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Travelling Wave Solutions of a Nonlinear Diffusion Equation with Integral Term

L. v. WOLFERSDORF

Dedicated to Professor L. Berg on the occasion of his 60th birthday

Für eine von T. Nagai und M. Mimura eingeführte Klasse nichtlinearer, nichtlokaler, ausgearteter Diffusionsgleichungen werden Lösungen vom Typ laufender Wellen mit kompakten Träger durch Zurückführung auf ein Randwertproblem für eine Integrodifferentialgleichung zweiter Ordnung und weiter eine Integralgleichung mit nichtnegativem Kern untersucht. Es werden Existenzbeweise bei drei Typen der Potenz-Nichtlinearitäten mit allgemeinem Integralterm und geschlossene Lösungen für einen einfachen Integralterm angegeben.

Исследуются решения типа бегущей волны с компактным носителем для класса нелинейных нелокальных вырожденных диффузионных уравнений, введенных Т. Нагай и М. Мимура. Задача сводится к граничной задаче для интегро-дифференциального уравнения второго порядка и дальше к интегральному уравнению с неотрицательным ядром. Даются доказательства существования для трех типов степенных нелинейностей с общим интегральным членом и замкнутые решения для простого интегрального члена.

Travelling wave solutions with compact support are investigated for a class of nonlinear nonlocal degenerate diffusion equations introduced by T. Nagai and M. Mimura. The problem is reduced to a boundary value problem for an integro-differential equation of second order and in turn to an integral equation with nonnegative kernel. There are given existence proofs for three types of power nonlinearities in case of a general integral term and closed solutions for a simple integral term.

Introduction. For describing diffusion processes with aggregation effects NAGAI and MIMURA [6–8] introduced a class of nonlinear degenerate diffusion equations with integral terms as given by equation (1) below. NAGAI [6] and NAGAI and MIMURA [7] studied the general Cauchy problem for these equations. Further IKEDA [2, 3] and MIMURA and SATSUMA [5] (cf. also NAGAI and MIMURA [8]) constructed explicit equilibrium and travelling wave solutions with compact support, respectively, for particular integral terms in the diffusion equations, especially for the case $m = 2$ in (1).

In this paper we investigate the existence of travelling wave solutions with compact support for the general equation (1) with a sufficiently smooth integral term. We reduce this problem to a two-point boundary value problem for an integro-differential equation of second order and this one in turn to an integral equation with a nonnegative kernel. Utilizing the well-known general theory of such equations by KRASNOSELSKII [4] (cf. also [11]) and also the special treatment by BUSHELL [1], we prove three general existence theorems for the cases $m > 2$, $m = 2$, $1 < m < 2$ in (1), respectively. Besides for the special case of a piecewise constant integral kernel we derive travelling wave solutions in closed form which contain the sine-solutions for $m = 2$ found by MIMURA and SATSUMA [5] in another way and a new explicit solution expressed by an elliptic Jacobian function for $m = 4/3$. We remark that travelling wave solutions with compact support for the special case of a piecewise constant integral kernel were also constructed in a more general context and in more involved form by NAGAI and MIMURA [8, 9].

1. Statement of problem. We deal with the equation of Nagai and Mimura

$$u_t = (u^m)_{xx} - \left[\left\{\int_{-\infty}^{\infty} K(x-y)\, u(y,t)\, dy\right\} u\right]_x, \tag{1}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$ and $1 < m < \infty$. We are looking for travelling wave solutions of (1) with compact support $u = \varphi(x - ct)$, where φ is a continuous function on $\mathbb{R}$ which is positive and twice continuously differentiable in the interior of some compact interval $(0, a)$ and vanishes outside of this interval. The kernel K has the form

$$K(x) = \begin{cases} K_1(x) & \text{as } x < 0, \\ K_2(x) & \text{as } x > 0, \end{cases} \tag{2}$$

where K_1, K_2 are continuously differentiable functions on $(-\infty, 0]$ and $[0, \infty)$, respectively, having nonpositive derivatives K_1', K_2' there and a positive limit

$$\Delta = \lim_{x\to -0} K_1(x) - \lim_{x\to +0} K_2(x) > 0. \tag{3}$$

In the applications as a rule there holds $K_1 > 0$, $K_2 < 0$ in $(-\infty, 0]$ and $[0, \infty)$, respectively, but this is not necessary to assume.

Substituting the ansatz $u = \varphi(x - ct)$ in (1) and taking the vanishing of φ at infinity into account (thereby assuming the continuity of $(\varphi^m)_x$ on $\mathbb{R}$), we obtain the equation for φ

$$-c\varphi(x - ct) = m\varphi^{m-1}\varphi'(x - ct) - \int_{-\infty}^{\infty} K(x-y)\, \varphi(y - ct)\, dy \cdot \varphi(x - ct).$$

Introducing the variables $\xi = x - ct$, $\eta = y - ct$, this equation writes

$$m\varphi^{m-1}(\xi)\, \varphi'(\xi) - \varphi(\xi)\left[\int_{-\infty}^{\infty} K(\xi - \eta)\, \varphi(\eta)\, d\eta - c\right] = 0,$$

which in virtue of $\varphi = 0$ on $(-\infty, 0) \cup (a, \infty)$ yields the *integro-differential equation* for φ

$$m\varphi^{m-2}(\xi)\, \varphi'(\xi) - \int_0^a K(\xi - \eta)\, \varphi(\eta)\, d\eta + c = 0, \qquad 0 < \xi < a,$$

with the boundary conditions $\varphi(0) = \varphi(a) = 0$. Here the length of the supporting interval a and the speed of the wave c act as additional unknown parameters.

We further introduce $\Phi = \varphi^{m-1}$ as an unknown function getting

$$\lambda\Phi'(\xi) - \int_0^a K(\xi - \eta)\, \Phi^p(\eta)\, d\eta + c = 0, \qquad 0 < \xi < a, \tag{4}$$

with the given parameters $p = 1/(m-1)$, $\lambda = m/(m-1) = 1 + p$ and the boundary conditions $\Phi(0) = \Phi(a) = 0$. Differentiating (4) leads to the *two-point boundary value problem*

$$\lambda\Phi''(\xi) + \Delta\Phi^p(\xi) - \int_0^a K'(\xi - \eta)\, \Phi^p(\eta)\, d\eta = 0, \qquad 0 < \xi < a, \tag{5}$$

with $\Phi(0) = \Phi(a) = 0$. This problem in turn is equivalent to the *integral equation*

$$\Phi(x) = \int_0^a G(x, \xi)\, \Phi^p(\xi)\, d\xi, \qquad 0 \leqq x \leqq a, \tag{6}$$

with the kernel

$$G(x, \xi) = \frac{\Delta}{\lambda} G_0(x, \xi) - \frac{1}{\lambda} \int_0^a G_0(x, \eta)\, K'(\eta - \xi)\, d\eta, \tag{7}$$

where

$$G_0(x, \xi) = \begin{cases} x(a - \xi)/a & \text{as } 0 \leqq x \leqq \xi \leqq a, \\ (a - x)\xi/a & \text{as } 0 \leqq \xi \leqq x \leqq a \end{cases}$$

is the well-known Green's function.

Our problem is now equivalent to seek continuous solutions of (6) in some finite interval $[0, a]$, which are positive in the open interval $(0, a)$. After knowing Φ from (5) or (6), the velocity constant c can be determined from (4), for instance as $\xi \to +0$

$$c = \int_0^a K_1(-\eta)\, \Phi^p(\eta)\, d\eta - \lambda \Phi'(0). \tag{8}$$

2. Particular case. At first we consider the special case of a *piecewise constant kernel*

$$K(x) = \begin{cases} \alpha & \text{as } x < 0, \\ \beta & \text{as } x > 0, \end{cases} \tag{9}$$

where $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha > \beta$. Then the two-point boundary value problem (5) writes

$$\lambda \Phi''(x) + \Delta \Phi^p(x) = 0, \qquad \Delta = \alpha - \beta, \tag{10}$$

with $\Phi(0) = \Phi(a) = 0$. The differential equation (10) has the first integral $\Phi'^2(x) + (2\Delta/\lambda^2)\, \Phi^\lambda(x) = C$ with a free constant C or

$$\Phi'(x) = \pm \sqrt{C - (2\Delta/\lambda^2)\, \Phi^\lambda(x)}. \tag{11}$$

Putting $A = 2\Delta/\lambda^2$, integration of (11) yields the implicit functions

$$x = \pm \int_0^{\Phi(x)} \frac{dy}{\sqrt{C - Ay^\lambda}} + C_0 \quad \text{or} \quad A^{1/2}B^{1-\lambda/2}x = \pm \int_0^{B\Phi(x)} \frac{ds}{\sqrt{1 - s^\lambda}} + C_1$$

with free constants C_0, C_1 as solutions, where we introduced the new free constant $B = (A/C)^{1/\lambda}$ instead of C. Finally, the boundary conditions $\Phi(0) = \Phi(a) = 0$ and the positiveness of Φ in $(0, a)$ require that

$$A^{1/2}B^{1-\lambda/2}x = \begin{cases} \int_0^{B\Phi(x)} (1 - s^\lambda)^{-1/2}\, ds & \text{if } 0 \leqq A^{1/2}B^{1-\lambda/2}x \leqq D_\lambda, \\ D_\lambda + \int_{B\Phi(x)}^1 (1 - s^\lambda)^{-1/2}\, ds & \text{if } D_\lambda \leqq A^{1/2}B^{1-\lambda/2}x \leqq 2D_\lambda, \end{cases}$$

where

$$D_\lambda = \int_0^1 (1 - s^\lambda)^{-1/2}\, ds = \frac{1}{\lambda} B\left(\frac{1}{\lambda}, \frac{1}{2}\right) = \sqrt{\pi}\, \Gamma\left(1 + \frac{1}{\lambda}\right) \Big/ \Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right).$$

Hence we have the relation for the parameter a

$$A^{1/2}B^{1-\lambda/2}a = 2D_\lambda \tag{12}$$

and the solution Φ is given by

$$(2D_\lambda/a)\, x = \begin{cases} \int\limits_0^{B\Phi(x)} (1 - s^\lambda)^{-1/2}\, ds & \text{if} \quad 0 \leqq x \leqq a/2, \\ D_\lambda + \int\limits_{B\Phi(x)}^1 (1 - s^\lambda)^{-1/2}\, ds & \text{if} \quad a/2 \leqq x \leqq a. \end{cases} \tag{13}$$

As a referee kindly points out, the right-hand side of (13) can be represented by means of the incomplete Beta function and this one in turn by the Gauss hypergeometric function.

The solutions Φ (and therefore φ) are symmetric functions with respect to $x = a/2$, further by (10) Φ (and for $m \geqq 2$ therefore also $\varphi = \Phi^p$) are concave functions, too.

Finally, by (8) and the corresponding relation (4) as $\xi \to a - 0$ we have $c = \alpha I - \lambda\Phi'(0) = \beta I - \lambda\Phi'(a)$, where $I = \int\limits_0^a \Phi^p\, dx = \int\limits_0^a \varphi\, dx$ is the invariant integral over u, and since $\Phi'(a) = -\Phi'(0)$ the simple formula for the speed c

$$c = \lambda\Delta^{-1}(\alpha + \beta)\, \Phi'(0) \tag{14}$$

follows. In particular, for $\beta = -\alpha$ the functions $\varphi = \Phi^p$ represent steady state (or equilibrium) solutions to equation (1). Also for the integral I there holds the relation $I = (2\lambda/\Delta)\, \Phi'(0)$ so that $c = (1/2)\, (\alpha + \beta)\, I$. (Cf. also [6, p. 198].)

There are two different situations. In case $m = 2$, i.e., $\lambda = 2$, $p = 1$, the differential equation (10), is *linear*, $a = A^{-1/2}\pi$ with $A = \Delta/2$ is uniquely determined by (12) and the (positive) constant B is arbitrary. $\varphi = \Phi$ is the positive eigenfunction of (10):

$$\varphi(x) = b \sin \sqrt{A}\, x = b \sin \sqrt{\Delta/2}\, x, \qquad 0 \leqq x \leqq \pi/\sqrt{\Delta/2}, \tag{15}$$

which is determined up to an arbitrary positive constant factor $b = 1/B$. The corresponding speed c is given by (14) as $c = b\sqrt{2/\Delta}\, [\alpha + \beta]$.

Remark: The case (9) with $\alpha = 1 + \theta$, $\beta = -1$, where θ is a nonnegative parameter, has been dealt with by MIMURA and SATSUMA [5] with the help of the auxiliary "potential" function $w(x, t) = \int\limits_{-\infty}^x u(y, t)\, dy$ (Mimura and Satsuma write the kernel (9) with $\alpha = 1$, $\beta = -(1 + \theta)$ but indeed consider the case $\alpha = 1 + \theta$, $\beta = -1$.) In this case $\Delta = 2 + \theta$, $c = b\theta/\sqrt{1 + \theta/2}$, $I = 2b/\sqrt{1 + \theta/2}$, i.e., $b = (1/2)\sqrt{1 + \theta/2}\, I$, and $c = (1/2)\, \theta I$.

In case $m \neq 2$, i.e. $\lambda \neq 2$, $p \neq 1$, the differential equation (10) is *nonlinear* and we get a solution for arbitrary positive a with uniquely determined constant $B = (2D_\lambda \times A^{-1/2}a^{-1})^{1/(1-\lambda/2)}$ from (12). I.e., for any $a > 0$ there exists a unique solution Φ and $\varphi = \Phi^p$. For instance, in case $m = 4/3$, i.e. $\lambda = 4$, $p = 3$, we have (cf. [10, p. 524]) $\Phi(x) = b \sin \operatorname{lemn}(A^{1/2}bx) = b \cos \operatorname{lemn}\left(K_0/\sqrt{2} - A^{1/2}bx\right) = b \operatorname{cn}\left(K_0 - \sqrt{2}\, A^{1/2}bx\right)$ with $A = \Delta/8$, $K_0 = \left(1/4\sqrt{\pi}\right) \Gamma^2(1/4)$, $b = 1/B = 4K_0a^{-1}/\sqrt{\Delta}$, where sin lemn, cos lemn are the lemniscate functions and cn is the Jacobian elliptic cosine function with modulus $k = 1/\sqrt{2}$. Therefore

$$\varphi(x) = (64K_0^3/\Delta^{3/2}a^3)\, \operatorname{cn}^3(K_0 - 2K_0x/a) \tag{16}$$

with the corresponding speed $c = 16\sqrt{2}\, (K_0^2/a^2)\, (\alpha + \beta)/\Delta^{3/2}$ following from (14) (cf. [10, p. 493]).

3. The basic integral equation. We substitute $x = at$, $\xi = as$ and $\Psi(t) = \Phi(at)$ in equation (6) and obtain the *basic integral equation*

$$\Psi(t) = \int_0^1 k(t, s; a)\, \Psi^p(s)\, ds, \qquad 0 \leqq t \leqq 1, \tag{17}$$

with the kernel $k(t, s; a) = aG(at, as)$, i.e., $k(t, s; a) = k_0(t, s; a) + k_1(t, s; a)$, where

$$k_0(t, s; a) = (a^2\Delta/\lambda)\, g_0(t, s), \tag{18}$$

$$k_1(t, s; a) = (-a^3/\lambda) \int_0^1 g_0(t, \sigma)\, K'\big(a(\sigma - s)\big)\, d\sigma \tag{19}$$

with the normalized Green's function

$$g_0(t, s) = \begin{cases} t(1 - s) & \text{as } 0 \leqq t \leqq s \leqq 1, \\ s(1 - t) & \text{as } 0 \leqq s \leqq t \leqq 1. \end{cases}$$

Like k_0 and k_1 the kernel k is a nonnegative continuous function on $[0, 1] \times [0, 1] \times [0, \infty)$ satisfying the conditions

$$k(0, s; a) = k(1, s; a) = 0 \qquad \text{and} \qquad k(t, s; 0) = 0. \tag{20}$$

Therefore, for any $a > 0$ the corresponding linear operator A defined by

$$Ay(t) = \int_0^1 k(t, s; a)\, y(s)\, ds$$

is a compact mapping in the Banach space $C[0, 1]$ of continuous functions, which leaves the cone $K[0, 1]$ of nonnegative functions from $C[0, 1]$ invariant and moreover maps any function $y \in C[0, 1]$ into a function $z = Ay \in C[0, 1]$ satisfying $z(0) = z(1) = 0$.

Further, for any $y \in K[0, 1]$, $y \not\equiv 0$, there exist positive numbers α and β such that the inequality

$$\alpha g(t) \leqq Ay(t) \leqq \beta g(t), \qquad t \in [0, 1], \tag{21}$$

with the function $g(t) = t(1 - t)$ holds. Namely, by the assumption $\Delta > 0$ and by Lemma 7.6 in [4, p. 302] this inequality is valid for the operator A_0 defined by

$$A_0 y(t) = \int_0^1 k_0(t, s; a)\, y(s)\, ds.$$

Since $Ay(t) \geqq A_0 y(t)$, $t \in [0, 1]$, the left-hand side of (21) follows. Besides

$$A_1 y(t) = \int_0^1 k_1(t, s; a)\, y(s)\, ds$$

$$\leqq M \left(\int_0^1 y(s)\, ds\right)\left(\int_0^1 g_0(t, \sigma)\, d\sigma\right) = \frac{M}{2}\left(\int_0^1 y(s)\, ds\right) g(t),$$

$$M = \frac{a^3}{\lambda} \sup\{|K'(x)| : x \in [-a, a]\},$$

so that also the right-hand side of (21) holds. The inequality (21) means that the operator A is *u_0-positive* with $u_0 = g$ in the sense of KRASNOSELSKII [4, Chap. 2, § 1].

Remark: The assumption of continuity of K_j', $j = 1, 2$, on $(-\infty, 0]$ and $[0, \infty)$, respectively, may be relaxed. What is really needed is that the (nonnegative) kernel k_1 leads to a compact operator in $C[0, 1]$ (cf. [11, Chap. 5, Th. 1.4]) and that the inequality (21) holds. For example, k_1 is even continuous in (t, s) and a if K_j', $j = 1, 2$, are continuous in $(-\infty, 0)$ and $(0, \infty)$, respectively, and satisfy inequalities of the type $|K_j'(x)| \leqq \text{const } |x|^\nu$, $\nu > -1$. And the inequality (21) may be fulfilled also if $\Delta = 0$, so that the kernel k_0 vanishes.

We consider a typical example:

$$K(x) = \begin{cases} c_1(-x)^\gamma & \text{as } x < 0, \\ -c_2 x^\delta & \text{as } x > 0, \end{cases}$$

where $c_1, c_2 \geqq 0$ with $c_1 + c_2 > 0$ and $0 < \gamma, \delta < 1$. The operator A_1 then writes

$$A_1 y(t) = \frac{a^3}{\lambda} \left[(1 - t) \int_0^t \sigma I_1(\sigma)\, d\sigma + t \int_t^1 (1 - \sigma)\, I_1(\sigma)\, d\sigma \right],$$

where $I_1(\sigma) = c_2 \delta a^{\delta - 1} \int_0^\sigma (\sigma - s)^{\delta - 1} y(s)\, ds + c_1 \gamma a^{\gamma - 1} \int_\sigma^1 (s - \sigma)^{\gamma - 1} y(s)\, ds$. As $\sigma \to 0$ and $\sigma \to 1$ there holds

$$I_1(\sigma) \sim \begin{cases} c_1 \gamma a^{\gamma - 1} \int_0^1 s^{\gamma - 1} y(s)\, ds =: C_1 & \text{as } \sigma \to 0, \\ c_2 \delta a^{\delta - 1} \int_0^1 (1 - s)^{\delta - 1} y(s)\, ds =: C_2 & \text{as } \sigma \to 1, \end{cases}$$

so that

$$A_1 y(t) \sim \begin{cases} \dfrac{a^3}{\lambda} \left[t \displaystyle\int_0^1 (1 - \sigma)\, I_1(\sigma)\, d\sigma + \frac{1}{2} C_1 t^2 \right] & \text{as } t \to +0, \\ \dfrac{a^3}{\lambda} \left[(1 - t) \displaystyle\int_0^1 \sigma I_1(\sigma)\, d\sigma + \frac{1}{2} C_2 (1 - t)^2 \right] & \text{as } t \to 1 - 0. \end{cases}$$

I.e., we have

$$A_1 y(t) \sim \begin{cases} D_1 t & \text{as } t \to +0, \\ D_2 (1 - t) & \text{as } t \to 1 - 0 \end{cases}$$

with the positive constants

$$D_1 = \frac{a^3}{\lambda} \int_0^1 (1 - s)\, I_1(s)\, ds, \quad D_2 = \frac{a^3}{\lambda} \int_0^1 s I_1(s)\, ds.$$

But this asymptotic behaviour together with $A_1 y(t) > 0$ in $(0, 1)$ implies an inequality of the form (21).

The function $F(\Psi) = \Psi^p$ fulfils the condition $F(0) = 0$ and is concave for $0 < p < 1$, i.e. $1 < m < 2$, linear for $p = 1$, i.e. $m = 2$, and convex for $p > 1$, i.e. $m > 2$.

Our aim is to prove the existence of a nontrivial solution $\Psi \in K[0, 1]$ to the equation (17) for any $a > 0$ if $p \neq 1$, i.e. $m \neq 2$, and for a certain $a > 0$ if $p = 1$, i.e. $m = 2$.

4. The case $m \neq 2$. At first we deal with the subcase $m > 2$, where $0 < p < 1$. In this case we can immediately employ a modification of a theorem of BUSHELL [1, Th. 2.2] saying that for a continuous non-negative kernel k if there exists a $g \in C[0, 1]$ with $g(0) = g(1) = 0$ and $g(t) > 0$ in $(0, 1)$ such that

$$\alpha g(t) \leqq \int_0^1 k(t, s)\, g^p(s)\, ds \leqq \beta g(t), \qquad t \in [0, 1], \tag{22}$$

with $0 < \alpha \leqq \beta < \infty$, the integral equation $y(t) = \int_0^1 k(t, s)\, y^p(s)\, ds$ has a unique continuous solution y with the property $\alpha_0 g(t) \leqq y(t) \leqq \beta_0 g(t)$, $t \in [0, 1]$, where $0 < \alpha_0 \leqq \beta_0 < \infty$. The proof of this theorem follows as for Theorem 2.2 in [1] using there the cone $K_g = \{y : \inf y(t)/g(t) \geqq 0,\ t \in (0, 1)\}$ of the Banach space $C_g = \{y \in C[0, 1] : \|y\|_g = \sup_t |y(t)|/g(t) < \infty\}$.

Since the assumption (22) is fulfilled by (21), we have

Theorem 1: *In the case $m > 2$ for any $a > 0$ the integral equation* (17) *has a unique continuous solution Ψ_0 with the property*

$$\alpha_0 t(1 - t) \leqq \Psi_0(t) \leqq \beta_0 t(1 - t), \qquad t \in [0, 1], \tag{23}$$

where $0 < \alpha_0 \leqq \beta_0 < \infty$.

Remark: The theory of Bushell uses Hilbert's projective metric and the Banach's contracting mapping theorem in the interior K_g^0 of the cone K_g so that it also yields the possibility of computing the solution Ψ_0 of (17) by successive approximations. The existence of a nontrivial solution $\Psi_0 \in K[0, 1]$ to (17) also follows from the general considerations of KRASNOSELSKII [4, Chap. 7, § 4, 6] who uses the cone of concave (= convex from above) functions y in $C[0, 1]$ satisfying the conditions $y(0) = y(1) = 0$. The assumption on the nonlinearity F there that it possesses a suitable sufficiently large linear minorant for small positive values of the variable is obviously fulfilled for the function $F(y) = y^p$, $0 < p < 1$. We further remark that the solutions Ψ of (17) must be concave functions as follows from the connection with equation (5). Also since the function $F(y) = y^p$, $0 < p < 1$, is concave, the operator T in (17), i.e.

$$Ty(t) = \int_0^1 k(t, s; a)\, y^p(s)\, ds, \qquad 0 < p < 1,$$

is u_0-*concave* with $u_0 = g$ in the sense of KRASNOSELSKII [4, Chap. 6, § 1, 3] for any $a > 0$. This implies the general uniqueness of the nontrivial nonnegative continuous solution Ψ_0 of the equation (17) (cp. [4, Chap. 6, § 1, Th. 6.3 and Chap. 7, § 4, p. 309]) and also again the possibility of computing Ψ_0 by successive approximations (cp. [4, Chap. 6, § 1, Th. 6.7]).

Corollary: *Theorem* 1 *also holds for the more general equation*

$$u_t = \left(uf(u)\, u_x\right)_x - \left[\left\{\int_{-\infty}^{\infty} K(x - y)\, u(y, t)\, dy\right\} u\right]_x,$$

where the function f is continuous on $[0, \infty)$ *and continuously differentiable on* $(0, \infty)$ *satisfying the conditions $f(0) = 0$, $f(u) > 0$, and $f'(u) > 0$ on* $(0, \infty)$. *The corresponding integral equation* (6) *writes*

$$\Phi(x) = \int_0^a G(x, \xi)\, H(\Phi)(\xi)\, d\xi, \qquad 0 \leqq x \leqq a,$$

where $\Phi = F(\varphi) = \int\limits_0^{\varphi} f(w)\,dw$ *and* H *is the inverse function to* F. *Without the assumption* $f'(u) > 0$ *on* $(0, \infty)$ *we have existence of a solution without uniqueness.*

In the other subcase $1 < m < 2$ we have $p > 1$ and the existence of a nontrivial nonnegative continuous solution Ψ_0 to (17) follows from the example in [4, Chap. 7, § 4,6]. Only the uniqueness of this solution does not result from the general theory in [4] (cf. the corresponding remark in [4, Chap. 7, § 4,4]) although uniqueness of the solution (eventually under additional assumptions on K) is conjectured in comparison with the above particular case of a piecewise constant kernel K.

Theorem 2: *In the case* $1 < m < 2$ *for any* $a > 0$ *the integral equation* (17) *has a nontrivial nonnegative continuous solution* Ψ_0.

Remark: Of course, by (21) the solution Ψ_0 again fulfils an inequality of the form (23).

5. The case $m = 2$. Here $p = 1$ and (17) is a homogeneous linear Fredholm integral equation of the second kind. We want to prove that there exists an $a > 0$ such that the *eigenvalue problem*

$$A\Psi_a(t) \equiv \int\limits_0^1 k(s, t; a)\,\Psi_a(s)\,ds = \mu(a)\,\Psi_a(t) \tag{24}$$

has the eigenvalue $\mu(a) = 1$ with a corresponding nonnegative continuous eigenfunction Ψ_a. Since the operator $A = A(a)$ is u_0-positive with $u_0 = g$ in the cone $K[0, 1]$ of $C[0, 1]$ for any $a > 0$ there exists a unique simple positive eigenvalue $\mu(a)$ of A with nonnegative eigenfunction Ψ_a satisfying (23) (cf. [4, Chap. 2] and [11, Chap. 4, § 1]). In the sequel we normalize this eigenfunction by the condition

$$\int\limits_0^1 \Psi_a(t)\,dt = 1. \tag{25}$$

Firstly, it can be shown that $\mu = \mu(a)$ is a continuous function on $(0, \infty)$.

Namely, let $a \in (0, \infty)$ be fixed and $a > a' > 0$ for definiteness. Then

$$\int\limits_0^1 k(t, s; a')\,\Psi_a(s)\,ds = \int\limits_0^1 [k(t, s; a') - k(t, s; a)]\,\Psi_a(s)\,ds + \int\limits_0^1 k(t, s; a)\,\Psi_a(s)\,ds$$
$$= K_0(t) + K_1(t) + \mu(a)\,\Psi_a(t),$$

where

$$K_0 = \int\limits_0^1 [k_0(t, s; a') - k_0(t, s; a)]\,\Psi_a(s)\,ds = (a'^2 - a^2)\,\frac{A}{\lambda}\int\limits_0^1 g_0(t, s)\,\Psi_a(s)\,ds$$
$$\geqq -C(a^2 - a'^2)\,t(1 - t), \qquad C > 0,$$

by (21) for the operator A_0 and

$$K_1 = \int\limits_0^1 [k_1(t, s, a') - k_1(t, s; a)]\,\Psi_a(s)\,ds = \frac{1}{\lambda}\left(\int\limits_0^1 g_0(t, \sigma)\,d\sigma\right)\varphi(a, a'; \sigma)$$
$$\geqq -\frac{1}{2\lambda}\,t(1 - t)\,\varphi_0(a, a')$$

with

$$\varphi(a, a'; \sigma) = a'^3 \int_0^1 K'(a'(\sigma - s)) \Psi_a(s)\, ds - a^3 \int_0^1 K'(a(\sigma - s)) \Psi_a(s)\, ds,$$

$$\varphi_0(a, a') = \max_{\sigma\in[0,1]} |\varphi(a, a'; \sigma)| \to 0 \quad \text{as} \quad a' \to a$$

so that on account of (23) we have $\int_0^1 k(t, s; a') \Psi_a(s)\, ds \geqq [\mu(a) - \omega_1(a, a')] \Psi_a(t)$, where $\omega_1(a, a')$ is some continuous function tending to zero as $a' \to a$. This by [4, Chap. 2, Th. 2.5] implies $\mu(a) - \omega_1(a, a') \leqq \mu(a')$. In the same way we obtain $\mu(a') - \omega_2(a, a') \leqq \mu(a)$ with an analogous function $\omega_2(a, a')$. This yields the continuity of μ on $(0, \infty)$.

Further, $\mu(0) := \lim_{a\to+0} \mu(a) = 0$ as follows by (25) from

$$\mu(a) = \int_0^1 \int_0^1 k(t, s; a) \Psi_a(s)\, ds\, dt \leqq \max_{s\in[0,1]} \int_0^1 k(t, s; a)\, dt,$$

which goes to zero as $a \to +0$ in virtue of $k(t, s; a) \to 0$ as $a \to +0$ uniformly in (t, s) (cf. (20)). Finally, there holds

$$\begin{aligned}\mu(a) \int_0^1 t(1 - t) \Psi_a(t)\, dt &\geqq \int_0^1 t(1 - t) \int_0^1 k_0(t, s; a) \Psi_a(s)\, ds\, dt \\ &= \frac{a^2 \Lambda}{\lambda} \int_0^1 \Psi_a(s)\, ds \int_0^1 t(1 - t)\, g_0(t, s)\, dt \\ &= \frac{a^2 \Lambda}{12\lambda} \int_0^1 s(1 - s)(1 + s - s^2) \Psi_a(s)\, ds \\ &\geqq \frac{a^2 \Lambda}{12\lambda} \int_0^1 s(1 - s) \Psi_a(s)\, ds,\end{aligned}$$

so that $\mu(a) \geqq (a^2\Lambda)/(12\lambda)$ and $\mu(a) \to +\infty$ as $a \to +\infty$. Therefore, there exists an $a > 0$ with $\mu(a) = 1$ and we have

Theorem 3: *In the case $m = 2$ there exists an $a > 0$ such that the integral equation* (17) *possesses a unique nontrivial nonnegative continuous solution Ψ_a normalized by* (25) *and again fulfilling an inequality of the form* (23).

Corollary 1: *If like k_0 the kernel $k(t, s; a)$ is an increasing function with respect to a, the function $\mu(a)$ is an increasing function of a in* $(0, \infty)$, *too* (cf. [4, Chap. 2, § 5,5]). *In this case the value of a with $\mu(a) = 1$ and hence the solution Ψ_a in Theorem 3 with* (25) *is unique at all. The monotonicity of k with respect to a is e. g. fulfilled if the sectionally kernels K_j, $j = 1, 2$, of K have continuous second derivatives in $(-\infty, 0)$ and $(0, \infty)$, respectively, satisfying besides $K_j'(x) \leqq 0$ additionally the inequalities $3K_j'(x) + xK_j''(x) \leqq 0$ there. This holds for instance if $K_1''(x) \geqq 0$ in $(-\infty, 0)$ and $K_2''(x) \leqq 0$ in $(0, \infty)$; i.e., if K_1 is a convex and K_2 a concave function.*

Corollary 2: *If instead of* (1) *for $m = 2$ we have the more general equation*

$$u_t = (u^2)_{xx} + \left[\nu u^2 - \left\{\int_{-\infty}^{\infty} K(x - y)\, u(y, t)\, dy\right\} u\right]_x, \qquad \nu \in \mathbb{R},$$

we obtain in place of (5) *for* $\Phi = \varphi$ *the equation*

$$2\varphi''(\xi) + 2\nu\varphi'(\xi) + \Delta\varphi(\xi) - \int_0^a K'(\xi - \eta)\,\varphi(\eta)\,d\eta = 0.$$

This equation can be transformed to a corresponding equation as (5) *by the well-known substitution* $\varphi(x) = \psi(x)\exp[-(\nu/4)\,x]$ *leading to*

$$2\psi''(\xi) + (\Delta - \nu^2/8)\,\psi(\xi) - \int_0^a K'(\xi - \eta)\,e^{(\nu/4)(\xi-\eta)}\,\psi(\eta)\,d\eta = 0$$

together with $\psi(0) = \psi(a) = 0$. *This equation can be dealt with in an analogous way as* (5) *if the assumption* $\Delta > 0$ *is sharpened to* $\Delta > \nu^2/8$.

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VERFASSER:

Prof. Dr. LOTHAR v. WOLFERSDORF
Sektion Mathematik der Bergakademie Freiberg
Bernhard-v.-Cotta-Str. 2
DDR-9200 Freiberg