Zeitschrift für Analysis
und ihre Anwendungen
Bd. 9 (3) 1990, S. 277 – 288

Monotonicity Properties of Oscillatory Solutions of Second Order Differential Equations

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Es wird untersucht, wie sich Monotonieeigenschaften der Koeffizienten gewöhnlicher Differentialgleichungen zweiter Ordnung auf oszillierende Lösungen u solcher Gleichungen übertragen. Zum Beispiel werden Aussagen gemacht über die Abstände der Nullstellen von u, u' und der von u und u' untereinander.

Исследуется, как переносятся свойства монотонности коэффициентов обыкновенных дифференциальных уравнений второго порядка на осциллирующие решения u таких уравнений. Делаются, например, высказывания о расстоянии нулей функций u, u' и тех от u и u' между собой.

It is proved in what way monotonicity properties of the coefficients of ordinary second order differential equations are transmitted to oscillatory solutions u of such equations. For instance, there are statements on the distances of the zeros of u, u', and u and u' mutually.

This paper generalizes the following theorem of P. HARTMAN and A. WINTNER [2]: Consider the equation

$$-u'' + Q(x) u = 0, \quad x_1 \leqq x \leqq x_3, \quad Q \in C, \quad Q \leqq 0,$$

and let u be a solution with three consecutive zeros x_1, x_2, x_3, and relative extrema at x_1', $x_1 < x_1' < x_2$, and x_2', $x_2 < x_2' < x_3$. Let A_j be the area bounded by the x-axis, the straight line $x = x_j'$, and the graph of u belonging to the interval $[x_j, x_j']$, $j = 1$, 2, and let A_j' be the area bounded by the x-axis, the straight line $x = x_j'$, and the graph of u belonging to the interval $[x_j', x_{j+1}]$, $j = 1, 2$ (Fig. 1).

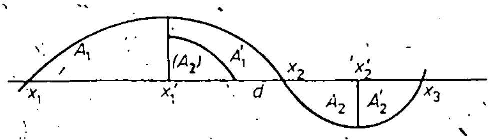

Fig. 1

If Q is monotone decreasing on $[x_1, x_3]$, then A_j' can be placed into A_j by reflection at the straight line $x = x_j'$, $j = 1, 2$, and A_2 can be placed into A_1' by rotation through 180° about the point x_2 and translation about $d = 2x_2 - x_1' - x_2'$ to the left placing the abscissae of the extrema ordinates in coincidence. (After this rotation and translation of A_2 the new position of A_2 is denoted by (A_2) in Fig. 1.) Concerning the monotonicity of the quarter-waves A_j, A_j' and the half-waves $A_j \cup A_j'$ if Q is monotone decreasing or increasing compare also the papers of E. MAKAI [9] and I. BIHARI [1].

In the following we consider the non-selfadjoint differential equations ($r, R \in C[a, b]$)

$$-(Pu')' + Ru' + Qu = 0 \quad (0 < P \in C^1[a, b], 0 > Q \in C[a, b]), \tag{1}$$

and

$$-(pv')' + rv' + qv = 0 \quad (0 < p \in C^1[a, b], 0 > q \in C[a, b]). \tag{2}$$

Let u be an oscillatory solution to (1) and throughout this paper denote the zeros of u by $x_1, x_2, \ldots$, and the zeros of u' being greater than x_1 by $x_1', x_2', \ldots$, so that $0 \leqq x_1 < x_1' < x_2 < x_2' < \ldots$. Define the areas A_j, A_j' of the quarter-waves of u as above. Further, let the following definitions hold throughout the paper.

i) $A_j \supseteqq A_j'$ means that A_j' can be placed into A_j by reflection at the straight line $x = x_j'$.

ii) $A_j' \supseteqq A_{j+1}$ means that A_{j+1} can be placed into A_j' by rotation through 180° about the point x_{j+1} (Fig. 2; after the 180°-rotation the position of A_{j+1} is denoted by $[A_{j+1}]$).

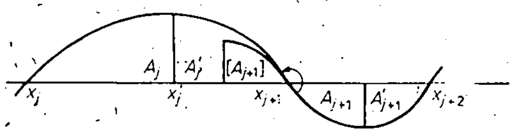

Fig. 2

iii) $A_j \supseteqq A_{j+1}$ means that A_{j+1} can be placed into A_j by reflection at the x-axis and translation about $x_{j+1}' - x_j'$ to the left.

iv) $A_j' \supseteqq A_{j+1}'$ means that A_{j+1}' can be placed into A_j' by reflection at the x-axis and translation about $x_{j+1}' - x_j'$ to the left.

v) $A_j \cup A_j' \supseteqq A_{j+1} \cup A_{j+1}'$ means that $A_{j+1} \cup A_{j+1}'$ can be placed into $A_j \cup A_j'$ by reflection at the x-axis and translation about $x_{j+1}' - x_j'$ to the left.

The inclusions $A_j \supset A_j'$, $A_j \subseteqq A_j'$, $A_j \subset A_j'$ and so on are analogously defined.

Comparison theorems

To compare oscillatory solutions of (1) and (2) these equations are to be transformed into Riccati differential equations.

Lemma 1: *Let u be a solution to* (1) *which doesn't vanish on* (t_1, t_2), *and choose any positive function* $\Phi \in C^1[t_1, t_2]$. *Then the function*

$$y = -\Phi P u^{-1} u' \tag{3}$$

is a solution to the Riccati differential equation

$$y' = \Phi^{-1} P^{-1} y^2 + (\Phi^{-1}\Phi' + P^{-1}R)\, y - \Phi Q \quad \text{on} \quad (t_1, t_2). \tag{4}$$

This assertion follows by an easy calculation. Analogously, if v is a solution to (2) which doesn't vanish on (t_1, t_2) and $\varphi \in C^1[t_1, t_2]$ is any positive function, then

$$z = -\varphi p v^{-1} v' \tag{5}$$

is a solution to

$$z' = \varphi^{-1} p^{-1} z^2 + (\varphi^{-1}\varphi' + p^{-1} r)\, z - \varphi q \quad \text{on} \quad (t_1, t_2). \tag{6}$$

Theorem 1: *Let u be a solution to* (1) *with* $u(a) = u'(c) = u(b) = 0$, $a < c < b$, $u > 0$ *on* (a, b), *and consider the solution v to* (2) *determined by* $v(c) = u(c)$, $v'(c) = 0$.

If there exist a number $\eta > 0$ *and a point* $x_0 \in [a, b]$ *such that*

$$\eta P(x) \exp\left(-\int_{x_0}^{x} RP^{-1}\,dt\right) \geqq p(x) \exp\left(-\int_{x_0}^{x} rp^{-1}\,dt\right), \quad x \in [a, b], \tag{7}$$

$$\eta Q(x) \exp\left(-\int_{x_0}^{x} RP^{-1}\,dt\right) \geqq q(x) \exp\left(-\int_{x_0}^{x} rp^{-1}\,dt\right), \quad x \in [a, b], \tag{8}$$

then there exist points $\alpha \in [a, c)$ *and* $\beta \in (c, b]$ *with* $v(\alpha) = 0 = v(\beta)$ *and* $0 < v \leqq u$ *on* (α, β). *If in* (7) *and* (8) *the signs* $\geqq$ *are replaced by* $\leqq$, *then* $v \geqq u$ *on* $[a, b]$. *If in* (7) *and* (8) *at least one of the signs* $\geqq$ *is replaced by* $>$, *then* $\alpha \in (a, c)$, $\beta \in (c, b)$ *and* $0 < v < u$ *on* $(\alpha, c) \cup (c, \beta)$. *If in* (7) *and* (8) *the signs* $\leqq$ *are valid and at least one of the arising inequalities is strict, then* $v > u$ *on* $[a, c) \cup (c, b]$.

Proof: By setting

$$\Phi(x) = \eta \exp\left(-\int_{x_0}^{x} RP^{-1}\,dt\right), \quad x \in [a, b], \tag{9}$$

and

$$\varphi(x) = \exp\left(-\int_{x_0}^{x} rp^{-1}\,dt\right), \quad x \in [a, b], \tag{10}$$

it follows from (4) and (6) that

$$y' = \Phi^{-1}P^{-1}y^2 - \Phi Q \quad \text{on} \quad (a, b),\ y(c) = 0, \tag{11}$$

and

$$z' = \varphi^{-1}p^{-1}z^2 - \varphi q, \quad z(c) = 0, \tag{12}$$

respectively. (12) holds on every interval (t_1, t_2), $c \in (t_1, t_2)$, where z exists. By (7) and (8) we obtain $z' \geqq y'$ everywhere in the strip $S_{(t_1,t_2)} = \{(x, y) \mid x \in (t_1, t_2), y \in (-\infty, \infty)\}$. Thus, concerning the solution to (11) and (12) it follows that $z \leqq y$ on $(t_1, c]$ and $z \geqq y$ on $[c, t_2)$ (cf. [3, p. 91]). Therefore, and by $y(x) \to -\infty$ as $x \downarrow a$, $y(x) \to +\infty$ as $x \uparrow b$ it follows that there exist points $\alpha \in [a, c)$ and $\beta \in (c, b]$ such that $z(x) \to -\infty$ as $x \downarrow \alpha$, $z(x) \to +\infty$ as $x \uparrow \beta$. Hence, we have $v(\alpha) = 0 = v(\beta)$. Further, $y' > 0$ holds on (α, β) because of (11) and $Q < 0$. This implies $z \leqq y < 0$ on (α, c) and $0 < y \leqq z$ on (c, β). Thus, by (3), (5), (7), and (8), we obtain $v^{-1}v' \geqq u^{-1}u' > 0$ on (α, c) and $v^{-1}v' \leqq u^{-1}u' < 0$ on (c, β). Finally, integration leads to $v \leqq u$ on $[\alpha, \beta]$. If in (7) and (8) the signs $\geqq$ are replaced by the signs $\leqq$ the assertion $v \geqq u$ on $[a, b]$ can analogously be proved.

Assume now that in (7) or (8) at least one of the signs $\geqq$ is replaced by $>$. Then, in view of (11) and (12), it follows that $z' > y'$ everywhere in the strip $S_{(t_1,t_2)}$ whenever $(x, y) \neq (c, y)$. Thus, we obtain $z < y < 0$ on (α, c), $0 < y < z$ on (c, β), and, consebuently, $0 < v < u$ on $(\alpha, c) \cup (c, \beta)$. To prove that $\beta \in (c, b)$, for instance, choose a point $x_1 \in (c, \beta)$ and consider the solution y_1 to (11) determined by $y_1(x_1) = z(x_1)$ $(> y(x_1))$. The function $w = (y_1 - y)^{-1}$ is a solution to

$$w' + 2y\Phi^{-1}P^{-1}w + \Phi^{-1}P^{-1} = 0 \quad \text{on } (a, b) \tag{13}$$

(cf. [3, 10]). Hence

$$w(x) = \exp\left(-2\int_{x_1}^{x} y\Phi^{-1}P^{-1}\,dt\right)\left(w(x_1) - \int_{x_1}^{x}\Phi^{-1}P^{-1}\exp\left(2\int_{x_1}^{t} y\Phi^{-1}P^{-1}\,d\tau\right)dt\right)$$

$$= \frac{u^2(x)}{u^2(x_1)}\left(w(x_1) - u^2(x_1)\int_{x_1}^{x}\Phi^{-1}P^{-1}u^{-2}\,dt\right), \tag{14}$$

$$w(x_1) = \left(y_1(x_1) - y(x_1)\right)^{-1} > 0.$$

It follows from (14) and

$$\lim_{x\uparrow b}\int_{x_1}^{x}\Phi^{-1}P^{-1}u^{-2}\,dt = \infty$$

that there exists a point $\xi_1 \in (x_1, b)$ such that $w(x) \to 0$ as $x \uparrow \xi_1$. Hence, we have $y_1(x) \to \infty$ as $x \uparrow \xi_1$. Since $z' > y_1'$ in $S_{(t_1,t_2)}$, it follows that $z > y_1$ on (x_1, t_2). Therefore, there exists a point $\beta \in (c, b)$ such that $z(x) \to \infty$ as $x \uparrow \beta$. This proves that $v(\beta) = 0$, $\beta \in (c, b)$. The assertion $\alpha \in (a, c)$ can analogously be proved ∎

In the selfadjoint case $R \equiv r \equiv 0$ concerning the location of the zero β of v Theorem 1 is essentially a result by LEIGHTON [5].

Theorem 2: *Let u be a solution to* (1) *with $u(a) = u'(b) = 0$, $u' > 0$ on $[a, b)$ and consider the solution v to* (2) *determined by $v(a) = 0$, $v'(a) = u'(a) > 0$. If there exist a number $\eta > 0$ and a point $x_0 \in [a, b]$ such that* (7) and (8) *are fulfilled, then there exists a point $\beta \in (a, b]$ with $v'(\beta) = 0$* and *$v' > 0$ on $[a, \beta)$. If in* (7) *and* (8) *the signs $\geqq$ are replaced by $\leqq$, then $v' > 0$ on $[a, b)$. If* (7) *and* (8) *hold and one of these inequalities is strict, then $\beta \in (a, b)$.*

Proof: Use the functions (9), (10), and, consequently, the equations (11) and (12). It follows from the hypotheses of the theorem that $y(x) \to -\infty$ and $z(x) \to -\infty$ as $x \downarrow a$, $y' > 0$ on (a, b), $y(b) = 0$, and $y < 0$ on (a, b). Let the conditions (7) and (8) be fulfilled. We prove that $z \geqq y$ on all intervals (a, β'), $\beta' \leqq b$, where z exists. Assuming the contrary suppose that there exists a point $x_1 \in (a, \beta')$ with $z(x_1) < y(x_1)$. Consider the solution y_1 to (11) determined by $y_1(x_1) = z(x_1)$. The function $w = (y_1 - y)^{-1}$ is a solution to (13). w is given by (14) with $w(x_1) = \left(y_1(x_1) - y(x_1)\right)^{-1} < 0$ and we have

$$\lim_{x\downarrow a}\int_{x_1}^{x}\Phi^{-1}P^{-1}u^{-2}\,dt = -\infty.$$

Hence, there exists a point $\xi_1 \in (a, x_1)$ such that $w(x) \to 0$ as $x \downarrow \xi_1$ and, consequently, $y_1(x) \to -\infty$ as $x \downarrow \xi_1$. Because of $y' \leqq z'$ everywhere in the strip $S_{(a,\beta')} = \{(x, y) \mid x \in (a, \beta'), y \in (-\infty, \infty)\}$ we obtain $z \leqq y_1$ on the left-hand side of x_1. Hence, there exists a point $\xi_2 \in [\xi_1, x_1)$ with $z(x) \to -\infty$ as $x \downarrow \xi_2$. This, however, contradicts the fact that z exists on (a, β'). This proves $z \geqq y$ on (a, β'). Since $y(b) = 0$, there exists a point $\beta \in (a, b]$ with $z(\beta) = 0$ and $-\infty < y \leqq z < 0$ on (a, β). Thus, we obtain $v'(\beta) = 0$ and $v' > 0$ on $[a, \beta)$. If (7) and (8) hold, but one of these inequalities is strict, we obtain $y' < z'$ in $S_{(a,\beta)}$ and, consequently, $y < z$ on (a, β). It now follows from $y' < z'$ on (a, β) that $\beta < b$.

If (7) and (8) with $\leqq$ in place of $\geqq$ are considered, we prove that $z \leqq y$ on $(a, b]$. Assume the contrary and let $x_1 \in (a, b)$ be a point with $z(x_1) > y(x_1)$. The solution y_1 to (11) determined by $y_1(x_1) = z(x_1)$ is monotone increasing on $(a, x_1]$. The case $y_1(x) \to -\infty$ as $x \downarrow a$ is impossible, because, by assuming this case, it would follow that there exists a point $\xi_1 \in (a, x_1)$ with $y(x) \to -\infty$ as $x \downarrow \xi_1$ as it is seen from above.

Hence, we have $y_1(x) \to \omega$ as $x \downarrow a$, $\omega \in (-\infty, y_1(x_1))$. Further, by the hypotheses under consideration, it follows that $y' \geqq z'$ in the strip $S_{(a,b)}$ which implies that $z \geqq y_1 > \omega$ on $(a, x_1]$. This, however, is impossible because of $z(x) \to -\infty$ as $x \downarrow a$. Hence, we have $z \leqq y$ on $(a, b]$. This proves that $v' > 0$ on $[a, b)$ ∎

In the selfadjoint case $R \equiv r \equiv 0$ and $\eta = 1$ Theorem 2 is due to LEIGHTON [5, 6]. The next theorem can similarly be proved.

Theorem 3: *Let u be a solution to* (1) *with $u(b) = u'(a) = 0$, $u' < 0$ on $(a, b]$, and consider the solution v to* (2) *determined by $v(b) = 0$, $v'(b) = u'(b) < 0$. If there exist a number $\eta > 0$ and a point $x_0 \in [a, b]$ such that* (7) *and* (8) *are satisfied, then there exists a point $\alpha \in [a, b)$ with $v'(\alpha) = 0$ and $v' < 0$ on $(\alpha, b]$. If, additionally, one of these inequalities is strict, then $\alpha \in (a, b)$. If in* (7) *and* (8) *the signs $\geqq$ are replaced by $\leqq$, then $v' < 0$ on (a, b).*

By setting $P \equiv p \equiv \Phi \equiv \varphi \equiv 1$ in (4) and (6) we obtain the Riccati equations

$$y' = y^2 + Ry - Q, \quad y = -u^{-1}u', \tag{15}$$

and

$$z' = z^2 + rz - q, \quad z = -v^{-1}v', \tag{16}$$

which lead to the following theorem.

Theorem 4: *Let u and v be solutions to*

$$-u'' + Ru' + Qu = 0, \quad (R, Q \in C[a, b], Q < 0) \tag{17}$$

and

$$-v'' + rv' + qv = 0, \quad (r, q \in C[a, b], q < 0) \tag{18}$$

with $u(a) = v(a) = u'(b) = 0$, $u'(a) = v'(a) > 0$, $u' > 0$ on $[a, b)$. If

$$r \leqq R, \quad q \leqq Q \quad \text{on } [a, b], \tag{19}$$

then there exists a point $\beta \in (a, b]$ with $v'(\beta) = 0$, $v' > 0$ on $[a, \beta)$, and $0 < v \leqq u$ on $(a, \beta]$. The point β is equal to b only if the equations (17) *and* (18) *are identical. If*

$$r \geqq R, \quad q \geqq Q \quad \text{on } [a, b], \tag{20}$$

then $v \geqq u$ on $[a, b]$, $v' > 0$ on $[a, b)$, and $v'(b) = 0$ only if (17) *and* (18) *are identical.*

Proof: The function $y = -u^{-1}u'$ is defined on $(a, b]$. By the hypotheses on u it follows that $y(x) \to -\infty$ as $x \downarrow a$, $y(b) = 0$, and $y < 0$ on (a, b). Let the hypothesis (19) be fulfilled. Assume that there doesn't exist a zero of v' on (a, b). Then the function $z = -v^{-1}v'$ is negative on (a, b). By (15), (16), and (19) it follows that $y' \leqq z'$ everywhere in the half-strip $H_{(a,b)} = \{(x, y) \mid x \in (a, b), y \in (-\infty, 0)\}$. Thus, as in the proof of Theorem 2, we obtain $y \leqq z$ on intervals $(a, \beta']$, $\beta' \in (a, b]$, where $-\infty < z \leqq 0$. If the functions y and z are not identical, there exists a point $x_1 \in (a, b)$ with $y(x_1) < z(x_1)$ (<0). The solution y_1 to (15) determined by the initial value $y_1(x_1) = z(x_1)$ must cross the x-axis because of $y(b) = 0$ and the uniqueness of solutions to (15). Now, it follows from $y' \leqq z'$ in $H_{(a,b)}$ that $y_1 \leqq z$ to the right of x_1 and for points (x, y_1) and (x, z) which are placed in $H_{(a,b)}$. Hence, the graph of z must also cross the x-axis. This, however, contradicts the assumption that v' does not vanish on (a, b). Hence, we have $y = z$ on (a, b). Thus, by (15), (16), and (19) it follows that $Q \equiv q$ and $R \equiv r$, i.e. the equations (17) and (18) are identical if v' does not vanish on (a, b). Let $\beta \in (a, b]$ be the first zero of v'. Then we have $y \leqq z < 0$ on (a, β), which implies that $u^{-1}u' \geqq v^{-1}v' > 0$ on (a, β). By integration we obtain $0 < v \leqq u$ on $(a, \beta]$.

This proves the first part of the theorem. If (20) is supposed exchange the parts of the equations (17) and (18) ∎

The different parts of the following theorem can analogously be proved.

Theorem 5: *Consider the differential equations* (17) *and* (18) *on* $[a, b]$.

i) *Let* u *and* v *be solutions to* (17) *and* (18), *respectively, with* $u(a) = v(a) > 0$, $u'(a) = v'(a) = u(b) = 0$, $u > 0$ *on* $[a, b)$. *If*

$$r \geqq R, \quad q \leqq Q < 0 \quad \text{on } [a, b], \tag{21}$$

then $u' < 0$ *on* $[a, b]$, *and there exists a point* $\beta \in (a, b]$ *such that* $v(\beta) = 0$, $0 < v \leqq u$ *on* $[a, \beta)$, *and* $v' < 0$ *on* $(a, \beta]$, *where* β *is equal to* b *only if* (17) *and* (18) *are identical. I,*

$$r \leqq R, \quad Q \leqq q < 0 \quad \text{on } [a, b],$$

then $v \geqq u$ *on* $[a, b]$, *where* $v(b) = 0$ *only if* (17) *and* (18) *are identical.*

ii) *Let* u *and* v *be solutions to* (17) *and* (18), *respectively, with* $u(b) = v(b) > 0$, $u'(b) = v'(b) = u(a) = 0$, *and* $u > 0$ *on* $(a, b]$. *If*

$$r \leqq R, \quad q \leqq Q < 0 \quad \text{on } [a, b],$$

then $u' > 0$ *on* $[a, b]$ *and there exists a point* $\alpha \in [a, b)$ *such that* $v(\alpha) = 0$, $0 < v \leqq u$ *on* $(\alpha, b]$ *and* $v' > 0$ *on* $[\alpha, b)$, *where* α *is equal to* a *only if* (17) *and* (18) *are identical. If*

$$r \geqq R, \quad Q \leqq q < 0 \quad \text{on } [a, b],$$

then $v \geqq u$ *on* $[a, b]$, *where* $v(a) = 0$ *only if* (17) *and* (18) *are identical.*

iii) *Let* u *and* v *be solutions to* (17) *and* (18), *respectively, with* $u(b) = v(b) = u'(a) = 0$, $u'(b) = v'(b) < 0$, $u' < 0$ *on* $(a, b]$. *If*

$$r \geqq R, \quad q \leqq Q < 0 \quad \text{on } [a, b],$$

then there exists a point $\alpha \in [a, b)$ *with* $v'(\alpha) = 0$, $0 < v \leqq u$ *on* $[\alpha, b)$ *and* $v' < 0$ *on* $(\alpha, b]$, *where* α *is equal to* a *only if* (17) *and* (18) *are identical. If*

$$r \leqq R, \quad Q \leqq q < 0 \quad \text{on } [a, b],$$

then $v > u$ *on* $[a, b]$, $v' < 0$ *on* $(a, b]$, *and* $v'(a) = 0$ *only if* (17) *and* (18) *are identical.*

Proof: i) The assertion $u' < 0$ on $(a, b]$, for instance, easily follows from (15): Since $y(a) = 0$, $y'(a) = -Q(a) > 0$, the function y is positive in a neighbourhood of a. Further, the graph of y cannot touch the x-axis at a point $x_0 \in (a, b)$ as can be proved as follows. Assume that x_0 is the smallest point to the right of a with $y(x_0) = 0$. Since $y > 0$ on (a, x_0), we have $y'(x_0) \leqq 0$, contradictory to $y'(x_0) = -Q(x_0) > 0$. $y > 0$ on (a, b) implies $u' < 0$ on (a, b). To prove the other assertions of the theorem compare the proof of Theorem 4 ∎

Of course, the Theorems 4 and 5 can easily be applied to the selfadjoint equations (1) and (2) with $R \equiv 0$ and $r \equiv 0$, respectively, to obtain analogous comparison theorems.

Monotonicity properties of solutions

In the following the comparison theorems from above are used to study monotonicity properties of oscillatory solutions of second order differential equations implied by corresponding monotonicity behaviour of the coefficients of the differential equations.

Theorem 6: *Let u be an oscillatory solution to* (1) *on* $[0, \infty)$. *Denote the zeros of u by $x_1, x_2, \ldots$, and the zeros of u' by $x_1', x_2', \ldots$, so that $0 \leqq x_1 < x_1' < x_2 < x_2' < \ldots$*[1] *If there exists $c \in \mathbb{R}$ such that the functions*

$$P(x) \exp\left(cx - \int_0^x RP^{-1}\,dt\right), \quad Q(x) \exp\left(cx - \int_0^x RP^{-1}\,dt\right) \tag{22}$$

are monotone decreasing (increasing) on $[0, \infty)$, then, for $j \in \mathbb{N}$,

$$x_j' - x_j \underset{(\leqq)}{\geqq} x_{j+1}' - x_{j+1}, \quad x_{j+1} - x_j' \underset{(\leqq)}{\geqq} x_{j+2} - x_{j+1}' \tag{23}$$

and, consequently,

$$x_{j+1} - x_j \underset{(\leqq)}{\geqq} x_{j+2} - x_{j+1}, \quad x_{j+1}' - x_j' \underset{(\leqq)}{\geqq} x_{j+2}' - x_{j+1}'. \tag{24}$$

If, additionally, one of the functions (22) *is strictly monotone, the inequalities* (23) *and* (24) *are also strict.*

Proof: The function $\hat{u}(x) = u(x_{j+1}' - x_j' + x)$ $(x \in [x_j, x_{j+1}];\ j \in \mathbb{N})$ is a solution to the differential equation

$$-\left(P(x_{j+1}' - x_j' + x)\hat{u}'\right)' + R(x_{j+1}' - x_j' + x)\,\hat{u}' + Q(x_{j+1}' - x_j' + x)\,\hat{u} = 0.$$

Assume that the functions (22) are monotone decreasing. Then

$$\eta_j P(x) \exp\left(cx - \int_0^x RP^{-1}\,dt\right)$$

$$\geqq \eta_j P(x_{j+1}' - x_j' + x) \exp\left(c(x_{j+1}' - x_j' + x) - \int_0^{x_{j+1}' - x_j' + x} RP^{-1}\,dt\right)$$

$$= P(x_{j+1}' - x_j' + x) \exp\left(cx - \int_0^x R(x_{j+1}' - x_j' + \tau)\,P^{-1}(x_{j+1}' - x_j' + \tau)\,d\tau\right)$$

with $\eta_j = \exp\left(c(x_j' - x_{j+1}') + \int_0^{x_{j+1}' - x_j'} RP^{-1}\,dt\right)$. Hence,

$$\tilde{\eta}_j P(x) \exp\left(-\int_{x_j'}^x RP^{-1}\,dt\right)$$

$$\geqq P(x_{j+1}' - x_j' + x) \exp\left(-\int_{x_j}^x R(x_{j+1}' - x_j' + \tau)\,P^{-1}(x_{j+1}' - x_j' + \tau)\,d\tau\right),$$

where

$$\tilde{\eta}_j = \eta_j \exp\left(-\int_0^{x_j'} RP^{-1}\,dt + \int_0^{x_j'} R(x_{j+1}' - x_j' + \tau)\,P^{-1}(x_{j+1}' - x_j' + \tau)\,d\tau\right),$$

[1]) Here and in the following a possible zero $x_0' \in [0, x_1)$ of u' is disregarded.

$x \in [x_j, x_{j+1}]$, $j \in \mathbb{N}$. Analogously,

$$\tilde{\eta}_j Q(x) \exp\left(-\int_{x_j}^{x} RP^{-1}\,dt\right)$$

$$\geqq Q(x'_{j+1} - x_j' + x) \exp\left(-\int_{x_j}^{x} R(x'_{j+1} - x_j' + \tau)\, P^{-1}(x'_{j+1} - x_j' + \tau)\,d\tau\right),$$

$x \in [x_j, x_{j+1}]$, $j \in \mathbb{N}$. To finish the first part of Theorem 6 set

$$P(x'_{j+1} - x_j' + x) = p(x), \quad R(x'_{j+1} - x_j' + x) = r(x),$$
$$Q(x'_{j+1} - x_j' + x) = q(x), \quad v(x) = u(x_j')\, \tilde{u}^{-1}(x_j')\, \tilde{u}(x),$$
$$\tilde{x}_j = c = x_0, \quad x_j = a, \quad x_{j+1} = b,$$

and apply Theorem 1. The part of Theorem 6 described by the brackets can analogously be proved. If, additionally, one of the functions (22) is strictly monotone decreasing (increasing), the remaining part of Theorem 6 follows from the last part of Theorem 1 ∎

In the special case $P \equiv 1$, $R \equiv 0$, $c = 0$ and concerning the inequalities $x'_{j+1} - x_j' \underset{(<)}{>} x'_{j+2} - x'_{j+1}$ Theorem 6 is due to A. LAFORGIA [4].

Theorem 7: *Let u be an oscillatory solution to* (1) *on* $[0, \infty)$. *If the functions*

$$P(x) \exp\left(-\int_0^x RP^{-1}\,dt\right), \quad Q(x) \exp\left(-\int_0^x RP^{-1}\,dt\right) \tag{25}$$

are monotone decreasing (increasing) on $[0, \infty)$, *then* $x_j' - x_j \underset{(\leqq)}{\geqq} x_{j+1} - x_j' \underset{(\leqq)}{\geqq} x'_{j+1} - x_{j+1}$ *and* $A_j \underset{(\subseteqq)}{\supseteqq} A_j'$ $(j \in \mathbb{N})$. *If, additionally, one of the functions* (25) *is strictly monotone decreasing (increasing), then the asserted inequalities and inclusions are also strict.*

Proof: We prove that $A_j \supseteqq A_j'$ if the functions (25) are monotone decreasing. The function $\tilde{u}(x) = u(2x_j' - x)$, $x \in [x_j, x_j']$, is a solution to the differential equation

$$-\big(P(2x_j' - x)\, \tilde{u}'\big)' - R(2x_j' - x)\, \tilde{u}' + Q(2x_j' - x)\, \tilde{u} = 0$$

with $\tilde{u}(x_j') = u(x_j')$, $\tilde{u}'(x_j') = -u'(x_j') = 0$. Since $P(x) \exp\left(-\int_0^x RP^{-1}\,dt\right)$ is monotone decreasing, we have

$$P(x) \exp\left(-\int_{x_j'}^{x} RP^{-1}\,dt\right) = \exp\left(\int_0^{x_j'} RP^{-1}\,dt\right) P(x) \exp\left(-\int_0^x RP^{-1}\,dt\right)$$

$$\geqq \exp\left(\int_0^{x_j'} RP^{-1}\,dt\right) P(2x_j' - x) \exp\left(-\int_0^{2x_j'-x} RP^{-1}\,dt\right)$$

$$= P(2x_j' - x) \exp\left(-\int_{x_j'}^{2x_j'-x} RP^{-1}\,dt\right)$$

$$= P(2x_j' - x) \exp\left(-\int_{x_j'}^{x} \big(-R(2x_j' - \tau)\big)\, P^{-1}(2x_j' - \tau)\,d\tau\right),$$

$x \in [x_j, x_j']$. Analogously

$$Q(x) \exp\left(-\int_{x_j'}^{x} RP^{-1}\,dt\right) \geqq Q(2x_j' - x) \exp\left(-\int_{x_j'}^{x} \left(-R(2x_j' - \tau)\right) P^{-1}(2x_j' - \tau)\,d\tau\right),$$

$x \in [x_j, x_j']$. By setting

$$P(2x_j' - x) = p(x), \quad -R(2x_j' - x) = r(x), \quad Q(2x_j' - x) = q(x),$$

$$\tilde{u}(x) = v(x), \quad x_j' = x_0 = c, \quad x_j = a, \quad \eta = 1,$$

and applying Theorem 1 we obtain $A_j' \subseteqq A_j$ and, consequently, $x_j' - x_j \geqq x_{j+1} - x_j'$. To prove $x_{j+1} - x_j' \geqq x_{j+1}' - x_{j+1}$ define the function $\tilde{u}(x) = -u(2x_{j+1} - x)$, $x \in [x_j', x_{j+1}]$. This function is a solution to

$$-\left(P(2x_{j+1} - x)\,\tilde{u}'\right)' - R(2x_{j+1} - x)\,\tilde{u}' + Q(2x_{j+1} - x)\,\tilde{u} = 0$$

with $\tilde{u}(x_{j+1}) = -u(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+1})$. Now, conclude as above and apply Theorem 3. Thus, we obtain $x_{j+1} - x_j' \geqq x_{j+1}' - x_{j+1}$. If, additionally, one of the functions (25) is strictly monotone decreasing, by using the Theorems 1 and 3 the corresponding assertions can analogously be proved. The same holds in the case that the functions (25) are monotone increasing ∎

Theorem 8: *Consider the differential equation*

$$-u'' + Ru' + Qu = 0 \quad \left(R, Q \in C[0, \infty), Q < 0\right) \tag{26}$$

and let u be an oscillatory solution.

i) *If the coefficients are monotone decreasing (increasing) on* $[0, \infty)$, *then* $x_j' - x_j \underset{(\leqq)}{\geqq} x_{j+1}' - x_{j+1}$, $j \in \mathbb{N}$. *If, additionally, one of these coefficients is strictly decreasing (increasing), then* $x_j' - x_j \underset{(<)}{>} x_{j+1}' - x_{j+1}$, $j \in \mathbb{N}$.

ii) *If R is monotone decreasing (increasing) and Q is monotone increasing (decreasing), then* $x_{j+1} - x_j' \underset{(\geqq)}{\leqq} x_{j+2} - x_{j+1}'$, $j \in \mathbb{N}$. *If, additionally, one of these functions is strictly monotone, then* $x_{j+1} - x_j' \underset{(>)}{<} x_{j+2} - x_{j+1}'$, $j \in \mathbb{N}$.

Proof: Let R and Q be monotone decreasing on $[0, \infty)$. The solution $v = u^{-1}(x_{j+1}') \times u(x_j')\,u$ to (26) has the properties $v(x_{j+1}') = u(x_j')$ and $v'(x_{j+1}') = u'(x_j') = 0$. By translating the graphs of v, R, and Q belonging to the interval $[x_{j+1}, x_{j+1}']$ to the left about $x_{j+1}' - x_j'$ and applying Theorem 5/ii), we obtain $x_j' - x_j \geqq x_{j+1}' - x_{j+1}$. Additionally, we have $x_j' - x_j > x_{j+1}' - x_{j+1}$ if one of the coefficients is strictly decreasing. By the help of Theorem 5 the other assertions of Theorem 8 can analogously be proved ∎

Theorem 9: *Let u be an oscillatory solution to* (26) *and as in the introduction denote the areas of the quarter-waves by* A_j *and* A_j', *respectively.*

i) *If* $R \underset{(\leqq)}{\geqq} 0$ *and Q is monotone decreasing (increasing) on* $[0, \infty)$, *then* $A_j' \underset{(\supseteqq)}{\subseteqq} A_j$, $j \in \mathbb{N}$. *If, additionally,* $R \underset{(<)}{>} 0$ *or Q is strictly decreasing (increasing), then* $A_j' \underset{(\supset)}{\subset} A_j$, $j \in \mathbb{N}$.

ii) *If* $R \underset{(\geqq)}{\leqq} 0$ *and Q is monotone decreasing (increasing) on* $[0, \infty)$, *then* $A_j' \underset{(\subseteqq)}{\supseteqq} A_{j+1}$, $j \in \mathbb{N}$. *If, additionally,* $R \underset{(>)}{<} 0$ *or Q is strictly decreasing (increasing), then* $A_j' \underset{(\subset)}{\supset} A_{j+1}$, $j \in \mathbb{N}$.

Proof: Assume first that $R \geqq 0$ and Q is monotone decreasing on $[0, \infty)$. The function $\tilde{u}(x) = u(2x_j' - x)$, $x \in [x_j, x_j']$, is a solution to the differential equation $-\tilde{u}'' - R(2x_j' - x)\,\tilde{u}' + Q(2x_j' - x)\,\tilde{u} = 0$ with the initial values $\tilde{u}(x_j') = u(x_j')$, $\tilde{u}'(x_j') = -u'(x_j') = 0$. Hence, by Theorem 5/ii), we have $\tilde{u} \leqq u$ on $[2x_j' - x_{j+1}, x_j']$ if $-R(2x_j' - x) \leqq R(x)$ and $Q(2x_j' - x) \leqq Q(x)$, $x \in [x_j, x_j']$. But, in the present case, these conditions are satisfied.

We else discuss the case $R \leqq 0$ and Q is monotone decreasing. The function $\tilde{u}(x) = -u(2x_{j+1} - x)$, $x \in [x_j', x_{j+1}]$, is a solution to the differential equation

$$-\tilde{u}'' - R(2x_{j+1} - x)\,\tilde{u}' + Q(2x_{j+1} - x)\,\tilde{u} = 0 \quad \text{on } [x_j', x_{j+1}]$$

with $\tilde{u}(x_{j+1}) = 0$, $\tilde{u}'(x_{j+1}) = u'(x_{j+1})$. Hence, by Theorem 5/iii), we obtain $A_{j+1} \subseteqq A_j'$ if $-R(2x_{j+1} - x) \geqq R(x)$ and $Q(2x_{j+1} - x) \leqq Q(x)$, $x \in [x_j', x_{j+1}]$. By the assumptions on the coefficients in the present case these conditions are fulfilled. The remaining assertions of the theorem can analogously be proved ∎

By joining the Theorems 5 and 9 the following theorem is obtained.

Theorem 10: *Let u be an oscillatory solution to* (26).

i) *If $R \ (\overset{\leqq}{\geqq})\ 0$ and both functions R and Q are monotone decreasing (increasing), then $A_j \ (\overset{\supseteqq}{\subseteqq})\ A_{j+1}$, $j \in \mathbb{N}$. If, additionally, $R \ (\overset{<}{>})\ 0$ or one of the functions R and Q is strictly decreasing (increasing), then $A_j \ (\overset{\supset}{\subset})\ A_{j+1}$, $j \in \mathbb{N}$.*

ii) *If $R \ (\overset{\leqq}{\geqq})\ 0$, R is monotone increasing (decreasing), and Q is monotone decreasing (increasing), then $A_j' \ (\overset{\supseteqq}{\subseteqq})\ A_{j+1}'$, $j \in \mathbb{N}$. If, additionally, $R \ (\overset{<}{>})\ 0$ or R or Q is strictly monotone increasing (decreasing), then $A_j' \ (\overset{\supset}{\subset})\ A_{j+1}'$, $j \in \mathbb{N}$.*

Proof: We handle the case that $R \leqq 0$ and both functions R and Q are monotone decreasing. By Theorem 9/ii), it follows that $A_j' \supseteqq A_{j+1}$, $j \in \mathbb{N}$. Hence, we have $|u(x_j')| \geqq |u(x_{j+1}')|$, $j \in \mathbb{N}$. The function $v(x) = -u(x_{j+1}' - x_j' + x)$ is a solution to

$$-v'' + R(x_{j+1}' - x_j' + x)\,v' + Q(x_{j+1}' - x_j' + x)\,v = 0 \quad \text{on } [x_j, x_j']$$

with $|v(x_j')| = |u(x_{j+1}')| \leqq |u(x_j')|$ and $v'(x_j') = -u'(x_{j+1}') = 0$. Since $R(x_{j+1}' - x_j' + x) \leqq R(x)$ and $Q(x_{j+1}' - x_j' + x) \leqq Q(x)$, $x \in [x_j, x_j']$, in view of Theorem 5/ii), it follows that $A_j \supseteqq A_{j+1}$, $j \in \mathbb{N}$. The remaining cases of the theorem can analogously be handled ∎

Remark: Consider the differential equation $-u'' + \varrho u' + Qu = 0$ $(0 > Q \in C[0, \infty)$, $\varrho = \text{const})$ and let u be an oscillatory solution. Then by Theorems 9 and 10 the following holds:

i) If $\varrho \leqq 0$ and Q is monotone decreasing, then $A_j \supseteqq A_{j+1}$ and $A_j' \supseteqq A_{j+1}'$, $j \in \mathbb{N}$. Hence, concerning the half-waves $A_j \cup A_j'$ and $A_{j+1} \cup A_{j+1}'$ we have $A_j \cup A_j' \supseteqq A_{j+1} \cup A_{j+1}'$. If, additionally, $\varrho < 0$ or Q is strictly monotone decreasing, then $A_j \supset A_{j+1}$ and $A_j' \supset A_{j+1}'$, $j \in \mathbb{N}$.

ii) If $\varrho \geqq 0$ and Q is monotone increasing, then $A_j \subseteqq A_{j+1}$ and $A_j' \subseteqq A_{j+1}'$, $j \in \mathbb{N}$. If, additionally, $\varrho > 0$ or Q is strictly monotone increasing, then $A_j \subset A_{j+1}$ and $A_j' \subset A_{j+1}'$, $j \in \mathbb{N}$.

iii) If $\varrho = 0$ and Q is decreasing (increasing), then $A_j \ (\overset{\supseteqq}{\subseteqq})\ A_j' \ (\overset{\supseteqq}{\subseteqq})\ A_{j+1}$, $j \in \mathbb{N}$. If $\varrho = 0$ and Q is strictly monotone decreasing (increasing), then $A_j \ (\overset{\supset}{\subset})\ A_j' \ (\overset{\supset}{\subset})\ A_{j+1}$, $j \in \mathbb{N}$.

Concerning the selfadjoint equation

$$-(Pu')' + Qu = 0, \quad (0 < P \in C^1[0, \infty),\ 0 > Q \in C[0, \infty)) \tag{27}$$

we have the following situation.

Theorem 11: *Let u be an oscillatory solution to* (27).

i) *If P is monotone increasing (decreasing) and $P^{-1}Q$ is monotone decreasing (increasing), then $A_j' \underset{(\subseteq)}{\supseteq} A_{j+1}$, $j \in \mathbb{N}$. If, additionally, P is strictly monotone increasing (decreasing) or $P^{-1}Q$ is strictly monotone decreasing (increasing), then $A_j' \underset{(\subset)}{\supset} A_{j+1}$, $j \in \mathbb{N}$.*

ii) *If P, $P^{-1}Q$ are monotone decreasing (increasing), then $A_j' \underset{(\supseteq)}{\subseteq} A_j$, $j \in \mathbb{N}$. If, additionally, P or $P^{-1}Q$ is strictly monotone decreasing (increasing), then $A_j' \underset{(\supset)}{\subset} A_j$, $j \in \mathbb{N}$.*

iii) *If P, $P^{-1}P'$ are monotone increasing (decreasing), and $P^{-1}Q$ is monotone decreasing (increasing), then $A_j \underset{(\subseteq)}{\supseteq} A_{j+1}$, $j \in \mathbb{N}$. If, additionally, one of these functions is strictly monotone, then $A_j \underset{(\subset)}{\supset} A_{j+1}$, $j \in \mathbb{N}$.*

iv) *If P is monotone increasing (decreasing), and $P^{-1}P'$, $P^{-1}Q$ are monotone decreasing (increasing), then $A_j' \underset{(\subseteq)}{\supseteq} A_{j+1}'$, $j \in \mathbb{N}$. If, additionally, one of these functions is strictly monotone, then $A_j' \underset{(\subset)}{\supset} A_{j+1}'$, $j \in \mathbb{N}$.*

v) *If $P(x) = e^{cx}$, $c \underset{(\leqq)}{\geqq} 0$, and $e^{-cx}Q(x)$ is monotone decreasing (increasing), then $A_j \underset{(\subseteq)}{\supseteq} A_{j+1}$ and $A_j' \underset{(\subseteq)}{\supseteq} A_{j+1}'$, $j \in \mathbb{N}$. If, additionally, $c \underset{(<)}{>} 0$ or $e^{-cx}Q(x)$ is strictly monotone decreasing (increasing), then $A_j \underset{(\subset)}{\supset} A_{j+1}$ and $A_j' \underset{(\subset)}{\supset} A_{j+1}'$, $j \in \mathbb{N}$.*

Proof: By considering that the equation (27) can be written as $-u'' - P^{-1} \times P'u' + P^{-1}Q = 0$ Theorem 11 directly follows from the Theorems 9 and 10 ∎

Finally, we apply the Theorems 9 and 10 to the Bessel differential equation

$$-u'' - x^{-1}u' - (1 - x^{-2}\nu^2)\,u = 0, \quad x \in (0, \infty). \tag{28}$$

The Riccati differential equation (15) belonging to (28) calls

$$y' = y^2 - x^{-1}y + (1 - x^{-2}\nu^2), \quad x \in (0, \infty), \quad y = -u^{-1}u'. \tag{29}$$

By means of (29) one can easily see that a non-trivial solution u to (28) possesses at most one zero on $(0, |\nu|]$. Additionally, if the first zero x_1 of u is placed in $(0, |\nu|)$, the first zero x_1' ($> x_1$) of u' is greater than $|\nu|$. Hence, $Q(x) = -(1 - x^{-2}\nu^2) < 0$, $x \in [x_1', \infty)$, and the Theorems 8–10 can be applied to (28) if x is restricted by $x_1' \leqq x < \infty$.

Theorem 12: *Let $\mathscr{C}_\nu$, $\nu \neq 0$, be a non-trivial solution to* (28) *and denote its quarter-waves by A_j and A_j', $j \in \mathbb{N}$, respectively. Then, $A_j' \supset A_{j+1}$ and $A_j' \supset A_{j+1}'$, $j \in \mathbb{N}$. Further we have $x_{j+1} - x_j' > x_{j+1}' - x_{j+1}$, $x_{j+1} - x_j' > x_{j+2} - x_{j+1}'$, and $|\mathscr{C}_\nu(x_j')| > |\mathscr{C}_\nu(x_{j+1}')|$, $j \in \mathbb{N}$. In the special case $\nu = 0$ the inclusions $A_j' \supset A_{j+1}$, $A_j' \supset A_{j+1}'$, $A_{j+1}' \supset A_{j+1}$ and the inequalities $x_{j+1} - x_j' > x_{j+1}' - x_{j+1}$, $x_{j+1} - x_j' > x_{j+2} - x_{j+1}'$, $x_{j+2} - x_{j+1}' > x_{j+1}' - x_{j+1}$, $|\mathscr{C}_0(x_j')| > |\mathscr{C}_0(x_{j+1}')|$, $j \in \mathbb{N}$, hold.*

Proof: Apply the Theorems 9 and 10 ∎

The inequalities $|\mathscr{C}_\nu(x_j')| > |\mathscr{C}_\nu(x_{j+1}')|$ ($\nu \in \mathbb{R}$, $j \in \mathbb{N}$) are due to L. Lorch, M. E. Muldoon, and P. Szegö. Additionally, they proved that the sequence $\{\mathscr{C}_\nu^2(x_j')\}_{j\in\mathbb{N}}$ is completely monotonic (cf. [8]). Furthermore, they proved that the sequence $\{x_{j+1}' - x_j'\}_{j\in\mathbb{N}}$ is also completely monotonic (cf. [8]).

Corollary: *Assume $|\nu| \leqq 1/2$ and let $\mathscr{C}_\nu$ be a non-trivial solution to* (28). *Then, concerning the zeros x_j and x_j' of $\mathscr{C}_\nu$ and $\mathscr{C}_\nu'$, respectively, the inequalities $x_{j+1} - x_j' > x_{j+1}' - x_{j+1}$, $x_{j+1} - x_j' > x_{j+2} - x_{j+1}'$, and $x_{j+1} - x_j' > x_j' - x_j$, $j \in \mathbb{N}$, hold.*

Proof: In view of Theorem 12 we only have to show that $x_{j+1} - x_j' > x_j' - x_j$. It is well-known that $x_{j+1} - x_j \leqq x_{j+2} - x_{j+1}$ (cf. [7], for instance). Hence, together with the inequalities of Theorem 12, we obtain

$$x_{j+1} - x_j' + x_j' - x_j \leqq x_{j+2} - x_{j+1}' + x_{j+1}' - x_{j+1} < 2(x_{j+1} - x_j')$$

and, consequently, the assertion $x_j' - x_j < x_{j+1} - x_j'$, $j \in \mathbb{N}$ ∎

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Manuskripteingang: 28. 06. 1988; in revidierter Fassung 14. 04. 1989

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