion of the following corollary under the hypothesis

(i). The other assertions can analogously be proved.
 \therefore Corollary 9: An upper bound b for the first conjugate point of $x = 0$ with re (i). The other assertions can analogously be proved.

(i). The other assertions can analogously be proved.
 i. Córollary 9: An upper bound b for the first conjugate point of $x =$
 to the equation (1) considered on [0,

$$
= \frac{\pi^2}{8a^2} \left(ap(a) + \int_{-a}^{0} p(x) dx \right) + \frac{1}{2} \left(aq(0) + \int_{0}^{a} q(x) dx \right).
$$

This estimate proves the assertion of the following corollary under the hypothesis
(i). The other assertions can analogously be proved.
Corollary 9: An upper bound b for the first conjugate point of $x = 0$ with respect
to the equation (1) considered on [0, \bar{X}), $X \le \infty$, is given by the smallest roots of the
following equations:
(i) $\frac{\pi^2}{b^2} \left(\frac{b}{2} p(b) + \int_{0}^{b/2} p dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_{b/2}^{b} q dx = 0$ when p, q are monotone in-
creasing;

-

Conjugate Points for Sturm-Liouville Diff. Equ. 163
\n(i)
$$
\frac{\pi^2}{b^2} \left(\frac{b}{2} p(b) + \int_0^{b/2} p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_0^{b/2} q \, dx = 0
$$
 when p is monotone increasing

and q ismonotone decreasing; ^I

(i)
$$
\frac{\pi^2}{b^2} \left(\frac{b}{2} p(b) + \int_p p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_q q \, dx = 0
$$
 when p is monotone increasing

\n(ii)
$$
\frac{\pi^2}{b^2} \left(\frac{b}{2} p(0) + \int_{b/2}^b p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_{b/2}^b q \, dx = 0
$$
 when p is monotone decreasing

\n(iii)
$$
\frac{\pi^2}{b^2} \left(\frac{b}{2} p(0) + \int_{b/2}^b p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_{b/2}^b q \, dx = 0
$$
 when p is monotone decreasing

\n(iv)
$$
\frac{\pi^2}{b^2} \left(\frac{b}{2} p(0) + \int_{b/2}^b p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_0^{b/2} q \, dx = 0
$$
 when p , q are monotonically

creasing and q is monotone increasing;

(11)
$$
\frac{1}{b^2} \left(\frac{1}{2} p(b) + \int_a^b p \, dx \right) + \frac{1}{2} q \left(\frac{1}{2} \right) + \int_a^b q \, dx = 0
$$
 when p is monotonic increasing
\nand q is monotone decreasing; (2)
\n(31) $\frac{\pi^2}{b^2} \left(\frac{b}{2} p(0) + \int_{b/2}^b p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_{b/2}^b q \, dx = 0$ when p is monotone de-
\ncreasing and q is monotone increasing;
\n $\int \left(\text{iv} \right) \frac{\pi^2}{b^2} \left(\frac{b}{2} p(0) + \int_{b/2}^b p \, dx \right) + \frac{b}{2} q \left(\frac{b}{2} \right) + \int_a^b q \, dx = 0$ when p, q are monotone
\ndecreasing.
\nExamples: a) Consider the differential equation
\n $-u'' + (x^2 - 7) u = 0 \qquad (0 \le x < \infty)$.
\nThe interesting solution defined by the boundary condition $u(0) = 0$ is the Hermite function,
\n $\int_{H_3}(x) = c e^{x/2} (c^{-x})^{(3)}$. The first conjugate point of $x = 0$ is $x_1 = \sqrt{3/2} = 1,2247...$. By means
\nof Corollary $S/(i)$ we obtain the upper bound $b = 1,2328...$ for x_1 .
\n $\int \left[(1 - x^2) u' \right]' + \left(\frac{1}{1 - x^2} - 12 \right) u = 0 \quad (-a \le x \le a < 1)$.
\n \int is concave and q is convex. By applying Corollary 7/(iii), it is seen that there exists a pair of
\nconjugate points on $[-a, a]$ with $a = 0,4584...$ (18) is a Legendre differential equation the

decreasing.

-(

$$
-u'' + (x^2 - 7) u = 0 \qquad (0 \leq x < \infty).
$$

The interesting solution defined by the boundary condition $u(0) = 0$ is the Hermite function. **Examples: a)** Consider the differential equation
 $-u'' + (x^2 - 7) u = 0$ $(0 \le x < \infty)$.

The interesting solution defined by the boundary condition $u(0) = 0$ is the Hermite function,
 $H_3(x) = c e^{x^2/2} (e^{-x^2})^{(3)}$. The first $\begin{array}{c}\n1 \\
1 \\
0 \\
0\n\end{array}$ $\begin{array}{c}\n\text{dec}\n\text{E}\n\end{array}$
 $\begin{array}{c}\n\text{The}\n\text{H}_3(a)\n\text{of }C\n\text{b}\n\end{array}$
 $\begin{array}{c}\n\text{p is}\n\end{array}$ $-u'' + (x^2 - 7) u = 0$ $(0 \le x < \infty)$.

The interesting solution defined by the boundary condition $u(0) = 0$ is the Hermite function $H_3(x) = c e^{x^2/2} (e^{-x^4})^{(3)}$. The first conjugate point of $x = 0$ is $x_1 = \sqrt{3/2} = 1,2247...$ By

$$
-[(1-x^2) u']' + \left(\frac{1}{1-x^2}-12\right) u = 0 \quad (-a \leq x \leq a < 1).
$$

p is concave and *q* is convex. By applying Corollary 7/(iii), it is seen that there exists a pair of conjugate points on $[-a, a]$ with $a = 0,4584...$ (18) is a Legendre differential equation the interesting solution of wh conjugate points on $[-a, a]$ with $a = 0,4584...$ (18) is a Legendre differential equation the interesting solution of which is equal to $H_3(x) = c e^{x/2} (c^{-x^2})^{(3)}$. The first conjugate point of $x = 0$ is $x_1 = \sqrt{3}/2 = 1,2247$.

of Corollary 8/(i) we obtain the upper bound $b = 1,2328...$ for x_1 .

b) Consider the equation
 $-[(1 - x^2) u']' + (1 - x^2 - 12) u = 0 \quad (-a \le x$ 5
5
5
5 the upper bound $b = 1,2328...$ for x_1 .
 $\left(\frac{1}{1-x^2}-12\right)u = 0 \quad (-a \le x \le a < 1).$

By applying Corollary 7/(iii), it is seen that there exists a pair

with $a = 0,4584...$ (18) is a Legendre differential equation

is equal to

$$
P_3^{\,1}(x) = c_1 \sqrt{1-x^2} \left((1-x^2)^3 \right)^{(4)} = c_2 \sqrt{1-x^2} \left(5x^2 - 1 \right)
$$

Eq. (1, pp. 94, 96, 344]) with zeros at $x_{1,2} = \pm 5$, $T = \pm 0.4472...$
Both examples show that the calculated upper bounds are good approximate values for the act numbers. exact numbers. interesting solution of which is eq.
 $P_3^{-1}(x) = c_1 \sqrt{1 - x^2}$ (1)

(cf. [1, pp. 94, 96, 344]) with zero

Both examples show that the c

exact numbers.

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1, pp. 94, 96, 344]) with zeros at $x_{1,2} = \pm 5^{-1/2} = \pm 0.4472...$

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