Monotonicity Properties of Oscillatory Solutions of Second Order **Differential Equations**

E. MÜLLER-PFEIFFER

Es wird untersucht, wie sich Monotonieeigenschaften der Koeffizienten gewöhnlicher Diffe rentialgleichungen zweiter Ordnung auf oszillierende Lösungen u solcher Gleichungen übertragen. Zum Beispiel werden Aussagen gemacht über die Abstände der Nullstellen von u. u und der von u und u' untereinander.

Исследуется, как переносятся свойства монотонности коэффициентов обыкновенных дифференциальных уравнений второго порядка на осциллирующие решения и таких уравнений. Делаются, например, высказывания, о расстоянии нулей функций и, и и тех от и и и' между собой.

It is proved in what way monotonicity properties of the coefficients of ordinary second order differential equations are transmitted to oscillatory solutions u of such equations. For instance, there are statements on the distances of the zeros of u, u' , and u and u' mutually.

This paper generalizes the following theorem of P. HARTMAN and A. WINTNER [2]: Consider the equation

$$
-u'' + Q(x)u = 0, \quad x_1 \leq x \leq x_3, \quad Q \in C, \quad Q \leq 0,
$$

and let u be a solution with three consecutive zeros x_1, x_2, x_3 , and relative extrema at $x_1', x_1 < x_1' < x_2$, and $x_2', x_2 < x_2' < x_3$. Let A_j be the area bounded by the x-axis, the straight line $x = x_j'$, and the graph of u belonging to the interval $[x_j, x_j']$, $j = 1$, 2, and let A_i be the area bounded by the x-axis, the straight line $x = x_i$, and the graph of u belonging to the interval $[x_j, x_{j+1}], j = 1, 2$ (Fig. 1).

If Q is monotone decreasing on $[x_1, x_3]$; then A_i can be placed into A_i , by reflection at the straight line $x = x_j$, $j = 1, 2$, and A_2 can be placed into A_1 by rotation through 180° about the point x_2 and translation about $d = 2x_2 - x_1' - x_2'$ to the left placing. the abscissae of the extrema ordinates in coincidence. (After this rotation and translation of A_2 the new position of A_2 is denoted by (A_2) in Fig. 1.) Concerning the monotonicity of the quarter-waves A_j , A_j and the half-waves $A_j \cup A_j$ if Q is monotone decreasing or increasing compare also the papers of E. MAKAI [9] and I. BIHARI [1].

In the following we consider the non-selfadjoint differential equations (r, R, ϵ) $C[a, b]$

$$
-(Pu')' + Ru' + Qu = 0 \quad (0 < P \in C^{1}[a, b], 0 > Q \in C[a, b]),
$$

(1)

• 278 E. **MULLER-PFEIFFEB**

and

$$
-(pv')' + rv'^{n} + qv = 0 \quad (0 < p \in C^{1}[a, b], 0 > q \in C[a, b]). \tag{2}
$$

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E. MÜLLER-PFEIFFER
 $-(pv')' + rv'' + qv = 0 \quad (0 < p \in C¹[a, b], 0 > q \in C[a, b])$. (2)

an oscillatory's olution to (1) and throughout this paper denote the zeros of Let *u* be an oscillatory' solution to (1) and throughout this paper denote the zeros of *u* by x_1, x_2, \ldots , and the zeros of *u'* being greater than x_1 by x_1 ', x_2 ', ..., so that $0 \leq x_1 < x_1' < x_2 < x_2' < \ldots$ Define the areas A_j , A_j' of the quarter-waves of *u* as **278 E. MÜLLER-PEEIFFER**
 and
 $-(pv')' + rv' + qv = 0 \quad (0 < p \in C^1[a, b], 0 > q \in C[a, b])$. (2)

Let *u* be an oscillatory solution to (1) and throughout this paper denote the zeros of
 u by $x_1, x_2, ...$, and the zeros of *u'* being g

above. Further, let the following definitions hold throughout the paper.
i) $A_j \supseteq A_j'$ means that A_j' can be placed into A_j by reflection at the straight. $\lim_{x \to \infty} x = x$

ii) $A_i \supseteq A_{i+1}$ means that A_{i+1} can be placed into A_i' by rotation through 180° about the point x_{i+1} (Fig. 2; after the 180°-rotation the position of A_{i+1} is denoted by $[A_{j+1}].$

iii) $A_j \supseteq A_{j+1}$ means that A_{j+1} can be placed into A_j by reflection at the x-axis and translation about $x'_{j+1} - x'_j$ to the left.

iv) $A_j' \supseteq A_{j+1}'$ means that A_{j+1}' can be placed into A_j' by reflection at the x-axis and translation about $x'_{i+1} - x_i'$ to the left.

iii) $A_j \supseteq A_{j+1}$ means that A_{j+1} can be placed into A_j by reflection at the *x*-axis and
inslation about $x'_{j+1} - x'_{j}$ to the left.
iv) $A_j \supseteq A'_{j+1}$ means that A'_{j+1} can be placed into A_j by reflection ¹; v) $A_j \cup A_j' \supseteq A_{j+1} \cup A_{j+1}'$ means that $A_{j+1} \cup A_{j+1}'$ can be placed into $A_j \cup A_j'$ by reflection at the *x*-axis and translation about $x'_{j+1} - x_j'$ to the left.
The inclusions $A_j \supseteq A_j'$, $A_j \subseteq A_j'$, $A_j \subseteq A_j'$ iv) $A_j' \supseteq A_{j+1}'$ means that A_{j+1}' can be placed into A_j' by reflection at the x-a
and translation about $x'_{j+1} - x'_j$ to the left.
 \downarrow v) $A_j \cup A_j' \supseteq A_{j+1} \cup A_{j+1}'$ means that $A_{j+1} \cup A_{j+1}'$ can be placed int *positive in A_i* \equiv *A_i* \pm *p p p p p p p p p c f_j i x j t p c j p c j c j i y i p c j i y i p c j c j i y i p d j c i* $\equiv A_{j+1}$ means that A_{j+1} can be placed into A_j by reflection at the *x*-axis

blation about $x'_{j+1} - x'_j$ to the left.
 $y A_j' \equiv A_{j+1} \cup A_{j+1}'$ means that $A_{j+1} \cup A_{j+1}'$ can be placed into $A_j \cup A_j'$ by

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Comparison theorems

To compare oscillatory solutions of (1) and (2) these equations are to be transformed into Riccati differential equations. **Comparison theorems**
 i To compare oscillatory solutions of (1) and (2) these equations are to be transform

into Riccati differential equations.
 \hat{L} em ma 1: Let u be a solution to (1) which doesn't vanish on $(t_1$ *[±]***(10'** *+ P'R)y* -'Q *on (t1 , t2). '* (4) (1) and (2) t

1) which doe

the function
 l equation
 $P^{-1}R$) y –

calculation
 $\varphi \in C^1[t_1, t_2]$

Lemma 1: Let u be a solution to (1) which doesn't vanish on (t_1, t_2) , and choose any

$$
y=-\Phi P u^{-1} u'
$$

 i s a sol i

$$
y' = \Phi^{-1}P^{-1}y^2 + (\Phi^{-1}\Phi' + P^{-1}R)y - \Phi Q \quad on \quad (t_1, t_2).
$$
 (4)

This assertion follows by an-easy calculation. Analogously, if *v'is* a solution to (2) which doesn't vanish on (t_1, t_2) and $\varphi \in C^1[t_1, t_2]$ is any positive function, then *Zoon to the Riccati differentia*
 $y' = \Phi^{-1}P^{-1}y^2 + (\Phi^{-1}\Phi' + \Phi^{-1})y^2 + (\Phi^{-1}\Phi' - \Phi^{-1})y^2$
 Z = $-\varphi p v^{-1} v'$
 Z = $-\varphi p v^{-1} v'$
 Z = φ

$$
z = -\varphi p v^{-1} v' \tag{5}
$$

is a solution to

$$
z' = \varphi^{-1} p^{-1} z^2 + (\varphi^{-1} \varphi' + p^{-1} r) z - \varphi q \quad \text{on } (t_1, t_2).
$$
 (6)

1: Let u be a solution to (1) which doesn't vanish on (t_1, t_2) , and choose any

ction $\Phi \in C^1[t_1, t_2]$. Then the function
 $= -\Phi P u^{-1} u'$ (3)

to the Riccati differential equation
 $= \Phi^{-1}P^{-1}y^2 + (\Phi^{-1}\Phi' + P^{-1}R) y - \Phi Q$ on **•** This assertion follows by an easy calculation. Analogously, if v'is a solution to (2) which doesn't vanish on (t_1, t_2) and $\varphi \in C^1[t_1, t_2]$ is any positive function, then
 $z = -\varphi p v^{-1} v' \cdot$;

is a solution to
 z' **Theorem 1:** Let u be a solution to (1) with $u(a) = u'(c) = u(b) = 0, a < c < b$, $u > 0$ on (a, b) , and consider the solution v to (2) determined by $v(c) = u(c)$, $v'(c) = 0$.

Monotonicity Properties of Oscillatory Solutions 279.

Monotonicity Properties of Oscillatory Solutions
\nIf there exist a number
$$
\eta > 0
$$
 and a point $x_0 \in [a, b]$ such that
\n
$$
\eta P(x) \exp \left(-\int_{x_0}^{x} \bar{R}P^{-1} dt\right) \geq p(x) \exp \left(-\int_{x_0}^{x} r p^{-1} dt\right), \quad x \in [a, b],
$$
\n(7)

exist a number
$$
\eta > 0
$$
 and a point $x_0 \in [a, b]$ such that

\n
$$
\eta P(x) \exp\left(-\int_{x_0}^x \tilde{R}P^{-1} \, dt\right) \geq p(x) \exp\left(-\int_{x_0}^x r p^{-1} \, dt\right), \quad x \in [a, b], \quad (7)
$$
\n
$$
\eta Q(x) \exp\left(-\int_{x_0}^x R P^{-1} \, dt\right) \geq q(x) \exp\left(-\int_{x_0}^x r p^{-1} \, dt\right), \quad x \in [a, b], \quad (8)
$$

then there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ with $v(\alpha) = 0 = v(\beta)$ and $0 < v \leq u$ on *If in (7) and (8) the signs* \geq *are replaced by* \leq *, then* $v \geq u$ *-on [a, b]. If in (7) and* (8) *at least one of the signs* \geq *is replaced by* $>$, *then* $\alpha \in (a, c), \beta \in (c, b)$ *and* $0 < v$ **Example 1998**
 uhen there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ with $v(\alpha) = 0 = v(\beta)$ and $0 < v \leq u$ on
 (α, β) . If in (7) and (8) the signs \geq are replaced by \leq , then $v \geq u$ on $[a, b]$. If in (7)
 $x = u$ on $(\$ *nQ(x)* exp $\left(-\int_{x_1}^{x} RP^{-1} dt\right) \geq q(x) \exp\left(-\int_{x_2}^{x} rp^{-1} dt\right), \quad x \in \mathbb{R}$

then there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ with $v(\alpha) = 0 = v(\beta)$
 (α, β) . If in (7) and (8) the signs \geq are replaced by \leq , then $\eta Q(x) \exp \left\{ \begin{array}{cc} \hbox{nthen there exist points}\ (\alpha,\beta). \ \hbox{If in (7) and (8) at least one of}\ \alpha=u\ \hbox{on}\ (\alpha,c)\cup (c,\beta). \ \hbox{ing inequalities is str}\ \text{Proof: By setting}\ \Phi(x)=\eta\ \hbox{ex} \end{array} \right.$ $Q(x) \exp\left(-\int_{x_0}^{x} RP^{-1} dt\right) \geq q(x) \exp\left(-\int_{x_0}^{x} rp^{-1} dt\right), \quad x \in R.$
 xist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ with $v(\alpha) = 0 = v(\beta)$ *and* (3) the signs \geq are replaced by \leq , then $v \geq u$ *i* east one of the signs \geq α is α , α , α) α (c, β). If in (7) and (8) the signs \leq are valid and at least one of the aris-

Proof: By setting
\n
$$
\Phi(x) = \eta \exp\left(-\int_{x}^{x} RP^{-1} dt\right), \quad x \in [a, b],
$$
\n(9)
\nand
\n
$$
\varphi(x) = \exp\left(-\int_{x_0}^{x} rp^{-1} dt\right), \quad x \in [a, b],
$$
\n(10)
\nit follows from (4) and (6) that
\n
$$
y' = \Phi^{-1}P^{-1}y^2 - \Phi Q \quad \text{on} \quad (a, b), y(c) = 0,
$$
\n(11)
\nand
\n
$$
z' = \varphi^{-1}p^{-1}z^2 - \varphi q, \quad z(c) = 0,
$$
\n(12)
\nrespectively. (12) holds on every interval (t_1, t_2) , $c \in (t_1, t_2)$, where z exists. By (7)
\nand (8) we obtain $z' \geq y'$ everywhere in the strip $S_{(t_1, t_2)} = \{(x, y) \mid x \in (t_1, t_2)\}$,
\n $\varphi(x) = \frac{S_{(t_1, t_2)}(x, y)}{S_{(t_1, t_2)}} = \frac{S_{(t_1, t_2)}(x, y)}{S_{(t_1, t_2)}} = \frac{S_{(t_1, t_2)}}{S_{(t_1, t_2)}} = \frac{S_{(t_1, t_2)}}{S$

and

•

$$
\varphi(x) = \exp\left(-\int\limits_{x_0}^x r p^{-1} \, dt\right), \quad x \in [a, b], \tag{10}
$$

$$
y' = \Phi^{-1} P^{-1} y^2 - \Phi Q \quad \text{on} \quad (a, b), \, y(c) = 0, \qquad \qquad \longrightarrow \qquad (11)
$$

$$
z' = \varphi^{-1} p^{-1} z^2 - \varphi q, \quad z(c) = 0, \tag{12}
$$

respectively. (12) holds on every interval (t_1, t_2) , $c \in (t_1, t_2)$, where *z* exists. By (7) and (8) we obtain $z' \geq y'$ everywhere in the strip $S_{(t_1, t_1)} = \{(x, y) | x \in (t_1, t_2)\}$. and $y' = \Phi^{-1}P^{-1}y^2 - \Phi Q$ on $(a, b), y(c) = 0,$ (11)

and $z' = \varphi^{-1}p^{-1}z^2 - \varphi q$, $z(c) = 0$, (12)

respectively. (12) holds on every interval (t_1, t_2) , $c \in (t_1, t_2)$, where z exists. By (7)

and (8) we obtain $z' \geq y'$ ever $y \in (-\infty, \infty)$. Thus, concerning the solution to (11) and (12) it follows that $z \leq y$ on $(t_1, c]$ and $z \geq y$ on $[c, t_2)$ (cf. $[3, p: 91]$). Therefore, and by $y(x) \rightarrow -\infty$ as $x \downarrow a$, $y(x) \to +\infty$ as $x \uparrow b$ it follows that there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ such that $z(x) \to -\infty$ as $x \downarrow \alpha$, $z(x) \to +\infty$ as $x \uparrow \beta$. Hence, we have $v(\alpha) = 0 = v(\beta)$. $y \in (-\infty, \infty)$. Thus, concerning the solution to (11) and (12) it follows that $z \leq y$
on $(t_1, c]$ and $z \geq y$ on $[c, t_2)$ (cf. [3, p. 91]). Therefore, and by $y(x) \to -\infty$ as $x \downarrow a$,
 $y(x) \to +\infty$ as $x \uparrow b$ it follows that on $(t_1, c]$ and $z \geq y$ on $[c, t_2)$ (cf. $[3, p.91]$). Therefore, and by $y(x) \to -\infty$ as $x \downarrow a$,
 $y(x) \to +\infty$ as $x \uparrow b$ it follows that there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ such

that $z(x) \to -\infty$ as $x \downarrow \alpha$, z $y(x) \rightarrow +\infty$ as $x \uparrow b$ it follows that there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ such that $z(x) \rightarrow -\infty$ as $x \downarrow \alpha$, $z(x) \rightarrow +\infty$ as $x \uparrow \beta$. Hence, we have $v(\alpha) = 0 = v(\beta)$.
Further, $y' > 0$ holds on (α, β) because o > 0 on (α, c) and $v^{-1}v' \le u^{-1}u' < 0$ on (c, β) . Finally, integration leads to $v \le u$ on $[\alpha, \beta]$. If in (7) and (8) the signs \ge are replaced by the signs \le the assertion $v \ge u$ on $[a, b]$ can analogously be proved. $y(x) \rightarrow +\infty$ as $x \uparrow b$ it follows that there exist points $\alpha \in [a, c)$
that $z(x) \rightarrow -\infty$ as $x \downarrow \alpha$, $z(x) \rightarrow +\infty$ as $x \uparrow \beta$. Hence, we have
Further, $y' > 0$ holds on (α, β) because of (11) and $Q < 0$. This im
 (α, c) an as $x \uparrow \beta$. Hence, we have $v(x) = 0 = v(\beta)$.

of (11) and $Q < 0$. This implies $z \leq y < 0$ on

(3), (5), (7), and (8), we obtain $v^{-1}v' \geq u^{-1}u'$

(c, β). Finally, integration leads to $v \leq u$ on

replaced by the signs

Assume now that in (7) or (8) at least one of the signs \geq is replaced by \geq . Then, in 'view of (11) and (12), it follows that $z' > y'$ everywhere in the strip $S_{(i_1,i_2)}$ whenever $(x,y) + (c, y)$. Thus, we obtain $z < y < 0$ on (x, c) , $0 < y < z$ on (c, β) , and, consebuently, $0 < v < u$ on $(\alpha, c) \cup (c, \beta)$. To prove that $\beta \in (c, b)$, for instance, choose Assume now that in (7) or (8) at least one of the signs \geq is replaced by $>$. Then, in view of (11) and (12), it follows that $z' > y'$ everywhere in the strip $S_{(i_1, i_1)}$ whenever $(x, y) + (c, y)$. Thus, we obtain $z < y <$ $\begin{aligned}\n\mathbf{a} \times \mathbf{b} &= \mathbf{a} \times \mathbf{b} \\
\mathbf{b} \times \mathbf{b} &= \mathbf{b} \times \mathbf{c} \\
\mathbf{c} \times \mathbf{c} &= \mathbf{b} \times \mathbf{c} \\
\mathbf{d} \times \mathbf{c} &= \mathbf{b} \times \mathbf{c} \\
\mathbf{d} \times \mathbf{c} &= \mathbf{b} \times \mathbf{c} \\
\mathbf{d} \times \mathbf{c} &= \mathbf{b} \times \mathbf{c} \\
\mathbf{c} \times \mathbf{c} &$

$$
w' + 2y\Phi^{-1}P^{-1}w + \Phi^{-1}P^{-1} = 0 \quad \text{on } (a, b)
$$

$$
(13
$$

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E. MÜLLER-PFEIFFER

$$
\begin{aligned}\n\text{(cf. [3, 10]). Hence} \\
w(x) &= \exp\left(-2\int_{x_1}^x y\Phi^{-1}P^{-1} \, dt\right) \left(w(x_1) - \int_{x_1}^x \Phi^{-1}P^{-1} \exp\left(2\int_{x_1}^t y\Phi^{-1}P^{-1} \, dx\right) \, dt\right) \\
&\quad \text{(if } x_1 \text{ and } y_1 \text{ is } y_1 \text{ and } y_2 \text{ is } y_1 \text{ and } y_1 \text{ is } y_1 \text{ and } y_2 \text{ is } y_1 \text{ and } y_2 \text{ is } y_1 \text{ and } y_1 \text{ is } y_1 \text{ and } y_2 \text{ is } y_1 \text
$$

It follows from (14)'and

$$
\lim_{x \uparrow b} \int_{x_1}^{\infty} \varphi^{-1} P^{-1} u^{-2} dt = \infty
$$

that there exists a point $\xi_1 \in (x_1, b)$ such that $w(x) \to 0$ as $x \uparrow \xi_1$. Hence, we have $y_1(x) \to \infty$ as $x \uparrow \xi_1$. Since $z' > y_1'$ in $S_{(t_1,t_1)}$, it follows that $z > y_1$ on (x_1, t_2) . Therefore, there exists a point $\beta \in (c, b)$ such that $z(x) \to \infty$ as $x \uparrow \beta$. This proves that $v(\beta) = 0, \beta \in (c, b)$. The assertion $\alpha \in (a, c)$ can analogously be proved

In the selfadjoint case $R \equiv r \equiv 0$ concerning the location of the zero β of v Theorem 1 is essentially a result by LEIGHTON [5].

Theorem 2: Let u be a solution to (1) with $u(a) = u'(b) = 0$, $u' > 0$ on [a, b) and consider the solution v to (2) determined by $v(a) = 0$, $v'(a) = u'(a) > 0$. If there exist a number $\eta > 0$ and a point $x_0 \in [a, b]$ such that (7) and (8) are fulfilled, then there exists a point $\beta \in (a, b]$ with $v'(\beta) = 0$ and $v' > 0$ on $[a, \beta)$. If in (7) and (8) the signs \geq are replaced by \leq , then $v' > 0$ on [a, b). If (7) and (8) hold and one of these inequalities is strict, then $\beta \in (a, b)$.

Proof: Use the functions (9), (10), and, consequently, the equations (11) and (12). It follows from the hypotheses of the theorem that $y(x) \rightarrow -\infty$ and $z(x) \rightarrow -\infty$ as $x \downarrow a, y' > 0$ on $(a, b), y(b) = 0$, and $y < 0$ on (a, b) . Let the conditions (7) and (8) be fulfilled. We prove that $z \geq y$ on all intervals $(a, \beta'), \beta' \leq b$, where z exists. Assuming the contrary suppose that there exists a point $x_1 \in (a, \beta')$ with $z(x_1) < y(x_1)$. Consider the solution y_1 to (11) determined by $y_1(x_1) = z(x_1)$. The function $w = (y_1 - y)^{-1}$ is a solution to (13). w is given by (14) with $w(x_1) = (y_1(x_1))^{\frac{1}{2}} - y(x_1)^{-1} < 0$ and we have

$$
\lim_{x \downarrow a} \int_{x_1}^{x} \Phi^{-1} P^{-1} x^{-2} dt = -\infty.
$$

Hence, there exists a point $\xi_1 \in (a, x_1)$ such that $w(x) \to 0$ as $x \downarrow \xi_1$ and, consequently, $y_1(x) \to -\infty$ as $x \downarrow \xi_1$. Because of $y' \leq z'$ everywhere in the strip $S_{(a, \beta')} = \{(x, y)\}$ $x \in (a, \beta')$, $y \in (-\infty, \infty)$ we obtain $z \leq y_1$ on the left-hand side of x_1 . Hence, there exists a point $\xi_2 \in [\xi_1, x_1)$ with $z(x) \to -\infty$ as $x \downarrow \xi_2$. This, however, contradicts the fact that z exists on (a, β') . This proves $z \geq y$ on (a, β') . Since $y(b) = 0$, there exists a point $\beta \in (a, b]$ with $z(\beta) = 0$ and $-\infty < y \leq z < 0$ on (a, β) . Thus, we obtain $v'(\beta) = 0$ and $v' > 0$ on [a, β). If (7) and (8) hold, but one of these inequalities is strict, we obtain $y' < z'$ in $S_{(a,\beta)}$ and, consequently, $y < z$ on (a, β) . It now follows from $y' < z'$ on (a, β) that $\beta < b$.

If (7) and (8) with \leq in place of \geq are considered, we prove that $z \leq y$ on (a, b) . Assume the contrary and let $x_1 \in (a, b)$ be a point with $z(x_1) > y(x_1)$. The solution y_1 to (11) determined by $y_1(x_1) = z(x_1)$ is monotone increasing on $(a, x_1]$. The case $y_1(x)$ $\rightarrow -\infty$ as $x \downarrow a$ is impossible, because, by assuming this case, it would follow that there exists a point $\xi_1 \in (a, x_1)$ with $y(x) \to -\infty$ as $x \downarrow \xi_1$ as it is seen from above.

280

Monotonicity Properties of Oscillatory Solutions 281

nce, we have $y_1(x) \to \omega$ as $x \downarrow a$, $\omega \in (-\infty, y_1(x_1))$. Further, by the hypotheses Hence, we have $y_1(x) \to \omega$ as $x \downarrow a$, $\omega \in (-\infty, y_1(x_1))$. Further, by the hypotheses under consideration, it follows that $y' \geq z'$ in the strip $S_{(a,b)}$ which implies that Monotonicity Properties of Oscillatory Solutions 281

Hence, we have $y_1(x) \to \omega$ as $x \downarrow a$, $\omega \in (-\infty, y_1(x_1))$. Further, by the hypotheses

under consideration, it follows that $y' \geq z'$ in the strip $S_{(a,b)}$ which impli *z z z <i>z <i>z z w w as x z a w* \in $(-\infty, y_1(x_1))$. Further, by the hypotheses under consideration, it follows that $y' \ge z'$ in the strip $S_{(a, b)}$ which implies that $z \ge y_1 > \omega$ on $(a, x_1]$. This, Hence, we have $z \leq y$ on (a, b) . This proves that $v' > 0$ on $[a, b)$

In the selfadjoint case $R \equiv r \equiv 0$ and $\eta = 1$ Theorem 2 is due to LET UP 1 The next theorem can similarly be proved.

Theorem 3: Let u be a solution to (1) with $u(b) = u'(a) = 0$, $u' < 0$ on $(a, b]$, and *consider the solution v to (2)'determined by* $v(b) = 0$ *,* $v'(b) = u'(b) < 0$ *. If there exist a number* $\eta > 0$ and a point $x_0 \in [a, b]$ such that (7) and (8) are satisfied, then there *exists a point* $\alpha \in [a, b)$ with $v'(\alpha) = 0$ and $v' < 0$ on $(\alpha, b]$. If, additionally, one of The increase is strict, then α (1) with $u(b) = u'(a) = 0$, $u' < 0$ on $(a, b]$, and
consider the solution v to (2) determined by $v(b) = 0$, $v'(b) = u'(b) < 0$. If there exist a
number $\eta > 0$ and a point $x_0 \in [a, b]$ such that (7 \leq *, then v'* $<$ 0 *on* (a, b) *. y*₁(*y*⁻ *+ Q**y*₁(*y*⁻ *+ b*² *y***₁(***y***⁻** *+ x*² *y*₁(*y*⁻ *+y*¹(*y*⁻ *y*₁(*y*⁻) *+ y*¹(*y*⁻ *y*₂) *x y*¹(*y*-*x*) *y y i y*¹(*y*² *x*) *y i y*¹(*y*² $z \ge y_1 > \omega$ on $(a, x_1]$. Th

Hence, we have $z \le y$ on

In the selfadjoint case

The next theorem can sin

Theorem 3: Let u be a

consider the solution v to (

number $\eta > 0$ and a point

exists a point $\alpha \in [a, b)$ u

thes *zelfadjoint case* $R \equiv r \equiv 0$ *and* $\eta = 1$ *Theorem 2 is due to LETGHTON* [5, 6].
 Leonem can similarly be proved.
 Zeonem 3: Let u be a solution to (1) *with* $u(b) = u'(a) = 0$, $u' < 0$ *on* (a, b), and
 lhe solution v to In the selladjoint case $R \equiv r \equiv 0$ and $\eta = 1$ Theorem 2 is due to LEIGHTON [5, 6].

The next theorem can similarly be proved.

Theorem 3: Let u be a solution to (1) with $u(b) = u'(a) = 0$, $u' < 0$ on $(a, b]$, and

consider *au solution v to* (2) determined by $v(b) = 0$, $v'(b) = u'(b) < 0$. If there exist a
 > 0 and a point $x_0 \in [a, b]$ such that (7) and (8) are satisfied, then there ω *c* ω *(a, b)* with $v'(x) = 0$ and $v' < 0$ *on* $(x, b]$ *•and a point x*₀ \in [*a*
 exists a point $\alpha \in$ *[<i>a, b) with v'*(α)
 these inequalities is strict, then $\alpha \in$ (\leq , *then* $v' < 0$ *on* (*a, b*).
 By setting $P \equiv p \equiv \Phi \equiv \varphi \equiv 1$
 $y' = y^2 + Ry - Q$, $y =$

and
 g $\Phi = \varphi \equiv 1$ in (4) and (6) we obtain the Riccati equations
 $\Phi = \varphi \equiv 1$ in (4) and (6) we obtain the Riccati equations
 $-Q, y = -u^{-1}u'$, (15)
 $q, z = -v^{-1}v'$, (16)

wing theorem.
 nd v be solutions to
 $Qu = 0, (R, Q \in C[a$

$$
y'=y^2+Ry-Q, \quad y=-u^{-1}u', \qquad (15)
$$

$$
z' = z^2 + rz - q, \quad z = -v^{-1}v', \tag{16}
$$

Theorem *4: Let u and be solutions to*

which lead to the following theorem.
\nTheorem 4: Let
$$
u
$$
 and v be solutions to
\n
$$
-u'' + Ru' + Qu = 0, \quad (R, Q \in C[a, b], Q < 0)
$$
\nand
\n
$$
-v'' + rv' + qv = 0, \quad (r, q \in C[a, b], q < 0)
$$
\n
$$
with \quad u(a) = v(a) = u'(b) = 0, \quad u'(a) = v'(a) > 0, \quad u' > 0 \text{ on } [a, b). \quad If
$$
\n(18)

$$
-v'' + rv' + qv = 0, \quad (r, q \in C[a, b], q < 0)
$$
 (18)

with
$$
u(a) = v(a) = u'(b) = 0
$$
, $u'(a) = v'(a) > 0$, $u' > 0$ on [a, b). If

$$
0 \leq R, \quad q \leq Q \quad on \ [a, b],
$$

 \leq , then $v' < 0$ on (a, b)

By setting $P \equiv p \equiv$
 $y' = y^2 + Ry$

and
 $z' = z^2 + rz$

which lead to the follo

Theorem 4: Let u a
 $-u'' + Ru' +$

and
 $-v'' + rv' +$

with $u(a) = v(a) = u'(b)$
 $r \leq R, q \leq 0$

then there exists a point *y'* = *y*² + *Ry* - *Q*, *y* = $-u^{-1}u'$, (15)

z' = $z^2 + rz - q$, $z = -v^{-1}v'$, (16)

d to the following theorem.

em 4: Let *u* and *v* be solutions to
 $-u'' + Ru' + Qu = 0$, $(R, Q \in C[a, b], Q < 0)$ (17)
 $-v'' + rv' + qv = 0$, $(r, q \in C[a, b], q < 0$ *then there* $ev'' + rv' + qv = 0$, $(r, q \in C[a, b], q < 0)$
 then there $ev(a) = v'(b) = 0$, $u'(a) = v'(a) > 0$, $u' > 0$ on [a, b). If
 $r \leq R$, $q \leq Q$ on [a, b], (19)
 then there exists a point $\beta \in (a, b]$ with $v'(\beta) = 0$, $v' > 0$ on [a, *• (a, b) (a, b) iih v'(b)* = 0, *v'* > 0 *on* [*a, b)*, *and* $0 < v \le u$ on (a, β) . The point β is equal to b only if the equations (17) and (18) are identical. If $r \ge R$, $q \ge Q$ on [a, b], *r* and to the following theorem.
 $- u'' + Ru' + Qu = 0$, $(R, Q \in C[a, b], Q < 0)$. (17)
 $- v'' + rv' + qu = 0$, $(r, q \in C[a, b], q < 0)$. (17)
 $- v'' + rv' + qu = 0$, $(r, q \in C[a, b], q < 0)$. (18)
 $= v(a) = u'(b) = 0, u'(a) = v'(a) > 0, u' > 0$ on [a, b). If
 $r \leq R$, $q \leq$

then $v \ge u$ on [a, b], $v' > 0$ on [a, b), and $v'(b) = 0$ only if (17) and (18) are identical.

Proof: The function $y = -u^{-1}u'$ is defined on (a, b) . By the hypotheses on *u* it *follows that* $y(x) \rightarrow -\infty$ as $x \downarrow a$; $y(b) = 0$, and $y < 0$ on (a, b) . Let the hypothesis (19) be fulfilled. Assume that there doesn't exist a zero of v' on (a, b) . Then the function $z = -v^{-1}v'$ is negative on (a, b) . By (15), (16), and (19) it follows that $y' \leq z'$ everywhere in the half-strip $H_{(a,b)} = \{(x, y) \mid x \in (a, b), y \in (-\infty, 0)\}$. Thus, as in the proof of Theorem 2, we obtain $y \leq z$ on intervals (a, β') , $\beta' \in (a, b]$, where $-\infty < z \leq 0$. If the functions y and z are not identical, there exists a point $x_1 \in (a, b)$ (19) be fulfilled. Assume that there doesn't exist a zero of v' on (a, b) . Then the function $z = -v^{-1}v'$ is negative on (a, b) . By (15), (16), and (19) it follows that $y' \leq z'$ everywhere in the half-strip $H_{(a,b)} = \{(x, y)$ with $y(x_1) < z(x_1)$ (<0). The solution y_1 to (15) determined by the initial value $y_1(x_1) = z(x_1)$ must cross the x-axis because of $y(b) = 0$ and the uniqueness of solutions to (15). Now, it follows from $y' \leq z'$ in $H_{(a,b)}$ that $y_1 \leq z$ to the right of x_1 and for points (x, y_1) and (x, z) which are placed in $H_{(a,b)}$. Hence, the graph of *z* must also cross the x-axis. This, however, contradicts the assumption that *v'* does not vanish on (a, b) . Hence, we have $y = z$ on (a, b) . Thus, by (15), (16), and (19) it follows that tions to (15). Now, it follows from $y' \leq z \cdot \ln H_{(a,b)}$ that $y_1 \leq z$ to the right of x_1 and
for points (x, y_1) and (x, z) which are placed in $H_{(a,b)}$. Hence, the graph of z must also
cross the x-axis. This, however, *(a, b).* Let $\beta \in (a, b]$ be the first zero of *v'*. Then we have $y \le z < 0$ on (a, β) , which implies that $u^{-1}u' \ge v^{-1}v' > 0$ on (a, β) . By integration we obtain $0 < v \le u$ on (a, β) . cross the *x*-axis. This, however, contradicts the assumption that *v'* does not vanish on (a, b) . Hence, we have $y = z$ on (a, b) . Thus, by (15), (16), and (19) it follows that $Q \equiv q$ and $R \equiv r$, i.e. the equations (17) a

This proves the first part of the theorem. If (20) is supposed exchange the parts of the 282 E. MÜLLER-PFEIFFER
This proves the first part of the
equations (17) and (18) \blacksquare .
The different parts of the foll

The different parts of the following theorem can analogously be proved.

Theorem 5: *Consider the differential equations* (17) *and* (18) *on [a, b].*

i) Let u and v be solutions to, (17) and (18), respectively, with $u(a) = v(a) > 0$, This proves the first part of the theorem. If (20) is equations (17) and (18) \blacksquare
The different parts of the following theorem can
Theorem 5: *Consider the differential equations*
i) Let u and v be solutions to (17) an

E. MÜLLER-PFEIFFER
 rR is the first part of the theorem. If (20) is supposed exchange the parts of the
 fierent parts of the following theorem can analogously be proved.
 em 5: *Consider the differential equations then* $u' \geq R$, $q \leq Q < 0$ on $[a, b]$,
then $u' < 0$ on $[a, b]$, and there exists a point $\beta \in (a, b]$ such that $v(\beta) = 0$, $0 < v \leq u$
on $[a, \beta)$, and $v' < 0$ on $(a, \beta]$, where β is equal to b only if (17) and (18) are ide **282 b.** MÜLLER PFEIFFER
 on This proves the first part of the theorem. If (20) is supposed exchange the parts of the

equations (17) and (18) **a**
 on Equations 5: Consider the differential equations (17) and (18) o The different parts of the following theorem can analogously be proved.

Theorem 5: *Consider the differential equations* (17) and (18), a, b]

i) Let u and v be solutions to (17) and (18), respectively, with $u(a) = v(a)$

$$
r\leq R, \quad Q\leq q<0 \quad on\ [a,b],
$$

then $v \geq u$ on [a, b], where $v(b) = 0$ only if (17) and (18) are identical.

ii) Let u and x be solutions to (17) and (18), respectively, with $u(b) = v(b) > 0$ *,* $u'(b) = v'(b) = u(a) = 0$ *, and* $u > 0$ *on* (a, b) *. If* on $[a, \beta)$; and $v' < 0$ on (a, β) , where β is equal to b only if (17) an
 $r \leq R$, $Q \leq q < 0$ on $[a, b]$,

then $v \geq u$ on $[a, b]$, where $v(b) = 0$ only if (17) and (18) are ide

ii) Let u and v be solutions to (17), a

 $r \leq R$, $q \leq Q < 0$ on [a, b],

then $u' > 0$ *on* [a, b] and there exists a point $\alpha \in [a, b)$ such that $v(\alpha) = 0, 0 < v \leq u$ then $u' > 0$ on [a, b] and there exists a point $\alpha \in [a, b)$ such that $v(\alpha) = 0, 0 < v \le u$

on $(\alpha, b]$ and $v' > 0$ on $[\alpha, b)$, where α is equal to a only if (17) and (18) are identical. If
 $r \ge R$, $Q \le q < 0$ on [a, b],

t

$$
r\geq R, \quad Q\leq q<0 \quad on\ [a,b]\}
$$

iii) Let u and v be solutions to (17), and (18), respectively, with $u(b) = v(b) = u'(a)$

 $r \geq R$, $q \leq Q < 0$ on [a, b],

then there exists a point $\alpha \in [a, b)$ with $v'(\alpha) = 0$, $0 < v \leq u$ on $[\alpha, b)$ and $v' < 0$ on (α, b) , where α is equal to a only if (17) and (18) are identical. If

 $r \leq R$, $Q \leq q < 0$ on [a, b],

then $v > u$ *on* [a, b], $v' < 0$ *on* (a, b], and $v'(a) = 0$ *only if* (17) *and* (18) *are identical.*

Proof: i) The assertion $u' < 0$ on $(a, b]$, for instance, easily follows from (15): Since $y(a) = 0$, $y'(a) = -Q(a) > 0$, the function *y* is positive in a neighbourhood of *a*. Further, the graph of *y* cannot touch the *x*-axis at a point $x_0 \in (a, \bar{b})$ as can be proved as follows. Assume that x_0 is the smallest point to the right of *a* with $y(x_0) = 0$. Froof: i) The assertion $u' < 0$ on $(a, b]$, and $v(u) = 0$ only $i \in (1, 1)$ and (15) are dientical:
Proof: i) The assertion $u' < 0$ on $(a, b]$, for instance, easily follows from (15):
Since $y(a) = 0$, $y'(a) = -Q(a) > 0$, the fun compare the proof of Theorem 4 I (α , b], where α is equal to a only if (17) and (
 $r \leq R$, $Q \leq q < 0$ on [a , b],

then $v > u$ on [a , b], $v' < 0$ on (a , b], and v'

Proof: i) The assertion $u' < 0$ on (a , b]

Since $y(a) = 0$, $y'(a)$ then $v > u$ on $[a, b], v' < 0$ on $(a, b],$ and $v'(a) = 0$ only if (17) and (18) are identical.

Proof: i) The assertion $u' < 0$ on $(a, b]$, for instance, easily follows from (15):

Since $y(a) = 0$, $y'(a) = -Q(a) > 0$, the function y is

y > 0 on (a, x_0) , we have *y* $(x_0) \ge 0$, contradictory to *y* $(x_0) = -Q(x_0) > 0$.
y > 0 on (a, b) implies $u' < 0$ on (a, b) . To prove the other assertions of the theorem compare the proof of Theorem 4 \blacksquare
Of course Of course, the Theorems 4 and 5 can easily be applied to the selfadjoint equations

In the following the comparison theorems from above are used to study monotonicity properties of oscillatory solutions of second order differential equations implied by corresponding monotonicity behaviour of the coefficients of the differential proved as follows. Assume that x_0 is the smallest point to the right of a with $y(x_0)$

Since $y > 0$ on (a, b) , we have $y'(x_0) \le 0$, contradictory to $y'(x_0) = -Q(x_0)$
 $y > 0$ on (a, b) implies $u' < 0$ on (a, b) . To pro

Theorem 6: Let u be an oscillatory solution to (1) on $[0, \infty)$. Denote the zeros of $u \setminus$ *by* $x_1, x_2, ...,$ and the zeros of u' by $x_1', x_2', ...,$ so that $0 \le x_1 < x_1' < x_2 < x_2' < ...$ ¹ *If there exists* $c \in \mathbb{R}$ *such that the functions*

Monotonicity Properties of Oscillatory Solutions

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\nTheorem 6: Let
$$
u
$$
 be an oscillatory solution to (1) on $[0, \infty)$. Denote the zeros of u by x_1, x_2, \ldots , and the zeros of u' by x'_1, x'_2, \ldots , so that $0 \leq x_1 < x'_1 < x_2 < x'_2 < \ldots$.)

\nIf there exists $c \in \mathbb{R}$ such that the functions

\n
$$
P(x) \exp\left(cx - \int_0^x RP^{-1} dt\right), \quad Q(x) \exp\left(cx - \int_0^x RP^{-1} dt\right) \qquad (22)
$$
\nare monotone decreasing (increasing) on $[0, \infty)$, then, for $j \in \mathbb{N}$,

\n
$$
x'_j - x_j \geq x'_{j+1} - x_{j+1}, \quad x_{j+1} - x_j \geq x_{j+2} - x'_{j+1}
$$
\nand, consequently,
$$
x_{j+1} - x_j \geq x_{j+2} - x_{j+1}, \quad x'_{j+1} - x_j \geq x'_{j+2} - x'_{j+1}
$$
\nIf, additionally, one of the functions (22) is strictly monotone, the inequalities (23) and (24) are also strict.

\nProof: The function $\tilde{u}(x) = u(x'_{j+1} - x_j' + x)$ ($x \in [x_j, x_{j+1}]; j \in \mathbb{N}$) is a solution to the differential equation

\n
$$
- (P(x'_{j+1} - x_j' + x)\tilde{u}') + R(x'_{j+1} - x_j' + x) \tilde{u}' + Q(x'_{j+1} - x_j' + x) \tilde{u} = 0.
$$

\nAssume that the functions (22) are monotone decreasing. Then

 \hat{a} are monotone decreasing (increasing) on [0, ∞), then, for $j \in \mathbb{N}$,

$$
x_{j}^{\prime} - x_{j} \geq x_{j+1}^{\prime} - x_{j+1}, \quad x_{j+1} - x_{j}^{\prime} \geq x_{j+2} - x_{j+1}^{\prime}
$$
\n
$$
equently, \quad \ldots
$$
\n
$$
x_{j+1} - x_{j} \geq x_{j+2} - x_{j+1}, \quad x_{j+1}^{\prime} - x_{j}^{\prime} \geq x_{j+2}^{\prime} - x_{j+1}^{\prime}
$$
\n
$$
(24)
$$

and, consequently,

$$
x_{j+1}-x_j\underset{1\leq j}{\leq}x_{j+2}-x_{j+1},\quad x'_{j+1}-x'_j\underset{1\leq j}{\leq}x'_{j+2}-x'_{j+1}.
$$
 (24)

1/, additionally, one of the functions (22) *is strictly monotone, the inequalities* (23) *and -*

Proof: The function $\tilde{u}(x) = u(x'_{j+1} - x'_j + x)$ ($x \in [x_j, x_{j+1}]$; $j \in \mathbb{N}$) is a solution to the differential equation

$$
x_{j+1} - x_j \, (\underline{\xi}) \, x_{j+2} - x_{j+1}, \quad x_{j+1} - x_j \, (\underline{\xi}) \, x_{j+2} - x_{j+1}.
$$
\n(24)

\nIditionally, one of the functions (22) is strictly monotone, the inequalities (23) and

\nare also strict.

\nsof: The function $\tilde{u}(x) = u(x'_{j+1} - x'_j + x)$ $(x \in [x_j, x_{j+1}]; j \in \mathbb{N})$ is a solution

\n $- \left(P(x'_{j+1} - x'_j + x)\tilde{u}' \right)' + R(x'_{j+1} - x'_j + x) \, \tilde{u}' + Q(x'_{j+1} - x'_j + x) \, \tilde{u} = 0.$

\nthe that the functions (22) are monotone decreasing. Then

Assume that the functions (22) are monotone decreasing. Then

1P (x) evp *(cx* _f*RP' dt) (* ^ *11P(x - x1 '* +x)exp *(c(x; ^l - x* + *x) - f RP' dt)* ⁼*P(x, - x, + x)* ex *(cx _fR(x ,. - x7 + r) P ^I (x)fl —x,* **S ;.F1—X'** with j ⁼exp (c(' - *x1)+ f- RP:' dt).* Hence, **- .5** *Ix - ,P(x)* exp (*-f RP-' dt* **\ x** *P(x - x' + x)* exp *(-1* R(x ¹ • ¹ —x¹ ' + r) *P'(x 1 -* xj' *+ r) dr)* **Xj S - ⁴ .1)** Here and in the following a possible zero -z0 ' E CO, x1)of *u'* is disregarded. *1 ,- -* **^S**

'whe

- S

0

$$
\eta_j = \eta_j \exp \left(-\int\limits_0^{x_j'} R P^{-1} \, dt + \int\limits_0^{x_j'} R(x_{j+1}' - x_j' + \tau) \, P^{-1}(x_{j+1}' - x_j' + \tau) \, d\tau\right),
$$

•
• .5
• .5 S

 $\frac{1}{2}$ **here and in the following a** $\frac{1}{2}$ ¹) Here and in the following a possible zero $x_0' \in [0, x_1]$ of u' is disregarded.

S

1 - '• '

 $x \in [x_j, x_{j+1}], j \in \mathbb{N}$. Analogously,

284 E. MÜLER-PFEIFPER
\n
$$
x \in [x_j, x_{j+1}], j \in \mathbb{N}. \text{ Analogously,}
$$
\n
$$
\tilde{\eta}_j Q(x) \exp\left(-\int_{x_j}^x RP^{-1} dt\right)
$$
\n
$$
\geq Q(x'_{j+1} - x'_j + x) \exp\left(-\int_{x_j}^x R(x'_{j+1} - x'_j + \tau) P^{-1}(x'_{j+1} - x'_j + \tau) d\tau\right),
$$
\n
$$
x \in [x_j, x_{j+1}], j \in \mathbb{N}. \text{ To finish the first part of Theorem 6 set}
$$
\n
$$
P(x'_{j+1} - x'_j + x) = p(x), \quad R(x'_{j+1} - x'_j + x) = r(x),
$$
\n
$$
Q(x'_{j+1} - x'_j + x) = q(x), \quad v(x) = u(x'_j) \tilde{u}^{-1}(x'_j) \tilde{u}(x),
$$
\n
$$
\tilde{x}_j = c = x_0, \quad x_j = a, \quad x_{j+1} = b,
$$

 $x \in [x_j, x_{j+1}], j \in \mathbb{N}$. To finish the first part of Theorem 6 set

$$
P(x'_{j+1} - x'_{j} + x) = p(x), \quad R(x'_{j+1} - x'_{j} + x) = r(x),
$$

\n
$$
Q(x'_{j+1} - x'_{j} + x) = q(x), \quad v(x) = u(x'_{j}) \tilde{u}^{-1}(x'_{j}) \tilde{u}(x),
$$

\n
$$
\tilde{x}_{j} = c = x_{0}, \quad x_{j} = a, \quad x_{j+1} = b,
$$

and apply Theorem 1. The part of Theorem 6 described by the brackets can analogously, be proved. If, additionally, one of the functions (22) is strictly, monotone decreasing (increasing), the remaining part of Theorem 6 follows from the last part of $x \in [x_j, x_{j+1}], j \in \mathbb{N}$. To finish
 $x \in [x_j, x_{j+1}], j \in \mathbb{N}$. To finish
 $P(x'_{j+1} - x'_j + x) =$
 $\hat{Q}(x'_{j+1} - x'_j + x) =$
 $\hat{x}_j = c = x_0, \quad x_j =$

and apply Theorem 1. The p

gously be proved. If, addition

creasing (increasing), In the special case $P \equiv 1$, $R \equiv 0$, $c = 0$ and concerning the inequalities $x'_{j+1} - x_j$

In the special case $P \equiv 1$, $R \equiv 0$, $c = 0$ and concerning the inequalities $x'_{j+1} - x_j$
 $x'_{j+2} - x'_{j+1}$ Theorem 6 is due to A $x) = p(x), R(x'_{j+1} - x_j' + x) = r(x),$
 $x) = q(x), v(x) = u(x'_j) u^{-1}(x'_j) u(x),$
 $x_j = a, x_{j+1} = b,$

The part of Theorem 6 described by the brackets can analo-

ditionally, one of the functions (22) is strictly monotone de-

remaining part of Th

 $-x'_{i+1}$ Theorem 6 is due to A. LAFORGIA [4].

Theorem 7: Let u be an oscillatory solution to (1) on $[0, \infty)$. If the functions

(increasing), the remaining part of Theorem 6 follows from the last part of

\n1

\nspecial case
$$
P = 1
$$
, $R = 0$, $c = 0$ and concerning the inequalities $x'_{j+1} - x_j$ and $x'_{j+1} - x_j$.

\n x'_{j+1} Theorem 6 is due to A. LATOR (1) on [0, ∞). If the functions

\n $P(x) \exp\left(-\int_{0}^{x} R P^{-1} dt\right)$, $Q(x) \exp\left(-\int_{0}^{x} R P^{-1} dt\right)$.

\n(25)

are monotone decreasing (increasing) on $[0, \infty)$ *, then* $x_i' - x_i \geq x_{i+1} - x_i' \geq x_{i+1}'$ $P(x) \exp\left(-\int_{0}^{x} RP^{-1} dt\right), \quad Q(x) \exp\left(-\int_{0}^{x} RP^{-1} dt\right)$ (25)

are monotone decreasing (increasing) on $[0, \infty)$, then $x_j' - x_j \geq x_{j+1} - x_j' \geq x_{j+1}'$
 $-x_{j+1}$ and $A_j \geq A_j'$ ($j \in \mathbb{N}$). If, additionally, one of the functi *monotone, decreasing (increasing), then the asserted inequalities and inclusions are* Theorem 7: Let u be an oscillatory solution to (1) on $[0, \infty)$. If the functions
 $P(x) \exp \left(-\int_0^x RP^{-1} dt\right)$, $Q(x) \exp \left(-\int_0^x RP^{-1} dt\right)$ (25)

are monotone decreasing (increasing) on $[0, \infty)$, then $x_i' - x_i \geq x_{i+1} - x_i' \geq x_{$

Proof: We prove that $A_j \supseteq A_j'$ if the functions (25) are monotone decreasing. The function $\tilde{u}(x) = u(2x_j' - x)$, $x \in [x_j, x_j']$, is a solution to the differential equation

$$
- (P(2x_1' - x) \tilde{u}')' - R(2x_1' - x) \tilde{u}' + Q(2x_1' - x) \tilde{u} = 0
$$

monotone decreasing (increasing), then the asserted inequalities and inclusions ar

also strict.

Proof: We prove that $A_j \supseteq A'_j$ if the functions (25) are monotone decreasing. The

function $\tilde{u}(x) = u(2x'_j - x), x \in [x_j, x'_j$ and inclusions are
 $\begin{aligned} \text{the decreasing. The}\ \text{initial equation}\ \text{R} \ P^{-1} \ dt \end{aligned}$ is mono-
 $\begin{aligned} \text{R} \ P^{-1} \ dt \end{aligned}$

1
\n
$$
x_{i+1}
$$
 and $A_i \in A_i$ for *i* is a *i*, *i* is a *ii i i i i i i i i i i i i i i i i i i i i i i i i i i i ii ii i i i i ii ii i i i i ii ii i ii i i i ii ii i ii i ii ii*

 $\mathbf{V} = \begin{bmatrix} \mathbf{V} & \mathbf{V} & \mathbf{V} \\ \mathbf{V} & \mathbf{V} & \mathbf{V} \end{bmatrix}$

Monotonicity Properties of Oscillatory Solutions
\n
$$
x \in [x_i, x_i']
$$
. Analogously
\n
$$
Q(x) \exp \left(-\int_{x_i'}^{x} RP^{-1} dt\right) \geq Q(2x_i' - x) \exp \left(-\int_{x_i'}^{x} (-R(2x_i' - \tau)) P^{-1}(2x_i' - \tau) d\tau\right),
$$
\n $x \in [x_i, x_i']$. By setting
\n
$$
P(2x_i' - x) = p(x), \quad -R(2x_i' - x) = r(x), \quad Q(2x_i' \stackrel{\sim}{=} x) = q(x),
$$
\n
$$
\tilde{u}(x) = v(x), \quad x_i' = x_0 = c, \quad x_i = a, \quad \eta = 1,
$$
\nand applying Theorem 1 we obtain $A_i' \subseteq A_i$ and, consequently, $x_i' - x_i \geq x_{i+1} - x_i'$.
\nTo prove $x_{i+1} - x_i' \geq x_{i+1}' - x_{i+1}$ define the function $\tilde{u}(x) = -u(2x_{i+1} - x), x \in [x_i', x_{i+1}']$. This function is a solution to
\n
$$
-(P(2x_{i+1} - x) \tilde{u}') - R(2x_{i+1} - x) \tilde{u}' + Q(2x_{i+1} - x) \tilde{u} = 0
$$
\nwith $\tilde{u}(x_{i+1}) = -u(x_{i+1}) = 0, \tilde{u}'(x_{i+1}) = u'(x_{i+1})$. Now, conclude as above and apply Theorem 3. Thus, we obtain $x_{i+1} - x_i' \geq x_{i+1}' - x_{i+1}$. If, additionally, one of the functions (95).

$$
P(2x_i'-x)=p(x), -R(2x_i'-x)=r(x), Q(2x_i'-x)=q(x),
$$

$$
\tilde{u}(x) = v(x), \quad x_j' = x_0 = c, \quad x_j = a, \quad \eta = 1,
$$

 $P(2x_i' - x) = p(x), \quad -R(2x_i' - x) = r(x), \quad Q(2x_i' - x) = q(x),$
 $\tilde{u}(x) = v(x), \quad x_i' = x_0 = c, \quad x_i = a, \quad \eta = 1,$

and applying Theorem 1 we obtain $A_i' \subseteq A_i$ and, consequently, $x_i' - x_i \ge x_{i+1} - x_i'.$

To prove $x_{i+1} - x_i' \ge x_{i+1}' - x_{i+1}$ define th $\tilde{u}(x) = v(x), \quad x_j' = x_0 = c, \quad x_j = a, \quad \eta = 1,$

and applying Theorem 1 we obtain $A_j' \subseteq A_j$ and, consequently, $x_j' - x_j \ge x_{j+1} - x_j'$.

To prove $x_{j+1} - x_j' \ge x_{j+1}' - x_{j+1}$ define the function $\tilde{u}(x) = -u(2x_{j+1} - x), x \in [x_j', x_{j+1}]$ To prove $x_{j+1} - x_j \ge x_{j+1}' - x_{j+1}$ define the function $\tilde{u}(x) = -u(2x_{j+1} - x), x \in [x_j],$

$$
-(P(2x_{j+1}-x) \tilde{u}')' - R(2x_{j+1}-x) \tilde{u}' + Q(2x_{j+1}-x) \tilde{u} = 0
$$

with $\tilde{u}(x_{j+1}) = -u(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+1})$. Now, conclude as above and apply Theorem 3. Thus, we obtain $x_{j+1} - x_j' \ge x_{j+1}' - x_{j+1}$. If, additionally, one of the functions (25) is strictly monotone decreasing, by using the Theorems 1 and 3 the corresponding assertions can analogously be proved. The same holds in the case that the Monotonicity Properties of Oscillatory Solutions 225
 $z \in [z_1, z_1']$. Analogously
 $Q(z) \exp\left(-\int\limits_{z_1}^z RP^{-1} dt\right) \geq Q(2z_1' - z) \exp\left(-\int\limits_{z_1'}^z \left(-R(2z_1' - z)\right) P^{-1}(2z_1' - z) d\tau\right),$
 $z \in [z_1, z_1']$. By setting
 $P(2z_1' - x) = p(x), \$ Frove $x_{j+1} - x_j \leq x_{j+1} - x_{j+1}$ define the function $u(x) = -u(2x_{j+1} - x)$, x_{j+1} . This function is a solution to
 $-(P(2x_{j+1} - x) u')' - R(2x_{j+1} - x) u' + Q(2x_{j+1} - x) u = 0$

th $\tilde{u}(x_{j+1}) = -u(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+$ **p**lying Theorem 1 we obtain $A_i' \subseteq A$, and, consequently, $x_i' - x_i \ge x_{i+1} - x_i'$.

We $x_{i+1} - x_i' \ge x_{i+1}' - x_{i+1}$ define the function $\tilde{u}(x) = -u(2x_{i+1} - x), x \in [x_i',$

This function is a solution to
 $-\left(P(2x_{i+1} - x) \tilde{u}'\right)'$ $-(P(2x_{j+1} - x) u')' - R(2x_{j+1} - x) u' + Q(2x_{j+1} - x)$

with $\tilde{u}(x_{j+1}) = -u(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+1})$. Now, conclude

Theorem 3. Thus, we obtain $x_{j+1} - x_j' \ge x'_{j+1} - x_{j+1}$. If, additionis (25) is strictly monotone d $-(P(2x_{j+1}-x) \tilde{u}')' - R(2x_{j+1}-x) \tilde{u}' + Q(2x_{j+1}-x) \tilde{u}$

with $\tilde{u}(x_{j+1}) = -u(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+1})$. Now, conclude a

Theorem 3. Thus, we obtain $x_{j+1} - x_j' \ge x'_{j+1} - x_{j+1}$. If, additional

tions (25) is **monotone** decreasing, by using the Theorem
 monotone decreasing, by using the Theorem

sponding assertions can analogously be proved. The same holds,

functions (25) are monotone increasing \blacksquare

Theorem 8: Consider

$$
-u'' + Ru' + Qu = 0 \quad (R, Q \in C[0, \infty), Q < 0)
$$

and let u be an oscillatory solution.

i) If the coefficients are monotone decreasing (increasing) on $[0, \infty)$, then $x'_j - x_j$ and let u be an oscillatory solution.

(i) If the coefficients are monotone decreasing (increasing) on $[0, \infty)$, then $x'_i - x_i \leq x'_{i+1} - x_{i+1}$, $j \in \mathbb{N}$. If, additionally, one of these coefficients is strictly decrea

For the monotone increasing \mathbf{F} and \mathbf{F} and \mathbf{F} and \mathbf{F} and \mathbf{F} are the set that the metions (25) are monotone increasing \blacksquare

Theorem 8: Consider the differential equation
 $-\mathbf{u}'$, $\vdash Ru' + Qu =$ *ii)* If R is monotone decreasing (increasing) and Q is monotone increasing (decreasing), then $x_{i+1} - x_i' \geq x_{i+2} - x'_{i+1}$, $j \in \mathbb{N}$. If, additionally, one of these functions is strictly monotone, then $x_{i+1} - x_i' \leq$

Proof: Let *R* and *Q* be monotone decreasing on [0, ∞). The solution $v = u^{-1}(x_{i+1}')$ hen $x_{j+1} - x_j'$ $(\geq x_{j+2} - x_{j+1}, j \in \mathbb{N}$. If, additionally, one of these functions is strictly
monotone, then $x_{j+1} - x_j'$ $(\leq x_{j+2} - x_{j+1}', j \in \mathbb{N}$.
Proof: Let R and Q be monotone decreasing on $[0, \infty)$. The solut then $x_{j+1} - x_j'$ $(\xi_1 x_{j+2} - x'_{j+1}, j \in \mathbb{N}$. If, additionally, one of these functions is strictly
monotone, then $x_{j+1} - x_j'$ $(\xi_1 x_{j+2} - x'_{j+1}, j \in \mathbb{N}$.
Proof: Let R and Q be monotone decreasing on $[0, \infty)$. The • decreasing. By the help of Theorem 5 the other assertions of Theorem 8 can analo- **^V** i) If the coefficients are monoton
 $(\frac{3}{2}, x'_{j+1} - x_{j+1}, j \in \mathbb{N}$. If, addition

(increasing), then $x'_j - x_j$, $\geq x'_{j+1} -$

ii) If R is monotone decreasing (in

then $x_{j+1} - x'_j$, $\geq x_{j+2} - x'_{j+1}, j \in \mathbb{N}$

monoton **m** 5 **Proof:** Let *R* and *Q* be monotone decreasing on $[0, \infty)$. The solution $\times u(x_i') u$ to (26) has the properties $v(x_{j+1}') = u(x_i')$ and $v'(x_{j+1}') =$
translating the graphs of *v*, *R*, and *Q* belonging to the interval $[x_{j+1}, x$ **Proof:** Let *R* and *Q* be monotone decreasing on $[0, \infty)$. The solution $v = u^{-1}(x'_{j+1})$
 $\times u(x'_j) u$ to (26) has the properties $v(x'_{j+1}) = u(x_j)$ and $v'(x'_{j+1}) = u'(x_j) = 0$. By

translating the graphs of *v*, *R*, and *Q* b monotone, then $x_{j+1} = x_j \cdot (s_j x_{j+2} - x_{j+1}, j \in \mathbb{N}$.

Proof: Let R and Q be monotone decreasing on $[0, \infty)$. The solution $v = u^{-1}(x'_{j+1}) \times u(x'_j)$ u to (26) has the properties $v(x'_{j+1}) = u(x'_j)$ and $v'(x'_{j+1}) = u'(x'_j) = 0$.

Theorerri 9: *Let u be an oscillatory solution to* (26) *and as in the introduction denote*

i) If R ⁰*and Q is monotone dec ^reasing (increasing) on* [0, cc), *theiA' i*) *If* $R \ge 0$ and *Q* is monotone decreasing (increasing) on $[0, \infty)$, then $A_i' \ge A_i$, $\in \mathbb{N}$. If, additionally, $R \ge 0$ or *Q* is strictly decreasing (increasing), then $A_i' \subseteq A_i$, $\in \mathbb{N}$. If, additionally, *i*) If $R \ge 0$ and Q is monotone decreasing (increasing) on $[0, \infty)$, then $A_i' \ge 0$, A_j , $j \in \mathbb{N}$. If, additionally, $R \ge 0$ or Q is strictly decreasing (increasing), then $A_i' \subseteq A_j$, $j \in \mathbb{N}$.

ii) If $R \ge$

 $V = \frac{1}{2}$ **S S V I S V I S V I S V I S V I S V I S V I S V I S V I S V I S V I E I I S I -** ^V

V V :

^V V

V

 $\frac{1}{2}$

Proof: Assume first that $R \ge 0$ and *Q* is monotone decreasing on [0, ∞). The function $\tilde{u}(x) = u(2x, ' - x), x \in [x_i, x_i'],$ is a solution to the differential equation $-\tilde{u}''$ tion $\tilde{u}(x) = u(2x_i - x)$, $x \in [x_i, x_j]$, is a solution to the differential equation $-u^{(1)}$, $R(2x_i' - x) \tilde{u}' + Q(2x_i' - x) \tilde{u} = 0$ with the initial values $\tilde{u}(x_i') = u(x_i')$, $\tilde{u}'(x_i')$ **Proof:** Assume first that $R \ge 0$ and Q is monotone decreasing on $[0, \infty)$. The function $\tilde{u}(x) = u(2x'_1 - x)$, $x \in [x_1, x'_1]$, is a solution to the differential equation $-\tilde{u}''$
 $-R(2x'_1 - x)\tilde{u}' + Q(2x'_1 - x)\tilde{u} = 0$ wit Proof: Assume first that $R \ge 0$ and Q is monotone decreasing on $[0, \infty)$. The func-
tion $\tilde{u}(x) = u(2x'_1 - x)$, $x \in [x_1, x'_1]$, is a solution to the differential equation $-\tilde{u}''$
 $-R(2x'_1 - x) \tilde{u}' + Q(2x'_1 - x) \tilde{u} = 0$ $-R(2x'_j - x) \le R(x)$ and $Q(2x'_j - x) \le Q(x)$, $x \in [x_j, x'_j]$. But, in the present case, these conditions are satisfied. **E.** MÜLLER-PFEIFFER
 E. MÜLLER-PFEIFFER
 $f:$ Assume first that $R \ge 0$ and Q is monotone decreasing on $[0, \infty)$. The func-
 $x_j - x_j \tilde{u}' + Q(2x'_j - x) \tilde{u} = 0$ with the initial values $\tilde{u}(x_j) = u(x_j)$, $\tilde{u}(x_j)$
 $(x_j \$ Proof: Assume first that $R \ge 0$ and Q is monotone decreasing on [(
tion $\tilde{u}(x) = u(2x'_i - x)$, $x \in [x_i, x'_i]$, is a solution to the differential
 $-R(2x'_i - x) \tilde{u}' + Q(2x'_i - x) \tilde{u} = 0$ with the initial values $\tilde{u}(x'_i)$
 $R(2x_j - x) \tilde{u}' + Q(2x_j' - x) \tilde{u} = 0$ with the initial values $\tilde{u}(x_j) = u(x_j)$, $\tilde{u}'(x - u'(x_j)) = 0$. Hence, by Theorem 5/ii), we have $\tilde{u} \leq u$ on $[2x_j' - x_{j+1}, x_j]$, $R(2x_j' - x) \leq R(x)$ and $Q(2x_j' - x) \leq Q(x)$, $x \in [x_j, x_j']$.

We else discuss the case $R \leq 0$ and *Q* is monotone decreasing. The function $\tilde{u}(x)$ $x = -u(2x_{i+1} - x), x \in [x_i', x_{i+1}],$ is a solution to the differential equation

$$
-\tilde{u}'' - R(2x_{i+1} - x) \tilde{u}' + Q(2x_{i+1} - x) \tilde{u} = 0 \text{ on } [x_i', x_{i+1}]
$$

with $\tilde{u}(x_{i+1}) = 0$, $\tilde{u}'(x_{i+1}) = u'(x_{i+1})$. Hence, by Theorem 5/iii), we obtain $A_{i+1} \subseteq A_i'$ if $-R(2x_{i+1}-x) \ge R(x)$ and $Q(2x_{i+1}-x) \le Q(x), x \in [x_i', x_{i+1}]$. By the assumptions on the coefficients in the present case these conditions are fulfilled. The remaining

By joining the Theorems 5-and 9 the following theorem is obtained.

i) If $R \leq 0$ *and both functions* R *and* Q *are monotone decreasing (increasing), then* Theorem 10: Let u be an oscillatory solution to (26).
 i) *If* $R \underset{i \in \mathbb{Z}}{\geq} 0$ and both functions R and Q are monotone decreasing (increasing), then
 $A_i \underset{i \in \mathbb{Z}}{\geq} A_{i+1}$, $j \in \mathbb{N}$: *If*, additionally, R Theorem 10: Let u be an oscillatory solutio
 i) If $R_{\{\leq\atop\leq\}}$ of and both functions R and Q are
 $A_j \subseteq A_{j+1}, j \in \mathbb{N}$: If, additionally, $R_{\{>0\}}$ or α

decreasing (increasing), then $A_j \subseteq A_{j+1}, j \in \mathbb{N}$.
 ii) *If* $R_{(\frac{2}{m})}$ *i* and both functions R and Q are monotone decreasing (increasing), then $\sum_{i=1}^m A_{i+1}$, $j \in \mathbb{N}$: *If*, *additionally*, $R_{(\frac{2}{m})}$ or one of the functions R and Q is strictly cre

By joining the Theorem 5 and 9 the following theorem is obtained.

Theorem 10: Let u be an oscillatory solution to (26).

i) If $R \geq 0$ and both functions R and Q are monotone decreasing (increasing), then
 $A_j \geq A_{j+1}, j$ *monotone increasing (decreasing), then* $A_i \subseteq A'_{i+1}$ *,* $j \in \mathbb{N}$ *.* $\left\{ \begin{array}{ll} I_f'.I \ \text{\emph{res}}\ \text{\emph{non}}\ \text{\emph{non}}\ \text{\emph{non}}\ \text{\emph{non}}\ \text{\emph{res}}\ \text{\emph{in}}\ \text{\emph{$

Proof:-We handle the case that $\dot{R} \leqq 0$ and both functions R and Q are monotone (increasing), then $A'_j \in A'_{j+1}$, $j \in \mathbb{N}$. If, additionally, $R_{\{0\}} \in \mathbb{N}$ or R or Q is strictly
monotone increasing (decreasing), then $A'_j \in A'_{j+1}$, $j \in \mathbb{N}$.
Proof:-We handle the case that $R \le 0$ and monotone increasing (decreasing), then $A_j'(\frac{1}{c_1}, A'_{j+1}, j \in \mathbb{N}$.

Proof: We handle the case that $R \le 0$ and both functions R and Q are monotone

decreasing. By Theorem 9/ii), it follows that $A_j' \supseteq A_{j+1}$, j $|u(x_j')| \ge |u(x'_{j+1})|, j \in \mathbb{N}$. The function $v(x) = -u(x'_{j+1} - x'_j + x)$ is a solution to
 $-v'' + R(x'_{j+1} - x'_j + x)v' + Q(x'_{j+1} - x'_j + x)v = 0$ on $[x_j, x'_j]$

Froot: we handle the case that $R \ge 0$ and both functions R and Q are induction
decreasing. By Theorem 9/ii), it follows that $A'_i \supseteq A_{j+1}$, $j \in \mathbb{N}$. Hence, we have
 $|u(x'_i)| \ge |u(x'_{j+1})|$, $j \in \mathbb{N}$. The function $= -u(x_i) = 0$. thence, by 1neoregy $\int f(x) dx$ we have $u \le u$
 $=R(2x_j - x) \le R(x)$ and $Q(2x_j' - x) \le Q(x)$, $x \in \{x_j, x_j\}$. But, in the p

these conditions are satisfied.

We else discuss the case $R \le 0$ and Q is monotone decreasi

Example 12. **A** *d* $Q(x_{j+1} - x_j' + x) \leq Q(x_j, x \in [x_j, x_j']$, in view of Theorem 3/11), it
follows that $A_j \supseteqeq A_{j+1}$, $j \in \mathbb{N}$. The remaining cases of the theorem can analogously
be handled \blacksquare .
Remark: Consider the dif Remark: Consider the differential equation $-u'' + \varrho u' + Qu = 0$ (0 > *Q*) $\in C[0,\infty)$, $\varrho = \text{const}$) and let *u* be an oscillatory solution. Then by Theorems 9 and 10 the following holds;

i) If $\varrho \leq 0$ and Q is monotone decreasing, then $A_j \supseteq A_{j+1}$ and $A_j' \supseteq A_{j+1}'$, $j \in \mathbb{N}$.
Hence, concerning the half-waves $A_j \cup A_j'$ and $A_{j+1} \cup A_{j+1}'$ we have $A_j \cup A_j'$ $\supseteq A_{j+1} \cup A'_{j+1}$. If, additionally, $\varrho < 0$ or Q is strictly monotone decreasing, then $A_j \supseteq A_{j+1}$ and $A_j' \supseteq A'_{j+1}$, $j \in \mathbb{N}$. decreasing. By Theorem 9(ii), it follows that $A_i' \supseteq A_{i+1}$, $j \in \mathbb{N}$. Hence, we have $|u(x_j)| \geq |u(x'_{j+1})|$, $j \in \mathbb{N}$. The function $v(x) = -u(x'_{j+1} - x_j' + x)$ is a solution to $-y'' + R(x'_{j+1} - x_j' + x)v' + Q(x'_{j+1} - x_j' + x)v' = Q$ o i) If $\varrho \leq 0$ and Q is monotone decreasing, then $A_j \supseteq A_{j+1}$ and $A_j' \supseteq A_{j+1}'$, $j \in \mathbb{N}$.
ence, concerning the half-waves $A_j \cup A_j'$ and $A_{j+1} \cup A_{j+1}'$ we have $A_j \cup A_j'$
 $A_{j+1} \cup A_{j+1}'$. If, additionally,

If, additionally, $\rho > 0$ or *Q* is strictly monotone increasing, then $A_j \subseteq A_{j+1}$, $j \in \mathbb{N}$.
It, additionally, $\rho > 0$ or *Q* is strictly monotone increasing, then $A_j \subseteq A_{j+1}$ and A'_j .

If $\rho = 0$ and Q is strictly monotone decreasing (increasing), then $A_i \circ A_i' \circ A_{i+1}$, $i \in \mathbb{N}$. $\subset A'_{j+1}, j \in \mathbb{N}$.
iii) If $\rho = 0$ and Q is decreasing (increasing), then $A_j \stackrel{\equiv}{\subseteq} A'_j \stackrel{\equiv}{\subset} A'_{j+1}, j \in \mathbb{N}$.

Concerning the selfadjoint equation

$$
-(Pu')' + Qu = 0, \ (0 < P \in C^1[0, \infty), 0 > Q \in C[0, \infty))
$$
\n⁽²⁷⁾

we have the following situation.

•
•
•
••
•• Pheorem 11: *Let u be an oscillatory-solution to (27).
 •• i) If P is monotone increasing (decreasing) and P⁻¹

<i>creasing*), then $A_j^T \subseteq A_{j+1}$, $j \in \mathbb{N}$. If, additionally, P *i)* If P is monotone increasing (decreasing) and $P^{-1}Q$ is monotone decreasing (in-Theorem 11: Let u be an oscillatory solution to (27).
 i) If P is monotone increasing (decreasing) and $P^{-1}Q$ is monotone decreasing (increasing), then A'_i $\overline{\overline{A}}_iA_{j+1}$, $j \in \mathbb{N}$. *If*, additionally, *• (decreasing) or P- IQ is ' strictly monotone decreasing (increasing), then A,'* (c) *'j€ [N..* Theorem 11: Let u be an oscillatory solution to (27).

i) If P is monotone increasing (decreasing) and P⁻¹Q is monotone

creasing), then $A_i'(\vec{\xi}, A_{i+1}, j \in \mathbb{N}$. If, additionally, P is strictly mono

(decreasing) or P

ii) If P, $P^{-1}Q$ are monotone decreasing (increasing), then $A_j \oplus A_j$, $j \in \mathbb{N}$. *If*, *additionally, P or P⁻¹Q is strictly monotone decreasing (increasing), then* $A_i \subset A_j$ *,* $∈ N$.
iii) *If P*, *P*⁻¹*P'* are monotone increasing (decreasing), and *P*⁻¹*Q* is monotone de-

is strictly monotone, then A_j $\overline{A_j}$ $\overline{A_{j+1}}$, $j \in \mathbb{N}$.

creasing), then $A_i \in A_{i+1}$ $j \in \mathbb{N}$. If, additionally, P is strictly monotone increasing

(decreasing) or $P^{-1}Q$ is strictly monotone decreasing (increasing), then $A_i' \in A_{i+1}$,
 $j \in \mathbb{N}$.

ii) If P, $P^{-1}Q$ are iv) If P is monotone increasing (decreasing), and P⁻¹P', P⁻¹Q are monotone decreasing (increasing), then A_i' , $\overline{\xi}$, A'_{i+1} , $j \in \mathbb{N}$. If, additionally, one of these functions is

v) *If* $P(x) = e^{cx}$, $c \ge 0$, d'_{i+1} , $j \in \mathbb{N}$.
 v) *If* $P(x) = e^{cx}$, $c \ge 0$, and $e^{-cx}Q(x)$ *is monotone decreasing (increasing), then A A*_{*J*} *A*_{*J*} *A*_{*i*} *A*_{*j*} *m i n i 1 i 1 i i a i monotone <i>n i*_s *a*_{*i*} *a* *(increasing), then* $A_j \subseteq A_{j+1}, j \in \mathbb{N}$. If, additionally, one of these functions
 monotone, then $A_j \subseteq A_{j+1}, j \in \mathbb{N}$.
 P is monotone increasing (decreasing), and $P^{-1}P'$, $P^{-1}Q$ are monotone decreas-
 assing) v) If $P(x) = e^{cx}$, $c \ge 0$, and $e^{-cx}Q(x)$ is monotone decreasing (in $A_i \in A_{i+1}$ and $A'_i \in A_{i+1}$, $j \in \mathbb{N}$. If, additionally, $c \ge 0$ or e^{-cx} monotone decreasing (increasing), then $A_i \in A_{i+1}$ and $A'_i \in A'_{i+1}$,

Proof: By considering that the equation (27) can be written as $-u'' - P^{-1}$ $\chi P'u' + P^{-1}Q = 0$ Theorem 11 directly follows from the Theorems 9 and 10⁻¹

Finally, we apply the Theorems 9 and 10 to the Bessel differential equation

$$
-u'' - x^{-1}u' - (1 - x^{-2}v^2) u = 0, \quad x \in (0, \infty).
$$
 (28)

The Riccati differential equation (15) belonging to (28) calls

2 *- ^x ¹ ^y ⁺*(1 - *^x 2v2), x* € (0, co), *y* = —u'u'. . '. (29),

By means of (29) one can easily see that a non-trivial solution *u* to (28) possesses at most one zero on $(0, |\nu|]$. Additionally, if the first zero^x₁ of u is placed in $(0, |\nu|)$, the first zero x_1' ($> x_1$) of *u'* is greater than |v|. Hence, $Q(x) = -(1 - x^{-2}y^2) < 0$, x
first zero x_1' ($> x_1$) of *u'* is greater than |v|. Hence, $Q(x) = -(1 - x^{-2}y^2) < 0$, x $\in [x_1', \infty)$, and the Theorems 8-10 can be applied to (28) if x is restricted by x_1' $\leq x < \infty$.

Theorem 12: Let \mathcal{E}_r , $v = 0$, be a non-trivial solution to (28) and denote its quarter*waves by* A_j , and A'_j , $j \in \mathbb{N}$, *respectively. Then,* $A'_j \supseteq A'_{j+1}$ and $A'_j \supseteq A'_{j+1}$, $j \in \mathbb{N}$.
Further we have $x_{j+1} - x'_j > x'_{j+1} - x_{j+1}$, $x_{j+1} - x'_j > x_{j+2} - x'_{j+1}$, and $|\mathcal{E}_i(x'_j)| > |\mathcal{E}_i(x'_{j+1$ *Further we have* $x_{j+1} - x_j' > x_{j+1}$ *and the inequalities* $x_{j+1} - x_j' > x_{j+1}$ *and the inequalities* $x_{j+1} - x_j' > x_{j+1}$ *and the special case* $y_i \in \{x_i, x_i\}$ *and* f_i *,* $j \in \mathbb{N}$ *, respectively. Thence,* $Q(x) = -(1 - x^{-2}v^2) < 0$ The Riccati differential equation (15) belonging to (28) calls
 $y' = y^2 - x^{-1}y + (1 - x^{-2}v^2)$, $x \in (0, \infty)$, $y = -u^{-1}u'$.

By means of (29) one can easily see that a non-trivial solution u to (28) possesse

most one zero on \Rightarrow A_{j+1} and the inequalities $x_{j+1} - x_j > x'_{j+1} - x_{j+1}$, $x_{j+1} - x_j > x_{j+2}' - x'_{j+1}$,
 $- x'_{j+1} > x'_{j+1} - x_{j+1}$, $|\mathcal{E}_0(x_j')| > |\mathcal{E}_0(x'_{j+1})|$, $j \in \mathbb{N}$, hold.

Proof: Apply the Theorems 9 and 10

The inequalities $|\mathscr{E}(x_i)| > |\mathscr{E}(x_{i+1})|$ $(v \in \mathbb{R}, j \in \mathbb{N})$ are due to L. LORCH, M. E. MULDOON, and P. SzEGO. Additionally, they proved that the sequence $\{\mathcal{E}, \mathcal{E}(x_j)\}_{j \in \mathbb{N}}$ is completely monotonic (cf. [8])., Furthermore, they proved that the sequence $(x'_{j+1} - x_j')_{j \in \mathbb{N}}$ is also completely monotonic (cf. [8]).

Corollary: Assume $|v| \leq 1/2$ and let ℓ , be a non-trivial solution to (28). Then, *concerning the zeros* x'_i and x'_i of \mathcal{E}_r and \mathcal{E}_r' , respectively, the inequalities $x_{i+1} - x'_i$ $> x'_{j+1} - x_{j+1}, x_{j+1} - x_j' > x_{j+2} - x'_{j+1},$ and $x_{j+1} - x_j' > x_j' - x_j, j \in \mathbb{N}$, hold.

Proof: In view of Theorem 12 we only have to show that $x_{j+1} - x_j' > x_j' - x_j$.
It is well-known that $x_{j+1} - x_j \le x_{j+2} - x_{j+1}$ (cf. [7], for instance). Hence, together **288** E. MÜLLER-PFEIFFER

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 $x_{j+1} - x$ 288 E. MÜLLER-PFEIFFER

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with the inequalities of Theorem 12, we obtain
 $x_{j+1} - x_j' + x_j' - x_j \le x_{j+2} - x_{j+1}' +$

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$$
x_{j+1}-x_{j}'+x_{j}'-x_{j} \leq x_{j+2}-x_{j+1}'+x_{j+1}'-x_{j+1} < 2(x_{j+1}-x_{j'})
$$

and, consequently, the assertion $x'_i - x_i < x_{i+1} - x'_i$, $j \in \mathbb{N}$

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