Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (3) 1990, S. 221–233

Factoring Compact Operators and Approximable Operators

I. M. Popovici and D. T. Vuza

In unserer Arbeit werden zwei Gegenstände behandelt. Erstens wird eine Version des Figiel-Johnson-Theorems über Faktorisierungen kompakter Operatoren für geordnete Banachräume angegeben. Genauer wird gezeigt, daß jeder kompakte Operator, der einen Banachraum in einen geordneten Banachraum mit abgeschlossenem, erzeugendem Kegel (bzw. in einen Banachverband) abbildet, mit kompakten Faktoren durch einen reflexiven geordneten Banachraum mit abgeschlossenem, erzeugendem Kegel (bzw. durch einen verbandsgeordneten reflexiven Banachraum mit stetigem Absolutbetrag) faktorisiert werden kann. Dabei kann der zweite Faktor positiv (bzw. als Verbandshomomorphismus) gewählt werden.

Zweitens werden Faktorisierungen approximierbarer Operatoren U zwischen Banachverbänden behandelt. Wir zeigen, daß ein jeder solcher Operator U mit kompakten Faktoren durch einen reflexiven Banachverband mit unbedingter Basis faktorsiert werden kann, wobei einer der Faktoren positiv gewählt werden kann. Darüber hinaus geben wir eine Bedingung an U an, die notwendig und hinreichend dafür ist, daß beide Faktoren in dieser Zerlegung als Differenzen von positiven kompakten Operatoren gewählt werden können.

Наша работа посвящена двум темам. Первая тема — новая версия теоремы Фигеля-Джонсона о факторизации компактных операторов для полуупорядоченных банаховых пространств. А именно, показывается, что любой компактный оператор из банахова пространства в полуупорядоченное банахово пространство со замкнутым производящим конусом (соответственно, в банахову решетку) факторизуется с компактными факторами через рефлексивное полуупорядоченное банахово пространство со замнутым производящим конусом (соответственно, через рефлексивную банахову решетку с непрерывным модулем). При этом второй фактор может быть выбран положительным (соответственно, решеточным гомоморфизмом).

Вторая тема представляет собой дискуссию о факторизации аппроксимируемых операторов действующих между банаховыми решетками. Показывается, что любой такой оператор U факторизуется с компактными факторами через рефлексивную банахову решетку с безусловным базисом, при чем один из факторов может быть выбран положительным. Более того, дается условие на U, необходимое и достаточное для того чтобы обе факторы в этом разложении могут быть выбраны как разность положительных компактных операторов.

Our paper is concerned with two topics. The first one is represented by a version of Figiel's and Johnson's theorem on the factorization of compact operators adapted to the framework of ordered Banach spaces. Namely, we prove that every compact operator from a Banach space to an ordered Banach space with closed generating cone (respectively, a Banach lattice) factors, with compact factors, through a reflexive ordered Banach space with closed generating cone, the second factor being positive (respectively, a reflexive lattice-ordered Banach space with continuous modulus, the second factor being a Riesz homomorphism).

The second topic is provided by a discussion of the factorization of approximable operators between Banach lattices. We prove that every such operator U factors through a reflexive Banach lattice with an unconditional basis, the factors being compact and one of them being positive. We also give a necessary and sufficient condition on U under which both factors in the mentioned factorization can be taken to be differences of positive compact operators.

1. Introduction

All operators in this paper which act between Banach spaces will be assumed to be linear and bounded.

The classical factorization theorem due to T. Figiel [5] and W. B. Johnson [7] asserts that every compact operator U from a Banach space E to a Banach space F factors according the scheme

$$E \xrightarrow{U_1} G \xrightarrow{U_2} F \tag{1.1}$$

where G is a reflexive Banach space and the factors U_1, U_2 are compact. This scheme means that $U = U_2U_1$. In the situation when E and/or F belong to a special class $\mathscr{C}$ of Banach spaces, it is natural to try to find the reflexive factorization space G in (1.1) among the members of a class more or less related to $\mathscr{C}$. In Section 2 of our paper we examine from this viewpoint two such classes $\mathscr{C}$, namely the class $\mathscr{C}_1$ of all ordered Banach spaces with closed generating cones and the class $\mathscr{C}_2$ of all Banach lattices. It is shown that $\mathscr{C}_1$ is stable under factorization, that is, the hypothesis that F belongs to $\mathscr{C}_1$ ensures the possibility of choosing G among the members of the same class. The situation is more involved for $\mathscr{C}_2$. Thus, the answer to the following problem seems to be still unknown:

Problem 1.1: Does every compact operator U from a Banach space E to a Banach lattice F factor according (1.1) with G a reflexive Banach lattice and U_1, U_2 compact?

C. D. Aliprantis and O. Burkinshaw [1] have given a partial answer to Problem 1.1: namely, they have proved that whenever a given compact operator from a Banach space to a Banach lattice factors, with compact factors, through a Banach lattice, then it also factors, with compact factors, through a reflexive Banach lattice.

In our paper we present an alternative scheme of factorization which applies to every compact operator U from a Banach space to a Banach lattice. Namely, we prove that U factors according (1.1) so that the reflexive space G belongs to the class $\mathscr{C}_3$ of so-called lattice-ordered Banach spaces with continuous moduli, the factors are compact and U_2 is a Riesz homomorphism. The class $\mathscr{C}_3$ contains all Banach lattices as well as some classical lattice-ordered Banach spaces which are not Banach lattices, such as Sobolev spaces and Besov spaces. We consider our factorization result as being independent of the answer to Problem 1.1. Indeed, the fact that the order structure on G in our scheme is weaker than the structure of a Banach lattice is not due to the lack of an answer to that problem; it is a logical consequence of the condition imposed to U_2 to be a compact Riesz homomorphism. Recall that a Riesz homomorphism is a linear operator T between two Riesz spaces G, F such that $|T(x)| = T(|x|)$ for every $x \in G$.

As an application of our factorization results we complete Figiel's operatorial characterization of Banach spaces without the approximation property with some additional statements corresponding to the situations when F belongs to $\mathscr{C}_1$ or $\mathscr{C}_2$.

It was communicated by W. B. Johnson to the authors of [1] that the answer to Problem 1.1 is affirmative provided that F has the approximation property; moreover, in this situation, G can even be taken to have an unconditional basis. Johnson's communication was the starting point for our results exposed in Section 3. In the first place, we present a proof of a stronger version of Johnson's result, showing that we always can take the compact factor U_2 to be positive; our proof makes use of a relatively recent principle of local reflexivity for ordered Banach spaces due to K. D. Kürsten [8]. Second, we consider the case when E is also a Banach lattice and we find a necessary and sufficient condition on U under which both factors in (1.1) can be choosen to be differences of positive compact operators and G to be a reflexive Banach lattice with an unconditional basis (the order relation on G being canonically defined by its basis).

A part of the results exposed in the present paper made the object of the first author's thesis [13].

The reader is supposed to be acquainted with some general facts about Riesz spaces, ordered Banach spaces, Banach lattices and operators between them; we refer him to the excellent monographs [3, 12, 14] for the needed information. We recall here some notions and notations to be used throughout Sections 2 and 3. For a Banach space E, we denote by $J_E: E \to E''$ the canonical map. Given an operator U between two Banach spaces E and F, we denote by $\|U\|$ its operator norm. U is called *approximable* provided that it belongs to the closure of the subspace of all finite-rank operators from E to F taken with respect to the operator norm.

2. The factorization of compact operators

In this section we study the factorization of a compact operator defined on a Banach space E and taking values in an ordered Banach space F. We shall be concerned with two types of order structures which can be introduced on the factorization space G, described by Definitions 2.1 and 2.2 below.

Definition 2.1: An *ordered Banach space with closed generating cone* is an ordered Banach space G with the property that the cone G_+ is closed and verifies the equality $G_+ - G_+ = G$.

Definition 2.2: A *lattice-ordered Banach space with continuous modulus* is a lattice-ordered Banach space G with the property that the map $x \mapsto |x|$ is continuous for the norm of G.

While the class of ordered Banach spaces indicated by Definition 2.1 has received consideration in the theory of ordered topological vector spaces (see for instance [12]), it seems to us that its subclass indicated by Definition 2.2 has not yet made the object of a special study. We hope we could emphasize its importance by underlining the rôle played by it in the factorization of compact operators. Obviously, every Banach lattice is a lattice-ordered Banach space with continuous modulus. However, our factorization theorem yields in general lattice-ordered Banach spaces which are not Banach lattices (see Proposition 2.2 below); for this reason, we think it is useful to know that there are examples of classical Banach spaces which are lattice-ordered Banach spaces with continuous moduli but not Banach lattices. We confine ourselves to mention the Sobolev spaces $L_1^p(\mathbf{R}^n)$ for $1 \leqq p < \infty$ and the Besov spaces $\Lambda_\alpha^{p,q}(\mathbf{R}^n)$ for $0 < \alpha < 1$, $1 \leqq p \leqq \infty$ and $1 \leqq q < \infty$. We draw the attention on the fact that there are lattice-ordered Banach spaces for which the map $x \mapsto |x|$ is not norm-continuous: examples are provided by the Sobolev space $L_1^\infty(\mathbf{R}^n)$ and by the spaces of Lipschitz functions $\Lambda_\alpha(\mathbf{R}^n)$ for $0 < \alpha < 1$ (see [15: Ch. V, §§ 2, 4, 5] for the definitions of the above mentioned spaces). In all examples above, we have assumed that the Banach spaces under consideration were endowed with the usual order relation attached to a function space.

The notion of an ordered Banach space having a principal latticial extension introduced by the second author will occur in the statement of the factorization theorem; consequently, we reproduce below the definition from [16]. The terminology "principal latticial extension" is motivated by the connexion existing between this notion and the theory of so-called "principal modules" developed by the second author; see [16] for details.

Definition 2.3: An ordered Banach space G is said to have a *principal latticial extension* provided that there is a Riesz space $\tilde{G}$ containing G as a vector subspace such that the following hold:

(i) $G_+ = \tilde{G}_+ \cap G$.

(ii) There is a dense vector subspace G_0 of G such that for every $\varepsilon > 0$ and $x \in G_0$ there are a linear operator $U: \tilde{G} \to \tilde{G}$ and $y \in G$ with the properties: $0 \leq U \leq I$ (I = the identity map on $\tilde{G}$), $U(G) \subset G$ and the restriction of U to G is norm-continuous, $U(x) \geqq 0$, $(I - U)(x_+) \leq y$ (the positive part of x being taken in $\tilde{G}$) and $\|y\| < \varepsilon$.

The importance of such ordered Banach spaces is justified by the following theorem, also reproduced from [16].

Let G be a reflexive ordered Banach space with closed generating cone and let F be a Banach lattice with order continuous norm. Suppose that G has a principal latticial extension. Then for every operators $S, T: G \to F$ such that T is compact and $0 \leqq S \leqq T$, it follows that S belongs to the closed two-sided algebraical ideal generated by T.

The proof of the factorization theorem in this section relies on the following proposition, which represents a version of the well-known lemma of W. J. Davis et al. [4] adapted to the situation of a compact subset of an ordered Banach space. It seems to us that the construction of the factorization space described in [4] (and also employed by C. D. Aliprantis and O. Burkinshaw [1]) cannot be immediately adapted to our purposes; for this reason, the construction of our factorization space is somewhat different.

Proposition 2.1: *Let F be a Banach space and K be a compact subset of F. There are a vector subspace G of F and a norm $|||\cdot|||$ on G such that the following hold:*

(i) *K is contained in G as a compact subset for $|||\cdot|||$.*

(ii) *$(G, |||\cdot|||)$ is a reflexive Banach space.*

(iii) *The inclusion map from the Banach space G to the Banach space F is compact.*

(iv) *If F is an ordered Banach space with closed generating cone, then G is an ordered Banach space with closed generating cone for the order relation induced by the order on F.*

(v) *Suppose that F is a Banach lattice. Then G is a Riesz subspace of F and the map $x \mapsto |x|$ on G is continuous for $|||\cdot|||$; if in addition, F is σ-order complete, then G has a principal latticial extension.*

Proof: For an arbitrary Banach space F, the assertions (i)—(iii) are consequences of the mentioned lemma in [4]. We are precisely interested in the additional properties (iv) and (v).

Let us therefore begin with the case when F is an ordered Banach space with closed generating cone. By [9: Prop. 1.e.2], we find a sequence $(x_n)_{n\geq 1} \subset F$ such that $K \subset \overline{\text{co}}\,(x_n)$ and $x_n \to 0$. By [12: Ch. II, Cor. 1.14], one can write $x_n = x_n' - x_n''$ with $x_n', x_n'' \in F_+$ and $x_n', x_n'' \to 0$. Let $(y_n)_{n\geq 1}$ be defined by $y_{2n-1} = x_n'$, $y_{2n} = x_n''$. As $y_n \to 0$, there is a sequence of integers $1 = n_1 < n_2 < \cdots$ such that $\sum_{k=1}^{\infty} 2^{2k} \sup_{n_k \leqq n < n_{k+1}} \|y_n\|^2 < \infty$. Let V_k be the vector subspace generated by $\{y_n \mid n_k \leqq n < n_{k+1}\}$. Define the norm p_k on V_k by $p_k(z) = \inf \{\|z'\| + \|z''\| \mid z', z'' \in V_k \cap F_+; z = z' - z''\}$; the set in the right side is nonvoid as $y_n \geqq 0$. We have

$$\|z\| \leqq p_k(z), \quad z \in V_k; \quad \|z\| = p_k(z), \quad z \in V_k \cap F_+. \tag{2.1}$$

Let V be the Banach space of all sequences $\zeta = (z_k)_{k\geq 1}$ such that $z_k \in V_k$ and $\|\zeta\|_V = \left(\sum_{k=1}^{\infty} 2^{2k} p_k(z_k)^2\right)^{1/2} < \infty$. V is a reflexive Banach space, as being an l_2-sum of finite-dimensional Banach spaces. Consider the operator $T: V \to F$ given by $T(\zeta) = \sum_{k=1}^{\infty} z_k$. T is approximable: indeed, if $T_n: V \to F$ is the finite-rank operator given by $T_n(\zeta) = \sum_{k=1}^{n} z_k$, then, taking into account (2.1), we have

$$\begin{aligned}\|(T - T_n)(\zeta)\| &\leqq \sum_{k=n+1}^{\infty} \|z_k\| = \sum_{k=n+1}^{\infty} 2^{-k}2^k \|z_k\| \\ &\leqq \left(\sum_{k=n+1}^{\infty} 2^{-2k}\right)^{1/2} \left(\sum_{k=n+1}^{\infty} 2^{2k} \|z_k\|^2\right)^{1/2} \\ &\leqq \left(\sum_{k=n+1}^{\infty} 2^{-2k}\right)^{1/2} \left(\sum_{k=n+1}^{\infty} 2^{2k} p_k(z_k)^2\right)^{1/2} \\ &\leqq \left(\sum_{k=n+1}^{\infty} 2^{-2k}\right)^{1/2} \|\zeta\|_V.\end{aligned}$$

The above calculation shows in particular that T is well defined. Let $V \xrightarrow{P} V/\mathrm{Ker}\, T \xrightarrow{\tilde{T}} F$ be the canonical factorization of T (the norm on $V/\mathrm{Ker}\, T$ being the quotient norm). Put $G = T(V)$ and define $|||\cdot|||$ by $|||z||| = \|\tilde{T}^{-1}(z)\|$. As $\tilde{T}$ is compact, the assertion (iii) in the statement of the proposition is proved. The assertion (ii) is also verified, as G is isometrically isomorphic to a quotient of a reflexive Banach space. For the proof of (i), remark that because of $|||y_n||| \leqq 2^k p_k(y_n) = 2^k \|y_n\|$ ($n_k \leqq n < n_{k+1}$), the closed absolutely convex hull of $\{2y_n \mid n \geqq 1\}$, taken in $(G, |||\cdot|||)$, is compact. Moreover, it is closed in F and contains, therefore, the set K. This proves that K is a compact subset of $(G, |||\cdot|||)$.

It remains to prove (iv). Clearly, $G \cap F_+$ is a convex cone in G closed for $|||\cdot|||$. To see that it generates G, let $z \in G$ be given. There is $\zeta = (z_k)_{k\geqq 1} \in V$ such that $z = T(\zeta)$. By the definition of p_k, there are $z_k', z_k'' \in V_k \cap F_+$ such that $\max\left(p_k(z_k'), p_k(z_k'')\right) \leqq 2p_k(z_k)$ and $z_k = z_k' - z_k''$. By letting $\zeta' = (z_k')_{k\geqq 1}$, $\zeta'' = (z_k'')_{k\geqq 1}$, one obtains that $\zeta', \zeta'' \in V$, $z' = T(\zeta') \in G \cap F_+$, $z'' = T(\zeta'') \in G \cap F_+$ and $z = z' - z''$.

Before undertaking the case of a Banach lattice F, we pause in order to establish some notations and to prove a lemma. Given a Banach lattice F and $x \in F_+$, we denote by F_x the order ideal in F generated by x, and by C_x, the set of components of x, that is, the set $\{y \in F \mid y \wedge (x - y) = 0\}$. C_x is a Boolean algebra with respect to the operations $\wedge$ and $\vee$. Given a Boolean subalgebra Σ of C_x, we shall denote by $\mathrm{Sp}(\Sigma)$ the vector subspace of F generated by Σ.

Lemma 2.1: *Let F be a σ-order complete Banach lattice and let K be a compact subset of F. There are $x \in F_+$ and a sequence $(x_n) \subset \mathrm{Sp}(C_x)$ such that $x_n \to 0$ and $K \subset \overline{\mathrm{co}}(x_n)$.*

Proof: We shall establish first the following assertion:

(A) Let $x \in F_+$. Every $y \in F_x$ can be written as $\sum_{n=1}^{\infty} 2^{-n} u_n$ with $u_n \in \mathrm{Sp}(C_x)$ and $\|u_n\| \leqq 2^{-n+2} \|y\|$.

Indeed, Freudenthal's theorem shows that $\mathrm{Sp}(C_x)$ is dense in F_x. Consequently, one can define inductively $x_n \in \mathrm{Sp}(C_x)$ so that $\|y - x_1 - \cdots - x_n\| \leqq 2^{-2n} \|y\|$ and $\|x_n\| \leqq 2^{-2(n-1)} \|y\|$. The decomposition we look for is obtained by taking $2^n x_n$ as u_n.

We prove now the lemma. By [9: Prop. 1.e.2], we find a sequence $(y_n)_{n\geqq 1} \subset F$ such that $y_n \to 0$ and $K \subset \overline{\mathrm{co}}(y_n)$. Let $x = \sum_{n=1}^{\infty} 2^{-n} |y_n|$. By (A) one can write $y_n = \sum_{m=1}^{\infty} 2^{-m} x_{nm}$ with $x_{nm} \in \mathrm{Sp}(C_x)$ and $\|x_{nm}\| \leqq 2^{-m+2} \|y_n\|$. We have $K \subset \overline{\mathrm{co}}(y_n) \subset \overline{\mathrm{co}}(x_{nm})_{n,m\geqq 1}$. On the other side, the double sequence (x_{nm}) has the property that for every $\varepsilon > 0$, the set $\{(n, m) \mid \|x_{nm}\| > \varepsilon\}$ is finite. Consequently, one can rearrange the x_{nm}'s as a sequence (x_n) converging to 0 ∎

We return now to the proof of Proposition 2.1 by considering the case of a Banach lattice F. Suppose first that F is σ-order complete. By Lemma 2.1, there are $x \in F_+$ and $(x_n)_{n\geq 1} \subset \mathrm{Sp}\,(C_x)$ so that $K \subseteq \overline{\mathrm{co}}\,(x_n)$ and $x_n \to 0$. We can find a sequence of integers $1 = n_1 < n_2 < \cdots$ such that $\sum_{k=1}^{\infty} 2^{2k} \sup_{n_k \leq n < n_{k+1}} \|x_n\|^2 < \infty$ and an increasing sequence $(\Sigma_k)_{k\geq 1}$ of finite Boolean subalgebras of C_x such that $x_n \in \mathrm{Sp}\,(\Sigma_k)$ for $n_k \leq n < n_{k+1}$. Let V be the Banach space of all sequences $\zeta = (z_k)_{k\geq 1}$ such that $z_k \in \mathrm{Sp}\,(\Sigma_k)$ and $\|\zeta\|_V = \left(\sum_{k=1}^{\infty} 2^{2k} \|z_k\|^2\right)^{1/2} < \infty$. Consider the operator $T\colon V \to F$ given by $T(\zeta) = \sum_{k=1}^{\infty} z_k$ and let $V \xrightarrow{P} V/\mathrm{Ker}\,T \xrightarrow{\tilde{T}} F$ be the canonical factorization of T. Put $G = T(V)$ and define $|||\cdot|||$ by $|||z||| = \|\tilde{T}^{-1}(z)\|$. As in the first part of the proof one verifies that (i) – (iii) hold. It remains to prove (v). To this purpose, call a sequence $\zeta = (z_k)_{k\geq 0}$ admissible for $z \in F$ if $z_0 = 0$, $z_k \in \mathrm{Sp}\,(\Sigma_k)$ for $k \geq 1$, $z_k \to z$ and $\lambda_\zeta = \left(\sum_{k=1}^{\infty} 2^{2k} \|z_k - z_{k-1}\|^2\right)^{1/2} < \infty$. It is easily seen that $z \in G$ if and only if there is a sequence admissible for z; in this case, $|||z|||$ equals the infimum of the numbers λ_ζ, where ζ runs over all sequences which are admissible for z. It follows immediately that G is a Riesz subspace of F: indeed, if $z \in G$ and if (z_k) is an admissible sequence for z, then $(|z_k|)$ is an admissible sequence for $|z|$ (remark that the $\mathrm{Sp}\,(\Sigma_k)$'s are Riesz subspaces). Let us prove that the map $z \mapsto |z|$ is continuous for $|||\cdot|||$. So let $z \in G$ be given and let (z_k) be an admissible sequence for z. Given $\varepsilon > 0$, choose k_ε so that

$$\sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \|z_k - z_{k-1}\|^2 < \varepsilon \tag{2.2}$$

and put $M = 2^{2k_\varepsilon}\left(\sum_{k=1}^{k_\varepsilon} 2^{-2k}\right)$. It suffices to prove that $|||\,|z + y| - |z|\,|||^2 < 11\varepsilon$ whenever $|||y|||^2 < M^{-1}\varepsilon$. So let $y \in G$ verify $|||y|||^2 < M^{-1}\varepsilon$ and let (y_k) be an admissible sequence for y such that

$$\sum_{k=1}^{\infty} 2^{2k} \|y_k - y_{k-1}\|^2 < M^{-1}\varepsilon. \tag{2.3}$$

We have

$$\|y_{k_\varepsilon}\| \leq \sum_{k=1}^{k_\varepsilon} \|y_k - y_{k-1}\| \leq \left(\sum_{k=1}^{k_\varepsilon} 2^{-2k}\right)^{1/2} \left(\sum_{k=1}^{k_\varepsilon} 2^{2k} \|y_k - y_{k-1}\|^2\right)^{1/2} \tag{2.4}$$

which implies $2^{2k_\varepsilon}\|y_{k_\varepsilon}\|^2 < \varepsilon$. The sequence $(y_k')_{k\geq 0}$ given by $y_k' = 0$ for $k < k_\varepsilon$, $y_k' = y_k$ for $k \geq k_\varepsilon$, is still admissible for y; the sequence $(|z_k + y_k'| - |z_k|)_{k\geq 0}$ is admissible for $|z + y| - |z|$. Consequently,

$$|||\,|z + y| - |z|\,|||^2 \leq \sum_{k=1}^{\infty} 2^{2k} \|\,|z_k + y_k'| - |z_k| - |z_{k-1} + y_{k-1}'| + |z_{k-1}|\,\|^2.$$

Divide the sum into two parts, namely up to k_ε and from $k_\varepsilon + 1$. The first equals

$$2^{2k_\varepsilon} \|\,|z_{k_\varepsilon} + y_{k_\varepsilon}| - |z_{k_\varepsilon}|\,\|^2 \leq 2^{2k_\varepsilon} \|y_{k_\varepsilon}\|^2 < \varepsilon.$$

The second is dominated by

$$2 \sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \| |z_k + y_k'| - |z_{k-1} + y_{k-1}'| \|^2 + 2 \sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \| |z_k| - |z_{k-1}| \|^2.$$

By virtue of (2.2), the second sum in the above expression is dominated by ε, while the first sum is dominated by

$$\sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \|z_k - z_{k-1} + y_k' - y_{k-1}'\|^2$$

$$\leqq 2 \sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \|z_k - z_{k-1}\|^2 + 2 \sum_{k=k_\varepsilon+1}^{\infty} 2^{2k} \|y_k - y_{k-1}\|^2 \leqq 2\varepsilon + 2\varepsilon = 4\varepsilon,$$

by virtue of (2.2) and (2.3). Finally, $||| \, |z + y| - |z| \, |||^2 < 11\varepsilon$.

The last thing to do is to construct a principal latticial extension for G. According to Definition 2.3, we take $\tilde{G} = F$ and $G_0 = \bigcup_{k=1}^{\infty} \operatorname{Sp}(\Sigma_k)$. Given $x \in G_0$, define $U: F \to F$ by $U(z) = \sup_n (z \wedge nx_+)$ for $z \in F_+$. Then $0 \leqq U \leqq I$, $U(x) \geqq 0$, $(I - U)(x_+) = 0$. It remains to prove that $U(G) \subset G$ and that the restriction of U to G is continuous for $|||\cdot|||$. Let k_0 be that integer for which $x \in \operatorname{Sp}(\Sigma_{k_0})$, let $z \in G$ and let (z_k) be an admissible sequence for z such that $\sum_{k=1}^{\infty} 2^{2k} \|z_k - z_{k-1}\|^2 \leqq 2\,|||z|||^2$. The sequence $(w_k)_{k \geqq 0}$ given by $w_k = 0$ for $k < k_0$, $w_k = U(z_k)$ for $k \geqq k_0$, is admissible for $U(z)$. Indeed, on one side we have $w_k \to U(z)$ and $w_k \in \operatorname{Sp}(\Sigma_k)$ for $k \geqq 1$, as $U(\operatorname{Sp}(\Sigma_k)) \subset \operatorname{Sp}(\Sigma_k)$ whenever $k \geqq k_0$. On the other side,

$$\sum_{k=1}^{\infty} 2^{2k} \|w_k - w_{k-1}\|^2 = 2^{2k_0} \|U(z_{k_0})\|^2 + \sum_{k=k_0+1}^{\infty} 2^{2k} \|U(z_k) - U(z_{k-1})\|^2$$

$$\leqq 2^{2k_0} \|z_{k_0}\|^2 + \sum_{k=k_0+1}^{\infty} 2^{2k} \|z_k - z_{k-1}\|^2$$

$$\leqq 2^{2k_0} \|z_{k_0}\|^2 + 2\,|||z|||^2.$$

A calculation similar to that which was done in (2.4) yields $2^{2k_0} \|z_{k_0}\|^2 \leqq M\,|||z|||^2$ with M depending only on k_0. Hence, $U(z) \in G$ and $|||U(z)||| \leqq (M + 2)^{1/2}\,|||z|||$.

In the situation when F is not σ-order complete, one applies the preceding construction to F'' and to its compact subset $J_F(K)$; one obtains the Riesz subspace G_1 of F'' endowed with the norm $|||\cdot|||_1$. Then one takes $J_F^{-1}(G_1)$ as G and one defines $|||\cdot|||$ by $|||z||| = |||J_F(z)|||_1$ ∎

We are now in position to state the main result in this section.

Theorem 2.1: *Let E, F be Banach spaces and let $U: E \to F$ be a compact operator. Then U factors according the scheme $E \xrightarrow{U_1} G \xrightarrow{U_2} F$ where G is a reflexive Banach space; the factors U_1, U_2 are compact and U_2 is one-to-one. Moreover, under additional assumptions on E, F and U, it follows that the space G and the factors U_1, U_2 can be taken to have the following additional properties:*

(i) *G is an ordered Banach space with closed generating cone and U_2 is positive provided that F is an ordered Banach space with closed generating cone.*

(ii) *G is a lattice-ordered Banach space with continuous modulus and U_2 is a Riesz homomorphism provided that F is a Banach lattice; if in addition, F is σ-order complete, then G can also be taken to have a principal lattical extension.*

In cases (i) *and* (ii) *above, U_1 can be taken positive if E is an ordered Banach space and U is positive.*

Proof: Apply Proposition 2.1 taking the closure of $\{U(x) \mid x \in E, \|x\| \leqq 1\}$ as K ∎

As it was already pointed out in Section 1, the factorization scheme proposed by Theorem 2.1 yields in general an order structure on G weaker than the structure of a Banach lattice. This is due to the condition imposed to U_2 to be a compact Riesz homomorphism. Indeed, it suffices to take any nondiscrete Banach lattice as F and any compact operator with dense range as U; the following proposition will show that the factorization space G cannot be a Banach lattice in this situation.

Proposition 2.2: *Let G, F be Banach lattices and let $U_2: G \to F$ be a compact Riesz homomorphism with dense range. Then F is discrete (that is, every nonzero order ideal in F contains a nonzero atomic element) and has order continuous norm.*

Proof: Let F_0 be the set of those $y \in F$ such that the order interval $[-|y|, |y|]$ is compact for the norm. By the well-known result of B. Walsh [17], it suffices to prove that $F_0 = F$.

F_0 is closed. Indeed, let $(y_n) \subset F_0$, $y_n \to y$. For $z \in [-|y|, |y|]$, let $z_n = \big(z \vee (-|y_n|)\big) \wedge |y_n|$. Then $z_n \in [-|y_n|, |y_n|]$ and $|z - z_n| \leqq |y - y_n|$; we easily infer from these relations that $[-|y|, |y|]$ is compact.

We have $U_2(G) \subset F_0$. To see this, let $x \in G$, let $I = [-|U_2(x)|, |U_2(x)|]$ and let $M = I \cap U_2(G)$. M is totally bounded. Indeed, if $y_n = U_2(x_n) \in M$, let x_n' be given by

$$x_n' = \big(x_n \vee (-|x|)\big) \wedge |x|. \tag{2.5}$$

Then $x_n' \in [-|x|, |x|]$ and $U_2(x_n') = y_n$. As U_2 is compact, the sequence (y_n) contains a convergent subsequence, the total boundedness of M being thus proved. M is dense in I: for if $y \in I$, there is by hypothesis $(x_n) \subset G$ such that $U_2(x_n) \to y$. Consider x_n' obtained from x_n via formula (2.5): we have $U_2(x_n') \in M$ and $U_2(x_n') \to y$. In conclusion, I is compact.

As F_0 is closed and $U_2(G)$ is dense in F, it follows that $F_0 = F$ ∎

T. Figiel [5] has proved that a Banach space F has not the approximation property if and only if there is a reflexive Banach space G and an operator $U: G \to F$ which is one-to-one, compact and nonapproximable. Our factorization theorem allows us to complete Figiel's statement in the situation when F is endowed with order structures.

Corollary 2.1: *Let F be a Banach space without the approximation property. There are a reflexive Banach space G and an operator $U: G \to F$ which is one-to-one, compact and nonapproximable. Moreover, under additional assumptions on F, it follows that G and U can be taken to have the following additional properties:*

(i) *G is an ordered Banach space with closed generating cone and U is positive provided that F is an ordered Banach space with closed generating cone.*

(ii) *G is a lattice-ordered Banach space with continuous modulus and U is a Riesz homomorphism provided that F is a Banach lattice; if in addition, F has order continuous norm, than G can also be taken to have the property that the convex cone $K_+(G, F)$ of all positive compact operators from G to F be a face of the convex cone of all positive operators from G to F, while the convex cone of all positive approximable operators from G to F be a (proper) face of $K_+(G, F)$.*

Proof: By Figiel's theorem, there are a Banach space H and a compact nonapproximable operator $V: H \to F$. By Theorem 2.1, V factors according the scheme $H \xrightarrow{V_1} G \xrightarrow{U} F$ where G is reflexive and U is one-to-one, compact and nonapproximable, as V is nonapproximable. The additional properties of G and U are consequences of the additional properties described in Theorem 2.1; the last statement in (ii) above follows from the fact that G has a principal latticial extension, by taking into account the theorem from [16] reproduced at the beginning of this section ∎

We recall that there are Banach lattices with order continuous norm and without the approximation property. For instance, A. Szankowski's Banach lattice without the approximation property is even reflexive; see [10].

3. The factorization of approximable operators and of regularly approximable operators

The paper [1] encloses the following lines:

"Finally, it should be mentioned that W. B. Johnson has pointed out to us that his techniques in [14] yield also the following result: *If $T: Z \to X$ is a compact operator and X is a Banach lattice with the approximation property, then T factors with compact factors (which can be taken positive if T is positive) through a reflexive Banach space with an unconditional basis.*" (The reference [14] is our reference [7].)

The above statement comprises two distinct parts:

a) the possibility of factoring every compact operator $U: E \to F$ (E a Banach space, F a Banach lattice with the approximation property) according the scheme

$$E \xrightarrow{U_1} G \xrightarrow{U_2} F \tag{3.1}$$

where G is a reflexive Banach space with an unconditional basis and the factors U_1, U_2 are compact;

b) the possibility of choosing both U_1 and U_2 to be positive in case U is positive (E also being a Banach lattice in this case).

The assertions a) and b) above are the starting point for our study in this section.

As concerns a), we shall give it a complete proof, replacing the hypothesis that F has the approximation property by the more general hypothesis that U is approximable. The proof is entirely elementary in case F is σ-order complete and a bit less elementary if F is not so, in which case an appeal to the local reflexivity principle is necessary. The classical form of that principle yields a factorization scheme (3.1) in which the factors have no positivity properties. However, K. D. Kürsten gave in the paper [8], which appeared in the same year as [1], an improved version of the local reflexivity principle which applies to the specific situation of ordered Banach spaces. By using this improved version we can prove in fact a stronger form of a): namely, any of the factors U_1; U_2 (but not both) can be choosen positive, irrespective to the fact whether U is positive or not.

As concerns b), it is not clear from [1] what is the order relation on G in (3.1) with respect to which the positivity of the factors is considered. In the present paper we shall be concerned only with the order relation canonically defined by an unconditional basis: namely, if $(e_n)_{n\geq 1}$ is an unconditional basis for G, call an element $\sum_{n=1}^{\infty} a_n e_n$ positive if $a_n \geq 0$ for every $n \geq 1$. It is well-known that such an order relation defines a structure of a Banach lattice on G (see for instance [9, 10]). The term "Banach lattice with an unconditional basis" will be exclusively employed to design a Banach lattice whose order relation is defined in the above indicated way. We shall see that, with respect to the lattice structure so defined, the positivity of U is not sufficient in order to ensure the possibility of choosing both factors U_1 and U_2 in (3.1) positive.

In fact, we shall find a necessary and sufficient condition on U under which both U_1 and U_2 can be taken to be differences of positive compact operators and we shall see that not every positive approximable operator U satisfies that condition.

Before stating our factorization theorems, let us recall some notions from the theory of operators between Banach lattices; for details, see [14].

Consider two Banach lattices E, F. An operator $U: E \to F$ is called *regular* if it can be written as a difference of positive operators. The vector space of all regular operators from E to F is a Banach space with respect to the regular norm $\|\cdot\|_r$ defined by $\|U\|_r = \inf \{\|V\| \mid V: E \to F, -V \leqq U \leqq V\}$. An operator $U: E \to F$ is called *regularly approximable* if it is regular and it belongs to the closure of the subspace of all finite-rank operators from E to F taken with respect to the regular norm. Every regularly approximable operator can be written as a difference of positive regularly approximable operators.

It is well known (see [11]) that the regularly approximable operators on a Hilbert lattice $L_2(\mu)$, with μ a σ-finite measure, are precisely those kernel operators defined by a kernel k with the property that the kernel $|k|$ defines a compact operator on $L_2(\mu)$.

The proofs of our factorization results rely on the lemma below. Recall that, given two Banach spaces E, F and a finite-rank operator $U: E \to F$, the finite nuclear norm $\nu_0(U)$ is defined as the infimum of $\sum_{i=1}^{n} \|x_i'\| \|y_i\|$ taken over all representations of U as $\sum_{i=1}^{n} x_i' \otimes y_i$ with $x_i' \in E'$, $y_i \in F$. The finite nuclear norm dominates the operator norm; it also dominates the regular norm, in case E and F are Banach lattices.

Lemma 3.1: *Let E be a Banach space, F be a Banach lattice, $V: E \to F$ be a finite-rank operator and let $\varepsilon > 0$. There are a finite-dimensional Riesz subspace L of F'' and operators $W: E \to L$, $P: L \to F$ with the following properties:*

(i) $P \geqq 0$ *and* $\|P\| \leqq 1 + \varepsilon$.

(ii) $\nu_0(J_F V - IW) \leqq \varepsilon$ (*I is the inclusion map*).

(iii) $\nu_0(V - PW) \leqq \varepsilon$.

Proof: By [14: Prop. III. 3.5], there are a finite-dimensional Riesz subspace L of F'' and an operator $W: E \to L$ such that $\nu_0(J_F V - IW) < 3^{-1}\varepsilon$, where $I: L \to F''$ denotes the inclusion map. The definition of ν_0 implies the existence of a finite-dimensional vector subspace M of F'' containing L and $J_F V(E)$ and having the property that, if we let $I_1: L \to M$ be the inclusion map and $V_1: E \to M$ be defined by $V_1(x) = J_F V(x)$, then

$$\nu_0(V_1 - I_1 W) < 3^{-1}\varepsilon. \tag{3.2}$$

L is generated by a set of mutually disjoint norm-one elements $z_1, \ldots, z_n \in F_+''$. Let $\delta = \min \left(2^{-1}, (n+1)^{-1}\varepsilon, (2n \|W\|)^{-1} \varepsilon\right)$. By K. D. Kürsten's version of the local reflexivity principle [8: Theorem 1], there is an operator $S: M \to F$ with the properties

$$\|S\| \leqq 1 + \delta, \tag{3.3}$$

$$\|S(z)_-\| \leqq \|z_-\| + \delta \|z\|, \quad z \in M, \tag{3.4}$$

$$S\big(J_F(y)\big) = y, \quad y \in J_F^{-1}(M). \tag{3.5}$$

Define $P: L \to F$ by $P = \sum_{i=1}^{n} z_i' \otimes S(z_i)_+$, where $(z_i')_{1 \leq i \leq n}$ denotes the basis in L' dual to $(z_i)_{1 \leq i \leq n}$. Obviously, $P \geqq 0$. To evaluate its norm, first remark that

$$\nu_0(P - SI_1) \leqq \sum_{i=1}^{n} \|z_i'\| \, \|S(z_i)_+ - S(z_i)\| = \sum_{i=1}^{n} \|z_i'\| \, \|S(z_i)_-\| \leqq n\delta \tag{3.6}$$

as $\|z_i'\| = 1$ and $\|S(z_i)_-\| \leqq \delta$ by (3.4). Consequently,

$$\|P\| \leqq \|SI_1\| + \|P - SI_1\| \leqq \|S\| + \nu_0(P - SI_1) \leqq 1 + \varepsilon$$

by (3.3) and (3.6). In order to evaluate $\nu_0(V - PW)$, write

$$\nu_0(V - PW) \leqq \nu_0(V - SV_1) + \nu_0(SV_1 - SI_1W) + \nu_0(SI_1W - PW).$$

The first term vanishes by (3.5). For the other two, we have: $\nu_0(SV_1 - SI_1W) \leqq \|S\| \, \nu_0(V_1 - I_1W) \leqq 2^{-1}\varepsilon$ by (3.2) and (3.3); $\nu_0(SI_1W - PW) \leqq \nu_0(SI_1 - P) \|W\| \leqq 2^{-1}\varepsilon$ by (3.6) ∎

Note that property (ii) in the above lemma entails $\|W\| \leqq \|V\| + \varepsilon$ and, in case E is a Banach lattice, $\|W\|_r \leqq \|V\|_r + \varepsilon$.

Theorem 3.1: *Let E be a Banach space and let F be a Banach lattice. For every $U: E \to F$ the following assertions are equivalent:*

(i) *U is approximable.*

(ii) *U factors according the scheme* (3.1) *where G is a reflexive Banach lattice with an unconditional basis and at least one of the factors U_1, U_2 is compact.*

(iii) *U factors according the scheme* (3.1) *where G is a reflexive Banach lattice with an unconditional basis, U_1 is approximable and U_2 is regularly approximable and positive.*

A similar statement is true for the situation when E is a Banach lattice and F is a Banach space, in which case the condition "U_2 is regularly approximable and positive" from (iii) *should be replaced by "U_1 is regularly approximable and positive".*

Proof: Clearly (iii) ⇒ (ii); (ii) ⇒ (i) is a consequence of the fact that G and G' have the approximation property. It remains to prove (i) ⇒ (iii). The hypothesis that U is approximable allows us to construct inductively, with the aid of Lemma 3.1, a sequence $(L_n)_{n \geq 1}$ of finite-dimensional Riesz subspaces of F'' and operators $W_n: E \to L_n$, $P_n: L_n \to F$ such that $P_n \geqq 0$, $\|P_n\| \leqq 2$, $\|W_n\| \leqq 2^{-2(n-1)} \|U\|$ and $\|U - P_1W_1 - \cdots - P_nW_n\| \leqq 2^{-2n} \|U\|$. Let G be the Banach lattice of all sequences $\zeta = (z_n)_{n \geq 1}$ such that $z_n \in L_n$ for $n \geqq 1$ and $\|\zeta\|_G = \left(\sum_{n=1}^{\infty} 2^{2n} \|z_n\|^2 \right)^{1/2} < \infty$. The factorization we look for with U_2 positive is obtained by defining U_1 and U_2 via the formulas $U_1(x) = (W_n(x))_{n \geq 1}$, $U_2(\zeta) = \sum_{n=1}^{\infty} P_n(z_n)$. Some calculations similar to those performed during the proof of Proposition 2.1 show that U_1 is approximable and U_2 is regularly approximable.

In the situation when E is a Banach lattice and F is a Banach space, first factor U according the scheme $E \xrightarrow{S_1} H_1 \xrightarrow{S_2} F$ where H_1 is a reflexive Banach space and S_1, S_2 are approximable; this can be done by using the method in the above part of the proof. Then factor S_1' according the scheme $H_1' \xrightarrow{T_1} H_2 \xrightarrow{T_2} E'$ where H_2 is a reflexive Banach lattice with an unconditional basis, T_1 is approximable and T_2 is regularly approximable and positive. Finally, let $G = H_2'$, $U_1 = T_2'J_E$ and $U_2 = S_2T_1'$ ∎

We remark that in the situation when F is σ-order complete, the construction of the L_n's can be done in F and no appeal to the local reflexivity principle is necessary.

Theorem 3.2: *Let E, F be Banach lattices. For every $U: E \to F$ the following assertions are equivalent:*

(i) *U is regularly approximable.*

(ii) *U factors according the scheme* (3.1) *where G is a reflexive Banach lattice with an unconditional basis, the factors U_1, U_2 are regular and at least one of them is a difference of positive compact operators.*

(iii) *U factors according the scheme* (3.1) *where G is a reflexive Banach lattice with an unconditional basis, the factors U_1, U_2 are regularly approximable and U_2 is positive.*

(iv) *Same as* (iii) *but with U_1 positive instead of U_2.*

Proof: Clearly (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Suppose that $U_1 = S - T$ with S, T positive and compact. Let (P_n) be the sequence of canonical finite-rank projections associated to the basis of G. As $S - P_nS \geq 0$ and $\|S - P_nS\| \to 0$, it follows that S is regularly approximable; similarly, T is regularly approximable, hence U_1 is so. U_2 being regular, it follows that U is regularly approximable. In case U_2 is a difference of positive compact operators, a similar reasoning applied to U_2' shows that U_2 is regularly approximable, hence U also is so.

(i) $\Rightarrow$ (iii): Repeat word by word the proof of (i) $\Rightarrow$ (iii) in Theorem 3.1 replacing the operator norm by the regular norm.

(i) $\Rightarrow$ (iv): This is deduced from (i) $\Rightarrow$ (iii) by an argument similar to the one employed in the proof of Theorem 3.1 ∎

We close the section with two comments.

First, there are positive approximable operators which are not regularly approximable. Indeed, D. H. Fremlin [6] has constructed a positive compact operator on $L_2(\lambda)$ (λ = the Lebesgue measure on [0, 1]) which is not a kernel operator. According to the result mentioned at the beginning of this section, such an operator cannot be regularly approximable.

Second, one cannot choose in general both factors in Theorem 3.2 positive, even for a positive regularly approximable operator U. To see this, consider any compact metrizable nondiscrete group together with its normalized Haar measure μ. Such a group always contains a closed subset M with void interior and such that $\mu(M) > 0$. Let χ_M be the characteristic function of M and let U be the operator on $L_2(\mu)$ defined by the convolution with χ_M. As U is a positive compact kernel operator, it is regularly approximable. On the other side, U cannot factor according the scheme $L_2(\mu) \xrightarrow{U_1} G \xrightarrow{U_2} L_2(\mu)$ with G a Banach lattice with an unconditional basis and both factors U_1, U_2 positive. Indeed, such a factorization would imply the existence of a nonzero rank-one operator S on $L_2(\mu)$ such that $0 \leqq S \leqq U$. Consequently, there would exist nonzero elements f, g in $L_2(\mu)_+$ such that $\chi_M(st^{-1}) \geqq f(s)\, g(t^{-1})$ $\mu \times \mu$-almost everywhere. As the map $(s, t) \mapsto (st, t)$ leaves $\mu \times \mu$ invariant, it follows that $\chi_M(s) \geqq f(st)\, g(t^{-1})$ $\mu \times \mu$-almost everywhere. Integrating with respect to t, one obtains $\chi_M(s) \geqq (f * g)(s)$ μ-almost everywhere, where $*$ denotes the operation of convolution. But $f * g$ is a positive continuous not identically zero function as being the convolution of two nonzero elements in $L_2(\mu)_+$; see [2]. We have thus arrived at a contradiction, as M is closed and has void interior.

REFERENCES

[1] Aliprantis, C. D., and O. Burkinshaw: Factoring compact and weakly compact operators through reflexive Banach lattices. Trans. Amer. Math. Soc. 283 (1984), 369–381.

[2] Bourbaki, N.: Intégration. Ch. 7: Mesure de Haar. Ch. 8: Convolution et représentations. Paris: Hermann 1963.

[3] Cristescu, R.: Ordered vector spaces and linear operators. Tunbridge Wells: Abacus Press 1976.

[4] DAVIS, W. J., FIGIEL, T., JOHNSON, W. B., and A. PELCZYNSKI: Factoring weakly compact operators. J. Funct. Anal. 17 (1974), 311–327.
[5] FIGIEL, T.: Factorization of compact operators and applications to the approximation property. Studia Math. 45 (1973), 191–210.
[6] FREMLIN, D. H.: A positive compact operator. Manuscripta Math. 15 (1975), 323–327.
[7] JOHNSON, W. B.: Factoring compact operators. Israel J. Math. 9 (1971), 337–345.
[8] KÜRSTEN, K. D.: Lokale Reflexivität und lokale Dualität von Ultraprodukten für halbgeordnete Banachräume. Z. Anal. Anw. 3 (1984), 245–265.
[9] LINDENSTRAUSS, J., and L. TZAFRIRI: Classical Banach spaces I. Berlin—Heidelberg—New York: Springer-Verlag 1977.
[10] LINDENSTRAUSS, J., and L. TZAFRIRI: Classical Banach spaces II. Berlin—Heidelberg—New York: Springer-Verlag 1979.
[11] NAGEL, R., and U. SCHLOTTERBECK: Zur Approximation kompakter Operatoren durch Operatoren endlichen Ranges. Archiv Math. 25 (1974), 514–515.
[12] PERESSINI, A. L.: Ordered topological vector spaces. New York—Evanston—London: Harper & Row 1967.
[13] POPOVICI, I. M.: Clase de spaţii liniare dirijate topologice. Thesis. Bucharest: University 1987.
[14] SCHWARZ, H. U.: Banach lattices and operators. Leipzig: B. G. Teubner Verlagsges. 1984.
[15] STEIN, E. M.: Singular integrals and differentiability properties of functions. Princeton, New Jersey: University Press 1970.
[16] VUZA, D.: Ideal properties of order bounded operators on ordered Banach spaces which are not Banach lattices. In: Advances in invariant subspaces and other results of Operator Theory. Proc. Conf. Timişoara/Herculane (Romania), June 4–14, 1984 (Ed.: R. G. Douglas et al.). Basel—Boston—Stuttgart: Birkhäuser Verlag 1986, p. 353–368.
[17] WALSH, B.: On characterizing Köthe sequence spaces as vector lattices. Math. Ann. 175 (1968), 253–256.

Manuskripteingang: 03. 05. 1988; in revidierter Fassung 16. 03. 1989

VERFASSER:

Dr. IOAN MIRCEA POPOVICI
High School Mircea cel Bătrîn
Str. Ştefan cel Mare 6
R-8700 Constantza

Dr. DAN TUDOR VUZA
Department of Mathematics
The National Institute for
Scientific and Technical Creation
Bd. Păcii 220
R-79622 Bucharest