Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (3) 1990, S. 203–212

Perturbation of Temperature Fields by a Small Inclusion

D. Göhde

In einem Gebiet, das ein kleines Teilgebiet Ω_ε enthält, wird ein Randwertproblem zweiter Ordnung mit Kopplungsbedingungen auf dem Rand von Ω_ε gestellt. Für seine Lösung wird eine asymptotische Entwicklung bezüglich des nach 0 gehenden Durchmessers von Ω_ε konstruiert.

В области содержащей малую подобласть Ω_ε ставится краевая задача второго порядка с переходным условием на границе Ω_ε. Для её решения строится асимптотика относительно диаметра Ω_ε стремящегося к нулю.

In a domain which contains a small subdomain Ω_ε a boundary value problem of second order with transition conditions on the boundary of Ω_ε is posed. For its solution an asymptotic expansion is constructed with respect to the diameter of Ω_ε tending to zero.

For the last decades, several phenomena in physical science and engineering have given raise to models of mathematical analysis which are concerned with domains with irregular boundaries (e.g., porous media or perforated walls) and therefore, especially since about ten years, to a literature referring to this. Within this topic, it seems A. M. Il'in had been the first to investigate, to some extent, a fundamental type of problems, namely, boundary value problems in domains with one small hole: From a bounded domain G, a small subset Ω_ε of diameter ε is removed, and the solution of a boundary value problem in G is perturbed by imposing additional boundary conditions on the boundary Γ_ε of Ω_ε. A. M. Il'in constructed representations of such perturbed solutions u_ε in $G_\varepsilon = G \setminus \bar{\Omega}_\varepsilon$ which are asymptotic with respect to ε tending to zero (e.g. [4–6]). A simplified method for simpler but most important cases in applications was proposed by the author in [2]. Also numerical solutions have been given for special cases (e.g. [8]).

Up to now, however, concerning the local asymptotic behaviour, apparently only fixed boundary conditions have been considered at the boundary of the hole, though transition (coupling) conditions could also be significant. We imagine, e.g., a temperature field in a domain occupied by a body with thermal conductivity λ^+ which contains but a little inclusion Ω_ε filled with material of another conductibility λ^-. Therefore we shall describe, by means of the first stages of approximation, a method to construct an asymptotic expansion, above all, for a three-dimensional model problem — Poisson's equation; further it is shown that the method will also work in the two-dimensional case and, on principle, for more general equations. Finally, in the special case of a spherical inclusion, a more direct procedure can be used, taking its pattern from [2, 3]. — For proving the asymptotic character of the expansions obtained a variant of the maximum principle is applied which admits jumps of the first derivatives.

It should be remarked that there are investigations of related problems for, e. g., bodies including thin sheets with material constants growing to infinity while their thickness tends to zero (e.g., [1, 8]); the very interesting results consist in global, rather weak convergence statements, but without precise description of the local behaviour, and, of course, the methods used are quite different from ours.

1. Formulation of the model problem

Let $G \subseteq \mathbb{R}^3$ be a bounded domain containing the origin 0, and h a smooth function defined in $\overline{G}$. The boundary $S = \partial G$ of G should be smooth enough to permit classical solutions of the Dirichlet problem

$$G: -\Delta w = h(x), \quad S: w = g(x) \tag{1}$$

for continuous boundary values g. Let, further, Ω be another bounded domain in $\mathbb{R}^3$ containing 0, with smooth boundary $\Gamma = \partial\Omega$. Set $\Omega_\varepsilon = \varepsilon\Omega = \{x : x/\varepsilon \in \Omega\}$, $\Gamma_\varepsilon = \varepsilon\Gamma = \partial\Omega_\varepsilon$, and $G_\varepsilon = G \setminus \overline{\Omega}_\varepsilon$, which is a domain with a small hole (provided ε is sufficiently small). (1) can be regarded as a system governing a stationary temperature distribution w in a body with constant thermal conductivity $\lambda^+ = 1$. Now, within the subset Ω_ε, the material is displaced by another one with different conductibility $\lambda^- = \lambda > 0$. In that way the temperature will vary, at least locally; the perturbed temperature $u = u_\varepsilon$ is subject to the conditions

$$\begin{aligned} &G_\varepsilon: -\Delta u = h(x), \quad S: u = g(x), \\ &\Omega_\varepsilon: -\lambda\, \Delta u = h(x), \\ &\Gamma_\varepsilon: u^- - u^+ = 0, \quad l(u) = 0, \end{aligned} \tag{2}$$

where

$$u = u_\varepsilon(x) = \begin{cases} u^+(x) & \text{for } x \in G_\varepsilon \\ u^-(x) & \text{for } x \in \Omega_\varepsilon \end{cases}, \qquad l(u) = \lambda \frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n};$$

n denotes the outer normal with respect to Ω_ε. (For existence confer, e.g., [9].)

2. Construction of an asymptotic expansion

We set $u = w + v$ and ask for an asymptotic expansion, for $\varepsilon \to +0$, of the difference v which is solution of the following problem:

$$\begin{aligned} &G_\varepsilon: -\Delta v = 0, \quad S: v = 0, \\ &\Omega_\varepsilon: -\lambda\, \Delta v = (\lambda - 1)\, \Delta w = (1 - \lambda)\, h, \\ &\Gamma_\varepsilon: v^- - v^+ = 0, \quad l(v) = (1 - \lambda)\, \partial w/\partial n. \end{aligned} \tag{3}$$

Now the boundary condition on S is omitted, we only demand the (extended) solution to be regular at infinity. So the exact difference v is changed to an approximation $\bar{v}$. Next the two inhomogenities in (3) are removed for a jump of $\bar{v}$ along Γ_ε by

$$\bar{v} = v_0 + \hat{w}, \quad \hat{w}(x) = \begin{cases} 0 & \text{for } x \in G_\varepsilon, \\ (1 - \lambda)\, \lambda^{-1}\, w(x) & \text{for } x \in \Omega_\varepsilon, \end{cases} \tag{4}$$

after which

$$\begin{aligned} &G_\varepsilon: -\Delta v_0 = 0, \quad \Omega_\varepsilon: -\Delta v_0 = 0 \\ &\Gamma_\varepsilon: v_0^- - v_0^+ = (\lambda - 1)\, \lambda^{-1} w, \quad l(v_0) = 0 \end{aligned} \tag{5}$$

and $v_0(\infty) = 0$. As third step we set

$$v_0(x) = \begin{cases} W[\psi](x) & \text{for} \quad x \in G_\varepsilon, \\ \lambda^{-1} W[\psi](x) & \text{for} \quad x \in \Omega_\varepsilon, \end{cases}$$

where $W[\psi]$ denotes the double layer potential with density ψ on Γ_ε, so that all conditions of (5), except the third one, will be satisfied automatically. As it will be more favourable for the following, we introduce the stretched variables

$$\xi = x/\varepsilon \tag{6}$$

and, corresponding, $|\xi| = \varrho = r/\varepsilon = |x|/\varepsilon$, so that the potential W is now given on the fixed boundary Γ of the prototype domain Ω by

$$W[\varphi](\xi) = \oint\!\!\oint_\Gamma \varphi(\tau)\, \partial\varrho^{-1}/\partial n_\tau\, d\sigma_\tau, \tag{7}$$

where $\varphi(\tau) = \psi(\varepsilon\tau)$. Adequately, we also rewrite the function v_0 as a function of ξ, too: $v_0(\xi) = v_0(x/\varepsilon)$. Using the jump property of W at Γ

$$W^+(\xi) = 2\pi\varphi(\xi) + W(\xi), \quad W^-(\xi) = -2\pi\varphi(\xi) + W(\xi),$$

where W^+, W^- are the limits from the outside and from the inside, respectively, while $W(\xi)$ denotes the value on Γ itself, we obtain the integral equation for the density φ,

$$2\pi\varphi(\xi) - \frac{1-\lambda}{1+\lambda} W[\varphi](\xi) = \frac{1-\lambda}{1+\lambda} w(\varepsilon\xi) \tag{8}$$

on the surface Γ, with weakly singular kernel. Because the homogeneous adjoint integral equation turns out to provide, but as a single layer potential V, a solution of (5) with w replaced by 0, regular at infinity, from the lemma below (maximum principle) it will follow that this solution is necessarily the trivial one. Therefore (8) will always be solvable; the solution will depend continuously on the right-hand side. Since a double layer potential will decrease for $\varrho = |\xi| \to \infty$, according to ϱ^{-2}, we obtain for

$$v_0(\xi) = v_0(x/\varepsilon) = \begin{cases} W[\varphi](x/\varepsilon) & \text{for} \quad x \in G_\varepsilon, \\ \lambda^{-1} W[\varphi](x/\varepsilon) & \text{for} \quad x \in \Omega_\varepsilon \end{cases} \tag{9}$$

the estimate

$$|v_0(x/\varepsilon)| \leqq C\varepsilon^2/r^2 \tag{10}$$

with a constant C independent of ε, for φ will be bounded from the uniform boundedness of the right-hand side of (8). With the function v_0 just constructed the approximate difference $\bar{v}$ in (4) satisfies the conditions (3) exactly except that on S: there will be $\bar{v} = v_0 = O(\varepsilon^2)$. By the lemma already mentioned it will turn out that

$$u(x) = \overline{w}(x) + \hat{w}(x) + v_0(x/\varepsilon) + \varepsilon^2 z_1(x) \tag{11}$$

with z_1 bounded in G, independently of ε.

The further procedure in order to get higher approximations is apparent: The function z_1 will be expanded

$$z_1 = w_1 + \hat{w}_1 + v_1 + \varepsilon^2 z_2, \tag{12}$$

where w_1 is the solution of (1) with $h = 0$, $g = -v_0/\varepsilon^2$ (in order to compensate the deviation of the boundary values on S induced by v_0), $\hat{w}_1$ is defined analogously by

(4) with w_1 instead of w, and v_1 is obtained by (5), (8), and (9) with w replaced by w_1 just as done for v_0, and so on. Summing up we have

Theorem 1: *The solution u of the perturbed problem (2) admits an asymptotic expansion*

$$u(x) = \sum_{k=0}^{m} \varepsilon^{2k}\big(w_k(x) + \hat{w}_k(x) + v_k(x/\varepsilon)\big) + \varepsilon^{2(m+1)} z_{m+1}(x), \tag{13}$$

where $w_0 = w$ is the solution of the unperturbed problem (1), w_k ($k \geqq 1$) the solution of (1) with $h = 0$, $g = -v_{k-1}/\varepsilon^2$,

$$\hat{w}_k(x) = \begin{cases} 0 & \text{for } x \in G_\varepsilon, \\ (1-\lambda)\,\lambda^{-1} w_k(x) & \text{for } x \in \Omega_\varepsilon, \end{cases}$$

$$v_k(\xi) = \begin{cases} W(\xi) & \text{for } \varepsilon\xi \in G_\varepsilon, \\ \lambda^{-1} W(\xi) & \text{for } \varepsilon\xi \in \Omega_\varepsilon, \end{cases}$$

$W = W[\varphi_k]$ the double layer potential on Γ with density φ_k which is a solution of the integral equation

$$2\pi\varphi_k(\xi) - \frac{1-\lambda}{1+\lambda} W[\varphi_k](\xi) = \frac{1-\lambda}{1+\lambda} w_k(\varepsilon\xi), \quad \xi \in \Gamma,$$

and z_{m+1} is bounded in all of G, independently of ε.

Remark: The functions v_k depend continuously on ε because the corresponding solutions φ_k of the integral equation (8) will do. Of course one could expand the right-hand side of (8) and obtain, for each v_k, an expansion with powers of ε, the coefficient functions depending on $\xi = x/\varepsilon$ only.

3. The maximum principle

In order to estimate the remainder z_{m+1} of the expansion (13), and for the proof of solvability of the integral equation (8), we use the following variant of the maximum principle.

Lemma: *Let $z, Z \in C(\overline{G}) \cap C^2(G_\varepsilon) \cap C^1(\overline{G}_\varepsilon) \cap C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$,*

$$L^\pm u = -\lambda^\pm \Delta u + c(x)\, u, \quad lu = \lambda^- \frac{\partial u^-}{\partial n} - \lambda^+ \frac{\partial u^+}{\partial n}$$

where $c \in C(\overline{G})$ is nonnegative, λ^-, λ^+ are positive constants, n is the outside normal (with respect to Ω_ε). If

$$G_\varepsilon\colon |L^+ z| \leqq L^+ Z, \quad S\colon |z| \leqq Z,$$

$$\Omega_\varepsilon\colon |L^- z| \leqq L^- Z, \quad \Gamma_\varepsilon\colon |lz| \leqq lZ,$$

then $|z| \leqq Z$ in all of G.

The proof consists in the usual application of the following *proposition* to the differences $v = \pm z - Z$, with apparently corresponding notations:

Let $G_1 \subsetneqq G_2 \subsetneqq \cdots \subsetneqq G_m = G$ be bounded domains in $\mathbb{R}^n$ with smooth boundaries $S_i = \partial G_i$ which do not intersect each other. In the closure of each $D_i = G_i \setminus \overline{G}_{i-1}$ ($i = 1, \ldots, m$; G_0 empty set) a linear uniformly elliptic positive operator L_i of second order, with bounded coefficients, is defined for which the strong maximum principle is valid. Let v be continuous in $\bar{G}$, with $v \leq 0$ on $S = S_m = \partial G$, $v(x) = v_i(x)$ for $x \in \bar{D}_i$, and, in each D_i, be a classical solution of $L_i v_i \leq 0$ with first derivatives continuous up to the boundaries. In any point of a boundary S_i separating D_i and D_{i+1}, for the outer normal derivatives the transition condition shall be fulfilled:

(*) If $\partial v_{i+1}/\partial n < 0$, then $\partial v_i/\partial n < 0$.

Then v cannot assume a positive maximum in the interior of G. Apparently it is only to be shown that v cannot attain a positive maximum at a point x_i of a boundary S_i ($i \leq m - 1$). But if this should be the case, and v_{i+1} not a constant, then $\partial v_{i+1}/\partial n < 0$ in x_i, and (*) would entail still greater values of $v = v_i$ in D_i. If $v_{i+1} =$ const, the argument may be repeated with respect to the boundary S_{i+1}, and so on ∎

Supplement: The assertion of the lemma will be maintained if there is a surface (e.g., a sphere) in G_ε where Z has only continuous derivatives of first order.

In order to apply the lemma to prove that z_1 in (11) is bounded we state the continuity of z_1 at Γ_ε and

$$\begin{aligned} &G_\varepsilon\colon -\Delta z_1 = 0, \quad S\colon |z_1| = |-v_0/\varepsilon^2| \leq C, \\ &\Omega_\varepsilon\colon -\lambda\, \Delta z_1 = 0, \quad \Gamma_\varepsilon\colon l z_1 = 0. \end{aligned} \tag{14}$$

Apparently $Z = C =$ const will be an upper bound for $|z_1|$. Preceding a step ($m = 1$) we have to estimate z_2 from (12). Inserting the definitions or properties of w_1, $\hat{w}_1$, v_1, and z_1 we see that z_2 is continuous in G and satisfies a system like (14), with v_0 replaced by v_1. Thus, by induction, the general assertion can be proved.

As to the solvability of the integral equation (8), it is to be shown that a solution of (5), with $w = 0$, represented by a single layer potential, must vanish identically. For this sake, the domain is extended to a ball of arbitrary radius; on its surface the potential will be as small as one likes and, according to the lemma, also in the ball and in the included domain G.

4. The two-dimensional case

The assertions of Theorem 1 will apparently be maintained, with obvious modifications, for space dimensions greater than three. Also for dimension two one expects, for the present, the formal expansion procedure to run quite similar: In the expansion (13) the powers ε^{2k} are to be replaced by ε^k — corresponding to the fact that the $v_k(\xi)$, given by double layer potentials, will now decrease with order $O(|\xi|^{-1})$ only —, and in the integral equation (8) 2π is substituted by π. Further on, the existence of a uniform bound for z_{m+1} in the remainder $\varepsilon^{m+1} z_{m+1}$ can be proved as just done by the lemma. But it cannot be applied immediately, as at the end of Section 3, to the proof of the solvability of (8): The function v satisfying (5) (with $w = 0$), since generated by a single layer, cannot be supposed here, a priori, to tend to zero at infinity, which had been essential above.

But we can conclude as follows: If $v(\xi) = V[\chi](\xi)$ is a single layer potential which will fulfil $lv = 0$ at Γ for a solution χ of the adjoint equation $\pi\chi - (1 - \lambda)(1 + \lambda)^{-1} \times \partial V[\chi]/\partial n = 0$, then

$$\vartheta(\xi) = \begin{cases} v(\xi) + W[\varphi](\xi) & \text{for} \quad \xi \notin \Omega, \\ \lambda v(\xi) + W[\varphi](\xi) & \text{for} \quad \xi \in \Omega \end{cases} \tag{15}$$

with the double layer potential $W[\varphi]$, $\varphi = -(1/2\pi)(1 - \lambda)\, v$, passes Γ continuously including its first derivatives. Therefore ϑ is a potential function in all the plane $\mathbb{R}^2$, with logarithmic growth at infinity, at most, but bounded from above or below — and such a singularity must be a removable one: $\hat{v} = c = \text{const}$. Now (15) shows that $v(\xi) = c + O(|\xi|^{-1})$ as $|\xi| \to \infty$, and by the lemma $v(\xi) - c \equiv 0$ will follow. Especially, $\partial v/\partial n = 0$ on Γ_ε and, therefore, $\chi = 0$ guarantees solvability of the integral equation (8) also in this case.

Theorem 2: *In a plane domain G with inclusion Ω_ε the solution u of the perturbed problem* (2) *admits an asymptotic expansion*

$$u = \sum_{k=0}^{m} (w_k + \hat{w}_k + v_k)\, \varepsilon^k + \varepsilon^{m+1} z_{m+1} \tag{13'}$$

with functions w_k, $\hat{w}_k$, v_k defined like in Theorem 1, *but w_k ($k \geqq 1$) solution of* (1) *with $g = -v_{k-1}/\varepsilon$ and $v_k(\xi) = v_k(x/\varepsilon) = W[\varphi_k]\,(\xi)$ given by the integral equation on Γ*

$$\pi\varphi_k(\xi) - \frac{1-\lambda}{1+\lambda} W[\varphi_k]\,(\xi) = \frac{1-\lambda}{1+\lambda} w_k(\varepsilon\xi). \tag{8'}$$

5. Remarks on the case $-\Delta + c$

The problem will now read

$$G\colon -\Delta w + c(x)\, w = h(x), \quad S\colon w = g(x) \tag{16}$$

and the perturbed problem is

$$\begin{aligned} &G_\varepsilon\colon -\Delta u + c(x)\, u = h(x), \quad S\colon u = g(x),\\ &\Omega_\varepsilon\colon -\lambda\, \Delta u + c(x)\, u = h(x), \quad \Gamma_\varepsilon\colon u^- - u^+ = 0, \quad lu = 0, \end{aligned} \tag{17}$$

with a nonnegative continuous function c in $\bar{G}$, G, Ω are bounded domains in $\mathbb{R}^3$ as in Section 1, $\Omega_\varepsilon = \varepsilon\Omega$, $\Gamma_\varepsilon = \varepsilon\Gamma = \partial\Omega_\varepsilon$. In addition, for technical reasons, Ω is to be supposed star-shaped with respect to a ball centered at the origin. As leading terms of an asymptotic approximation, the same functions as defined in (4), (9) for the simpler model above turn out to be also suited in this case. The difference $v = u - w$ will here satisfy the conditions corresponding to (3) — the differential equations in G_ε and Ω_ε augmented by $c(x)\, v$ on the left-hand side. But as approximating difference $\bar{v}$ we will again choose the solution, regular at infinity, of

$$\begin{aligned} &\mathbb{R}^3 \setminus \bar{\Omega}_\varepsilon\colon -\Delta\bar{v} = 0, \quad \Omega_\varepsilon\colon -\lambda\, \Delta\bar{v} = (\lambda - 1)\, \Delta w\\ &\Gamma_\varepsilon\colon \bar{v}^- - \bar{v}^+ = 0, \quad l\bar{v} = (1 - \lambda)\, \partial w/\partial n. \end{aligned} \tag{18}$$

Therefore we set, combining (4) and (9),

$$\bar{v}(x) = \hat{w}(x) + v_0(x/\varepsilon) = \begin{cases} W[\varphi]\,(x/\varepsilon) \\ (1-\lambda)\, \lambda^{-1} w(x) + \lambda^{-1} W[\varphi]\,(x/\varepsilon) \end{cases} \tag{19}$$

for $x \in G_\varepsilon$ or $x \in \Omega_\varepsilon$, respectively, where φ is the solution of the integral equation (8). The remainder z_0 in $u = w + \hat{w} + v_0 + z_0$ will now be submitted to

$$\begin{aligned} &G_\varepsilon\colon -\Delta z_0 + cz_0 = -cv_0, \quad S\colon z = -v_0,\\ &\Omega_\varepsilon\colon -\lambda\, \Delta z_0 + cz_0 = -c[(1-\lambda)\, \lambda^{-1} w + v_0],\\ &\Gamma_\varepsilon\colon z_0^- - z_0^+ = 0, \quad lz_0 = 0. \end{aligned} \tag{20}$$

As v_0 is given by a double layer potential W with bounded density, we will have

$$\begin{aligned} |-cv_0| &\leqq C(\varepsilon/r)^2 && \text{in } G_\varepsilon, \\ |-c[(1-\lambda)\lambda^{-1}w + v_0]| &\leqq C && \text{in } \Omega_\varepsilon, \\ |-v_0| &\leqq C\varepsilon^2 && \text{on } S. \end{aligned} \tag{21}$$

If G, Ω_ε are contained in balls of radii R and — e.g. — 2ε, then

$$Z = \begin{cases} C\varepsilon^2 \ln (R/r) & \text{for } r > 2\varepsilon, \\ C[\varepsilon^2 \ln (R/2\varepsilon) + 1/2(4\varepsilon^2 - r^2)] & \text{for } r < 2\varepsilon \end{cases} \tag{22}$$

is continuously differentiable in $\mathbb{R}^3$ and, for suitable constant C, will majorize z_0 in the sense of the lemma, possibly with exception of the last condition on Γ_ε:

$$|lz_0| = 0 \leqq lz = C(1-\lambda)\, \varepsilon^2 r^{-1} s(x),$$

where $s(x)$ denotes the scalar product between the outer unit normal n at $x \in \Gamma_\varepsilon$ and the normed radius vector $\overrightarrow{0x}/r$. As Ω is assumed star-shaped, $s(x)$ will be strictly positive on Γ_ε, and therefore the lemma can be applied if $\lambda \leqq 1$.

The case $\lambda > 1$ requires a modification. Let q be the harmonic function defined outside Ω, regular at infinity, and equal to 1 on Γ. There exists a positive constant c_0 so that, on Γ, $\partial q/\partial n \leqq -c_0$. Then

$$Q(x) = \begin{cases} q(x/\varepsilon) & \text{for } x \in \mathbb{R}^3 \setminus \Omega_\varepsilon, \\ 1 & \text{for } x \in \Omega_\varepsilon \end{cases}$$

has the properties $\Delta Q = 0$ in Ω_ε and outside Ω_ε, $Q > 0$, bounded, and $lQ \geqq c_0/\varepsilon$. Therefore $Z' = Z + c_1\varepsilon^2 Q$ with Z from (22) has all majorizing properties claimed by the lemma, especially

$$lZ' \geqq \left(4C(1-\lambda)\, \varepsilon^2/r + c_0 c_1 \varepsilon\right) s(x) \geqq 0$$

if $c_1 \geqq 4(\lambda - 1)\, C/r_0$, where r_0 is the radius of a ball, centered at 0, which is contained in Ω. Thus we obtain, in either case,

$$|z_0| \leqq C\varepsilon^2 |\ln \varepsilon|. \tag{23}$$

In order to raise the degree of approximation, next we have to expand $z_0 = w_1 + \bar{v}_1 + z_1$, where w_1 is the solution of an unperturbed problem which compensates the values of v_0 on the outer boundary S. Looking at the problem (20) for z_0 we see that it can be interpreted as perturbation (like (2) in proportion to (1)) of the problem

$$G: -\Delta w_1 + cw_1 = -c(v_0 + \hat{w}), \quad S: w_1 = -v_0. \tag{24}$$

Therefore the construction just carried out can be repeated, with $-c(v_0 + \hat{w})$ instead of h, $-v_0$ instead of g, and in (18), (19) $\bar{v}$, w, $\hat{w}$, v_0 replaced by $\bar{v}_1$, w_1, $\hat{w}_1$ (cf. (4)), v_1, respectively. For the remainder z_1 in $z_0 = w_1 + \hat{w}_1 + v_1 + z_1$ we now obtain, analogously, again (20), with z_1, v_1, w_1 instead of z_0, v_0, $w_0 = w$. As the estimate (23) for z_0 will also hold, the more, for w_1 (using the lemma with $\lambda^- = \lambda^+ = 1$), the right-hand sides of (18), (19), and (8), rewritten with the new quantities, will be of order $O(\varepsilon^2 |\ln \varepsilon|)$, the estimation above, applied to z_1, yields

$$|z_1| \leqq C\varepsilon^4 (\ln \varepsilon)^2. \tag{25}$$

Proposition 1: *The solution u of the perturbed problem* (17) *admits expansions*

$$u = w_0 + \hat{w}_0 + v_0 + z_0$$
$$= w_0 + \hat{w}_0 + v_0 + w_1 + \hat{w}_1 + v_1 + z_1$$

with $w_0 = w$ solution of the unperturbed problem (16), *$\hat{w}_0 + v_0$ and $\hat{w}_1 + v_1$ (cf.* (19)) *corrections; previously in the vicinity of Ω_ε, while w_1 (cf.* (24)) *corrects on S; the remainders z_0, z_1 are uniformly bounded according to* (23), (25).

6. Spherical inclusions

Finally, it should be remarked that in the special case where Ω_ε is a ball K_ε of radius ε, centered at 0, an asymptotic expansion of the solution of the model problem (2) would better be established more directly, based on the Taylor expansion of the unperturbed solution w, at $x = 0$,

$$w = w^0 + w_i{}^0 x_i + \frac{1}{2!}\, w_{ij}^0 x_i x_j + \cdots,$$

where $w_i{}^0 = \partial w/\partial x_i$, taken at $x = 0$, and analogously. As a first approximation to the equations (3) for the difference $v = u - w$ turns out to be reasonable, with regard to the estimation of the remainder,

$$\begin{aligned} G_\varepsilon&: \quad -\Delta\tilde{v} = 0 \\ K_\varepsilon&: \quad -\lambda\,\Delta\tilde{v} = (1 - \lambda)\,h(0) = (\lambda - 1)\,\Delta w(0) \\ \partial K_\varepsilon&: \tilde{v}^- - \tilde{v}^+ = 0, \quad l\tilde{v} = (\lambda - 1)\,[w_i{}^0 x_i/r + w_{ij}^0 x_i x_j/r] \end{aligned} \tag{26}$$

with $\tilde{v}(\infty) = 0$. Suitable expansion functions will be, for $\tilde{v} = \tilde{v}^+$ in G_ε, linear combinations of derivatives of the principal solution, and for $\tilde{v} = \tilde{v}^-$ in the ball $\bar{K}_\varepsilon$, homogeneous polynomials, some of which solve the approximating equation on K_ε, and the other ones are harmonic. For the latter it is best to take those which arise in the nominators $H(x)$ of the derivatives just mentioned if they are represented by $H(x)/r^m$. After inserting these ansatzes into (26) and equating at $r = \varepsilon$ the polynomials, or nominator polynomials, respectively, of the same order, some calculations will provide the exact solution of (26), in the three-dimensional case:

$$\tilde{v} = \begin{cases} \tilde{v}^+ = \dfrac{1-\lambda}{3+2\lambda}\left[\dfrac{h(0)}{3}\,\varepsilon^2\bar{\varrho}^3 + \dfrac{3+2\lambda}{2+\lambda}\,\bar{\varrho}^3 w_i{}^0 x_i + \bar{\varrho}^5 w_{ij}^0 x_i x_j\right], \\ \tilde{v}^- = \dfrac{1-\lambda}{3+2\lambda}\left[\dfrac{h(0)}{3}\,\varepsilon^2 + \dfrac{3+2\lambda}{2+\lambda}\,w_i{}^0 x_i + w_{ij}^0 x_i x_j + \dfrac{h(0)}{2\lambda}\,\varepsilon^2(1-\bar{\varrho}^2)\right], \end{cases} \tag{27}$$

where $\bar{\varrho} = \varepsilon/r$. It should be pointed out that in $\tilde{v}^+$ the principal solution $1/r$ does not appear itself. This will hold also for other dimensions; especially for $n = 2$ no logarithmic term will trouble — and this will confirm the fact stated above, in Section 4, that the expansion procedure will also run in the two-dimensional case.

In order to estimate the degree of approximation to u given by $w + \tilde{v}$ it is established, for $z = u - (w + \tilde{v})$,

$$\begin{aligned} &G_\varepsilon: -\Delta z = 0, \quad S: z = O(\varepsilon^3) \\ &K_\varepsilon: -\lambda\Delta\, z = O(\varepsilon), \quad \partial K_\varepsilon: z^- - z^+ = 0, \quad lz = O(\varepsilon^2). \end{aligned} \tag{28}$$

A majorant function Z which satisfies the conditions of the lemma, except that on S, will be given by

$$Z = \begin{cases} Z^+ = C_0\varepsilon^4/r, \\ Z^- = 1/6C_1\varepsilon(\varepsilon^2 - r^2) + C_0\varepsilon^3 \end{cases}$$

with proper constants C_0, C_1. Merely to be able to apply the lemma, formally a function $\varepsilon^3 w_1$ must be added to $\tilde{v}$, where

$$G: -\Delta w_1 = 0, \quad S: w_1 = -\frac{1-\lambda}{2+\lambda}\frac{w_i{}^0 x_i}{r^3}.$$

Then in (28) we will have $z = O(\varepsilon^5)$ on S – the other conditions are not concerned –, and Z is now an admissible majorant function.

Proposition 2: *In the case $n = 3$ and if the inclusion occupies the ball K_ε, an asymptotic expansion of the solution of* (2) *is*

$$u(x) = w(x) + \tilde{v}(x) + \varepsilon^3 z_1(x, \varepsilon) \tag{29}$$

with $\tilde{v}$ given by (27) *and* $|z_1(x, \varepsilon)| \leqq C \min \{\varepsilon/r, 1\}$.

If higher approximations are desired, then, of course, the function w_1 compensating on the outer boundary S, will be essential and must, further, be expanded itself similarly as w.

Remark: The limit case $\lambda = \infty$ where the temperature u must be constant in K_ε has been treated in [2]; the result there proves to be the limit, for $\lambda \to \infty$, of (29), (27).

REFERENCES

[1] Acerbi, E., Buttazzo, G., and D. Percivale: Thin inclusions in linear elasticity: a variational approach. J. reine angew. Math. **386** (1988), 99–115.

[2] Göhde, D.: Singuläre Störung von Randwertproblemen durch ein kleines Loch im Gebiet. Z. Anal. Anw. **4** (1985), 467–477.

[3] Göhde, D.: Störung stationärer Temperaturfelder durch kleine Einschlüsse mit großer Wärmeleitfähigkeit. Wiss. Beiträge Ingenieurhochschule Zwickau **12** (1986) 5, 91–96.

[4] Ильин, А. М.: Краевая задача для эллиптического уравнения второго порядка в области с узкой щелью. 1. Двумерный случай. Мат. Сборник **99** (1976), 514–537.

[5] Ильин, А. М.: Краевая задача для эллиптического уравнения второго порядка в области с узкой щелью. 2. Область с малым отверстием. Мат. Сборник **103** (1977), 265–284.

[6] Ильин, А. М.: Исследование асимптотики решения эллиптической краевой задачи в области с малым отверстием. Труды Сем. им. Петровского **6** (1981), 57–82.

[7] Симоненко, И. Б.: Задачи электростатики в неоднородной среде. Случай тонкого диэлектрика с большой диэлектрической постоянной II. Дифф. ур-ия **11** (1975), 1870–1878.

[8] Шишкин, Г. И.: Решение краевой задачи для уравнения второго порядка с малым отверстием в случае круговой симметрии. В сб-е: Дифференциальные уравнения с малым параметром. Свертловск: Уральский Научный Центр Акад. Наук СССР 1984, стр. 104—118.
[9] Wanka, G., und J. Wanka: Ein Existenztheorem für eine Klasse von allgemeinen Dirichletschen Randkontaktproblemen bei linearen Differentialgleichungen zweiter Ordnung. Math. Nachr. 110 (1983), 215—229.

Manuskripteingang: 18. 05. 1988; in revidierter Fassung 26. 08. 1988

VERFASSER:

Prof. Dr. Dietrich Göhde
Sektion Math./Naturwiss./Inf. der Technischen Hochschule
Dr.-Friedrichs-Ring 2a
DDR-9541 Zwickau

Added in proof: Solving (8), at first, only for the constant term of Taylor's expansion of the right-hand side, the properties of double layer potential W, with constant density (Gauss' integral — e.g., vanishing outside Ω_ε!), will allow the constant C in the basic estimate (10) to be replaced by $C \cdot \varepsilon$, so that all powers of ε, in the sequel, can be lifted respectively; especially, (13) is true even with ε^{3k} instead of ε^{2k}, and in (13′) ε^k may be replaced by ε^{2k}.