Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (2) 1990, S. 165–176

Self-Duality and C^*-Reflexivity of Hilbert C^*-Moduli

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Gegenstand dieses Artikels sind eine neue Definition des Begriffes „selbstdualer Hilbert-C^*-Modul" als ein Kategoriebegriff der Banach-C^*-Moduln und die Bedingungen für einige Hilbert-C^*-Moduln, selbstdual oder C^*-reflexiv zu sein. Es wird die Isomorphie zweier beliebiger Hilbertstrukturen auf einem gegebenen selbstdualen Hilbert-C^*-Modul gezeigt, die äquivalente Normen zur gegebenen Norm induzieren. Ein topologisches Kriterium der Selbstdualität und C^*-Reflexivität von Hilbert-W^*-Moduln wird bewiesen. Weiterhin wird ein Kriterium der Selbstdualität des abzählbar erzeugten Hilbert-C^*-Moduls $l_2(\mathfrak{A})$ für beliebige C^*-Algebren $\mathfrak{A}$ gezeigt. Als eine Anwendung wird die Klassifikation der abzählbar erzeugten selbstdualen Hilbert-W^*-Moduln durch deren Struktur gegeben.

Предметом стати является новое определение понятия „автодуальный гильбертов C^*-модуль" как понятие категории банаховых C^*-модулей и условия для некоторых гильбертовых C^*-модулей быть автодуальными или C^*-рефлексивными. Показана изоморфность любых двух гильбертовых структур на заданном автодуальном гильбертовом C^*-модуле, если они индуцируют нормы эквивалентные к заданной. Доказан топологический критерий автодуальности и C^*-рефлексивности гильбертовых W^*-модулей. Далее, сформулирован критерий автодуальности счетно порожденного гильбертового $\mathfrak{A}$-модуля $l_2(\mathfrak{A})$ для любых C^*-алгебр $\mathfrak{A}$. В качестве приложения дана классификация счетно порожденных гильбертовых W^*-модулей их структурой.

The subject of this paper are a new definition of the notion "self-dual Hilbert C^*-module" as a categorical concept of Banach C^*-moduli, and the conditions for some Hilbert C^*-moduli to be self-dual or C^*-reflexive. The isomorphism of any two Hilbert structures on a given self-dual Hilbert C^*-module inducing equivalent norms to the given one is stated. A topological criterion of self-duality and C^*-reflexivity of Hilbert W^*-moduli is proved. A criterion of self-duality of the countably generated Hilbert $\mathfrak{A}$-module $l_2(\mathfrak{A})$ is stated for arbitrary C^*-algebras $\mathfrak{A}$. As an application the classification of countably generated Hilbert W^*-moduli by their structure is given.

§1 Introduction

At the beginning we fix some denotations and give certain facts and examples from the literature: All moduli in this paper are left moduli by definition. A *pre-Hilbert $\mathfrak{A}$-module* over a certain C^*-algebra $\mathfrak{A}$ is an $\mathfrak{A}$-module $\mathbf{M}$ equipped with a conjugate bilinear mapping $\langle\cdot,\cdot\rangle\colon \mathbf{M}\times\mathbf{M}\to\mathfrak{A}$ satisfying

(i) $\langle x, x\rangle \geqq 0$ for any $x \in \mathbf{M}$,

(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for any $x, y \in \mathbf{M}$,

(iv) $\langle ax, y\rangle = a\langle x, y\rangle$ for any $a \in \mathfrak{A}$, $x, y \in \mathbf{M}$.

The map $\langle\cdot,\cdot\rangle$ is called the *$\mathfrak{A}$-valued inner product on* $\mathbf{M}$. Let us remark that we will write "pre-Hilbert C^*-module" instead of "pre-Hilbert $\mathfrak{A}$-module over the C^*-algebra $\mathfrak{A}$" whenever the concrete properties of the underlying C^*-algebra $\mathfrak{A}$ are unimportant in the context. A pre-Hilbert C^*-module is *Hilbert* if it is complete with respect to the norm $\|\cdot\| = \|\langle\cdot,\cdot\rangle\|_{\mathfrak{A}}^{1/2}$. Two Hilbert $\mathfrak{A}$-moduli $\{\mathbf{M}, \langle\cdot,\cdot\rangle_{\mathbf{M}}\}$, $\{\mathbf{N}, \langle\cdot,\cdot\rangle_{\mathbf{N}}\}$ over a certain fixed C^*-algebra $\mathfrak{A}$ are *isomorphic* if there exists a bijective $\mathfrak{A}$-linear bounded map $B\colon \mathbf{M} \to \mathbf{N}$ such that $\langle \mathbf{x}, \mathbf{y}\rangle_{\mathbf{M}} = \langle B(\mathbf{x}), B(\mathbf{y})\rangle_{\mathbf{N}}$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{M}$. A Hilbert $\mathfrak{A}$-module over a certain C^*-algebra $\mathfrak{A}$ is called *finitely* (resp., *countably*, *countably infinitely*) *generated* if it is finitely (resp., countably, countably infinitely) generated as an $\mathfrak{A}$-module, cf. [9]. A C^*-submodule $\mathbf{M}$ of a certain Hilbert C^*-module $\{\mathbf{N}, \langle\cdot,\cdot\rangle\}$ is a *Hilbert C^*-submodule of* $\mathbf{N}$ if $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ is a Hilbert C^*-module. A pre-Hilbert C^*-submodule $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ of a certain pre-Hilbert C^*-module $\{\mathbf{N}, \langle\cdot,\cdot\rangle\}$ is a *direct summand of* $\mathbf{N}$ if any element of $\mathbf{N}$ has a (unique) decomposition into the sum of an element of $\mathbf{M}$ and an element of the orthogonal with respect to $\langle\cdot,\cdot\rangle$ complement of $\mathbf{M}$.

Let us consider some examples.

(i) Any C^*-algebra $\mathfrak{A}$ becomes a Hilbert $\mathfrak{A}$-module with the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{A}}$ being defined by $\langle a, b\rangle_{\mathfrak{A}} = ab^*$ for any $a, b \in \mathfrak{A}$.

(ii) Any Hilbert space is a Hilbert C^*-module over the C^*-algebra $\mathbb{C}$.

(iii) Let I be an index set and let $\{\mathfrak{D}_\alpha\}_{\alpha\in I}$ be a collection of left ideals of a certain C^*-algebra $\mathfrak{A}$ indexed by I. Then the set of all I-tuples $\mathbf{x} = \{x_\alpha\}_{\alpha\in I}$ for which the sum $\sum x_\alpha x_\alpha^*$ converges with respect to the $\mathfrak{A}$-norm is a pre-Hilbert $\mathfrak{A}$-module.

(iv) Let H be a Hilbert space and $\mathfrak{A}$ be a C^*-algebra. The algebraic tensor product $\mathfrak{A} \otimes H$ becomes a pre-Hilbert $\mathfrak{A}$-module with the inner product $\langle\cdot,\cdot\rangle$ defined on elementary tensors by $\langle a \otimes \xi, b \otimes \eta\rangle = ab^*\langle \xi, \eta\rangle_H$. Obviously, if the Hilbert space H is finite-dimensional, then $\mathfrak{A} \otimes \mathbb{C}^n$ is isomorphic to the set $\mathfrak{A}^n$ of n-tuples of elements of $\mathfrak{A}$. Let us remark that the norm-closure of $\mathfrak{A} \otimes l_2$ is denoted by $l_2(\mathfrak{A})$, and it plays an important role describing properties of arbitrary countably generated Hilbert $\mathfrak{A}$-moduli, cf. [9: Th. 2].

(v) Let $\xi = (E, p, K, H)$ be a locally trivial Hilbert bundle over a compact space K. Denote by $\Gamma(\xi)$ the set of all continuous sections of this bundle. Then $\Gamma(\xi)$ becomes in a natural way a Hilbert $C(K)$-module splitting the inner products of the fibres H_x, $x \in K$, cf. [2: p. 48–49].

We denote by $\mathbf{M}'$ the set of all bounded module maps $f\colon \mathbf{M} \to \mathfrak{A}$. Following W. L. PASCHKE [14] a Hilbert C^*-module $\mathbf{M}$ is called *self-dual* if every map $r \in \mathbf{M}'$ is of the form $\langle\cdot, \mathbf{a}_r\rangle$ for some $\mathbf{a}_r \in \mathbf{M}$. W. L. Paschke proved that in the case of $\mathfrak{A}$ being a W^*-algebra the $\mathfrak{A}$-valued inner product on a pre-Hilbert $\mathfrak{A}$-module $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ lifts to an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle_D$ on the Banach $\mathfrak{A}$-module $\mathbf{M}'$ turning $\{\mathbf{M}', \langle\cdot,\cdot\rangle_D\}$ into a self-dual Hilbert $\mathfrak{A}$-module, cf. [14: Th. 3.2]. A. S. MIŠČENKO showed that every finitely generated Hilbert C^*-module is self-dual, cf. [12]. P. P. SAWOROTNOW got the result that every Hilbert $\mathfrak{A}$-module over a finite-dimensional C^*-algebra $\mathfrak{A}$ is self-dual, cf. [19: Th. 3].

Denote by $\mathbf{M}''$ the $\mathfrak{A}$-module of all bounded module maps from $\mathbf{M}'$ into $\mathfrak{A}$. Let q be the module map $q\colon \mathbf{M} \to \mathbf{M}''$ defined by $q(\mathbf{m})[r] = (r(\mathbf{m}))^*$ for each $\mathbf{m} \in \mathbf{M}$, any $r \in \mathbf{M}'$. A Banach C^*-module $\mathbf{M}$ over $\mathfrak{A}$ is called *C^*-reflexive* (or *$\mathfrak{A}$-reflexive*) if the module map q is a module isomorphism, cf. [13]. For a Hilbert C^*-module the map q is automatically an isometry [13: Cor. 1.1]. It turns out that for Hilbert W^*-moduli the C^*-reflexivity is equivalent to the self-duality [14: Th. 3.2]. W. L. PASCHKE [16] proved that for any Hilbert $\mathfrak{A}$-module $\mathbf{M}$ over a certain C^*-algebra $\mathfrak{A}$ the $\mathfrak{A}$-valued inner product can be extended to the $\mathfrak{A}$-bidual Banach $\mathfrak{A}$-module $\mathbf{M}''$ turning it into a C^*-reflexive Hilbert $\mathfrak{A}$-module.

The paper is organized as follows: The second part is concerned with the definition of the notion "self-dual Hilbert C^*-module". We show that the property of a Hilbert C^*-module to be self-dual (in the sense of [14]) depends not on the *structure* of the given inner product, but only on the *existence* of an inner product on the underlying Banach C^*-module inducing an equivalent Hilbert norm and realizing the condition of self-duality (Proposition 2.2). A new definition of this notion is given (Definition 2.1) describing it on the category of Banach C^*-moduli. As a consequence we get that on self-dual Hilbert C^*-moduli any two Hilbert structures inducing equivalent norms to the given one are isomorphic (Theorem 2.6). We prove that any self-dual Hilbert C^*-submodule of an arbitrary pre-Hilbert C^*-module is a direct summand (Theorem 2.7).

In the third part Hilbert W^*-moduli are treated. We characterize C^*-reflexive (and hence, self-dual) Hilbert W^*-moduli by their inner topological properties (Theorem 3.2). Also an example of a non-C^*-reflexive Hilbert $\mathfrak{A}$-module over a certain commutative unital W^*-algebra $\mathfrak{A}$ is given (Example 3.6) contradicting [13: Th. 2.1].

The fourth part of this paper is concerned with the Hilbert $\mathfrak{A}$-module $l_2(\mathfrak{A})$ over certain C^*-algebras $\mathfrak{A}$, $l_2(\mathfrak{A})$ being standard for all countably generated Hilbert $\mathfrak{A}$-moduli in the sense of [9: Th. 2]. We give a criterion of self-duality of $l_2(\mathfrak{A})$ (Theorem 4.3). Moreover, we show that every Hilbert $\mathfrak{A}$-module over a certain C^*-algebra $\mathfrak{A}$ is self-dual if and only if $\mathfrak{A}$ is finite-dimensional (Proposition 4.4). As an application we get the classification of all countably generated self-dual Hilbert W^*-moduli by their structure (Proposition 4.7).

§2 The notion "self-dual Hilbert C^*-module" — a category concept

W. L. Paschke [14] and other authors [12, 18] have defined that a Hilbert $\mathfrak{A}$-module $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ over a certain C^*-algebra $\mathfrak{A}$ is self-dual if and only if every bounded module map $f \in \mathbf{M}'$ is of the form $\langle\cdot, \mathbf{a}_f\rangle$ for some $\mathbf{a}_f \in \mathbf{M}$. We give another definition.

Definition 2.1: A Banach $\mathfrak{A}$-module $\mathbf{M}$ over a certain C^*-algebra $\mathfrak{A}$ is called a *self-dual Hilbert $\mathfrak{A}$-module over $\mathfrak{A}$* if there exists an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{M}$ with the properties:

(i) The norm induced on $\mathbf{M}$ by the $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle$ is equivalent to the given norm on $\mathbf{M}$.

(ii) The map $\varphi: \mathbf{M} \to \mathbf{M}'$ defined by the formula $\varphi(\mathbf{a}) = \langle\cdot, \mathbf{a}\rangle$ ($\mathbf{a} \in \mathbf{M}$) is surjective.

This definition seems to be weaker than the other one. In the following, however, we prove the equivalence of both definitions. As a result we can show the categorical sense of the notion for the category of Banach C^*-moduli.

Proposition 2.2: *Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathbf{M}$ be an $\mathfrak{A}$-module turning into a self-dual (in the sense of [14]) Hilbert $\mathfrak{A}$-module with the $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle_1$, and turning into a Hilbert $\mathfrak{A}$-module with the $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle_2$. We suppose the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{M}$ and the completeness of $\mathbf{M}$ with respect to them.*

Then $\{\mathbf{M}, \langle\cdot,\cdot\rangle_2\}$ is a self-dual (in the sense of [14]) Hilbert $\mathfrak{A}$-module and there exists a bounded $\mathfrak{A}$-linear operator $B: \mathbf{M} \to \mathbf{M}$ with the following properties:

(i) $\langle\mathbf{b}, \mathbf{a}\rangle_2 = \langle\mathbf{b}, B(\mathbf{a})\rangle_1$ *for any* $\mathbf{a}, \mathbf{b} \in \mathbf{M}$.

(ii) *B is one-to-one and, in addition, self-adjoint and positive on both* $\{\mathbf{M}, \langle\cdot,\cdot\rangle_1\}$ *and* $\{\mathbf{M}, \langle\cdot,\cdot\rangle_2\}$.

(iii) *B has an inverse B^{-1}, which is bounded and $\mathfrak{A}$-linear. For B^{-1} the properties of item* (ii) *are valid and* $\langle\mathbf{b}, \mathbf{a}\rangle_1 = \langle\mathbf{b}, B^{-1}(\mathbf{a})\rangle_2$ *for any* $\mathbf{a}, \mathbf{b} \in \mathbf{M}$.

Proof: (i) Since $\{\mathbf{M}, \langle\cdot,\cdot\rangle_1\}$ is self-dual, for each $\mathbf{a} \in \mathbf{M}$ there exists an element $B(\mathbf{a}) \in \mathbf{M}$ such that $\langle\cdot, \mathbf{a}\rangle_2 = \langle\cdot, B(\mathbf{a})\rangle_1$ on $\mathbf{M}$. The map B is $\mathfrak{A}$-linear. Since the inequality

$$\|\mathbf{a}\|_1 \leqq k\,\|\mathbf{a}\|_2 \leqq l\,\|\mathbf{a}\|_1 \quad (k, l \in (0, +\infty)) \tag{1}$$

is valid for any $\mathbf{a} \in \mathbf{M}$ by supposition, and because of the inequality

$$\|B(\mathbf{a})\|_1^2 = \|\langle B(\mathbf{a}), B(\mathbf{a})\rangle_1\|_{\mathfrak{A}} = \|\langle B(\mathbf{a}), \mathbf{a}\rangle_2\|_{\mathfrak{A}}$$
$$\leqq \|B(\mathbf{a})\|_2 \, \|\mathbf{a}\|_2 \leqq l^2 \, \|B(\mathbf{a})\|_1 \, \|\mathbf{a}\|_1 \qquad (2)$$

(cf. [14: Prop. 2.3] and (1)) we get the boundedness of B. It does not depend on the inner product.

(ii) We state the equalities ($\mathbf{a}, \mathbf{b} \in \mathbf{M}$)

$$\langle B(\mathbf{a}), \mathbf{b}\rangle_1 = \langle \mathbf{b}, B(\mathbf{a})\rangle_1^* = \langle \mathbf{b}, \mathbf{a}\rangle_2^* = \langle \mathbf{a}, \mathbf{b}\rangle_2 = \langle \mathbf{a}, B(\mathbf{b})\rangle_1,$$
$$\langle B(\mathbf{a}), \mathbf{b}\rangle_2 = \langle B(\mathbf{a}), B(\mathbf{b})\rangle_1 = \langle \mathbf{a}, B^2(\mathbf{b})\rangle_1 = \langle \mathbf{a}, B(\mathbf{b})\rangle_2.$$

This is enough to show $B = B^*$ with respect to both inner products. The other properties of B are trivial deductions now. In particular, we get that B is a one-to-one mapping.

(iii) Because of the inequality

$$\|\mathbf{a}\|_1^2 \leqq k^2 \, \|\mathbf{a}\|_2^2 = k^2 \, \|\langle \mathbf{a}, \mathbf{a}\rangle_2\|_{\mathfrak{A}} = k^2 \, \|\langle \mathbf{a}, B(\mathbf{a})\rangle_1\|_{\mathfrak{A}} \leqq k^2 \, \|B(\mathbf{a})\|_1 \, \|\mathbf{a}\|_1$$

being valid for any $\mathbf{a} \in \mathbf{M}$ (cf. (1) and [14: Prop. 2.3]), we get the connection

$$\|\mathbf{a}\|_1 \leqq k^2 \, \|B(\mathbf{a})\|_1 \leqq k^2 \, \|B\|_{op,1} \, \|\mathbf{a}\|_1 \quad \text{for any} \quad \mathbf{a} \in \mathbf{M}. \qquad (3)$$

Since B is bounded this means that every norm-fundamental sequence of the range of B has a (unique) norm-fundamental sequence of $\mathbf{M}$ as its pre-image. Moreover, B maps the limit of this pre-image sequence into the limit of the sequence taken in the range of B. Consequently, since B is $\mathfrak{A}$-linear the range Im (B) is a norm-closed $\mathfrak{A}$-submodule of $\mathbf{M}$ independent of the inner product. The Banach $\mathfrak{A}$-module Im (B) becomes a Hilbert $\mathfrak{A}$-submodule of $\mathbf{M}$ with both inner products $\langle \cdot, \cdot\rangle_1$ and $\langle \cdot, \cdot\rangle_2$. Now there are three possibilities how Im (B) can be related to the Hilbert $\mathfrak{A}$-module $\mathbf{M}$:

(a) $\mathrm{Im}\,(B) \not\equiv \mathbf{M}$, $\quad \mathrm{Im}\,(B)^\perp \not\equiv \{0\}$.

(b) $\mathrm{Im}\,(B) \not\equiv \mathbf{M}$, $\quad \mathrm{Im}\,(B)^\perp \equiv \{0\}$.

(c) $\mathrm{Im}\,(B) \equiv \mathbf{M}$.

We will show that, in fact, only (c) can be. To rule out the first possibility we take an element $\mathbf{b} \in \mathrm{Im}\,(B)^\perp$ with respect to $\langle \cdot, \cdot\rangle_1$, $\mathbf{b} \neq \mathbf{0}$. Then we get $\langle \mathbf{b}, \mathbf{b}\rangle_2 = \langle \mathbf{b}, B(\mathbf{b})\rangle_1 = 0$ and, therefore, $\mathbf{b} = \mathbf{0}$. This is a contradiction. The same happens if we take $\mathbf{b} \in \mathrm{Im}\,(B)^\perp$ with respect to $\langle \cdot, \cdot\rangle_2$, $\mathbf{b} \neq \mathbf{0}$. We get $0 = \langle B(\mathbf{b}), \mathbf{b}\rangle_2 = \langle B(\mathbf{b}), B(\mathbf{b})\rangle_1$ and, therefore, $B(\mathbf{b}) = \mathbf{0}$. Since B is injective, $\mathbf{b} = \mathbf{0}$ in contradiction to our choice of $\mathbf{b}$.

To drop the second possibility we use the fact that the canonical embedding of a Hilbert $\mathfrak{A}$-module $\{\mathbf{N}; \langle \cdot, \cdot\rangle\}$ over a certain C^*-algebra $\mathfrak{A}$ into its $\mathfrak{A}$-bidual Banach $\mathfrak{A}$-module $\mathbf{N}''$ does not depend on the structure of the $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot\rangle$ by definition. Using on $\mathbf{M}$ the inner product $\langle \cdot, \cdot\rangle_1$ we get $\mathbf{M}'' \equiv \mathbf{M}$ and $\mathrm{Im}\,(B)'' \equiv \mathbf{M}$ as Banach $\mathfrak{A}$-moduli. Now we define on Im (B) a third $\mathfrak{A}$-valued inner product by the formula $\langle \mathbf{a}, \mathbf{b}\rangle_3 = \langle B^{-1}(\mathbf{a}), B^{-1}(\mathbf{b})\rangle_1$ for any $\mathbf{a}, \mathbf{b} \in \mathrm{Im}\,(B)$. It is well defined since B is a one-to-one, surjective, $\mathfrak{A}$-linear, bounded mapping from $\mathbf{M}$ onto Im (B). Because of (3) we get that $\{\mathrm{Im}\,(B), \langle \cdot, \cdot\rangle_3\}$ is a Hilbert $\mathfrak{A}$-module, which is, moreover, self-dual in the sense of [14]. Using the inner product $\langle \cdot, \cdot\rangle_3$ we consider $\mathrm{Im}\,(B)'' \equiv \mathrm{Im}\,(B)$. This means $\mathrm{Im}\,(B) \equiv \mathbf{M}$ in contradiction to (b).

Therefore, only the relation $\mathbf{M} \equiv \mathrm{Im}\,(B)$ is possible. Moreover, the bounded $\mathfrak{A}$-linear operator $B^{-1}\colon \mathbf{M} \to \mathbf{M}$, which is inverse to B, exists and satisfies the conditions of the items (ii) and (iii). Finally, we show the self-duality of $\{\mathbf{M}; \langle \cdot, \cdot\rangle_2\}$ in the sense of [14]. We choose an $r \in \mathbf{M}'$ arbitrarily. By supposition there exists an element $\mathfrak{b}_r \in \mathbf{M}$ such that $r(\cdot) \equiv \langle \cdot, \mathfrak{b}_r \rangle_1$ on $\mathbf{M}$. We define $\mathfrak{c}_r = B^{-1}(\mathfrak{b}_r) \in \mathbf{M}$ and get $r(\cdot) \equiv \langle \cdot, \mathfrak{c} \rangle_2$ on $\mathbf{M}$. Since $r \in \mathbf{M}'$ is choosen arbitrarily, we are done ∎

Corollary 2.3: *Let $\{\mathbf{M}, \langle \cdot, \cdot \rangle\}$ be a self-dual Hilbert $\mathfrak{A}$-module over a C^*-algebra $\mathfrak{A}$. Then for any $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle_0$ on $\mathbf{M}$ inducing an equivalent norm to the given one there exists one and only one bounded $\mathfrak{A}$-linear invertible positive operator B_0 on $\{\mathbf{M}, \langle \cdot, \cdot \rangle\}$ with the property $\langle \cdot, \cdot \rangle_0 \equiv \langle \cdot, B_0(\cdot) \rangle$ on $\mathbf{M}$. And vice versa.*

This follows from [10: Lemma 2, Th. 3] and from Proposition 2.2.

Corollary 2.4: *Under the suppositions formulated in Proposition 2.2 beside the inequality* (1) *there holds the inequality*

$$\langle \mathfrak{a}, \mathfrak{a} \rangle_1 \leqq C \langle \mathfrak{a}, \mathfrak{a} \rangle_2 \leqq D \langle \mathfrak{a}, \mathfrak{a} \rangle_1$$

for some constants $C, D \in (0, +\infty)$ and for any $\mathfrak{a} \in \mathbf{M}$.

Proof: We have $\langle \mathfrak{a}, \mathfrak{a} \rangle_2 = \langle \mathfrak{a}, B(\mathfrak{a}) \rangle_1 = \langle B^{1/2}(\mathfrak{a}), B^{1/2}(\mathfrak{a}) \rangle_1 \leqq \|B^{1/2}\|^2_{op,1} \langle \mathfrak{a}, \mathfrak{a} \rangle_1 = \|B\|_{op,1} \langle \mathfrak{a}, \mathfrak{a} \rangle_1$ for any $\mathfrak{a} \in \mathbf{M}$ (cf. [14: Prop. 2.8]). In the same way we get $\langle \mathfrak{a}, \mathfrak{a} \rangle_1 \leqq \|B^{-1}\|_{op,2} \langle \mathfrak{a}, \mathfrak{a} \rangle_2$ for any $\mathfrak{a} \in \mathbf{M}$ ∎

Corollary 2.5: *The operator described in Proposition 2.2 has the property $k^{-2} \leqq \|B\|_{op,1} \leqq l^2$, where $k, l \in \mathbb{R}$ are taken from* (1).

This follows from (2) and (3). The following corollary is suggested from [13: Th. 1.1, Cor. 1.2], where a special case of it is stated.

Theorem 2.6: *Let $\mathfrak{A}$ be a C^*-algebra and $\{\mathbf{M}, \langle \cdot, \cdot \rangle_1\}$ be a self-dual Hilbert $\mathfrak{A}$-module. Then every $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle_2$ on $\mathbf{M}$, the norm induced from which is equivalent to the given norm, defines a Hilbert structure on $\mathbf{M}$ isomorphic to the Hilbert structure given by the $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle_1$.*

Proof: By Proposition 2.2 there exists a bounded $\mathfrak{A}$-linear invertible self-adjoint positive operator $B: \mathbf{M} \to \mathbf{M}$ satisfying the equality $\langle \mathfrak{a}, \mathfrak{b} \rangle_2 = \langle \mathfrak{a}, B(\mathfrak{b}) \rangle_1$ for any $\mathfrak{a}, \mathfrak{b} \in \mathbf{M}$. The set of all bounded $\mathfrak{A}$-linear operators on a self-dual Hilbert $\mathfrak{A}$-module is a C^*-algebra by [14: Cor. 3.5], and an operator B is positive on the Hilbert $\mathfrak{A}$-module if and only if it is positive as an element of this C^*-algebra by [10: Lemma 2, Th. 3]. So we can find a bounded $\mathfrak{A}$-linear invertible self-adjoint positive operator $C: \mathbf{M} \to \mathbf{M}$ satisfying the equality $\langle \mathfrak{a}, \mathfrak{b} \rangle_2 = \langle C(\mathfrak{a}), C(\mathfrak{b}) \rangle_1$ for any $\mathfrak{a}, \mathfrak{b} \in \mathbf{M}$ ∎

Remark 2.7: If the underlying C^*-algebra $\mathfrak{A}$ is commutative, then any Hilbert $\mathfrak{A}$-module has a unique (up to isomorphism) Hilbert structure, i.e. the property to be self-dual is omittable in this case. This fact can be drawn from the investigations of M. J. Dupré and R. M. Gillette [3: pp. 48–49]. However, if we drop the commutativity condition on $\mathfrak{A}$, the analogous problem of uniqueness is still open. There seems to be some hope to solve it affirmatively.

Now we are able to obtain a very important property of self-dual Hilbert C^*-moduli: Let us previously remark that not any Hilbert C^*-submodule of a pre-Hilbert C^*-module has to be a direct summand, in general.

Theorem 2.8: *Let $\mathfrak{A}$ be a C^*-algebra, $\{\mathbf{N}, \langle \cdot, \cdot \rangle\}$ be any pre-Hilbert $\mathfrak{A}$-module and $\mathbf{M} \subseteqq \mathbf{N}$ be a self-dual Hilbert $\mathfrak{A}$-submodule. Then $\mathbf{N} \equiv \mathbf{M} \oplus \mathbf{M}^\perp$.*

Proof: We take the $\mathfrak{A}$-valued inner product on $\mathbf{M}$ given by that one on $\mathbf{N}$ reduced to $\mathbf{M} \subseteqq \mathbf{N}$. By [14: Prop. 3.4] the injective isometric $\mathfrak{A}$-linear embedding $T: \mathbf{M} \to \mathbf{N}$ has an adjoint $\mathfrak{A}$-linear bounded operator $T^*: \mathbf{N} \to \mathbf{M}$ defined on $\mathbf{N}$ such that

$$\langle T(\mathbf{m}), \mathbf{n}\rangle_{\mathbf{N}} = \langle \mathbf{m}, T^*(\mathbf{n})\rangle_{\mathbf{M}} \tag{4}$$

for any $\mathbf{m} \in \mathbf{M}$, any $\mathbf{n} \in \mathbf{N}$. Because of the choice of the Hilbert structure on $\mathbf{M}$ and since T is isometric we can rewrite (4) as $\langle T(\mathbf{m}), \mathbf{n} - T(T^*(\mathbf{n}))\rangle_{\mathbf{N}} = O$ for any $\mathbf{n} \in \mathbf{N}$, any $\mathbf{m} \in \mathbf{M}$. That is, any element $\mathbf{n} \in \mathbf{N}$ can be decomposed $\mathbf{n} = TT^*(\mathbf{n}) + (\mathbf{n} - TT^*(\mathbf{n}))$, where $TT^*(\mathbf{n}) \in \mathbf{M} \subseteq \mathbf{N}$ and $(\mathbf{n} - TT^*(\mathbf{n})) \in \mathbf{M}^{\perp}$. This decomposition is unique ∎

Finishing this paragraph we list some results from the literature to illustrate the importance of Theorem 2.8.

Corollary 2.9 [13: Cor. 1.4], [2: Prop. 1], [8]: *The following is true:*

(i) *Let* $\mathfrak{A}$ *be any* C^**-algebra. Let* $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ *be a finitely generated Hilbert* $\mathfrak{A}$*-submodule of an arbitrary pre-Hilbert* $\mathfrak{A}$*-module* $\{\mathbf{N}, \langle\cdot,\cdot\rangle\}$. *Then* $\mathbf{N} = \mathbf{M} \oplus \mathbf{M}^{\perp}$.

(ii) *Let* $\mathfrak{A}$ *be a finite-dimensional* C^**-algebra. Let* $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ *be a Hilbert* $\mathfrak{A}$*-submodule of an arbitrary pre-Hilbert* $\mathfrak{A}$*-module* $\{\mathbf{N}, \langle\cdot,\cdot\rangle\}$. *Then* $\mathbf{N} = \mathbf{M} \oplus \mathbf{M}^{\perp}$.

§3 A topological characterization of self-dual and C^*-reflexive Hilbert W^*-moduli

The aim of the present paragraph is to characterize self-duality and C^*-reflexivity for a special class of *Hilbert* C^**-moduli*, namely, for Hilbert W^*-moduli, by their inner topological properties. The possibility of such a characterization is based either on Theorem 2.6 for self-duality or on the definition of the notion for C^*-reflexivity. The ideas for the following investigations arise from the proving technics and from the mental background of two papers of W. L. PASCHKE [14, 16].

Definition 3.1: Let $\mathfrak{A}$ be a W^*-algebra, $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ be a pre-Hilbert $\mathfrak{A}$-module and P be the set of all normal states on $\mathfrak{A}$. The topology induced on $\mathbf{M}$ by the seminorms

$$f(\langle\cdot,\cdot\rangle)^{1/2}, \quad f \in P, \tag{5}$$

is denoted by τ_1. The topology induced on $\mathbf{M}$ by the linear functionals $f(\langle\cdot, \mathbf{y}\rangle)$, $f \in P$, $\mathbf{y} \in \mathbf{M}$, is denoted by τ_2.

Let us remark that the topology τ_2 was already explicitly defined by W. L. PASCHKE in [14: Remark 3.9], whereas the topology τ_1 was suggested to the author by the proving technics of [16: Lemma 2.3]. If we define an $\mathfrak{A}$-valued inner product on the W^*-algebra $\mathfrak{A}$ by the formula $\langle a, b\rangle_{\mathfrak{A}} = ab^*$, $a, b \in \mathfrak{A}$, the topology τ_2 coincides with the weak* topology on $\mathfrak{A}$. In the case of $\mathfrak{A}$ being $\mathbb{C}$ and $\mathbf{M}$ being an arbitrary Hilbert space the topology τ_1 is the Hilbert topology on $\mathbf{M}$, but the topology τ_2 is the weak and weak* topology on $\mathbf{M}$. That is, they do not coincide, in general.

Theorem 3.2: *Let* $\mathfrak{A}$ *be a* W^**-algebra and* $\{\mathbf{M}, \langle\cdot,\cdot\rangle\}$ *be a Hilbert* $\mathfrak{A}$*-module. The following conditions for* $\mathbf{M}$ *are equivalent:*

(i) $\mathbf{M}$ *is self-dual.*
(ii) $\mathbf{M}$ *is* $\mathfrak{A}$*-reflexive.*
(iii) *The unit ball of* $\mathbf{M}$ *is* τ_1*-complete.*
(iv) *The unit ball of* $\mathbf{M}$ *is* τ_2*-complete.*

Proof: (i) ⇔ (ii) follows from the definitions and from [14: Th. 3.2]. (i) ⇒ (iii): Assume that the unit ball of a self-dual Hilbert $\mathfrak{A}$-module $\mathbf{M}$ is not complete relative to the topology τ_1. Denote by $\mathbf{L}$ the linear hull of the completion of the unit ball of

$\mathbf{M}$ relative to the topology τ_1. For the extensions of the semi-norms (5) from $\mathbf{M}$ to $\mathbf{L}$ we use the same denotations. By assumption there exists an $r \in \mathbf{L} \setminus \mathbf{M}$ and a norm-bounded net $\{y_\alpha\}_{\alpha \in I} \subset \mathbf{M}$ such that for every $f \in P$ and for each $\varepsilon > 0$ there is an $\alpha \in I$ with $f(\langle r - y_\beta, r - y_\beta \rangle) < \varepsilon$ for any $\beta \geqq \alpha$. We fix $f \in P$, $\varepsilon > 0$, $\alpha \in I$ and an arbitrary $x \in \mathbf{M}$. Then

$$|f(\langle x, y_\beta \rangle) - f(\langle x, y_\gamma \rangle)| = |f(\langle x, y_\beta - y_\gamma \rangle)|$$
$$\leqq f(\langle x, x \rangle)^{1/2} f(\langle y_\beta - y_\gamma, y_\beta - y_\gamma \rangle)^{1/2} \leqq \left(2\varepsilon f(\langle x, x \rangle)\right)^{1/2}$$

for any $\beta, \gamma \geqq \alpha$. Consequently, there exists

$$w^* - \lim \{\langle x, y_\alpha \rangle : \alpha \in I\} \doteq R(x) \tag{6}$$

for each $x \in \mathbf{M}$. Furthermore, the inequality

$$|f(\langle x, y_\beta \rangle)| \leqq \|x\| \sup \{\|y_\alpha\| : \alpha \in I\}, \quad (\beta \in I)$$

shows the boundedness of the map $R: \mathbf{M} \to \mathfrak{A}$ defined by (6). The $\mathfrak{A}$-linearity of R is obvious. Thus, (6) defines a bounded module map R. By assumption there exists an element $z \in \mathbf{M}$ such that $R(x) = \langle x, z \rangle$ for any $x \in \mathbf{M}$. Consequently, we arrive at $w^* - \lim \{\langle x, y_\beta \rangle : \beta \in I\} = \langle x, z \rangle$ for any $x \in \mathbf{M}$, $z \in \mathbf{M}$ being the τ_1-limit of the norm-bounded net $\{y_\alpha\}$. This means $r = z \in \mathbf{M}$ in contradiction to our assumption.

(ii) $\Rightarrow$ (i): We take an arbitrary $r \in \mathbf{M}'$ and we suppose the τ_1-completeness of the unit ball of $\mathbf{M}$. By [14: Th. 3.2] we can lift the $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle$ from $\mathbf{M}$ to $\mathbf{M}'$ turning $\mathbf{M}'$ into a self-dual Hilbert $\mathfrak{A}$-module and satisfying the following properties for the lifted inner product $\langle \cdot, \cdot \rangle_D$:

$$\langle x, y \rangle = \langle \varphi(x), \varphi(y) \rangle_D \quad \text{if} \quad x, y \in \mathbf{M},$$
$$r(x) = \langle \varphi(x), r \rangle_D \quad \text{if} \quad x \in \mathbf{M}, r \in \mathbf{M}' \setminus \mathbf{M},$$

where $\varphi(y) = \langle \cdot, y \rangle$ for any $y \in \mathbf{M}$. Furthermore, $\langle \mathbf{M}, \mathbf{M} \rangle$ and $\langle \mathbf{M}', \mathbf{M}' \rangle_D$ are W^*-algebras, $\langle \mathbf{M}, \mathbf{M} \rangle$ being a two-sided $*$-ideal in $\langle \mathbf{M}', \mathbf{M}' \rangle_D$. Therefore, they coincide because of the properties of the lifted inner product, and $\langle r, r \rangle_D$ belongs to $\langle \mathbf{M}, \mathbf{M} \rangle$. First, let $\mathfrak{A}$ be σ-finite and let $g \in P$ be a faithful normal state on $\mathfrak{A}$, which exists according to [1: p. 94, Prop. 2.3.6]. Let $\{H, \pi, \Omega\}$ be the cyclic representation associated with g. The vector $\Omega \in H$ is both cyclic and separating. The linear space $\mathbf{M}$ equipped with the inner product $g(\langle \cdot, \cdot \rangle)$ turns into a pre-Hilbert space. The map $g(r(\cdot)): \mathbf{M} \to \mathbb{C}$ is a linear functional on it. Consequently, there exists an element r_g in the completion of $\mathbf{M}$ relative to the norm $g(\langle \cdot, \cdot \rangle)^{1/2}$ such that $g(\langle x, r_g \rangle) = g(r(x))$ for any $x \in \mathbf{M}$. That means there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subset \mathbf{M}$ such that

$$0 = \lim_{i \to \infty} g(\langle x_i - r_g, x_i - r_g \rangle)$$
$$= \lim_{i \to \infty} g\left(\langle \varphi(x_i) - r, \varphi(x_i) - r \rangle_D\right)$$
$$= \lim_{i \to \infty} \left\| [\pi(\langle \varphi(x_i) - r, \varphi(x_i) - r \rangle_D)]^{1/2} \Omega \right\|^2.$$

Since the vector $\Omega \in H$ is both cyclic and separating there exists $w^* - \lim_i \langle \varphi(x_i) - r, \varphi(x_i) - r \rangle_D = 0$ by [1: Lemma 2.5.38, Lemma 2.5.39]. This proves the implication in the case of $\mathfrak{A}$ being σ-finite.

If $\mathfrak{A}$ is not σ-finite, there exists an increasing directed net of projections $\{p_\alpha\}_{\alpha \in I} \subset \mathfrak{A}$ such that $p_\alpha \mathfrak{A} p_\alpha$ is a σ-finite W^*-algebra for each $\alpha \in I$ and $w^* - \lim_\alpha p_\alpha = 1_{\mathfrak{A}}$.

(cf. [1: p. 164]). Observing $p_\alpha \mathfrak{A} p_\alpha$ and $(p_\alpha r) \in \mathbf{M}'$ for each $\alpha \in I$ we conclude that $(p_\alpha r) \in \mathbf{M}$ for each $\alpha \in I$ since $\langle p_\alpha r, p_\alpha r\rangle_D = p_\alpha \langle r, r\rangle_D\, p_\alpha$ and the latter belongs to $p_\alpha \mathfrak{A} p_\alpha$ for each $\alpha \in I$. Consequently, the τ_1-limit of the bounded net $\{p_\alpha r\}_{\alpha \in I}$ belongs to $\mathbf{M}$ and it is equal to $r \in \mathbf{M}'$. So the self-duality of $\mathbf{M}'$ turns out.

(iii) $\Leftrightarrow$ (iv): First, if the unit ball of $\mathbf{M}$ is τ_1-complete, then $\mathbf{M}$ must be self-dual as shown above. By [14: Prop. 3.8, Remark 3.9] there follows that $\mathbf{M}$ is a conjugate space with weak* topology τ_2. Therefore, the unit ball of $\mathbf{M}$ is τ_2-complete. Secondly, let $\{\mathbf{x}_\alpha\}_{\alpha \in I} \subset \mathbf{M}$ be a norm-bounded τ_1-fundamental net and let the unit ball of $\mathbf{M}$ be τ_2-complete. Then, for any $\mathbf{y} \in \mathbf{M}$, $f \in P$, $\beta, \gamma \in I$,

$$|f(\langle \mathbf{x}_\beta, \mathbf{y}\rangle) - f(\langle \mathbf{x}_\gamma, \mathbf{y}\rangle)|^2 \leqq f(\langle \mathbf{x}_\beta - \mathbf{x}_\gamma, \mathbf{x}_\beta - \mathbf{x}_\gamma\rangle)\, f(\langle \mathbf{y}, \mathbf{y}\rangle). \tag{7}$$

Denote by $\mathbf{L}$ the linear hull of the τ_1-completion of the unit ball of $\mathbf{M}$. The limit $\tau_1 - \lim \mathbf{x}_\alpha = l$ exists in $\mathbf{L}$. From the inequality (7) we get that the net $\{\mathbf{x}_\alpha\}$ is also τ_2-fundamental and so the τ_2-limit $x \in \mathbf{M}$ exists by assumption. Recall that $\mathbf{L} = \mathbf{M}'$ and that the $\mathfrak{A}$-valued inner product lifts from $\mathbf{M}$ to $\mathbf{M}'$ turning $\mathbf{M}'$ into a self-dual Hilbert $\mathfrak{A}$-module. Thus, $l = \tau_1 - \lim \mathbf{x}_\alpha = \tau_2 - \lim \mathbf{x}_\alpha = \mathbf{x} \in \mathbf{M}$ ∎

Remark 3.3: Let $\mathfrak{A}$ be an infinite-dimensional σ-finite W^*-algebra and $\{\mathbf{M}, \langle \cdot, \cdot\rangle\}$ be a Hilbert $\mathfrak{A}$-module. If g is a faithful normal state on $\mathfrak{A}$, the Hilbert completion of the pre-Hilbert space $\{\mathbf{M}, g(\langle \cdot, \cdot\rangle)\}$ does not coincide with the Hilbert $\mathfrak{A}$-module $\mathbf{M}$, in general.

Corollary 3.4: *If $\mathfrak{A}$ is a W^*-algebra and $\mathbf{M}$ is a self-dual Hilbert $\mathfrak{A}$-module, then the C^*-algebra $\langle \mathbf{M}, \mathbf{M}\rangle$ is a W^*-subalgebra of $\mathfrak{A}$ and a two-sided ideal in $\mathfrak{A}$.*

The converse is not true, in general, as will be shown on the example of $l_2(\mathfrak{A})$ in § 4 of the present paper.

Corollary 3.5: *Let $\mathfrak{A}$ be an infinite-dimensional C^*-algebra having a W^*-subalgebra* $\mathfrak{B}$ as its two-sided ideal. Let $\mathbf{M}$ be a finitely generated Hilbert $\mathfrak{A}$-module and $\mathbf{N}$ be an arbitrary Hilbert $\mathfrak{B}$-module. *Then the direct sum $\mathbf{M} \oplus \mathbf{N}'$ becomes a self-dual Hilbert $\mathfrak{A}$-module.*

Proof: First, we note that $\mathbf{N}$ is a Hilbert $\mathfrak{A}$-module, too, since $\mathfrak{B}$ is a two-sided ideal in $\mathfrak{A}$. Furthermore, the set of bounded module maps $f\colon \mathbf{N} \to \mathfrak{A}$ coincides with $\mathbf{N}'$ because $f(\mathbf{n}) = f(1_{\mathfrak{B}}\mathbf{n}) = 1_{\mathfrak{B}} f(\mathbf{n}) \in \mathfrak{B}$ for any $\mathbf{n} \in \mathbf{N}$. We know from [12] that $\mathbf{M}$ is self-dual. $\mathbf{N}'$ is also self-dual as it was shown in [14: Th. 3.2] ∎

Example 3.6: Take $\mathfrak{A} = l_\infty$, $\mathbf{M} = c_0$ with the $\mathfrak{A}$-valued inner product $\langle a, b\rangle_{\mathfrak{A}} = ab^*$ for $a, b \in c_0$. Easy computations show that $\mathbf{M}' = l_\infty$ and $\mathbf{M}'' = l_\infty$. This is an elegant counter-example to [13: Th. 2.1].

For the completeness of the present paragraph we reproduce a result of W. L. PASCHKE concerning another criterion of self-duality and C^*-reflexivity of Hilbert W^*-moduli.

Definition 3.7 [14]: Let $\mathfrak{A}$ be a W^*-algebra, I be an index set and $\{\mathbf{M}_\alpha, \langle \cdot, \cdot\rangle\}_{\alpha \in I}$ be a collection of pre-Hilbert $\mathfrak{A}$-moduli indexed by I. Let F denote the set of finite subsets of I, directed upwards by inclusion. For I-tuples $\mathbf{x} = \{\mathbf{x}_\alpha\}$, $\mathbf{y} = \{\mathbf{y}_\alpha\}$ $(\mathbf{x}_\alpha, \mathbf{y}_\alpha \in \mathbf{M}_\alpha)$ and $S \in F$ we set

$$\langle \mathbf{x}, \mathbf{y}\rangle_S = \sum_{\alpha \in S} \langle \mathbf{x}_\alpha, \mathbf{y}_\alpha\rangle.$$

Let $\mathbf{M}$ denote the set of I-tuples $\mathbf{x} = \{\mathbf{x}_\alpha\}$ such that $\sup \{\langle \mathbf{x}, \mathbf{x}\rangle_S : S \in F\}$ is finite. Notice that for $\mathbf{x} \in \mathbf{M}$ the net $\{\langle \mathbf{x}, \mathbf{x}\rangle_S\}_{S \in F}$ is bounded in norm and increasingly directed. We let $\langle \mathbf{x}, \mathbf{x}\rangle$ denote its least upper bound. The net $\{\langle \mathbf{x}, \mathbf{y}\rangle_S\}_{S \in F}$ is also bounded and w^*-convergent for any $\mathbf{x}, \mathbf{y} \in \mathbf{M}$. We denote by $\langle \mathbf{x}, \mathbf{y}\rangle$ its w^*-limit. Under co-ordinatewise operations $\mathbf{M}$ is a left $\mathfrak{A}$-module, and $\langle \cdot, \cdot\rangle$ defined as above is an $\mathfrak{A}$-valued inner product on $\mathbf{M}$. We call the pre-Hilbert $\mathfrak{A}$-module $\{\mathbf{M}, \langle \cdot, \cdot\rangle\}$ the *ultraweak direct sum* of the moduli $\{\mathbf{M}_\alpha, \langle \cdot, \cdot\rangle\}$ and write $\mathbf{M} = \mathrm{UDS}\,\{\mathbf{M}_\alpha : \alpha \in I\}$.

Theorem 3.8 [14: Th. 3.12]: *Let $\mathfrak{A}$ be a W^*-algebra and $\mathbf{M}$ be a Hilbert $\mathfrak{A}$-module. Then the following two conditions for $\mathbf{M}$ are equivalent:*

(i) *$\mathbf{M}$ is self-dual.*

(ii) *There is a collection $\{p_\alpha\}_{\alpha \in I}$ of (not necessary distinct) non-zero projections of $\mathfrak{A}$ such that $\mathbf{M}$ and UDS $\{\mathfrak{A}p_\alpha : \alpha \in I\}$ are isomorphic as Hilbert $\mathfrak{A}$-moduli.*

Remark 3.9: This theorem suggests a possibility to construct other useful topologies on self-dual Hilbert W^*-moduli $\mathbf{M}$ in the following way: One must take a topology on the underlying W^*-algebra $\mathfrak{A}$ with respect to which the unit ball of $\mathfrak{A}$ is complete. Then one has to combine this topology either with the map $\langle \cdot, \cdot \rangle$ on $\mathbf{M} \times \mathbf{M}$ or with all $\mathfrak{A}$-linear bounded functionals of $\mathbf{M}'$. We could get topologies on $\mathbf{M}$ with respect to which the unit ball of $\mathbf{M}$ would be complete.

§4 A criterion of self-duality of $l_2(\mathfrak{A})$. Applications

Let $\mathfrak{A}$ be an arbitrary C^*-algebra. We consider the Hilbert $\mathfrak{A}$-module $l_2(\mathfrak{A})$ mentioned in the introduction. It is representable as the set of all sequences $\mathbf{a} = \{a_i\}_{i \in \mathbf{N}} \subset \mathfrak{A}$ for which the series $\sum a_i a_i^*$ converges relative to the norm topology in $\mathfrak{A}$. The inner product on it is defined as $\langle \mathbf{a}, \mathbf{b} \rangle = \sum a_i b_i^*$ for any $\mathbf{a}, \mathbf{b} \in l_2(\mathfrak{A})$. If $\mathfrak{A}$ has an identity then $l_2(\mathfrak{A})$ is countably generated. The Hilbert $\mathfrak{A}$-module is standard for all countably generated Hilbert $\mathfrak{A}$-moduli in the sense of G. G. Kasparov's Stabilisation theorem [9: Th. 2]. Let us describe the inner structure of $l_2(\mathfrak{A})$. Denote by $\{\mathbf{e}_i\}_{i \in \mathbf{N}}$ the canonical orthonormal basis of $l_2(\mathfrak{A})$.

Lemma 4.1: *Let $\mathfrak{A}$ be a C^*-algebra with identity and $l_2(\mathfrak{A})$ be the standard countably generated Hilbert $\mathfrak{A}$-module. Then the map $\psi : f \in l_2(\mathfrak{A})' \to \{f(\mathbf{e}_i)^*\}_{i \in \mathbf{N}}$ is a bijection between $l_2(\mathfrak{A})'$ and the set of all sequences $\mathbf{a} = \{a_i\}_{i \in \mathbf{N}} \subset \mathfrak{A}$ with the property $\sup_N \|a_1 a_1^* + \cdots + a_N a_N^*\| < +\infty$. Moreover, ψ maps all bounded module maps of the form $\varphi(\mathbf{a}) = \langle \cdot, \mathbf{a} \rangle$, $\mathbf{a} \in l_2(\mathfrak{A})$, into the characterizing element $\mathbf{a}$ and vice versa.*

Corollary 4.2: *We have* $\|f\|^2 = \lim_N \left\| \sum_{i=1}^{N} f(\mathbf{e}_i)^* f(\mathbf{e}_i) \right\|_{\mathfrak{A}}$ *for any* $f \in l_2(\mathfrak{A})'$.

The statements of Lemma 4.1 and Corollary 4.2 are mentioned by [16] and by [5] without proof. Since the proof is easy it will be omitted. The following theorem is an extension of [6: Prop. 3, Prop. 4] to the non-commutative case. It was first proved by the author [7: Th. 22] with global C^*-algebraical methods and, independently, by O. G. Filippov [5] considering maximal commutative C^*-subalgebras. We reproduce here the proof from [7] in an ameliorated variant.

Theorem 4.3: *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *$\mathfrak{A}$ is finite-dimensional.*

(ii) *$l_2(\mathfrak{A})$ is self-dual.*

(iii) *For each $\mathbf{a} \in l_2(\mathfrak{A})$ the series $\sum \|a_i\|^{\mathfrak{A}2}$ converges.*

Proof: We start with a simple observation. If the C^*-algebra $\mathfrak{A}$ is finite-dimensional, it contains an identity. If $l_2(\mathfrak{A})$ is self-dual, then the bounded module map h defined by the formula $h(\mathbf{a}) = a_1$ for any $\mathbf{a} = \{a_i\}_{i \in \mathbf{N}} \in l_2(\mathfrak{A})$ belongs to $l_2(\mathfrak{A}) \subseteq l_2(\mathfrak{A})'$. Thus, $\mathfrak{A}$ must contain an identity.

Assume now that $\mathfrak{A}$ contains an identity. The Hilbert $\mathfrak{A}$-module $l_2(\mathfrak{A})$ is self-dual if and only if any norm-bounded increasing directed sequence of self-adjoint positive elements of $\mathfrak{A}$ is fundamental relative to the norm-topology of $\mathfrak{A}$, cf. Lemma 4.1. Equivalent to this condition is that all linear positive functionals on $\mathfrak{A}$ are normal, i.e., that the universal representation of $\mathfrak{A}$ is normal and, equivalently, that $\mathfrak{A}$ is reflexive as a Banach space. The latter is true if and only if $\mathfrak{A}$ is finite-dimensional. The equivalence of the conditions (i) and (iii) follows from a proposition of A. DVORETZKY and C. A. ROGERS [4], which can be found in [17: Prop. 3.4.1, Prop. 1.6.2] ∎

As a corollary we can extend [19: Th. 3] to these criteria:

Proposition 4.4: *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *$\mathfrak{A}$ is finite-dimensional.*

(ii) *Any Hilbert $\mathfrak{A}$-module $\mathbf{M}$ is self-dual.*

Moreover, if the C^*-algebra $\mathfrak{A}$ is commutative and unital (or, respectively, is a W^*-algebra), there exists a third equivalent condition:

(iii) *Any Hilbert $\mathfrak{A}$-module $\mathbf{M}$ is $\mathfrak{A}$-reflexive.*

Proof: The first item follows from [19: Th. 2] and from Theorem 4.3 above. To prove the second one we consider a compact space K consisting of infinitely many points. We denote by $C(K)$ the set of all continuous complex-valued functions on K and, respectively, by $C_0(K)$ the set of all $f \in C(K)$ satisfying $f(x) = 0$ at a certain fixed accumulation point $x \in K$. The sets $C(K)$ and $C_0(K)$ are both C^*-algebras, where the latter is a two-sided ideal in $C(K)$. We define on $C(K)$ the usual inner product $\langle \cdot, \cdot \rangle_{\mathfrak{A}}$. Then $\mathbf{M} = \{C_0(K), \langle \cdot, \cdot \rangle_{\mathfrak{A}}\}$ turns into a Hilbert $C(K)$-module for which the connection $\mathbf{M}' = \mathbf{M}'' = C(K)$ holds. Hence, $C_0(K)$ is not $\mathfrak{A}$-reflexive.

If $\mathfrak{A}$ is an infinite-dimensional W^*-algebra, the counter-example is given by Theorem 3.2 and Theorem 4.3 ∎

Proposition 4.5: *Let $\mathfrak{A}$ be a W^*-algebra or, respectively, a commutative unital C^*-algebra. The following two conditions are equivalent:*

(i) *$\mathfrak{A}$ is finite-dimensional.*

(ii) *For any Hilbert $\mathfrak{A}$-module $\mathbf{N}$ and any Hilbert $\mathfrak{A}$-submodule $\mathbf{M} \subseteq \mathbf{N}$ there holds $\mathbf{N} = \mathbf{M} \oplus \mathbf{M}^\perp$.*

Proof: If $\mathfrak{A}$ is an infinite-dimensional W^*-algebra, both $l_2(\mathfrak{A})$ and $l_2(\mathfrak{A})'$ are non-coinciding Hilbert $\mathfrak{A}$-moduli, where $l_2(\mathfrak{A}) \subset l_2(\mathfrak{A})'$ and $l_2(\mathfrak{A})^\perp = \{0\}$. If $\mathfrak{A} = C(K)$ is a commutative unital C^*-algebra, the Hilbert $\mathfrak{A}$-module $\{C_0(K), \langle \cdot, \cdot \rangle_{\mathfrak{A}}\}$ described in the proof of Proposition 4.4 can be viewed as a Hilbert $\mathfrak{A}$-submodule of $\mathfrak{A}$. But, $C_0(K)^\perp = \{0\}$ in this case and they do not coincide. Referring to [19: Th. 1, Lemma 3] we finish the proof ∎

Corollary 4.6: *If $\mathfrak{A}$ is a C^*-algebra with an infinite-dimensional two-sided W^*-ideal, the standard countably generated Hilbert $\mathfrak{A}$-module $l_2(\mathfrak{A})$ is neither self-dual nor $\mathfrak{A}$-reflexive.*

This follows from Theorem 3.2 and Theorem 4.3. As a further application we describe below the structure of self-dual and C^*-reflexive countably generated Hilbert W^*-moduli. That any finitely generated Hilbert C^*-module is self-dual was recalled in Corollary 2.9. Similarly, any Hilbert C^*-module over a finite-dimensional C^*-algebra is self-dual, cf. Corollary 2.9.

Proposition 4.7: *Let $\mathfrak{A}$ be an W^*-algebra and $\mathbf{M}$ be a C^*-reflexive (and, hence, self-dual) countably generated Hilbert $\mathfrak{A}$-module. There are two possibilities for the structure of $\mathbf{M}$ and of $\mathfrak{A}$:*

(i) *$\mathbf{M}$ is finitely generated and $\mathfrak{A}$ is arbitrary.*

(ii) *$\mathbf{M}$ is the direct sum of a finitely generated Hilbert $\mathfrak{A}$-module and of a countably generated Hilbert $\mathfrak{B}$-module, where $\mathfrak{B}$ is a finite-dimensional two-sided C^*-ideal in $\mathfrak{A}$.*

If the W^*-algebra $\mathfrak{A}$ has no finite-dimensional two-sided C^*-ideals, any countably infinitely generated Hilbert $\mathfrak{A}$-module is non-self-dual and non-C^*-reflexive.

Proof: Let $\{\mathbf{x}_i\}_{i\in\mathbb{N}}$ be the system of generators of $\mathbf{M}$ as $\mathfrak{A}$-module. By [14: Prop. 3.11] and [18: Lemma 6.7] there exists another system of generators $\{\mathbf{y}_i\}_{i\in\mathbb{N}}$ of $\mathbf{M}$ deduced from the first one such that $\langle y_i, y_i\rangle = p_i = p_i^2 \neq 0$ and $\langle y_i, y_j\rangle = 0$ for any $i \neq j$. Denote by $L_{\mathfrak{A}}(\mathbf{y}_i)$ the norm-closed $\mathfrak{A}$-linear hull of $\mathbf{y}_i$. The Hilbert $\mathfrak{A}$-module $\{L_{\mathfrak{A}}(\mathbf{y}_i), \langle\cdot,\cdot\rangle\}$ is self-dual. Therefore, it is isomorphic to $\{\mathfrak{A}p_i, \langle\cdot,\cdot\rangle_{\mathfrak{A}}\}$ by Theorem 3.8 and $\langle L_{\mathfrak{A}}(\mathbf{y}_i), L_{\mathfrak{A}}(\mathbf{y}_i)\rangle$ is a two-sided W^*-ideal in $\mathfrak{A}$ for any $i \in \mathbb{N}$. Consequently, we get that $\mathbf{M}$ is isomorphic to the Hilbert $\mathfrak{A}$-module

$$\mathbf{N} = \left\{\mathbf{x} = \{x_i\}_{i\in\mathbb{N}}: x_i \in \mathfrak{A}p_i, \sum x_i x_i^* \text{ is } \|\cdot\|_{\mathfrak{A}}\text{-coverging}\right\}.$$

Now we try to reach a situation in which the product of any two projections p_i, p_j $(i < j)$ of our choice is a projection r if and only if $r = p_j \neq 0$. For this end we use an inductive process of construction. First, fix the projection p_1 and check all products $r_k = p_1 p_k$ $(k \in \mathbb{N})$. If r_k is a projection for a certain $k \in \mathbb{N}$ and if $r_k \neq 0$, then replace p_1 by the sum $(p_1 + p_k - r_k)$ and p_k by r_k. If $r_k = 0$, then replace p_1 by $(p_1 + p_k)$ and exclude p_k from our choice. Finishing this first step we deal with the pairwise products $r_k' = p_2 p_k$ $(k \in \mathbb{N})$ with the first factor p_2 in the same way. This process is continued by induction. We remark that the claimed inductive process on the projections $\{p_k: k \in \mathbb{N}\}$ of $\mathfrak{A}$ is compatible with the module operations inside $\mathbf{N}$ and $\mathbf{M}$, respectively.

Suppose now there exist more than finitely many two-sided W^*-ideals $\langle\mathfrak{A}p_i, \mathfrak{A}p_i\rangle$ of our reconstructed choice being infinite-dimensional. Then $\mathbf{M}' \neq \mathbf{M}$ by Theorem 3.8 and Theorem 4.3. Suppose there exists no finite-dimensional two-sided C^*-ideal $\mathfrak{B}$ in $\mathfrak{A}$ containing all finite-dimensional W^*-ideals $\langle\mathfrak{A}p_i, \mathfrak{A}p_i\rangle$ of our reconstructed choice. Then $\mathbf{M}' \neq \mathbf{M}$ by Theorem 3.8 and Theorem 4.3 ∎

Finally, we state the main problem arising if these results are to be extended to the case when $\mathfrak{A}$ is not necessarily a W^*-algebra. W. L. Paschke [14: Th. 3.2] noted without proof referring to [20] that the $\mathfrak{A}$-valued inner product of a Hilbert $\mathfrak{A}$-module lifts on to the dual Banach $\mathfrak{A}$-module even if $\mathfrak{A}$ is a commutative AW^*-algebra. The question is: What are the conditions needed that this can be done for any Hilbert $\mathfrak{A}$-module over a certain C^*-algebra $\mathfrak{A}$? One condition is that $\mathfrak{A}$ must be an AW^*-algebra [16: Prop. 1.1]. It seems to be necessary that $\mathfrak{A}$ must be monotonically complete and possess an analogy of the w^*-topology coinciding with the topology of order convergence on bounded directed nets of self-adjoint elements of $\mathfrak{A}$, cf. Theorem 4.3. Solving this problem one could get general criteria of self-duality and C^*-reflexivity of Hilbert C^*-moduli.

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Manuskripteingang: 22. 03. 1988; in revidierter Fassung 01. 03. 1989

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