Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (2) 1990, S. 187–188

A Characterization of Dobrushin's Coefficient of Ergodicity

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Es wird bewiesen, daß ein zu einer Vektornorm des $\mathbb{R}^n$ gehörender Ergodizitätskoeffizient τ genau dann die Ungleichung $\tau(P) \leqq 1$ für alle stochastischen Matrizen P der Ordnung n erfüllt, wenn er der Dobrushinsche Ergodizitätskoeffizient ist.

Доказывается, что для коэффициента эргодичности τ относительно некоторой нормы в $\mathbb{R}^n$ выполнено неравенство $\tau(P) \leqq 1$ для всех стохастических матриц P порядка n тогда и только тогда когда он является коэффициентом эргодичности Добружина.

It is proved that the ergodicity coefficient τ corresponding to any vector norm on $\mathbb{R}^n$ fulfils the inequality $\tau(P) \leqq 1$ for all $n \times n$ stochastic matrices P iff it is the Dobrushin ergodicity coefficient.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and S_n $(n \geqq 2)$ the set of all $n \times n$ stochastic matrices. In [4] E. Seneta has introduced a general concept of coefficients of ergodicity τ for $P \in S_n$ with respect to $\|\cdot\|$:

$$\tau(P) = \sup \{\|xP\| : x \in H, \|x\| = 1\} \quad (P \in S_n), \tag{1}$$

where $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. For the l_1-norm $\|\cdot\|_1$, $\|x\|_1 = |x_1| + \cdots + |x_n|$ $(x \in \mathbb{R}^n)$, the ergodicity coefficient denoted by τ_1 is the well-known Dobrushin coefficient

$$\tau_1(P) = \frac{1}{2} \max_{ij} \sum_{k=1}^{n} |p_{ik} - p_{jk}| \quad (P \in S_n) \tag{2}$$

with $P = (p_{ij})_{i,j=1}^n$ (see the remark following Lemma 2). The coefficients $\tau(P)$ are bounds on all non-unit eigenvalues of P. For τ_1 the inequality $\tau_1(P) \leqq 1$ $(P \in S_n)$ holds. In our note we show that τ_1 is the only ergodicity coefficient τ satisfying this inequality. The proof points out the role of certain extremal points.

Denote K, S, K_1 and S_1 the set of all $x \in \mathbb{R}^n$ with $\|x\| \leqq 1$, $\|x\| = 1$, $\|x\|_1 \leqq 1$ and $\|x\|_1 = 1$, respectively. For $i, j \in \{1, \ldots, n\}$, $i \neq j$, let $e_{ij} = (x_1, \ldots, x_n)$ with $x_i = -x_j = 1/2$ and $x_l = 0$ for $l \neq i, j$, and denote $E_1 = \{e_{ij}\}_{ij}$. Let $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ for $x \in \mathbb{R}$. Finally, for a linear subspace $L \subset \mathbb{R}^n$, $L \neq \{0\}$, and norms $|\cdot|_i$ on L denote B_i the corresponding unit balls and $\operatorname{Ex} B_i$ the set of their extremal points $(i = 1, 2)$.

Lemma 1: *For $\lambda > 0$ the following assertions are equivalent:*
(i) $|x|_1 = \lambda |x|_2$ $(x \in L)$; (ii) $\lambda = \sup_{B_2} |x|_1$ *and* $|e|_2 = 1/\lambda$ $(e \in \operatorname{Ex} B_1)$.

Proof: The implication (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (i): The relation $\lambda = \sup_{B_2} |x|_1$ yields $|x|_1 \leqq \lambda |x|_2$ $(x \in L)$. On the other hand, if $x \in L$ by the Krein-Milman Theorem, there are $e_i \in \operatorname{Ex} B_1$ and $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, k)$ with $\lambda_i \geqq 0$, $\lambda_1 + \cdots + \lambda_k = 1$ such that $x = |x|_1 \sum_i \lambda_i e_i$. Thus, by $|e_i|_2 = 1/\lambda$ the inequality $|x|_2 \leqq (1/\lambda) |x|_1$ follows ∎

Lemma 2: *The equality* $\mathrm{Ex}\,(K_1 \cap H) = E_1$ *is true.*

Proof: Clearly, $\mathrm{Ex}\,(K_1 \cap H) \subset S_1 \cap H$. Since $x = \left(\sum_i x_i^+\right)^{-2} \sum_{ij} x_i^+ x_j^- e_{ij}$ ($x \in S_1 \cap H$), each $x \in S_1 \cap H$ is a convex combination of elements of E_1. Therefore, we have $\mathrm{Ex}\,(K_1 \cap H) \subset E_1$. On the other hand, let $e_{kl} = \lambda x + (1 - \lambda)\, y$ with $x, y \in K_1 \cap H$ and $0 < \lambda < 1$. Since $\lambda x_k + (1 - \lambda)\, y_k = 1/2$, $\lambda x_l + (1 - \lambda)\, y_l = -1/2$ and $\sum_i x_i^+ = \sum_i x_i^- = 1/2\, \|x\|_1 \leqq 1/2$, $\sum_i y_i^+ = \sum_i y_i^- = 1/2\, \|y\|_1 \leqq 1/2$ we have $x_k = y_k = 1/2$, $x_l = y_l = -1/2$, and therefore $x = y = e_{kl}$. Thus, $E_1 \subset \mathrm{Ex}\,(K_1 \cap H)$ ∎

Remark: Using Lemma 2 and (1) one can easily prove (2). Indeed, for each $P \in S_n$ the norm $\|\cdot P\|_1$ is a convex functional on $K_1 \cap H$ assuming its maximum at a point of $\mathrm{Ex}\,(K_1 \cap H)$. Therefore, one obtains $\tau_1(P) = \sup_{K_1 \cap H} \|xP\|_1 = \max_{E_1} \|eP\|_1 = \frac{1}{2} \max_{ij} \sum_{k=1}^{n} |p_{ik} - p_{jk}|$.

Theorem: *Let* τ *be an ergodicity coefficient with respect to the norm* $\|\cdot\|$ *satisfying* $\tau(P) \leqq 1$ *for all* $P \in S_n$. *Then*

(i) $\tau(P) = \tau_1(P)$ *for all* $P \in S_n$; (ii) $\lambda\, \|x\| = \|x\|_1$ ($x \in H$) *for some* $\lambda > 0$.

Proof: Since $\|\cdot\|_1$ is a continuous functional on the compact set $S \cap H$, there is a $y \in S \cap H$ with $\|y\|_1 = \max_{S \cap H} \|x\|_1 = \lambda$. For $j, k \in \{1, \ldots, n\}$, $j \neq k$, let $P_{j,k} = (p_{ml}) \in S_n$ be defined by $p_{mj} = 1$ for $y_m^+ > 0$ and $p_{mk} = 1$ for $y_m^+ = 0$. Then $yP_{j,k} = 2 \sum_i y_i^+ e_{jk} = \|y\|_1\, e_{jk}$. Therefore, $\tau(P) \leqq 1$ ($P \in S_n$) yields $\|e_{jk}\| \leqq 1/\|y\|_1 = 1/\lambda$. By definition of λ, $1 = \|e_{jk}\|_1 \leqq \lambda\, \|e_{jk}\|$, so that $\|e_{jk}\| = 1/\lambda$. Thus, Lemmata 2 and 1 imply (ii), and (i) directly follows ∎

Remark: After we had finished the first form of this paper we has been informed by A. Lešanovský that he has obtained independently the result of the theorem [2]. However his proof does not use the aspect of extremal points.

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Manuskripteingang: 29. 06. 1988; in revidierter Fassung 25. 11. 1988

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