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A Stabilization Method for the Tricomi Problem

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Es wird die Existenz einer verallgemeinerten Lösung für das Tricomiproblem der Gleichung $L_0[u] = T[u] + \lambda l(u) := yu_{xx} + u_{yy} + \lambda l(u) = f$ gezeigt, wobei $l = \alpha^1\partial/\partial x + \alpha^2\partial/\partial y$ ein spezieller Differentialoperator und $\lambda \geqq 0$ eine Konstante ist. Sodann wird die Lösbarkeit eines Anfangs-Randwertproblems für die Evolutionsgleichung $L[u] = T[u] + \partial l(u)/\partial t = F$ durch eine Approximationsmethode bewiesen. Es wird gezeigt, daß die verallgemeinerte Lösung des Evolutionsproblems gegen die verallgemeinerte Lösung des Tricomiproblems $T[u] = f$ für $t \to \infty$ konvergiert. Die Konvergenzgeschwindigkeit wird abgeschätzt.

Доказывается существование обобщенного решения для проблемы Трикоми уравнения $L_0[u] = T[u] + \lambda l(u) := yu_{xx} + u_{yy} + \lambda l(u) = f$, где $l = \alpha^1\,\partial/\partial x + \alpha^2\,\partial/\partial y$ специальный дифференциальный оператор и $\lambda \geqq 0$ постоянная. Потом доказывается разрешимость смешанной краевой задачи для эволюционного уравнения $L[u] = T[u] + \partial l(u)/\partial t = F$ аппроксимационным методом. Показывается, что обобщенное решение эволюционной задачи стремится к обобщенному решению проблемы Трикоми $T[u] = f$ для $t \to \infty$, и оценивается скорость сходимости.

We prove the existence of a generalized solution of the Tricomi problem for the equation $L_0[u] = T[u] + \lambda l(u) := yu_{xx} + u_{yy} + \lambda l(u) = f$, where $l = \alpha^1\partial/\partial x + \alpha^2\partial/\partial y$ is a special differential operator and $\lambda \geqq 0$ is a constant. Then we show the solvability of an initial boundary value problem for the evolution equation $L[u] = T[u] + \partial l(u)/\partial t = F$ by an approximation method. It is shown that the generalized solution of the evolution problem converges to the generalized solution of the Tricomi problem $T[u] = f$ as $t \to \infty$. The rate of convergence is estimated.

1. Introduction

The paper consists of three main chapters. After giving the notations and some preliminary results in Chapter 2, we consider in Chapter 3 the Tricomi problem for the equation

$$L_0[u] := Tu + \lambda l(u) = yu_{xx} + u_{yy} + \lambda l(u) = f \tag{1.1}$$

in a region $G \subset \mathbb{R}^2$, where $l = \alpha^1\,\partial/\partial x + \alpha^2\,\partial/\partial y$, α^1, α^2 are some special functions and $\lambda \geqq 0$ is a constant. Hereby G is a simply connected region bounded by the curves Γ_0, Γ_1 and Γ_2. Γ_0 is a piecewise smooth curve lying in the half-space $y > 0$ which intersects the line $y = 0$ in the points $A(-1, 0)$ and $B(0, 0)$. Γ_1, Γ_2 are characteristics of (1.1) issued from A and B which intersect in the point $C(-1/2, y_c)$. We prove the existence of a generalized solution (Def. 2.1) of the problem $L_0[u] = f$, $u|_{\Gamma_0 \cup \Gamma_1} = 0$ (Theorem 3.3). By the method used in Theorem 3.3 the solution is uniquely determined. In Chapter 4 we show the solvability of the initial boundary value

[1]) This work was accomplished during the author's working as Alexander von Humboldt fellow in University of Karlsruhe.

problem for the evolution equation

$$L[u] = Tu + \partial l(u)/\partial t = F, \quad u|_{(\Gamma_0 \cup \Gamma_1) \times [0, T_0]} = 0, \quad u|_{t=0} = u_0 \tag{1.2}$$

in $\Omega = G \times (0, T_0)$ $(T_0 > 0)$. Hereby the existence Theorem 4.2 is proved by an approximation method using the results of Theorem 3.3, thus the generalized solution is uniquely determined. In the last Chapter 5 we then show that the generalized solution of the problem (1.2) converges to the generalized solution of the Tricomi problem $Tu = f$, $u|_{\Gamma_0 \cup \Gamma_1} = 0$ as $t \to \infty$. The rate of convergence is estimated in Theorem 5.2. Such stabilisation methods are widely used for numerical methods, for example, for transonical gasdynamic problems [9].

Boundary value problems for equation of mixed type with solutions in different function spaces were considered by many authors. In the references we only note those papers which have connections with methods used here. For constructive methods based on Galerkin method and finite-element method see [1—3] and the references cited there.

2. Notations, preliminary results

We introduce the function spaces

$$U = \{u \mid u \in C^\infty(\bar{G}), u|_{\Gamma_0 \cup \Gamma_1} = 0\}, \quad V = \{v \mid v \in C^\infty(\bar{G}), v|_{\Gamma_0 \cup \Gamma_2} = 0\}, \tag{2.1}$$

noting that $v|_{\Gamma_0 \cup \Gamma_2} = 0$ are the adjoint boundary conditions to $u|_{\Gamma_0 \cup \Gamma_1} = 0$ with respect to equations (1.1). If $u \in U$ and $v \in V$, the formal application of Green's theorem to (1.1) gives

$$\begin{aligned} B[u, v] &:= (L_0[u], v)_0 = \int_G vL_0[u]\,dx\,dy = (u, L_0^*[v])_0 \\ &:= -\int_G \{yu_xv_x + u_yv_y - \lambda l(u)\,v\}\,dx\,dy = (f, v)_0. \end{aligned} \tag{2.2}$$

As is known, (2.2) gives the basis for the definition of generalized solutions. To this end we introduce the spaces

$$W^{1,2}(G, \mathrm{bd}), \quad W^{1,2}(G, \mathrm{bd}^*) \tag{2.3}$$

which are obtained by the completion of the function spaces (2.1) with respect to the norm $\|w\|_{1,2} = \left(\int_G \{w_x^2 + w_y^2 + w^2\}\,dx\,dy\right)^{1/2}$.

Definition 2.1: A function $u \in W^{1,2}(G, \mathrm{bd})$ is called a *generalized solution of* $L_0[u] = f$, $u|_{\Gamma_0 \cup \Gamma_1} = 0$ if

$$B[u, v] = -\int_G \{yu_xv_x + u_yv_y - \lambda l(u)\,v\}\,dx\,dy = (f, v)_0$$

for all $v \in W^{1,2}(G, \mathrm{bd}^*)$, and a function $u \in L^2(0, T_0; W^{1,2}(G, \mathrm{bd}))$ with $u_t \in L^2(0, T_0; W^{1,2}(G, \mathrm{bd}))$, $l(u_t) \in L^\infty(0, T_0; L^2(G))$ is called a *generalized solution of the problem* (1.2) if

$$\tilde{B}[u, v] := -\int_G \{yu_xv_x + u_yv_y - \partial l(u)/\partial t\, v\}\,dx\,dy = (F, v)_0$$

a.e. in $[0, T_0]$ for all $v \in L^\infty(0, T_0; W^{1,2}(G, \mathrm{bd}^*))$.

Remark: Let X be a Banach space. $L^2(0, T_0; X)$ $(L^\infty(0, T_0; X))$ denotes the spaces of classes of functions u, strongly measurable on $[0, T_0]$, with range in X and such that

$$\|u\|_{L^2(0,T_0;X)} = \left(\int\limits_0^{T_0} \|u(t)\|_X^2 \, dt\right)^{1/2} < \infty \quad (\|u\|_{L^\infty(0,T_0;X)} = \operatorname*{vrai\,sup}_{t\in[0,T_0]} \|u(t)\|_X < \infty).$$

Using the Hölder inequality we have the following result.

Lemma 2.2: *If $\alpha^1, \alpha^2 \in C^0(\overline{G})$, then there exists a constant $c_0 > 0$ such that*

$$|B[u, v]| \leqq c_0 \|u\|_{1,2} \|v\|_{1,2} \; \forall\, u \in W^{1,2}(G, \mathrm{bd}),\; v \in W^{1,2}(G, \mathrm{bd}^*).$$

If we introduce the negative spaces $W^{-1,2}(G, \mathrm{bd})$, $W^{-1,2}(G, \mathrm{bd}^*)$ which are obtained to the corresponding positive spaces (2.3) by the completion of the function space $L^2(G)$ with respect to the norms

$$\|w\|_{-1,2} = \sup_{0 \neq u \in W^{1,2}(G,\mathrm{bd})} \{|(w, u)_0| / \|u\|_{1,2}\},$$

$$\|w\|^*_{-1,2} = \sup_{0 \neq v \in W^{1,2}(G,\mathrm{bd}^*)} \{|(w, v)_0| / \|v\|_{1,2}\}$$

we obtain from Lemma 2.2, with a suitable constant $c_0 > 0$,

$$\|L_0^*[v]\|_{-1,2} = \sup_{0 \neq u \in W^{1,2}(G,\mathrm{bd})} \{|B[u, v]| / \|u\|_{1,2}\} \leqq c_0 \|v\|_{1,2} \; \forall\, v \in W^{1,2}(G, \mathrm{bd}^*),$$

$$\|L_0[u]\|^*_{-1,2} = \sup_{0 \neq v \in W^{1,2}(G,\mathrm{bd}^*)} \{|B[u, v]| / \|v\|_{1,2}\} \leqq c_0 \|u\|_{1,2} \; \forall\, u \in W^{1,2}(G, \mathrm{bd}), \tag{2.4}$$

where L_0^* is the adjoint differential operator.

Following the same methods and arguments as in [5–6, 8] we have the following result.

Lemma 2.3: *Suppose*

(i) *Γ_0 is a piecewise smooth curve;*

(ii) *$\alpha^1 n_1 + \alpha^2 n_2 |_{\Gamma_0} \geqq 0$, where (n_1, n_2) is the outward normal vector and $\alpha^1 = -(-y_c)^{1/2} + 2x$, $\alpha^2 = 1 + y$;*

(iii) *$v \in V$.*

Then there exists a unique solution $u \in W^{1,2}(G, \mathrm{bd})$ of the boundary value problem

$$l(u) = \alpha^1 u_x + \alpha^2 u_y = v, \quad u|_{\Gamma_0 \cup \Gamma_1} = 0. \tag{2.5}$$

Sketch of the proof: (2.5) is a partial differential equation whose characteristics by condition (ii) cannot intersect the curves Γ_0 and Γ_1 more than once. The characteristics of (2.5) are parabola $-[2x - (-y_c)^{1/2}]/(1 + y)^2 = k_0^2$ which intersect in the point $\left(2^{-1}(-y_c)^{1/2}, -1\right) \notin \overline{G}$. Introducing the coordinates $\xi = x$, $\eta = -[2x - (-y_c)^{1/2}]/(1 + y)^2$ in $\overline{G}$, the equation (2.5) becomes

$$[-(-y_c)^{1/2} + 2\xi]\, u_\xi = v, \tag{2.6}$$

and $\Gamma_0\colon \xi = \psi_+(\eta)$, $\Gamma_1\colon \xi = \psi_-(\eta)$, where ψ_- is smooth and ψ_+ is a piecewise smooth curve. The solution of (2.6) is given by the formula

$$u(\xi, \eta) = \int\limits_{t=\psi_\pm(\eta)}^{\xi} [-(-y_c)^{1/2} + 2t]^{-1}\, v\left(t, -1 + \left[-\eta^{-1}(2\xi - (-y_c)^{1/2})\right]^{1/2}\right) dt \; ∎$$

3. Generalized solution of the Tricomi problem

We consider the Tricomi problem $L_0[u] = f, u|_{\Gamma_0 \cup \Gamma_1} = 0$ and prove an a priori estimate.

Lemma 3.1: *Suppose the conditions* (i) *and* (ii) *of Lemma* 2.3 *hold. Then, for all* $u \in W^{1,2}(G, \mathrm{bd})$, *we have the a priori estimate*

$$m_0 \|u\|_{1,2}^2 + 2\lambda \|l(u)\|_0^2 \leqq 2B[u, l(u)], \quad m_0 > 0 \text{ a constant}. \tag{3.1}$$

Proof: Using (2.2) and Green's theorem [7, p. 248] we have

$$\begin{aligned} 2B[u, l(u)] = 2\lambda \|l(u)\|_0^2 + \int_{\partial G} \{[\alpha^1 y u_x^2 - \alpha^2 u_y^2 + 2y\alpha^2 u_x u_y]\, dy \\ + [\alpha^2 y u_x^2 - \alpha^2 u_y^2 - 2\alpha^1 u_x u_y]\, dx\} - \int_G \{A u_x^2 + 2B u_x u_y \\ + C u_y^2\}\, dx\, dy, \end{aligned}$$

where $A = y(\alpha_x^1 - \alpha_y^2) - \alpha^2$, $B = y\alpha_x^2 + \alpha_y^1$, $C = -(\alpha_x^1 - \alpha_y^2)$. Now using the assumptions (i) and (ii) of Lemma 2.3 it follows that $(u|_{\Gamma_0 \cup \Gamma_1} = 0)$

$$\int_{\Gamma_0} \ldots = \int_{\Gamma_0} (y u_x^2 + y_y^2)(\alpha^1 n_1 + \alpha^2 n_2)\, ds \geqq 0, \quad \int_{\Gamma_1} \ldots = 0;$$

$$\int_{\Gamma_2} \ldots = -\int_{\Gamma_2} [(-y)^{1/2} u_x + u_y]^2 (\alpha^1 dy + \alpha^2 dx) \geqq 0,$$

$$A = C = -1, B = 0,$$

such that $2B[u, l(u)] \geqq \int_G \{u_x^2 + u_y^2\}\, dx\, dy + 2\lambda \|l(u)\|_0^2$. Since $u|_{\Gamma_0 \cup \Gamma_1} = 0$, using Friedrich's inequality [4, p. 305], we obtain (3.1) ∎

Theorem 3.2: *Suppose the assumptions* (i) *and* (ii) *of Lemma* 2.3 *hold. Then there exists a constant* $c_1 > 0$ *such that*

$$c_1 \|v\|_0 \leqq \|L_0^*(v)\|_{-1,2} = \sup_{0 \neq u \in W^{1,2}(G,\mathrm{bd})} \{|B[u, v]|/\|u\|_{1,2}\}. \tag{3.2}$$

Proof: From Lemma 2.3 we know that for a function $v \in V$ there exists a function $u \in W^{1,2}(G, \mathrm{bd})$ such that $l(u) = \alpha^1 u_x + \alpha^2 u_y = v$, $u|_{\Gamma_0 \cup \Gamma_1} = 0$ and $\|v\|_0 \leqq k_2 \|u\|_{1,2}$. From (3.1) we get, with a suitable constant $c_1 > 0$,

$$c_1 \|v\|_0 \leqq B[u, v]/\|u\|_{1,2} \leqq \sup_{0 \neq u \in W^{1,2}(G,\mathrm{bd})} \{|B[u, v]|/\|u\|_{1,2}\} \; ∎$$

We now give an existence theorem for the problem $L_0[u] = f, u|_{\Gamma_0 \cup \Gamma_1} = 0$.

Theorem 3.3: *Suppose*

(i) Γ_0 *is a piecewise smooth curve;*
(ii) $\alpha^1 n_1 + \alpha^2 n_2|_{\Gamma_0} \geqq 0$, *where* (n_1, n_2) *is the outward normal vector and* $\alpha^1 = -(-y_c)^{1/2} + 2x$, $\alpha^2 = 1 + y$.

Then there exists a generalized solution of the boundary value problem

$$L_0[u] = y u_{xx} + u_{yy} + \lambda l(u) = f(x, y), f \in L^2(G), \lambda \geqq 0,$$

$$l(u) = \alpha^1 u_x + \alpha^2 u_y, u|_{\Gamma_0 \cup \Gamma_1} = 0,$$

i.e., there exists a function $u \in W^{1,2}(G, \text{bd})$ *such that*

$$B[u, v] = -\int_G \{y u_x v_x + u_y v_y - \lambda l(u)\, v\}\, dx\, dy = (f, v)_0$$

for all $v \in W^{1,2}(G, \text{bd}^*)$ *and* $\|u\|_{1,2} \leqq c_1^{-1} \|f\|_0$.

Proof: For fixed $v \in W^{1,2}(G, \text{bd}^*)$, $\psi_v(\cdot) := B[\cdot, v]$ is a linear bounded functional on $W^{1,2}(G, \text{bd})$:

$$\|\psi_v\|_{W^{1,2}(G,\text{bd})} = \sup_{0 \neq u \in W^{1,2}(G,\text{bd})} \{|\psi_v(u)|/\|u\|_{1,2}\} \underset{(2.4)}{\leqq} c_0 \|v\|_{1,2}.$$

Thus we have a unique element $w \in W^{1,2}(G, \text{bd})$ such that $\psi_v(u) = B[u, v] = (u, w)_{W^{1,2}(G,\text{bd})}$ for all $u \in W^{1,2}(G, \text{bd})$, and $\|w\|_{1,2} \leqq c_0 \|v\|_{1,2}$. A linear operator $S: W^{1,2}(G, \text{bd}^*) \to W^{1,2}(G, \text{bd})$ is defined such that $B[u, v] = (u, Sv)_{W^{1,2}(G,\text{bd})}$ for all $u \in W^{1,2}(G, \text{bd})$, $v \in W^{1,2}(G, \text{bd}^*)$. Using (3.2) we obtain

$$c_1 \|v\|_0 \leqq \|Sv\|_{W^{1,2}(R,\text{bd})} \underset{(2.4)}{\leqq} c_0 \|v\|_{W^{1,2}(G,\text{bd}^*)}. \tag{3.3}$$

For a function $v \in W^{1,2}(G, \text{bd}^*)$ and $\tilde{l}(Sv) := (f, v)_0$ we have

$$|\tilde{l}(Sv)| \leqq \|f\|_0 \|v\|_0 \underset{(3.3)}{\leqq} (1/c_1) \|f\|_0 \|Sv\|_{W^{1,2}(G,\text{bd})}.$$

Since S is injective (see (3.3)), $\tilde{l}(\tilde{v}) = \tilde{l}(Sv) = (f, v)_0$ defines a linear bounded functional $\tilde{l}$ on $S(W^{1,2}(G, \text{bd}^*)) \subset W^{1,2}(G, \text{bd})$ which can be extended by the Hahn-Banach theorem to $\hat{l}$ on $W^{1,2}(G, \text{bd})$ preserving the norm. It follows, that there exists a unique function $u \in W^{1,2}(G, \text{bd})$ such that

$$\hat{l}(w) = (u, w)_{W^{1,2}(G,\text{bd})} \text{ for all } w \in W^{1,2}(G, \text{bd})$$

and

$$\|u\|_{W^{1,2}(G,\text{bd})} = \sup_{0 \neq w \in W^{1,2}(G,\text{bd})} \{|\hat{l}(w)|/\|w\|_{1,2}\} \leqq c_1^{-1} \|f\|_0.$$

For $v \in W^{1,2}(G, \text{bd}^*)$ we have $\hat{l}(Sv) = (u, Sv)_{W^{1,2}(G,\text{bd})} = B[u, v] = (f, v)_0$, i.e., u is a generalized solution ∎

Remark: We observe that the solution $u \in W^{1,2}(G, \text{bd})$ constructed in the way of Theorem 3.3 is uniquely determined. We do not know if the generalized solution (Def. 2.1) of the Tricomi problem for equation (1.1) is unique. In [5] and [8] for uniqueness there is the essential assumption that the coefficients of u_x and u_y in (1.1) are sufficiently small, but this is not true in our case.

4. Generalized solution of the evolution equation (1.2)

We consider in $\Omega = G \times (0, T_0)$ the problem

$$L[u] = y u_{xx} + u_{yy} + \partial l(u)/\partial t = F(x, y, t), \quad F \in L^2(0, T_0; L^2(G)),$$

$$l(u) = \alpha^1 u_x + \alpha^2 u_y, \quad \alpha^1 = -(-y_c)^{1/2} + 2x, \alpha^2 = 1 + y;$$

$$u|_{(\Gamma_0 \cup \Gamma_1) \times [0, T_0]} = 0, \quad u|_{t=0} =: u_0 \in W^{2,2}(G, \text{bd}).$$

Observe that $W^{2,2}(G, \text{bd})$ is the completion of the function space U (2.1) with respect to the norm $\|u\|_{2.2} = \left(\int_G \sum_{|\alpha| \leq 2} |D^\alpha u|^2 \, dx \, dy\right)^{1/2}$. We denote ($n = 1, \ldots, N$):

$$h = T_0/N, \; u^n(x, y) = u(x, y, nh), \quad u^0(x, y) = u_0(x, y),$$

$$l(u_h^0) = F(x, y, 0) - yu_{xx}^0 - u_{yy}^0, \; F^n(x, y) = h^{-1} \int_{(n-1)/h}^{nh} F(x, y, \tau) \, d\tau$$

and consider

$$L_h[u^n] := yu_{xx}^n + u_{yy}^n + h^{-1}\big(l(u^n) - l(u^{n-1})\big) = F^n.$$

Lemma 4.1: *Suppose the assumptions of Theorem 3.3 hold and* $u|_{t=0} = u_0 \in W^{2,2}(G, \text{bd})$, $F \in L^2(0, T_0; L^2(G))$. *Then there exists for any* n $(1 \leq n \leq N)$ *a generalized solution* $u^n \in W^{1,2}(G, \text{bd})$ *of the problem* $L_h[u^n] = F^n$, $u^n|_{\Gamma_0 \cup \Gamma_1} = 0$. *The functions* u^n *are uniquely determined by the method used in Theorem* 3.3.

Proof: The statement follows immediately from Theorem 3.3 if we notice

$$yu_{xx}^1 + u_{yy}^1 + h^{-1}l(u^1) = h^{-1} \int_0^h f(x, y, \tau) \, d\tau + h^{-1}l(u^0) \in L^2(G)$$

and

$$Tu^n + h^{-1}l(u^n) = h^{-1} \int_{(n-1)h}^{nh} F(x, y, \tau) \, d\tau + h^{-1}l(u^{n-1}) \in L^2(G) \; \blacksquare$$

Theorem 4.2: *Suppose*

(i) Γ_0 *is a piecewise smooth curve*;

(ii) $\alpha^1 n_1 + \alpha^2 n_2|_{\Gamma_0} \geq 0$, *where* (n_1, n_2) *is the outward normal vector*, $\alpha^1 = -(-y_c)^{1/2} + 2x$, $\alpha^2 = 1 + y$;

(iii) $u_0 \in W^{2,2}(G, \text{bd})$ *and* $F, F_t \in L^2(0, T_0; L^2(G))$.

Then there exists a generalized solution (*Def.* 2.1) u,

$$u, u_t \in L^2(0, T_0; W^{1,2}(G, \text{bd})), \; l(u_t) \in L^\infty(0, T_0; L^2(G))$$

of the initial boundary value problem (1.2)

$$L[u] = Tu + \partial l(u)/\partial t = F, \quad u|_{(\Gamma_0 \cup \Gamma_1) \times [0, T_0]} = 0, \quad u|_{t=0} = u_0.$$

Proof: For fixed $n \in \mathbb{N}$ we know from Lemma 4.1 the existence of a generalized solution of the problem $L_h[u^n] = Tu^n + h^{-1}\big(l(u^n) - l(u^{n-1})\big) = F^n$ such that

$$\big(L_h[u^n], l(u^n)\big)_0 = \big(Tu^n, l(u^n)\big)_0 + h^{-1}\big(l(u^n) - l(u^{n-1}), l(u^n)\big) = \big(F^n, l(u^n)\big)_0. \tag{4.1}$$

From Lemma 3.1 we conclude

$$m_0\|u^n\|_{W^{1,2}(G, \text{bd})}^2 \leq 2\big(Tu^n, l(u^n)\big)_0. \tag{4.2}$$

A calculation shows

$$(2h)^{-1}\big(\|l(u^n)\|_0^2 - \|l(u^{n-1})\|_0^2\big) \leq h^{-1}\big(l(u^n) - l(u^{n-1}), l(u^n)\big),$$

$$\|l(u^n)\|_0 \leq k_2\|u^n\|_{1.2},$$

thus we have

$$m_0\|u^n\|_{1.2}^2 + h^{-1}\big(\|l(u^n)\|_0^2 - \|l(u^{n-1})\|_0^2\big) \leq 2\varepsilon\|u^n\|_{1.2}^2 + 2c(\varepsilon) \, \|F^n\|_0^2.$$

Choosing $\varepsilon > 0$ in a suitable manner, there exists a constant $c_0 > 0$ independend of n, such that

$$m_0\|u^n\|_{1,2}^2 + h^{-1}\big(\|l(u^n)\|_0^2 - \|l(u^{n-1})\|_0^2\big) \leqq c_0 \|F^n\|_0^2$$

and (summing up over $1 \leqq n \leqq s$)

$$m_0 h \sum_{n=1}^{s} \|u^n\|_{1,2}^2 + \|l(u^s)\|_0^2 \leqq \|l(u^0)\|_0^2 + c_0 h \sum_{n=1}^{s} \|F^n\|_0^2. \tag{4.3}$$

We observe

$$\|F^n\|_0^2 = \int_G h^{-2}\left(\int_{(n-1)h}^{nh} F(x, y, \tau)\, d\tau\right)^2 dx\, dy \leqq \int_G h^{-1} \int_{(n-1)h}^{nh} F^2(x, y, \tau)\, d\tau\, dx\, dy,$$

$$\sum_{n=1}^{s} h\|F^n\|_0^2 \leqq \int_G \int_0^{t=sh} F^2(x, y, \tau)\, d\tau\, dx\, dy$$

and

$$\|u^n\|_0^2 = \int_G u^2(x, y, nh)\, dx\, dy,$$

$$h \sum_{n=1}^{s} \|u^n\|_0^2 = \int_G h \sum_{n=1}^{s} u^2(x, y, nh)\, dx\, dy \xrightarrow[h\to 0, sh=t]{} \int_G \int_0^t u^2(x, y, \tau)\, d\tau\, dx\, dy.$$

From (4.1) we conclude, taking the limit $h \to 0$, $sh = t$,

$$m_0 \int_0^t \|u\|_{1,2}^2(\tau)\, d\tau + \|l(u)\|_0^2\,(t) \leqq \|l(u^0)\|_0^2\,(0) + c_0 \int_0^t \|F\|_0^2\,(\tau)\, d\tau, \tag{4.4}$$

thus we know $u \in L^2(0, T_0; W^{1,2}(G, \mathrm{bd}))$, $l(u) \in L^\infty(0, T_0; L^2(G))$.

We denote ($n = 1, 2, \ldots$)

$$F_h^n = h^{-1}(F^n - F^{n-1}),\ u_h^n = h^{-1}(u^n - u^{n-1}),$$

$$\tilde{L}_{2h}[u^n] = h^{-1}(L_h[u^n] - L_h[u^{n-1}]) = h^{-1}(F^n - F^{n-1}) = F_h^n,$$

and consider

$$\tilde{L}_{2h}[u^n] := L_h[u_h^n] = Tu_h^n + h^{-1}\big(l(u_h^n) - l(u_h^{n-1})\big) = F_h^n.$$

From $\big(L_h[u_h^n], l(u_h^n)\big)_0 = \big(F_h^n, l(u_h^n)\big)_0$ we conclude, in the same manner as before,

$$m_0 h \sum_{n=1}^{s} \|u_h^n\|_{1,2}^2 + \|l(u_h^s)\|_0^2 \leqq \|l(u_h^0)\|_0^2 + c_1 h \sum_{n=1}^{s} \|F_h^n\|_0^2. \tag{4.5}$$

We observe

$$F_h^n = h^{-1}(F^n - F^{n-1}) = h^{-1} \int_{(n-1)h}^{nh} h^{-1}\big(f(x, y, \tau) - f(x, y, \tau - h)\big)\, d\tau,$$

$$\sum_{n=1}^{s} h\|F_h^n\|_0^2 \leqq \int_G \int_0^{t=sh} \big(h^{-1}(f(x, y, \tau) - f(x, y, \tau - h))\big)^2\, d\tau\, dx\, dy$$

$$\xrightarrow[h\to 0, sh=t]{} \int_0^t \|F_\tau\|_0^2\,(\tau)\, d\tau.$$

From (4.5) we get ($h \to 0$, $sh = t$)

$$m_0 \int_0^t \|u_\tau\|_{1,2}^2 (\tau)\, d\tau + \|l(u_t)\|_0^2 (t) \leqq \|l(u_t)\|_0^2 (0) + c_1 \int_0^t \|F_\tau\|_0^2 (\tau)\, d\tau \tag{4.6}$$

and

$$u_t \in L^2(0, T_0;\, W^{1,2}(G, \mathrm{bd})),\ l(u_t) \in L^\infty(0, T_0;\, L^2(G)).$$

For fixed $n \in \mathbb{N}$, $u^n \in W^{1,2}(G, \mathrm{bd})$ is a uniquely determined generalized solution of the problem $L_h[u^n] = Tu^n + h^{-1}(l(u^n) - l(u^{n-1})) = F^n$ which satisfies (4.3), (4.5). For a function $v \in L^\infty(0, T_0;\, W^{1,2}(G, \mathrm{bd}^*))$ we thus have a.e. in $[0, T_0]$

$$B[u^n, v] = -\int_G (yu_x{}^n v_x + u_y{}^n v_y)\, dx\, dy + \left(\frac{l(u^n) - l(u^{n-1})}{h}, v\right)_0 = (F^n, v)_0$$

and for the limit $h \to 0$, $nh = t$ we get

$$\tilde{B}[u, v] = -\int_G \{yu_x v_x + u_y v_y - \partial l(u)/\partial t v\}\, dx\, dy = (F, v)_0,$$

i.e., u is a generalized solution of the problem (1.2) ∎

Remark: If $F = F(x, y, t)$ and $u_0 = u_0(x, y)$ are sufficiently smooth, then the generalized solution of (1.2) has more derivatives in t. In case $F, F_t \in L^2(0, \infty; L^2(G))$ the solution of the problem (1.2) exists for all $t \in \mathbb{R}^+$.

5. Stability

From Theorem 3.3 it follows that the Tricomi problem

$$Tu = f_0 \in L^2(G), \quad u|_{\Gamma_0 \cup \Gamma_1} = 0 \tag{5.1}$$

has a generalized solution $u_T \in W^{1,2}(G, \mathrm{bd})$ which is uniquely determined. Further, we know from Theorem 4.2 that the evolution problem

$$\begin{aligned} &L[u] = Tu + \partial l(u)/\partial t = f_1 \in L^2(\mathbb{R}^+; L^2(G)), \\ &f_{1_t} \in L^2(\mathbb{R}^+, L^2(G)), \quad u|_{(\Gamma_0 \cup \Gamma_1) \times \mathbb{R}^+} = 0, \quad u|_{t=0} =: u_0 \in W^{2,2}(G, \mathrm{bd}) \end{aligned} \tag{5.2}$$

has a generalized solution u_E, $u_{Et} \in L^2(\mathbb{R}^+; W^{1,2}(G, \mathrm{bd}))$ which is likewise uniquely determined by the construction method. The function $Z := u_E - u_T$ thus is a generalized solution of

$$L[Z] = TZ + \partial l(Z)/\partial t = f_1 - f_0 =: \varphi,$$

which is obtained by the approximate functions $Z^n = u_E{}^n - u_T{}^n = u_E{}^n - u_T$ as $h \to 0$, $nh = t$. Thus we have, that the a priori estimates (4.4) and (4.6) hold for the function Z, and from (4.1) and (4.2) we conclude

$$m_0 \|Z\|_{1,2}^2(t) + (\partial l(Z)/\partial t, l(Z))_0 = (\varphi, l(Z))_0. \tag{5.3}$$

From the a priori estimates (4.4) and (4.6) we know, if $\|\varphi\|_0^2$, $\|\varphi_t\|_0^2 \in L^1(\mathbb{R}^+)$, then

$$\|Z\|_{1,2}, \|Z_t\|_{1,2} \in L^2(\mathbb{R}^+). \tag{5.4}$$

Lemma 5.1: *If (5.4) holds, then $\|Z\|_{1,2}(t) \to 0$ as $t \to \infty$.*

Proof: We know

$$\|Z\|_{1.2}^2(t) - \|Z\|_{1.2}^2(\tau) = \int_{\tau=s}^{t} \partial/\partial\tau \, \|Z\|_{1.2}^2 (\tau)\, d\tau \leqq \int_{\tau=s}^{t} \|Z\|_{1.2}(\tau)\, \|Z_\tau\|_{1.2}(\tau)\, d\tau$$

$$\leqq \left(\int_{\tau=s}^{t} \|Z\|_{1.2}^2 (\tau)\, d\tau \right)^{1/2} \left(\int_{\tau=s}^{t} \|Z_\tau\|_{1.2}^2 (\tau)\, d\tau \right) \underset{(5.4)}{<} \varepsilon \qquad (5.5)$$

for $t, s > N(\varepsilon)$. If $\|Z\|_{1.2}(t) \to c_2 > 0$ as $t \to \infty$, from (5.5) we get $\|Z\|_{1.2}^2 (s) > c_2 - \varepsilon =: c_3$ for all $s > N(\varepsilon)$ but this gives a contradiction to $\|Z\|_{1.2} \in L^2(\mathbb{R}^+)$ ∎

Using

$$\big(\partial/\partial t\, l(Z), l(Z)\big)_0 = \|l(Z)\|_0\, \partial/\partial t\, \|l(Z)\|_0,$$

$$k_1 \|Z\|_{L^2(G)} \leqq \|l(Z)\|_{L^2(G)} \leqq k_2 \|Z\|_{1.2},$$

from (5.3) it follows

$$(m_0/k_2)\, \|l(Z)\|_0\, (t) + \partial/\partial t\, \|l(Z)\|_0\, (t) \leqq \|\varphi\|_0,$$

$$\partial/\partial t\, \{\|l(Z)\|_0\, (t)\, e^{(m_0/k_2)t}\} \leqq e^{(m_0/k_2)t}\, \|\varphi\|_0\, (t),$$

$$k_1 \|Z\|_0\, (t) \leqq \|l(Z)\|_0\, (t) \leqq e^{-(m_0/k_2)t} \left\{ \|l(Z)\|_0\, (0) + \int_{\tau=0}^{t} e^{(m_0/k_2)\tau}\, \|\varphi\|_0\, (\tau)\, d\tau \right\}.$$

Thus, if $e^{(m_0/k_2)t} \|f_1 - f_0\|_0\, (t) \in L^1(\mathbb{R}^+)$ there exists a constant $c_4 > 0$, such that $\|Z\|_0\, (t) = \|u_E - u_T\|_0\, (t) \leqq c_4\, e^{-(m_0/k_2)t} \to 0$ as $t \to \infty$. We have the following result

Theorem 5.2: *Suppose*

(i) Γ_0 *is a piecewise smooth curve;*

(ii) $\alpha^1 n_1 + \alpha^2 n_2 \,|_{\Gamma_0} \geqq 0$, *where* (n_1, n_2) *is the outward normal vector,* $\alpha^1 = -(-y_c)^{1/2} + 2x$, $\alpha^2 = 1 + y$;

(iii) $u_T \in W^{1,2}(G, \mathrm{bd})$ *is the generalized solution of the problem* (5.1) $Tu = f_0 \in L^2(G)$ *constructed by the method used in Theorem* 3.3;

(iv) $u_E, u_{Et} \in L^2\big(\mathbb{R}^+; W^{1,2}(G, \mathrm{bd})\big)$ *is the generalized solution of the problem* (5.2) $L[u] = f_1 \in L^2\big(\mathbb{R}^+, L^2(G)\big)$, $f_{1_t} \in L^2\big(\mathbb{R}^+; L^2(G)\big)$ *constructed by the method used in Theorem* 4.2;

(v) $e^{(m_0/k_2)t} \|f_1 - f_0\|_0 \in L^1(\mathbb{R}^+)$, $\|f_{1_t}\|_0^2 \in L^1(\mathbb{R}^+)$.

Then for all solutions u_E *of the problem* (5.2) *constructed in the way of Theorem* 4.2, *we have*

$$\|u_E - u_T\|_{1.2}\, (t) \xrightarrow[t\to\infty]{} 0, \quad \|u_E - u_T\|_0\, (t) \leqq c_4\, e^{-(m_0/k_2)t} \xrightarrow[t\to\infty]{} 0.$$

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